MIT 18.06SC Problem Set 5

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Problem 5.2

Problem 5.2: Suppose A is a four by four matrix. How many entries of A can be chosen independently if:

a) A is symmetric?

b) A is *skew-symmetric*? ($A^T = -A$)

a. Symmetric 4×4 Matrices

A symmetric matrix satisfies $A^T = A$. This means that each entry below the main diagonal is determined by the corresponding entry above it.

The elements we can easily choose are:

- the entries on the principal diagonal, and
- the entries on any one side of the principal diagonal (usually the upper triangular part).

For a 4×4 matrix:

$$\text{Diagonal entries: } 4, \qquad \text{Upper triangular off-diagonal entries: } \frac{4(4-1)}{2} = 6.$$

Thus the total number of independent entries is

$$\mathbf{4 + 6 = 10.}$$

$$\Rightarrow \text{ The space of symmetric } 4 \times 4 \text{ matrices is 10-dimensional.}$$

b. Skew-Symmetric 4×4 Matrices

A skew-symmetric matrix satisfies $A^T = -A$. This implies two important conditions:

$$a_{ij} = -a_{ji}, \qquad \text{and} \qquad a_{ii} = 0.$$

For skew-symmetric matrices, our choice remains the same as for symmetric matrices **except** that:

- the entries on the principal diagonal are fixed (all zeros),
- the independent entries lie entirely in the upper triangular part.

Thus:

$$\text{Independent entries} = \frac{4(4-1)}{2} = 6.$$

6 independent entries.

$\Rightarrow$ The space of skew-symmetric 4×4 matrices is 6-dimensional.

Visual Representation:

For Symmetric:-

$$\begin{bmatrix} \circ & \circ & \circ & \circ \\ & \circ & \circ & \circ \\ & & \circ & \circ \\ & & & \circ \end{bmatrix}$$

For Skew-Symmetric:-

$$\begin{bmatrix} & \circ & \circ & \circ \\ & & \circ & \circ \\ & & & \circ \\ & & & \end{bmatrix}$$

So for an $n \times n$ Matrix,

For Symmetric:-

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

For Skew-Symmetric:-

$$1 + 2 + 3 + (n-1) = \frac{n(n-1)}{2}$$

Problem 5.3

Problem 5.3: (3.1 #18.) True or false (check addition or give a counterexample):

a) The symmetric matrices in M (with $A^T = A$) form a subspace.

b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

c) The unsymmetric matrices in M (with $A^T \neq A$) form a subspace.

In this solution we will consider a, b, c, d, e, ... , n represent normal numbers.

For a Matrix to be a sub-space, it must satisfy: Scaling and adding within that space must be closed within that sub-space.

a. Symmetric Matrix.
Let us consider two general Symmetric Matrices as:

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \text{ and } B = \begin{bmatrix} g & h & i \\ h & j & k \\ i & k & l \end{bmatrix}$$

If we take, $mA + nB = C$, we get,

$$C = mA + nB = m \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} + n \begin{bmatrix} g & h & i \\ h & j & k \\ i & k & l \end{bmatrix} = \begin{bmatrix} ma+ng & mb+nh & mc+ni \\ mb+nh & md+nj & me+nk \\ mc+ni & me+nk & mf+nl \end{bmatrix}.$$

Check that $C^T = C$. So C is Symmetric. So, The Symmetric Matrices forms a sub-space.

True

b. Skew-Symmetric Matrix.
Let us consider two general Symmetric Matrices as:

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{bmatrix}$$

If we take, $mA + nB = C$, we get,

$$C = mA + nB = m \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + n \begin{bmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{bmatrix} = \begin{bmatrix} 0 & ma+nd & mb+ne \\ -(ma+nd) & 0 & mc+nf \\ -(mb+ne) & -(mc+nf) & 0 \end{bmatrix}.$$

Check that $C^T = -C$. So C is Skew-Symmetric. So, The Skew-Symmetric Matrices forms a sub-space.

True

c.

Let us consider two Random Un-symmetric Matrices as:

$$A = \begin{bmatrix} g & \frac{a-b}{m} & \frac{c-d}{m} \\ \frac{a+b}{m} & h & \frac{e-f}{m} \\ \frac{c+d}{m} & \frac{e+f}{m} & i \end{bmatrix} \text{ and } B = \begin{bmatrix} j & \frac{a+b}{n} & \frac{c+d}{n} \\ \frac{a-b}{n} & k & \frac{e+f}{n} \\ \frac{c-d}{n} & \frac{e-f}{n} & l \end{bmatrix}$$

If we take, $mA + nB = C$, we get,

$$C = mA + nB = m \begin{bmatrix} g & \frac{a-b}{m} & \frac{c-d}{m} \\ \frac{a+b}{m} & h & \frac{e-f}{m} \\ \frac{c+d}{m} & \frac{e+f}{m} & i \end{bmatrix} + n \begin{bmatrix} j & \frac{a+b}{n} & \frac{c+d}{n} \\ \frac{a-b}{n} & k & \frac{e+f}{n} \\ \frac{c-d}{n} & \frac{e-f}{n} & l \end{bmatrix}$$

$$= \begin{bmatrix} mg + nj & (a-b) + (a+b) & (c-d) + (c+d) \\ (a+b) + (a-b) & mh + nk & (e-f) + (e+f) \\ (c+d) + (c-d) & (e+f) + (e-f) & mi + nl \end{bmatrix}.$$

$$C = \begin{bmatrix} mg + nj & 2a & 2c \\ 2a & mh + nk & 2e \\ 2c & 2e & mi + nl \end{bmatrix}.$$

Check that $C^T = C$. So C is Symmetric. So, The Un-Symmetric Matrices does not forms a sub-space.

False