8.022 Problem Set 3

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December 2025

Problem 6

6. Forces on conductors.

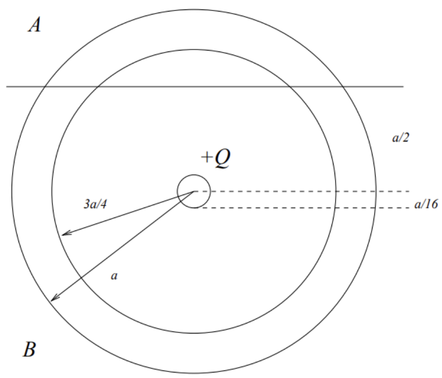

Figure 1: concentric conducting spheres.

A conducting spherical shell of outer radius a and inner radius $\frac{3a}{4}$ is cut in two pieces via a horizontal plane a distance $\frac{a}{2}$ above the center of the spherical shell as shown in figure 1. Let us label “A” the upper part of the shell and “B” the lower part of the shell. The shell is initially uncharged and the two pieces that result from the cutting procedure remain in perfect electrical contact. A new conducting sphere of radius $\frac{a}{16}$ and total charge $+Q$ is inserted in the shell and it is centered on the shell's center as shown in the same figure.

(a) Are there any charge densities on the inner ($r = \frac{3a}{4}$) and outer ($r = a$) surfaces of the shell as well as within it? If yes, derive them.
(b) What (if any) is the force per unit area on the inner and outer surfaces of the shell?
(c) From now on we focus only on the “A” part of the shell: Set up an integral that will yield the net force acting on the outer shell of the “A” part. What is its direction? Identify over which variables you are integrating and what are the limits of integration. (You are not asked to perform the integration!)
(d) Do the same as (c) for the inner shell of the “A” part of the shell.
(e) Do the same as (c) for the “A” part as a whole.

a.

Consider a Spherical Surface within the Metal with Radius r_1 as

$$\frac{3a}{4} < r_1 < a$$

Now Electric Field inside a Conductor is always Zero. So the Flux through this Spherical Surface is also Zero. So by Gauss Law, the net Charge enclosed inside this Sphere should be Zero. This implies that Charge on the Inner Surface of the Shell is Equal to the Negative of net Charge Inside.

Hence

$$\textbf{Surface Charge Density on the Inner Surface} = \frac{-Q}{4\pi(3a/4)^2} = -\frac{4Q}{9a^2\pi}$$

Now since the Conductor is Electrically Neutral,

$$Q_{OuterSurface} = -Q_{InnerSurface} = +Q$$

Hence

$$\textbf{Surface Charge Density on the Outer Surface} = \frac{+Q}{4\pi(a)^2} = +\frac{Q}{4a^2\pi}$$

b.

Force per unit are is given by

$$P = \frac{\sigma^2}{2\epsilon_0} \text{ here } \sigma \text{ is the Surface Charge Density}$$

Now,

$$\sigma^2_{InnerSurface} = \frac{16Q^2}{81a^4\pi^2} \text{ and } \sigma^2_{OuterSurface} = \frac{Q^2}{16a^4\pi^2}$$

Therefore

$$P_{InnerSurface} = \frac{8Q^2}{81a^4\pi^2\epsilon_0} \text{ and } P_{OuterSurface} = \frac{Q^2}{32a^4\pi^2\epsilon_0}$$

c.

The force on a conducting surface arises from the electrostatic pressure due to the electric field just outside the conductor. The pressure is given by

$$P = \frac{\sigma^2}{2\varepsilon_0},$$

where σ is the surface charge density on the outer surface.

The force element acting on a surface element dA is

$$d\vec{F} = P\,\hat{n}\,dA,$$

where $\hat{n}$ is the outward unit normal to the surface. We are interested in the net force in the vertical ($\hat{z}$) direction.

On a spherical surface, $\hat{n} \cdot \hat{z} = \cos\theta$, and the surface element is

$$dA = a^2 \sin\theta\, d\theta\, d\phi.$$

The outer surface corresponds to a sphere of radius a. The horizontal cut is located at $z = a/2$, which implies

$$a\cos\theta = \frac{a}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

Thus, the net force on the outer surface of part "A" is

$$F_z = \int_0^{2\pi} \int_0^{\pi/3} \frac{\sigma^2}{2\varepsilon_0} \cos\theta\, a^2 \sin\theta\, d\theta\, d\phi.$$

The force is directed upward along the $+\hat{z}$ direction.

d.

The force on a conducting surface is due to the electrostatic pressure arising from the electric field just outside the conductor. The pressure is given by

$$P = \frac{\sigma^2}{2\varepsilon_0},$$

where σ is the surface charge density on the inner surface.

The force element acting on a surface element dA is

$$d\vec{F} = P\,\hat{n}\,dA,$$

where $\hat{n}$ is the outward normal to the conducting material. For the inner surface of the shell, the outward normal points *radially inward*.

We are interested in the vertical ($\hat{z}$) component of the force. On a spherical surface,

$$\hat{n} \cdot \hat{z} = -\cos\theta,$$

and the surface element is

$$dA = \left(\frac{3a}{4}\right)^2 \sin\theta\, d\theta\, d\phi.$$

The inner surface corresponds to a sphere of radius $3a/4$. The horizontal cut is located at $z = a/2$, which implies

$$\frac{3a}{4}\cos\theta = \frac{a}{2} \quad \Rightarrow \quad \cos\theta = \frac{2}{3} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{3}\right).$$

Thus, the net force on the inner surface of part "A" is

$$F_z = \int_0^{2\pi}\int_0^{\cos^{-1}(2/3)} \frac{\sigma^2}{2\varepsilon_0}(-\cos\theta)\left(\frac{3a}{4}\right)^2 \sin\theta\, d\theta\, d\phi.$$

The force is directed downward along the $-\hat{z}$ direction.

e.

The total force acting on part "A" is obtained by summing the forces acting on its outer surface (from part (c)) and its inner surface (from part (d)). Since the conductor is in electrostatic equilibrium, the force on each surface arises from the electrostatic pressure acting normal to that surface.

The differential force on a surface element is

$$d\vec{F} = \frac{\sigma^2}{2\varepsilon_0}\,\hat{n}\, dA.$$

For the **outer surface**, the outward normal is radially outward, while for the **inner surface**, the outward normal to the conductor is radially inward. In both cases, only the vertical ($\hat{z}$) component contributes to the net force.

Therefore, the total force on part "A" is

$$\vec{F}_A = \vec{F}_{A,\textbf{outer}} + \vec{F}_{A,\textbf{inner}},$$

where

$$\vec{F}_{A,\textbf{outer}} = \int_0^{2\pi}\int_0^{\cos^{-1}(1/2)} \frac{\sigma^2_{\textbf{outer}}}{2\varepsilon_0}(\cos\theta)\, a^2 \sin\theta\, d\theta\, d\phi\,\hat{z},$$

and

$$\vec{F}_{A,\textbf{inner}} = \int_0^{2\pi}\int_0^{\cos^{-1}(2/3)} \frac{\sigma^2_{\textbf{inner}}}{2\varepsilon_0}(-\cos\theta)\left(\frac{3a}{4}\right)^2 \sin\theta\, d\theta\, d\phi\,\hat{z}.$$

The net force on the "A" part of the shell is directed along the vertical axis and points in the $-\hat{z}$ direction.

Problem 8

8. Energy of a conductor in a capacitor.

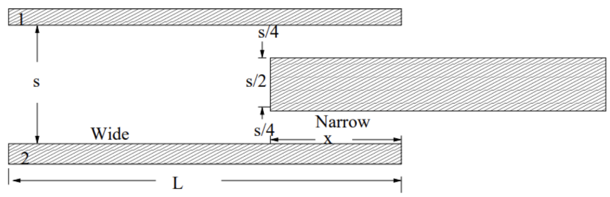

Figure 3: a capacitor of parallel square plates with a conducting square plate inserted.

Two flat square metal plates with sides of length L are arranged parallel to each other with a separation s; $s << L$. A total charge Q is on the bottom plate, $-Q$ on the top. Now, a third uncharged conducting plate of the same size with thickness $s/2$ is slipped between the original two plates. It is such placed that its distances to the top plate and the bottom plate are both $s/4$.
(a) When only a length x of the conducting plate is inside the metal plate, what are the surface charge densities σ_w and σ_n on the lower plate adjacent to the wide and narrow gaps? What is the electric field in the wide and narrow gaps? (Hint: think about $\oint \vec{E} \cdot d\vec{s}$ on certain paths.)
(b) What is the capacitance of this system?
(c) How much energy is stored in the electric field?
(d) What force must be exerted to keep the middle plate from moving?

a.

Since the inserted plate is conducting, the electric field inside it vanishes and the conductor is an equipotential.

We consider a closed rectangular loop that passes vertically through the wide gap, horizontally through the conductor, vertically through the narrow region (above and below the conductor), and closes through the lower plate.

Because electrostatics is conservative,

$$\oint \vec{E} \cdot d\vec{s} = 0.$$

Only the vertical segments contribute to the line integral. In the wide region, the field extends across a distance s, while in the narrow region the field crosses two gaps of thickness $s/4$ each, giving a total distance of $s/2$.

Therefore,

$$E_w\, s = E_n\, \frac{s}{2},$$

which gives

$$E_n = 2E_w.$$

Near a conducting surface, the normal electric field is related to the surface charge density by

$$E_{\perp} = \frac{\sigma}{\epsilon_0}.$$

Hence,

$$\sigma_n = 2\sigma_w.$$

The total charge Q on the lower plate is distributed over the two regions:

$$Q = \sigma_w \, L(L - x) + \sigma_n \, Lx.$$

Substituting $\sigma_n = 2\sigma_w$,

$$Q = \sigma_w L(L - x) + 2\sigma_w Lx = \sigma_w L(L + x).$$

Solving,

$$\sigma_w = \frac{Q}{L(L + x)}, \qquad \sigma_n = \frac{2Q}{L(L + x)}.$$

The electric fields in the two regions are therefore

$$E_w = \frac{\sigma_w}{\epsilon_0} = \frac{Q}{\epsilon_0 L(L + x)}, \qquad E_n = \frac{\sigma_n}{\epsilon_0} = \frac{2Q}{\epsilon_0 L(L + x)}.$$

b.

Capacitance is given by

$$C = \frac{\epsilon_0 (L - x)L}{s} + \frac{\epsilon_0 x(L)}{s/2} = \frac{\epsilon_0 L(L + x)}{s}$$

c.

Total Energy is given by

$$U = \frac{Q^2}{2C} = \frac{sQ^2}{2\epsilon_0 L(L + x)}$$

d.

The force on the conducting plate is obtained from the energy by

$$F = -\frac{dU}{dx} = \frac{sQ^2}{2\epsilon_0 L}\frac{1}{(L+x)^2}.$$

The force acts in the direction of increasing x, so the conducting plate is pulled further into the capacitor.

Hence the Force needed to prevent the Middle plate from moving is given by:-

$$F = \frac{sQ^2}{2\epsilon_0 L(L+x)^2}$$