MIT 18.06SC Problem Set 7

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Problem 7.1

Problem 7.1:

a) Find the row reduced form of:

$$A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix}$$

b) What is the rank of this matrix?

c) Find any special solutions to the equation $A\mathbf{x} = \mathbf{0}$.

We are given

$$A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix}.$$

a.

To get, Row reduced form of A
We perform Gaussian elimination.
First eliminate the entry in row 3, column 1 using row 1:

$$R_3 \leftarrow R_3 - 2R_1 : \quad \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & -12 & -3 & -21 \end{bmatrix}.$$

Next eliminate the entry in row 3, column 2 using row 2:

$$R_3 \leftarrow R_3 + 3R_2 : \quad \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & -12 & -3 & -21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is row echelon form. To reach reduced row echelon form, first scale row 2:

$$R_2 \leftarrow \frac{1}{4}R_2 : \quad \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then clear the entry above the pivot in column 2:

$$R_1 \leftarrow R_1 - 5R_2 : \quad \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{23}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the row reduced form (RREF) of A is

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{23}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

b.

Rank of A
The rank is the number of pivot columns in the row reduced form of A.
From

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{23}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we see pivots in columns 1 and 2 only. Therefore

$$\mathrm{rank}(A) = 2.$$

c.

Special solutions to $Ax = 0$
We solve the homogeneous system

$$Ax = 0, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Using the RREF of A, we have

$$\begin{bmatrix} 1 & 0 & \frac{23}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the equations

$$x_1 + \frac{23}{4}x_3 + \frac{1}{4}x_4 = 0, \qquad x_2 + \frac{1}{4}x_3 + \frac{7}{4}x_4 = 0.$$

Let x_3 and x_4 be free variables:

$$x_3 = s, \qquad x_4 = t.$$

Then

$$x_1 = -\frac{23}{4}s - \frac{1}{4}t, \qquad x_2 = -\frac{1}{4}s - \frac{7}{4}t.$$

So the general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -\frac{23}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

We can choose the **special solutions** by turning on one free variable at a time:

- Take $s = 1$, $t = 0$:

$$x^{(1)} = \begin{bmatrix} -\frac{23}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix}.$$

- Take $s = 0$, $t = 1$:

$$x^{(2)} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix}.$$

If we prefer integer entries, we can multiply each special solution by 4:

$$x_s^{(1)} = \begin{bmatrix} -23 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \qquad x_s^{(2)} = \begin{bmatrix} -1 \\ -7 \\ 0 \\ 4 \end{bmatrix}.$$

These two vectors span the nullspace of A and are the special solutions to $Ax = 0$.

Problem 7.2

Problem 7.2: (3.3 #17.b *Introduction to Linear Algebra:* Strang) Find A_1 and A_2 so that rank(A_1B) = 1 and rank(A_2B) = 0 for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Let B be a given matrix. We are asked to find matrices A_1 and A_2 such that:

$$\text{rank}(A_1B) = 1, \qquad \text{rank}(A_2B) = 0.$$

Recall that left multiplication by a matrix performs linear combinations of the rows of B. Therefore, the rank of AB depends on how the rows of A combine the rows of B.

Case 1: Constructing A_2 such that $\text{rank}(A_2B) = 0$

For A_2B to have rank zero, we must have

$$A_2B = 0.$$

This happens if every row of A_2 lies in the left nullspace of B, i.e.

$$\text{each row } r \text{ of } A_2 \text{ satisfies } rB = 0.$$

The simplest choice is the zero matrix:

$$A_2 = 0.$$

Then:

$$A_2B = 0 \quad \Rightarrow \quad \text{rank}(A_2B) = 0.$$

Thus such a matrix A_2 exists.

Case 2: Constructing A_1 such that $\text{rank}(A_1B) = 1$

To obtain rank 1, all rows of A_1B must be proportional to a single nonzero row vector.

This can be achieved by choosing A_1 so that all rows of A_1 are the same nonzero row vector.

For example, let

$$A_1 = \begin{bmatrix} r \\ r \\ \vdots \\ r \end{bmatrix}, \quad r \neq 0.$$

Then

$$A_1B = \begin{bmatrix} rB \\ rB \\ \vdots \\ rB \end{bmatrix}.$$

All rows of A_1B are scalar multiples of the same vector rB, so the row space is one-dimensional:

$$\operatorname{rank}(A_1B) = 1,$$

provided $rB \neq 0$.

Thus such a matrix A_1 exists.

Conclusion

We have shown that:

$$\exists\, A_1 \text{ such that } \operatorname{rank}(A_1B) = 1, \qquad \exists\, A_2 \text{ such that } \operatorname{rank}(A_2B) = 0.$$

Both constructions are possible.