MIT 18.06SC Problem Set 4

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28th November 2025

Problem 4.2

Problem 4.2: (2.6 #13. *Introduction to Linear Algebra:* Strang) Compute L and U for the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

Problem 4.2

We are given

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

We perform elimination to factor A into LU.

Step 1. Apply $R_4' \to R_4 - R_3$:

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ 0 & 0 & 0 & d-c \end{bmatrix}.$$

Step 2. Apply $R_3' \to R_3 - R_2$:

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ 0 & 0 & 0 & d-c \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}.$$

Step 3. Apply $R_2' \to R_2 - R_1$:

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}.$$

Thus, after elimination we obtain the upper triangular matrix

$$U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}.$$

Each of our row operations corresponds to left multiplication by an elementary elimination matrix. If we denote by E the product of these three elimination matrices, then

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$E_1 A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ 0 & 0 & 0 & d-c \end{bmatrix}, \quad E_2 E_1 A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix},$$

and

$$E_3 E_2 E_1 A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = U.$$

Thus the overall elimination matrix is

$$E = E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

$$EA = U.$$

Therefore,

$$A = E^{-1} U,$$

so $L = E^{-1}$ is the lower triangular factor.

Carrying out this multiplication (or observing the pattern of the row operations), we obtain

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Hence,

$$A = LU$$

with

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}.$$

For $A = LU$ to have four pivots, all diagonal entries of U must be nonzero:

$$a \neq 0, \quad b-a \neq 0, \quad c-b \neq 0, \quad d-c \neq 0.$$