8.022 Problem Set 11

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Problem 3

Poynting vector and capacitor: A current $I = dQ/dt$ delivers charge to a capacitor. This capacitor has radius a, and the plates are separated by a distance s.
(a)Find the Poynting vector due to the electric field and the magnetic field between the capacitor plates. Give both the magnitude and the direction.
(b) Calculate the total power, $P = \int \vec{S} \cdot d\vec{a}$, flowing into the capacitor. Given the Poynting vector found in (a), what is the correct surface to use for the integral?
(c)Integrate this power over time. Assuming that the capacitor has charge 0 at t = 0 and has some charge level Q at a later time t, show that the total energy that flows into the capacitor is given by $U = Q^2/2C$.

a.

The electric field between the capacitor plates is uniform and given by

$$\vec{E} = E(t)\,\hat{z} = \frac{\sigma(t)}{\epsilon}\,\hat{z} = \frac{It}{\pi a^2 \epsilon}\,\hat{z}$$

The magnetic field between the plates is azimuthal and is obtained from the Maxwell–Ampère Law. By symmetry,

$$\vec{B} = B(r,t)\,\hat{\phi}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu\epsilon \frac{\partial \Phi_E}{\partial t} \implies B(2\pi r) = \mu I \frac{r^2}{a^2}$$

$$\implies \vec{B} = \frac{\mu I r}{2\pi a^2}\,\hat{\phi}$$

The Poynting vector is defined as

$$\vec{S} = \frac{1}{\mu}\,\vec{E} \times \vec{B} = \frac{1}{\mu}\,E(t)B(r,t)\left(\hat{z} \times \hat{\phi}\right) = -\frac{1}{\mu}\,E(t)B(r,t)\,\hat{r}$$

Therefore,

$$\boxed{\vec{S}(r,t) = -\frac{I^2 tr}{2\pi^2 a^4 \epsilon}\,\hat{r}}$$

The Negative Sign indicates that Energy flows Radially Inward into the capacitor.

b.

The power flowing into the capacitor is given by

$$P = \oint \vec{S} \cdot d\vec{a}$$

For the curved cylindrical surface,

$$d\vec{a} = \hat{r}\,(a\,d\phi\,dz)$$

Thus,

$$P = \int_0^s \int_0^{2\pi} \left(-\frac{I^2 ta}{2\pi^2 a^4 \epsilon}\right)(a\,d\phi\,dz)$$

$$\implies P = -\frac{I^2 t}{\pi a^2 \epsilon}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^s dz$$

$$\implies P = -\frac{I^2 ts}{\pi a^2 \epsilon}$$

Therefore,

$$\boxed{\textbf{Power} = -\frac{I^2 ts}{\pi a^2 \epsilon}}$$

The Negative Sign indicates that Energy flows Radially Inward into the capacitor.

c.

The energy flowing into the capacitor is

$$U = -\int_0^{\tilde{t}} P\,dt$$

Let

$$I\tilde{t} = Q$$

So

$$\int_0^{\tilde{t}} \frac{I^2 ts}{\pi a^2 \epsilon}\,dt = \frac{I^2 \tilde{t}^2 s}{2\pi a^2 \epsilon} = \frac{1}{2}\frac{(I\tilde{t})^2}{\left(\frac{\pi a^2 \epsilon}{s}\right)} = \frac{1}{2}\frac{Q^2}{C}$$

Hence the Energy stored in a Capacitor is

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C}}$$

Problem 4

Poynting vector and coaxial cable:

A coaxial cable transmits DC power from a battery to a load. The cable consists of two concentric, long, hollow cylinders of zero resistance. The inner cylinder has radius a, the outer has b, and the length of both is l. The battery provides an EMF $\mathcal{E}$ between the two conductors at one end of the cable, and the load is a resistance R connected between the two conductors at the other end.

(a) How much power is dissipated in the resistor?
(b) What are $\vec{E}$ and $\vec{B}$ in the cable?
(c) What is the Poynting vector $\vec{S}$ in the cable?
(d) Show that $\int \vec{S} \cdot d\vec{a} = P$[from part (a)].
(e) Suppose the battery is now reversed. Does the direction of $\vec{S}$ change?

a.

The power dissipated in the resistor is given by circuit theory as

$$\boxed{P = \frac{\mathcal{E}^2}{R}}$$

b.

The electric field exists only in the region $a < r < b$ and is radial. By Gauss's law, for a cylindrical Gaussian surface of radius r and length l,

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon}$$

$$E(2\pi r l) = \frac{Q}{\epsilon} \implies \vec{E}(r) = \frac{\lambda}{2\pi\epsilon r}\,\hat{r}$$

where $\lambda = Q/l$ is the charge per unit length given by

$$\mathcal{E} = \frac{\lambda}{2\pi\epsilon}\ln(b/a) \implies \lambda = \frac{2\pi\epsilon\mathcal{E}}{\ln(b/a)}$$

Therefore,

$$\boxed{\vec{E}(r) = \frac{\mathcal{E}}{r\ln(b/a)}\,\hat{r}}$$

The magnetic field is azimuthal and follows from Ampère's law. For a circular Amperian loop of radius r,

$$\oint \vec{B}\cdot d\vec{l} = \mu I$$

$$B(2\pi r) = \mu I \implies \vec{B}(r) = \frac{\mu I}{2\pi r}\,\hat{\phi}$$

$$I = \frac{\mathcal{E}}{R}$$

Therefore,

$$\boxed{\vec{B}(r) = \frac{\mu\mathcal{E}}{2\pi r R}\,\hat{\phi}}$$

c.

The Poynting vector is

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{1}{\mu}\left(\frac{\mathcal{E}}{r\ln(b/a)}\,\hat{r}\right)\times\left(\frac{\mu\mathcal{E}}{2\pi r R}\,\hat{\phi}\right)$$

Thus,

$$\boxed{\vec{S}(r) = \frac{\mathcal{E}^2}{2\pi r^2 R\ln(b/a)}\,\hat{z}}$$

Energy flows along the cable in the $+\hat{z}$ direction, through the dielectric.

d.

The total power flowing down the cable is obtained by integrating $\vec{S}\cdot d\vec{a}$ over the annular cross section between the conductors:

$$P = \int \vec{S}\cdot d\vec{a}$$

Here,

$$d\vec{a} = \hat{z}\,(2\pi r\,dr)$$

So,

$$P = \int_a^b \frac{\mathcal{E}^2}{2\pi r^2 R\ln(b/a)}(2\pi r\,dr) = \int_a^b \frac{\mathcal{E}^2}{rR\ln(b/a)}\,dr = \frac{\mathcal{E}^2}{R\ln(b/a)}\ln(b/a)$$

Therefore,

$$\boxed{P = \frac{\mathcal{E}^2}{R}}$$

which agrees with part (a)

e.

If the battery is reversed, the electric field $\vec{E}$ reverses direction while $\vec{B}$ does not. Therefore, the Poynting vector $\vec{S} = \frac{1}{\mu}\vec{E} \times \vec{B}$ reverses direction.

Energy continues to flow from the battery toward the resistor, consistent with energy conservation.

Problem 6

Transmission line.

Two long conductors of width w and separation s form a waveguide as shown below.

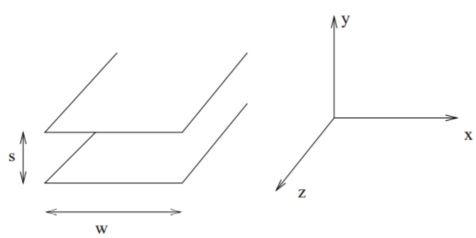

Figure 1: The waveguide.

(a) Find the inductance per unit length L_0 and the capacitance per unit length C_0 in terms of w,s and constants.
(b) One end of the transmission line is driven by a voltage $V(t) = V_0 f(zk - \omega t)$, Find the electric and magnetic fields $\vec{E}$ and $\vec{B}$ and the Poynting vector $\vec{S}$ in terms of w, s and V_0 and constants. (c) Find the impedance Z of the waveguide in terms of w, s and/or V_0 and constants.
(d) At z=10cm from the driven end of the waveguide, the waveguide changes to width $w + \Delta w$ ($\Delta w \ll w$), and separation $s + \Delta s \ll s$). What is the impedance Z' of the waveguide for $z > 10cm$? Use a Taylor expansion to express your answer as a linear function as Δs and Δw.
(e) How must Δs and Δw be related in order for $Z' = Z$?

a.

Let the current flowing through the two plates be I.

For $s \ll w$, the magnetic field between the plates is uniform. Applying Ampère's Law to a rectangular loop spanning the gap,

$$\oint \vec{B} \cdot d\vec{l} = \mu I \implies B = \frac{\mu I}{w}$$

The magnetic energy density is

$$u_B = \frac{B^2}{2\mu}$$

The magnetic energy per unit length is therefore

$$\frac{1}{2} L_0 I^2 = u_B(ws) = \frac{1}{2\mu}\left(\frac{\mu I}{w}\right)^2 (ws)$$

$$\implies L_0 = \frac{\mu s}{w}$$

Therefore,

$$\boxed{L_0 = \frac{\mu s}{w} \qquad C_0 = \frac{\epsilon w}{s}}$$

b.

The applied voltage on the transmission line is

$$V(z,t) = V_0 f(kz - \omega t)$$

The electric field between the plates is uniform and directed across the separation s:

$$\vec{E}(z,t) = \frac{V(z,t)}{s}\,\hat{y} = \frac{V_0}{s} f(kz - \omega t)\,\hat{y}$$

The wave propagates along the $+\hat{z}$ direction with speed

$$v = \frac{1}{\sqrt{L_0 C_0}} = \frac{1}{\sqrt{\mu \epsilon}}$$

For a traveling electromagnetic wave,

$$\vec{B} = \frac{1}{v}\,\hat{z} \times \vec{E}$$

Thus,

$$\vec{B}(z,t) = \frac{1}{v}\left(\hat{z} \times \frac{V_0}{s} f(kz - \omega t)\,\hat{y}\right) = \frac{V_0}{vs} f(kz - \omega t)\,\hat{x}$$

The Poynting vector is

$$\vec{S} = \frac{1}{\mu}\vec{E} \times \vec{B}$$

Substituting the fields,

$$\vec{S}(z,t) = \frac{1}{\mu}\left(\frac{V_0}{s} f(kz-\omega t)\,\hat{y}\right) \times \left(\frac{V_0}{vs} f(kz-\omega t)\,\hat{x}\right)$$

$$\implies \vec{S}(z,t) = \frac{V_0^2}{\mu v s^2} f^2(kz-\omega t)\,\hat{z}$$

Therefore,

$$\boxed{\vec{S}(z,t) = \frac{V_0^2}{\mu v s^2} f^2(kz-\omega t)\,\hat{z}}$$

Energy flows along the transmission line in the direction of wave propagation.

c.

The characteristic impedance of a transmission line is defined as

$$Z = \sqrt{\frac{L_0}{C_0}}$$

From part (a),

$$L_0 = \frac{\mu s}{w} \qquad \text{and} \qquad C_0 = \frac{\epsilon w}{s}$$

Substituting,

$$Z = \sqrt{\frac{\mu s/w}{\epsilon w/s}} = \sqrt{\frac{\mu}{\epsilon}}\,\frac{s}{w}$$

Therefore,

$$\boxed{Z = \sqrt{\frac{\mu}{\epsilon}}\,\frac{s}{w}}$$

d.

For $z > 10\,\text{cm}$, the transmission line changes geometry to width $w + \Delta w$ and separation $s + \Delta s$, where $\Delta w \ll w$ and $\Delta s \ll s$.

The new characteristic impedance is

$$Z' = \sqrt{\frac{\mu}{\epsilon}}\,\frac{s+\Delta s}{w+\Delta w}$$

Expanding to first order,

$$Z' = \sqrt{\frac{\mu}{\epsilon}}\,\frac{s}{w}\left(1 + \frac{\Delta s}{s}\right)\left(1 - \frac{\Delta w}{w}\right)$$

Neglecting second-order terms,

$$\boxed{Z' = Z\left(1 + \frac{\Delta s}{s} - \frac{\Delta w}{w}\right)}$$

e.

For there to be no reflection at $z = 10\,\text{cm}$, the impedance must be unchanged:

$$Z' = Z$$

Therefore,

$$\frac{\Delta s}{s} - \frac{\Delta w}{w} = 0$$

Hence the condition for impedance matching is

$$\boxed{\frac{\Delta s}{s} = \frac{\Delta w}{w}}$$