8.022 Problem Set 6

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Problem 6

Vector potential of an infinite solenoid.

Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I. (Hint: it is a bit hard to use Eq 35. of Purcell, since the solenoid is infinitely long here. Try thinking of

$$\oint \vec{A} \cdot d\vec{l} = \int (\nabla \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a} = \phi \tag{1}$$

where ϕ is the magnetic flux of $\vec{B}$ through the loop in question.)

The magnetic field of an infinite solenoid is known to be

$$\mathbf{B} = \begin{cases} \mu_0 n I \, \hat{z}, & r < R, \\ 0, & r > R. \end{cases}$$

Based on the System's Nature we will use Cylindrical Coordinate System $< \hat{r}, \hat{\phi}, \hat{z} >$

Let

$$A =< A_r(r, \phi, z), A_\phi(r, \phi, z), A_z(r, \phi, z) >$$

Based on Symmetry of the System, the magnitude of A is independent of ϕ and z. So A can be simplified into

$$A =< A_r(r), A_\phi(r), A_z(r) >$$

We know that

$$B = \nabla \times A$$

This gives the value of B to be

$$B =< (\frac{1}{r}(\frac{\partial A_z}{\partial \phi} - \frac{\partial r A_\phi}{\partial z})), (\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}), (\frac{1}{r}(\frac{\partial r A_\phi}{\partial r} - \frac{\partial A_r}{\partial \phi})) >$$

$$=< 0, -\frac{\partial A_z}{\partial r}, \frac{\partial A_\phi}{\partial r} + \frac{A_\phi}{r} >$$

Comparing with B of a Solenoid we get,

$$-\frac{\partial A_z}{\partial r} = 0 \implies A_z = c \text{ Here } c \text{ is a constant}$$

So, the A is of the form

$$A =< A_r(r), A_\phi(r), c > \text{ Here } A_r \text{ is arbitrary}$$

So we only need to calculate $A_\phi(r)$
The relation between A and ϕ is given by

$$\oint \mathbf{A} \cdot d\mathbf{l} = \Phi_B$$

which states that the circulation of **A** around a closed loop equals the magnetic flux through the loop.

Field inside the solenoid ($r < R$)

We choose a circular loop of radius $r < R$, centered on the axis.
The circulation of **A** around the loop is

$$\oint \mathbf{A} \cdot d\mathbf{l} = A_\phi(r)\,(2\pi r).$$

The magnetic flux through the loop is

$$\Phi_B = B\,(\pi r^2) = (\mu_0 n I)\pi r^2.$$

Equating circulation and flux,

$$A_\phi(r)\,(2\pi r) = \mu_0 n I \pi r^2.$$

Solving,

$$A_\phi(r) = \frac{\mu_0 n I}{2}\, r \qquad (r < R)$$

Field outside the solenoid ($r > R$)

We now choose a circular loop of radius $r > R$.
Since the magnetic field is nonzero only inside the solenoid, the flux through the loop is

$$\Phi_B = (\mu_0 n I)\pi R^2.$$

The circulation of **A** is again

$$\oint \mathbf{A} \cdot d\mathbf{l} = A_\phi(r)\,(2\pi r).$$

Equating,

$$A_\phi(r)\,(2\pi r) = \mu_0 n I \pi R^2.$$

Solving,

$$A_\phi(r) = \frac{\mu_0 n I R^2}{2r} \qquad (r > R)$$

Hence, the vector potential of the infinite solenoid is therefore

$$\boxed{\mathbf{A} = \begin{cases} \dfrac{\mu_0 n I}{2} r\,\hat{\boldsymbol{\phi}} + c\,\hat{\boldsymbol{z}} + A_r(r)\hat{\boldsymbol{r}} & r < R \\ \dfrac{\mu_0 n I R^2}{2r}\,\hat{\boldsymbol{\phi}} + c\,\hat{\boldsymbol{z}} + A_r(r)\hat{\boldsymbol{r}} & r > R \end{cases}}$$

This vector potential correctly reproduces the magnetic field of the solenoid via $\mathbf{B} = \nabla \times \mathbf{A}$ and illustrates that $\mathbf{A}$ may be nonzero even in regions where $\mathbf{B} = 0$.

Problem 7

Transformation of fields: A very large sheet of charge lies in the $x - y$ plane of the frame F. The charge per unit area of this sheet is σ. In the frame F', this sheet moves to the right with speed v.

(a) What is the electric field in the rest frame (above and below the sheet)?

(b) What is the electric field in the frame F' (above and below the sheet)?

(c) What is the magnetic field in the frame F' (above and below the sheet)?

(d) Show that the results of (b) and (c) are consistent with the general Lorentz transformations for electric and magnetic fields, Eq. (60) of Purcell Chapter 6.

a. Electric field in frame F

In the rest frame of the sheet, the charge distribution is static. By symmetry, the electric field is uniform and perpendicular to the sheet.

Using Gauss's law for an infinite charged sheet,

$$\mathbf{E} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\,\hat{z}, & z > 0, \\ -\dfrac{\sigma}{2\varepsilon_0}\,\hat{z}, & z < 0. \end{cases}$$

Since the charges are at rest in this frame, there is no magnetic field:

$$\mathbf{B} = 0.$$

b. Electric field in frame F'

The frame F' moves along the x direction with speed v. The electric field in frame F points along the z direction, which is perpendicular to the direction of motion.

Under a Lorentz transformation, the perpendicular component of the electric field transforms as

$$\mathbf{E}'_\perp = \gamma\,\mathbf{E}_\perp, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Hence,

$$\mathbf{E}' = \begin{cases} \dfrac{\gamma\sigma}{2\varepsilon_0}\,\hat{z}, & z > 0, \\ -\dfrac{\gamma\sigma}{2\varepsilon_0}\,\hat{z}, & z < 0. \end{cases}$$

c. Magnetic field in frame F'

Although $\mathbf{B} = 0$ in frame F, a magnetic field appears in frame F' because the charges are moving.

The Lorentz transformation for the magnetic field is

$$\mathbf{B}' = -\gamma\,\frac{\mathbf{v}}{c^2} \times \mathbf{E}.$$

With $\mathbf{v} = v\,\hat{x}$ and $\mathbf{E} = E_z\,\hat{z}$,

$$\mathbf{v} \times \mathbf{E} = v\,\hat{x} \times E_z\,\hat{z} = -vE_z\,\hat{y}.$$

Therefore,

$$\mathbf{B}' = \gamma\,\frac{vE_z}{c^2}\,\hat{y}.$$

Substituting $E_z = \sigma/(2\varepsilon_0)$,

$$\mathbf{B}' = \begin{cases} \dfrac{\gamma v\sigma}{2\varepsilon_0 c^2}\,\hat{y}, & z > 0, \\ -\dfrac{\gamma v\sigma}{2\varepsilon_0 c^2}\,\hat{y}, & z < 0. \end{cases}$$

d. Consistency with field transformations

In frame F, $\mathbf{B} = 0$, so the Lorentz transformation laws reduce to

$$\mathbf{E}'_\perp = \gamma\,\mathbf{E}_\perp, \qquad \mathbf{B}'_\perp = -\gamma\,\frac{1}{c^2}\,\mathbf{v} \times \mathbf{E}.$$

The fields obtained in parts (b) and (c) satisfy these relations exactly. Hence, the electromagnetic fields in frame F' are fully consistent with Lorentz transformations.

Problem 8

Electric and magnetic forces.

Two infinite lines of charges with charge per unit length λ_0 in their rest frame are separated by a distance d. These charges are moving in a direction parallel to their length with speed v.

(a) In the rest frame, what is the electric force per unit length that the top line feels due to the bottom line? Give both the direction and the magnitude.

(b) Repeat (a) in the lab frame.

(c) In the lab frame, what is the magnetic force per unit length that the top line feels due to the bottom line? Give both the direction and the magnitude.

(d) What is the total force per unit length in the lab frame?

a. Force per unit length in the rest frame

In the rest frame of the charges, both lines are stationary. The electric field due to an infinite line of charge is

$$E = \frac{\lambda_0}{2\pi\epsilon_0 r}.$$

At the position of the upper line ($r = d$),

$$E = \frac{\lambda_0}{2\pi\epsilon_0 d}.$$

The electric force per unit length is therefore

$$\frac{F}{L} = \lambda_0 E = \frac{\lambda_0^2}{2\pi\epsilon_0 d}.$$

Hence

$$\frac{F}{L} = \frac{\lambda_0^2}{2\pi\epsilon_0 d} \quad \textbf{(repulsive)}$$

b. Electric force per unit length in the lab frame

In the lab frame, the charges move with speed v. Due to length contraction,

$$\lambda = \gamma\lambda_0, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

The electric field produced by the lower line is

$$E = \frac{\lambda}{2\pi\epsilon_0 d} = \frac{\gamma\lambda_0}{2\pi\epsilon_0 d}.$$

The electric force per unit length on the upper line is

$$\frac{F_E}{L} = \lambda E = \frac{\gamma^2 \lambda_0^2}{2\pi\epsilon_0 d}.$$

Hence,

$$\frac{F_E}{L} = \frac{\gamma^2 \lambda_0^2}{2\pi\epsilon_0 d}$$

c. Magnetic force per unit length in the lab frame

Each moving line constitutes a current

$$I = \lambda v = \gamma \lambda_0 v.$$

The magnetic field due to the lower line at distance d is

$$B = \frac{\mu_0 I}{2\pi d} = \frac{\mu_0 \gamma \lambda_0 v}{2\pi d}.$$

The magnetic force per unit length on the upper line is

$$\frac{F_B}{L} = IB = \frac{\mu_0 \gamma^2 \lambda_0^2 v^2}{2\pi d}.$$

Using $\mu_0 = 1/(\epsilon_0 c^2)$,

$$\frac{F_B}{L} = \frac{\gamma^2 \lambda_0^2}{2\pi\epsilon_0 d} \frac{v^2}{c^2}.$$

Hence,

$$\frac{F_B}{L} = \frac{\gamma^2 \lambda_0^2}{2\pi\epsilon_0 d} \frac{v^2}{c^2} \quad \textbf{(attractive)}$$

d. Total force per unit length

The net force is the electric repulsion minus the magnetic attraction:

$$\frac{F_{\text{net}}}{L} = \frac{\gamma^2 \lambda_0^2}{2\pi\epsilon_0 d} \left(1 - \frac{v^2}{c^2}\right).$$

Since $\gamma^2(1 - v^2/c^2) = 1$,
Hence,

$$\frac{F_{\textbf{net}}}{L} = \frac{\lambda_0^2}{2\pi\epsilon_0 d}$$