# 8.022 Problem Set 8

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## Problem 6

Charge in series RLC.

In this problem, we will look at the behavior of $q(t)$, the charge on the capacitor in a series RLC circuit driven by a periodic EMF $\mathcal{E} = \mathcal{E}_0 \cos(\omega t)$.

(a) Derive a differential equation showing the time evolution of the charge on the capacitor. Don't solve it!

(b) Use the known solution for I(t) and the definition $I = \frac{dq}{dt}$ to find an expression for $q(t)$. Give the complex charge, $\tilde{q}(t)$, as well as the physical charge on the capacitor. What is the amplitude $q_0$ of q(t)?

(c) Show that the maximum charge amplitude is at $\omega = \sqrt{\omega_0^2 - R^2/2L^2}$

### a.

Let Charge stored in the Capacitor be $q$, So the Rate of Change of Charge in the Capacitor will be equal to the Current provided in the Circuit. Say the current be $i$.

$$i = \dot{q}$$

So the K.V.L. equation for this System will be

$$\mathcal{E} = iR + L\frac{di}{dt} + \frac{q}{C}$$

Hence, the required Differential Equation showing the Time Evolution of the Charge on the Capacitor is given by

$$\boxed{L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}_0 \cos(\omega\, t)}$$

**b.**

In the steady state, the current in a driven RLC circuit can be written in complex form as

$$I(t) = \Re\{\tilde{I}\, e^{i\omega t}\},$$

where the complex current amplitude is

$$\tilde{I} = \frac{\mathcal{E}_0}{R + i\left(\omega L - \frac{1}{\omega C}\right)}.$$

Since the current is related to the charge on the capacitor by

$$I = \dot{q},$$

we may write, in complex notation,

$$\tilde{I}(t) = \frac{d\tilde{q}(t)}{dt}.$$

Integrating with respect to time gives

$$\tilde{q}(t) = \int \tilde{I}\, e^{i\omega t}\, dt = \frac{\tilde{I}}{i\omega}\, e^{i\omega t}.$$

Substituting for $\tilde{I}$,

$$\boxed{\tilde{q}(t) = \frac{\mathcal{E}_0}{i\omega\left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right]}\, e^{i\omega t}}$$

The physical charge on the capacitor is the real part of $\tilde{q}(t)$, and can be written in the form

$$\boxed{q(t) = q_0 \cos(\omega t - \delta)}$$

$$\boxed{q_0 = \frac{\mathcal{E}_0}{\omega\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad \tan\delta = \frac{\omega L - \frac{1}{\omega C}}{R}}$$

**c.**

For $q_0$ to be maximum, $\omega\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ should be Minimum. This can also be found by Minimizing $R^2\omega^2 + (L\omega^2 - \frac{1}{C})^2$

$$\frac{d}{d\omega}(R^2\omega^2 + (L\omega^2 - \frac{1}{C})^2) = 0$$

$$\implies 2R^2\omega + 4\omega\, L(L\omega^2 - \frac{1}{C}) = 0 \implies \omega^2 = \frac{1}{LC} - \frac{R^2}{2L^2}$$

Putting Natural Frequency $\omega_0^2 = \frac{1}{LC}$, we get

$$\boxed{\omega = \sqrt{\omega_0^2 - \frac{R^2}{2L^2}}}$$

# Problem 7

Impedance of a RLC circuit.

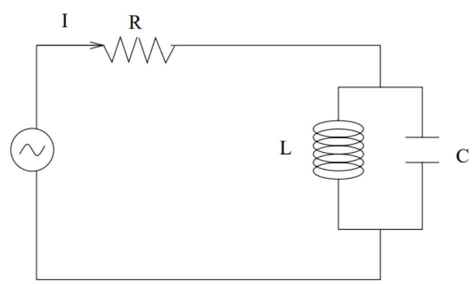


Figure 1: An AC driven RLC circuit.

(a) What is the complex impedance of the combination of the circuit elements, R, L, and C shown in Fig 1? The AC voltage is given as $V_0 cos(\omega t)$ Please rationalize the expressiion into separate real and imaginary parts. (Note: impedance in parallel behave in just the same way as resistor in parallel.)
(b) What is the current I, (the actual and not the complex current) flowing through the circuit? Give an expression for the phase angle.
(c)Explain the low and high frequency behavior of the phase shift of the current in terms of the currents through each of the circuit elements.

## a.

The inductor and capacitor are in parallel, so their combined impedance is

$$\frac{1}{Z_{LC}} = \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{1}{i\omega L} + i\omega C.$$

Thus,

$$Z_{LC} = \frac{1}{i\left(\omega C - \frac{1}{\omega L}\right)}$$

The total impedance of the circuit (with the resistor in series) is

$$Z = R + Z_{LC} = R + \frac{1}{i\left(\omega C - \frac{1}{\omega L}\right)}$$

Writing explicitly,

$$\boxed{Z = R - \frac{i}{\omega C - \frac{1}{\omega L}}}$$

## b.

The magnitude of the impedance is

$$|Z| = \sqrt{R^2 + \frac{1}{\left(\omega C - \frac{1}{\omega L}\right)^2}}$$

Hence the current amplitude is

$$I_0 = \frac{V_0}{|Z|} = \frac{V_0}{\sqrt{R^2 + \frac{1}{\left(\omega C - \frac{1}{\omega L}\right)^2}}}$$

The phase angle between the current and the voltage is given by

$$\boxed{\delta = \arctan\left(-\frac{1}{R\left(\omega C - \frac{1}{\omega L}\right)}\right)}$$

The physical current is therefore

$$\boxed{I(t) = I_0 \cos(\omega t - \delta)}$$

## c.

The phase of the total current relative to the applied voltage can be understood by examining the currents through each circuit element.

**Low-frequency limit ($\omega \to 0$):**
At low frequencies, the reactances are

$$X_L = \omega L \to 0, \qquad X_C = \frac{1}{\omega C} \to \infty.$$

Thus, the capacitor behaves like an open circuit and carries negligible current, while the inductor behaves like a short circuit. The current therefore flows primarily through the inductor and resistor.

Since the inductor current lags the voltage by $90^\circ$, the total current lags the applied voltage. Hence, the phase shift is negative in the low-frequency limit.

**High-frequency limit ($\omega \to \infty$):**
At high frequencies,

$$X_L = \omega L \to \infty, \qquad X_C = \frac{1}{\omega C} \to 0.$$

In this case, the inductor behaves like an open circuit, while the capacitor behaves like a short circuit. The current therefore flows primarily through the capacitor.

Since the capacitor current leads the voltage by $90^\circ$, the total current leads the applied voltage. Hence, the phase shift is positive in the high-frequency limit.

**Summary:**
At low frequencies, the circuit is inductive and the current lags the voltage. At high frequencies, the circuit is capacitive and the current leads the voltage. The transition between these regimes occurs near the resonant frequency, where the inductive and capacitive effects balance.

# Problem 8

Levitating ring demonstration.

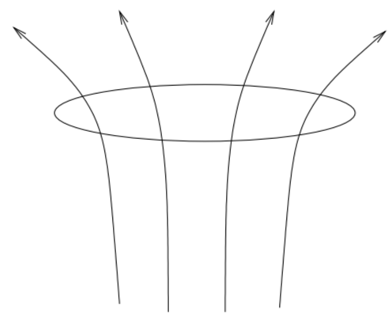

Figure 2: Sideview of the loop in the B field.

A conducting circular loop of radius $a$ is levitated by a magnetic field (created by a coil which is not shown). the loop is centered on the axis of symmetry of the coil, so you may consider the component of the field pointing radially outwards from the center of the loop, $B_r$, as being constant around the loop. Two cases are sonsidered here. First you will consider what happens when the field strength is increased at a constant rate. Second, you will determine what happens in the AC case.

(a) If the field strength is increasing at a constant reate, a current will start to flow in the loop. Draw a diagram of the "top view" to show the direction of the current.

(b) The increasing magnetic field creates an increasing magnetic flux $d\phi/dt$ through the loop. The inductance of the loop is L, and the resistance of the loop R. Write down the correct differential equation describing the current flow in the loop. Your equation should incolve $\frac{d\phi}{dt}$,$\frac{dI}{dt}$,I, L,R and c. (Besides the external magnetic flux, the current will also produce a magnetic flux through the loop.)

(c) Write down the value for the current around the loop as $t \to \infty$.

(d) Now consider that the magnetic field oscillates in time. This means that the flux through the loop oscillates as $\phi = \phi_0 cos(\omega t)$. What is the current in the loop? What is the phase of the current compared to the phase of the flux? (Hint: Use complex number and note that $\phi = Re[\phi_0 e^{i\omega t}] = Re[\tilde{\phi}]$.Also use the fact that $\frac{d\phi}{dt} = i\omega\tilde{\phi}$.

(e) What is the force on the loop? Which way does it point? Use the expression for the current that you derived in part (d). Remember that the magnetic field is oscillating so that $B_z$ oscillates as $B_z = B_{z0} cos(\omega t)$ and $B_r$ oscillates as $B_r = B_{r0} cos(\omega t)$. For what value of hte resistance is the force maximized?

## a.

The Direction of Current will **Clockwise** as seen from the top as given by Lenz Law.

## b.

By Faraday's Law

$$\mathcal{E} = -\frac{d\phi}{dt}$$

By K.V.L.

$$\mathcal{E} = IR + L\frac{dI}{dt}$$

Therefore, the Differential Equation is given by

$$\boxed{L\frac{dI}{dt} + RI = -\frac{d\phi}{dt}}$$

## c.

Solving,

$$\frac{dI}{dt} + \frac{R}{L}I = -\frac{1}{L}\frac{d\phi}{dt}$$

$$\implies I(t) = e^{-\frac{R}{L}t}\Big[c - \frac{1}{L}\int e^{\frac{R}{L}t}(\frac{d\phi}{dt})\,dt\Big] =$$

As

$$t \to \infty$$

$$ce^{-\frac{R}{L}t} \to 0 \qquad e^{-\frac{R}{L}t}\Big(-\frac{1}{L}\Big)\int^t e^{\frac{R}{L}t}\frac{d\phi}{dt}\,dt \to -\frac{1}{R}\frac{d\phi}{dt}$$

Therefore

$$\boxed{I(t)_{\langle t\to\infty\rangle} = -\frac{1}{R}\frac{d}{dt}\Big(\phi(t)\Big)}$$

## d.

$$\tilde{\phi} = \phi_0 e^{i\omega t} \implies \frac{d\tilde{\phi}}{dt} = i\omega\phi_0 e^{i\omega t}$$

This gives us

$$\tilde{I} = ce^{-\frac{R}{L}t} - \frac{e^{-\frac{R}{L}t}}{L}\int e^{\frac{R}{L}t} i\omega\phi_0 e^{i\omega t}\,dt$$

$$= ce^{-\frac{R}{L}t} - \frac{e^{-\frac{R}{L}t}}{L}\frac{i\omega\phi_0 e^{(\frac{R}{L}+i\omega)t}}{\frac{R}{L}+i\omega} = ce^{-\frac{R}{L}t} - \frac{i\omega\phi_0 e^{i\omega t}}{R+iL\omega}$$

$$\implies I = ce^{-\frac{R}{L}t} + \frac{\omega\phi_0}{R^2 + L^2\omega^2}\Big(R\sin(\omega t) - L\omega\cos(\omega t)\Big)$$

Therefore, Steady State Current is given by

$$\boxed{I(t) = \frac{\omega\phi_0}{R^2 + L^2\omega^2}\Big(R\sin(\omega t) - L\omega\cos(\omega t)\Big)}$$

## e.

The force on a current-carrying loop is given by

$$d\mathbf{F} = I\, d\boldsymbol{\ell} \times \mathbf{B}.$$

For a circular loop of radius $a$, the tangential current element $d\boldsymbol{\ell}$ interacts with the radial magnetic field $B_r$ to produce a force along the $z$ direction. Since $B_r$ is constant around the loop, the total force is

$$F(t) = \oint I(t)\, a B_r(t)\, d\phi = 2\pi a\, I(t)\, B_r(t).$$

With

$$B_r(t) = B_{r0}\cos(\omega t),$$

and using the steady-state current from part (d),

$$I(t) = \frac{\omega\phi_0}{R^2 + L^2\omega^2}\Big(R\sin\omega t - L\omega\cos\omega t\Big),$$

the time-averaged force over one cycle is

$$\boxed{\langle F\rangle = \frac{\pi a\, \omega^2 L\, \phi_0 B_{r0}}{R^2 + L^2\omega^2}}$$

directed upward, leading to levitation of the loop.

The force is maximized when

$$\boxed{R = 0}$$