8.012 Problem Set 1

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Problem 4

(Kleppner & Kolenkow, Problem 1.13)

1.13 An elevator ascends from the ground with uniform speed. At time T_1 a boy drops a marble through the floor. The marble falls with uniform acceleration $g = 9.8\ \mathrm{m/s^2}$, and hits the ground T_2 seconds later. Find the height of the elevator at time T_1.

Ans. clue. If $T_1 = T_2 = 4$ s, $h = 39.2$ m

When the marble is dropped from the elevator, it inherits the velocity of the elevator due to the property of Inertia.
Let velocity of the elevator be v.
The length traveled by the elevator in time T_1 is equal to the length traveled by the ball. Let that height be h
Considering the motion along downward direction as positive, using the Equations of Motion this gives us, $vT_1 = -vT_2 + \frac{1}{2}gT_2^2 => v = \frac{gT_2^2}{2(T_1+T_2)}$
Now $h = vT_1 = \frac{gT_1T_2^2}{2(T_1+T_2)} = \frac{9.8T_1T_2^2}{2(T_1+T_2)}$.
Checking with the given hint: $h = \frac{9.8\times4\times4^2}{2\times(4+4)} = 39.2$ (exact match). So our answer is correct.

Hence, $h = \frac{9.8T_1T_2^2}{2(T_1+T_2)}$

Problem 8

(Kleppner & Kolenkow, Problem 1.20)

1.20 A particle moves outward along a spiral. Its trajectory is given by $r = A\theta$, where A is a constant. $A = (1/\pi)$ m/rad. θ increases in time according to $\theta = \alpha t^2/2$, where α is a constant.

a. Sketch the motion, and indicate the approximate velocity and acceleration at a few points.

b. Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.

c. At what angles do the radial and tangential accelerations have equal magnitude?

b.

$$\theta = \frac{\alpha}{2}t^2$$

$$r = A\theta = \frac{\alpha}{2\pi}t^2$$

So, the equation of motion is given by

$$\overrightarrow{x} = \frac{\alpha}{2\pi}t^2\hat{r}$$

where $\overrightarrow{x}$ denotes the position of the particle.
Hence, acceleration,

$$\ddot{\overrightarrow{x}} = \left(\frac{d^2(\frac{\alpha}{2\pi}t^2)}{dt^2} - \frac{\alpha}{2\pi}t^2\left(\frac{d(\frac{\alpha}{2}t^2)}{dt}\right)^2\right)\hat{r} + \left(\frac{\alpha}{2\pi}t^2\frac{d^2(\frac{\alpha}{2}t^2)}{dt^2} + 2\frac{d(\frac{\alpha}{2\pi}t^2)}{dt}\frac{d(\frac{\alpha}{2}t^2)}{dt}\right)\hat{\theta}$$

$$=> \ddot{\vec{x}} = \left(\frac{\alpha}{\pi} - \frac{\alpha^3}{2\pi}t^4\right)\hat{r} + \left(\frac{5\alpha^2}{2\pi}t^2\right)\hat{\theta}.$$

To find when the radial acceleration vanishes, we set

$$a_r = \frac{\alpha}{\pi} - \frac{\alpha^3}{2\pi}t^4 = 0.$$

Solving,

$$\frac{\alpha}{\pi} = \frac{\alpha^3}{2\pi}t^4 \implies t^4 = \frac{2}{\alpha^2} \implies t = \left(\frac{2}{\alpha^2}\right)^{1/4}.$$

Plugging this value of t into

$$\theta(t) = \frac{\alpha}{2}t^2,$$

we obtain

$$\theta = \frac{\alpha}{2}\left(\frac{2}{\alpha^2}\right)^{1/2} = \frac{1}{\sqrt{2}}$$

c.

Equating the radial and tangential accelerations,

$$\frac{\alpha}{\pi} - \frac{\alpha^3}{2\pi}t^4 = \frac{5\alpha^2}{2\pi}t^2,$$

which simplifies to the biquadratic

$$\alpha^2 t^4 + 5\alpha t^2 - 2 = 0.$$

Letting $u = t^2$, we obtain the quadratic

$$\alpha^2 u^2 + 5\alpha u - 2 = 0.$$

Using Sridharacharya's formula,

$$u = \frac{-5\alpha \pm \sqrt{(5\alpha)^2 - 4\alpha^2(-2)}}{2\alpha^2} = \frac{-5\alpha \pm \sqrt{25\alpha^2 + 8\alpha^2}}{2\alpha^2} = \frac{-5\alpha \pm \sqrt{33}\,\alpha}{2\alpha^2} = \frac{-5 \pm \sqrt{33}}{2\alpha}.$$

Hence,

$$t^2 = \frac{-5 \pm \sqrt{33}}{2\alpha}.$$

Putting this value of t^2 into

$$\theta(t) = \frac{\alpha}{2}t^2,$$

we obtain

$$\theta = \frac{\alpha}{2}\left(\frac{-5 \pm \sqrt{33}}{2\alpha}\right) = \frac{-5 \pm \sqrt{33}}{4}.$$

Problem 9

9. **Two Trains and a Bee.** (15 points)

Consider two trains moving in opposite directions on the same track. The trains start simultaneously from two towns, Aville and Bville, separated by a distance d. Each train travels toward each other with constant speed v. A bee is initially located in front of the train in Aville. As the train departs Aville, the bee travels with speed $u > v$ along the track towards Bville. When it encounters the second train, it instantaneously reverses direction until it encounters the first train, then it reverses again, etc. The bee continues flying between the two trains until it is crushed between the trains impacting each other. The purpose of this problem is to compute the total distance flown by the bee until it is crushed. Assume that the bee is faster than the trains.

There are at least two good ways to solve this problem. One is to compute the distance for each flight leg and sum the resulting series. There is also another way to solve the problem with very little calculation. You are to do it both ways.

(a) (10 points) Find an expression for the distance d_n covered by the bee after its n^{th} encounter with a train. Define d_0 as the distance traveled during the first flight from Aville towards the train near Bville, d_1 the distance traveled by the bee during the first trip from the Bville train to the Aville train, etc. Sum the resulting series to get the final answer.

(b) (5 points) Devise another way to obtain the same answer using very little calculation.

a.
We will consider all distances and speeds in the ground frame.
The bee flies with speed u and the opposite train approaches the bee with speed v, so the closing speed is $u + v$. Let d_0 be the distance traveled by the bee during its first flight.
In the same time, the trains reduce their separation from d to $d - d_0$.
Let this time be t_0.
This gives

$$t_0 = \frac{d_0}{u} = \frac{d - d_0}{2v}.$$

Solving,

$$d_0 = \frac{u}{u + v} d.$$

Now let d_1 be the distance covered by the bee in the second flight. Again the closing speed is $u + v$, so

$$t_1 = \frac{d_1}{u} = \frac{d_0 - d_1}{2v}.$$

Solving,

$$d_1 = \frac{u - v}{u + v} d_0.$$

Observing the pattern, we obtain

$$d_n = \left(\frac{u - v}{u + v}\right) d_{n-1}$$

for $n \geq 1$, and $d_0 = \frac{u}{u+v} d$.

The total distance covered by the bee is

$$D = \sum_{n=0}^{\infty} d_n = \sum_{n=0}^{\infty} \left(\frac{u-v}{u+v} \right)^n d_0 = \sum_{n=0}^{\infty} \left(\frac{u-v}{u+v} \right)^n \frac{u}{u+v} d.$$

Since $\left| \frac{u-v}{u+v} \right| < 1$, we get

$$D = \frac{\frac{u}{u+v} d}{1 - \frac{u-v}{u+v}} = \frac{ud}{2v}.$$

b.

A very simple approach could be, to simply find the total time of flight i.e. $t = \frac{d}{2v}$ and then multiplying with the speed of the bee u.
So,

$$D = tu = \frac{ud}{2v}$$

Problem 10

10. **Dimensional Analysis.** (15 points)

You can often derive the solution of a problem by considering the dimensions of the relevant variables, including all related fundamental constants, and match the dimensions of the quantity you want to determine. This is called *dimensional analysis*, and it is a powerful approximation technique as well as a method of determining how quantities scale with different variables. The basic idea is to write the unknown quantity (X) as a factor of all your relevant variables (V_n) and constants (C_n):

$$X = V_1^a V_2^b V_3^c C_1^d C_2^e ... \tag{1}$$

and then solve for the powers $a, b, c, d, e,$ ... so that the dimensions on both sides of the equation work out.

(a) (3 points) Derive an expression for the vibration frequency of a star of mass M and radius R, if that vibration is caused by gravitational instabilities.

(b) (4 points) Derive an expression for the drag force on a ball of radius R and mass M moving with velocity v through a medium with mass density ρ.

(c) (4 points) Derive an expression for the terminal velocity of a falling ball of radius R and mass M close to the surface of the Earth, when it experiences a drag force of the form $F = bv^2$. Can you find an alternate way of deriving this velocity?

(d) (4 points) Derive an expression for the frequency of a pendulum of mass M, hanging from a rope of length L near the surface of the Earth, released from rest at an initial angle θ_0. Warning! θ_0 is dimensionless. Is it possible to constrain how the frequency depends on this variable?

a.

By dimensional analysis, the only combination of G, M, and R with dimensions of frequency is

$$f \sim \sqrt{\frac{GM}{R^3}}.$$

b.
Using $[\rho] = ML^{-3}$, $[v] = LT^{-1}$, $[R] = L$, dimensional analysis gives

$$F \sim \rho R^2 v^2.$$

c.
At terminal velocity, the drag force balances weight:

$$b v_T^2 \sim Mg \quad \Longrightarrow \quad v_T \sim \sqrt{\frac{Mg}{b}}.$$

d.
Since θ_0 is dimensionless and M does not affect dimensional scaling, the only length and time–determining quantities are g and L. Hence

$$f \sim \sqrt{\frac{g}{L}}\, F(\theta_0),$$

where $F(\theta_0)$ is an undetermined dimensionless function.