8.022 Problem Set 10

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Problem 5

5. Displacement Current.

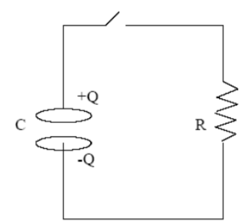

Figure 1: A RC circuit

A capacitor C with circular plates of radius b is charged to a voltage V_0. The space between the two plates is small compared to b so that we can safely ignore any fringing effects. At t = 0 the switch is closed and the capacitor discharges through the resistor R. In all the questions below give your answers in terms of C, b, V_0,R, t and any universal constants.

(a) Give an expression for the charge Q(t) as a function of time of the positively charged plate (upper one in the above figure) of the capacitor.
(b) Find the electric field $\vec{E}(t)$ between the two capacitor plates.
(c) Find the displacement current density $\vec{J}(t)$ between the two capacitor plates.
(d) Find the magnetic field $\vec{B}(t)$ anywhere in between the capacitor plates.

a.

$$\frac{Q}{C} + IR = 0 \implies \frac{Q}{C} - R\dot{Q} = 0$$

$$\implies \dot{Q} + \frac{1}{RC}Q = 0$$

Therefore,

$$\boxed{Q(t) = CV_0\, e^{-\frac{t}{RC}}}$$

b.

$$E = \frac{\sigma}{\epsilon} = \frac{Q}{\pi b^2 \epsilon}$$

Therefore,

$$\boxed{E(t) = \frac{CV_0}{\pi b^2 \epsilon}\, e^{-\frac{t}{RC}}}$$

c.

$$J_{\text{Displacement}} = \epsilon \frac{\partial E}{\partial t}$$

Therefore,

$$\boxed{J_{\textbf{Displacement}}(t) = -\frac{V_0}{\pi b^2 R}\, e^{-\frac{t}{RC}}}$$

d.

Consider a ring of Radius r ($r \leq b$) within the space between the plates of the Capacitor and $I_{\text{Displacement}}$ be the Displacement Current through the Circular cross section made by the ring.

So,

$$I_{\text{Displacement}} = \pi r^2 J = -\frac{r^2 V_0}{b^2 R}\, e^{-\frac{t}{RC}}$$

By Maxwell–Ampère Law,

$$\oint B\, dl = \mu \Big(I_{\text{Conduction}} + I_{\text{Displacement}} \Big)$$

$$\implies B 2\pi r = \mu \Big(0 - \frac{r^2 V_0}{b^2 R}\, e^{-\frac{t}{RC}} \Big)$$

The Direction of B is given by Right-Hand Rule

Therefore,

$$\boxed{B = -\frac{\mu r V_0}{2\pi b^2 R}\, e^{-\frac{t}{RC}}\, \hat{\phi}}$$

Problem 7

7. An infinite fat wire.

An infinite fat wire, with radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $\omega \ll a$, forms a parallel-plate capacitor, as shown in the figure. Find the magnetic field in the gap, at a distance $s < a$ from the axis.

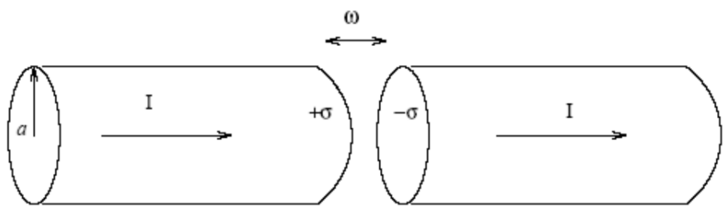

Figure 2: An infinite fatwire

By symmetry, the magnetic field in the gap is azimuthal and depends only on the distance s from the axis:

$$\vec{B} = B(s)\,\hat{\phi}$$

Choose a circular Amperian loop of radius $s < a$, lying entirely in the gap. In the gap,

$$I_{\text{conduction}} = 0$$

The current density in the wire is uniform:

$$J_{\text{Conduction}} = \frac{I}{\pi a^2}$$

Since the current is steady, Charge conservation requires that the rate of Surface Charge accumulation equals the Conduction Current Density:

$$\frac{\partial \sigma}{\partial t} = J_{\text{Conduction}}$$

The electric field in the gap is uniform, and the Displacement Current Density is

$$\vec{J}_{\text{Displacement}} = \epsilon \frac{\partial \vec{E}}{\partial t} = \epsilon \frac{\partial}{\partial t}\left(\frac{\sigma}{\epsilon}\right)\hat{n} = \frac{\partial \sigma}{\partial t}\hat{n} = \vec{J}_{\text{Conduction}} = \frac{I}{\pi a^2}$$

The displacement current through the surface bounded by the Amperian loop is

$$I_{\text{displacement}} = J\,(\pi s^2) = I\frac{s^2}{a^2}$$

Applying the Maxwell–Ampère Law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{\text{conduction}} + I_{\text{displacement}} \right)$$

$$\implies B(2\pi s) = \mu_0 I \frac{s^2}{a^2}$$

Solving for B,

$$\boxed{\vec{B}(s) = \frac{\mu_0 I}{2\pi a^2}\, s\, \hat{\phi}}$$

This is identical to the magnetic field inside a solid wire carrying a uniform current.