

8.022 Problem Set 9

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December 2025

Problem 5

Two capacitors and inductor.

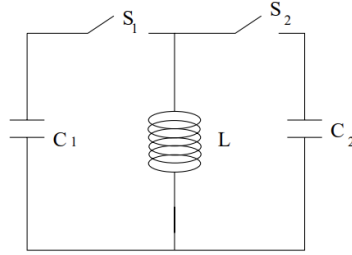


Figure 1: Two capacitors and an inductor.

Consider the following circuit where C_1 is initially charged to 75V. Supposed that C_1 is 10,000 μF , C_2 is 3000 μF , and L is 15H. Explain how to open and close the switches so as to discharge C_1 and charge C_2 . Starting at $t = 0$, you should give explicitly times for opening and closing each switch. What is the final voltage across C_2

Let S_1 be closed at $t = 0$ and charge stored in capacitor C_1 be q_1 .

We get the K.V.L. equation to become

$$\frac{q_1}{C_1} + L\dot{i} = 0$$

Since

$$i = \dot{q}_1,$$

this gives

$$L\ddot{q}_1 + \frac{1}{C_1}q_1 = 0$$

$$\Rightarrow q_1(t) = Q_{1,1} \cos\left(\frac{t}{\sqrt{LC_1}}\right) + Q_{1,2} \sin\left(\frac{t}{\sqrt{LC_1}}\right)$$

At $t = 0$,

$$q_1(0) = 75 \times 10^4 \times 10^{-6} \text{ C} = 0.75 \text{ C} \quad \dot{q}_1(0) = 0 \\ \Rightarrow Q_{1,1} = 0.75 \quad Q_{1,2} = 0$$

Therefore,

$$q_1(t) = 0.75 \cos\left(\frac{t}{\sqrt{LC_1}}\right)$$

All the energy is transferred from C_1 to L when

$$q_1(t) = 0$$

This occurs at

$$t = \frac{\pi}{2} \sqrt{LC_1} = \frac{\pi\sqrt{0.15}}{2}$$

Now let us consider the Inductor to be fully Charged, S_1 to be opened, S_2 to be closed and start out timing from the moment S_2 is closed. Let Charge stored in C_2 be q_2

So our Differential Becomes

$$L\ddot{q}_2 + \frac{q_2}{C_2} = 0 \\ \Rightarrow q_2 = Q_{2,1} \cos\left(\frac{t}{\sqrt{LC_2}}\right) + Q_{2,2} \sin\left(\frac{t}{\sqrt{LC_2}}\right)$$

At $t = 0$

$$q_2(0) = 0 \quad \dot{q}_2(0) = 3\sqrt{0.15} \\ \Rightarrow Q_{2,1} = 0 \quad Q_{2,2} = 0.135\sqrt{0.15}$$

Therefore,

$$q_2 = 0.135\sqrt{0.15} \sin\left(\frac{t}{\sqrt{LC_2}}\right)$$

All the Energy transferred from L to C_2 when

$$\dot{q}_2 = 0$$

This occurs at

$$t = \frac{\pi}{2} \sqrt{LC_2} = \frac{\pi\sqrt{0.075}}{2}$$

Final Voltage on C_2 will be

$$\frac{q_2^{<max>}}{C_2} = \frac{0.135\sqrt{0.15}}{0.003} = 45\sqrt{0.15}$$

For Transferring total Charge of C_1 to C_2 , S_1 and S_2 are operated as follows :-

- **Initially both S_1 and S_2 are Closed**

- S_1 is Closed at

$$t = 0$$

- S_1 is Opened and S_2 is Closed at

$$t = \frac{\pi\sqrt{0.15}}{2}$$

- S_2 is Opened at

$$t = \frac{\pi(\sqrt{0.15} + \sqrt{0.075})}{2}$$

The Final Voltage on C_2 is

$$45\sqrt{0.15}$$

Problem 6

Betatron accelerator.

A particle of mass M and charge Q is constrained by a magnetic field $B_0(R)$ to move on a circle of radius R which is perpendicular to the field. While $B_0(r)$ is constant in angle around a circle, the magnitude B_0 may depend on the radius. then one can define an average field B_{av} within the R -circle by;

$$B_{av} = \frac{1}{\pi R^2} 2\pi \int_0^R r B_0(r) dr \quad (3)$$

In the betatron $B_0(R)$ can be increased with time. At $t=0$, $B_0(R)$ is very small and particles from an ion source are injected with a low velocity just sufficient to follow a circular path. Then both B_{av} and $B_0(R)$ are increased with time, accelerating the particle by Faraday induction and keeping it on a circular orbit by the Lorentz force.

- Find the force on the charge Q due to the changing field.
 - Find the speed of the particle in its orbit as a function of $B_{av}(t)$
 - The particle will stay in the orbit of radius R so long as $B_0(r)$ and the speed v have the correct relation. However for the accelerator to work at all speeds, B_{av} and $B_0(R)$ must have the proper relationship. What is this relationship? Can it be satisfied by a uniform B'_0 , that is, one which does not depend on r ?
- (This problem is not very closely related to this chapter, but it illustrated an important principal.)

a.

Here the force qE acts tangentially to the circular path and changes the speed of the particle, while the force $q\mathbf{v} \times \mathbf{B}$ acts radially and provides the centripetal force required for circular motion.

By Faraday's Law,

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}$$

The magnetic flux through the circular orbit of radius R is

$$\Phi_B = \pi R^2 B_{\text{av}}(t)$$

Therefore,

$$2\pi R E = -\pi R^2 \frac{dB_{\text{av}}}{dt} \implies E = -\frac{R}{2} \frac{dB_{\text{av}}}{dt}$$

Hence, the tangential electric force acting on the particle is

$$F_{\text{Tangential}} = qE = -\frac{qR}{2} \frac{dB_{\text{av}}}{dt}$$

b.

The tangential acceleration of the particle is produced solely by the induced electric field. Thus,

$$M \frac{dv}{dt} = qE = -\frac{qR}{2} \frac{dB_{\text{av}}}{dt}$$

Rearranging,

$$\frac{dv}{dt} = -\frac{qR}{2M} \frac{dB_{\text{av}}}{dt}$$

Integrating with respect to time,

$$v(t) = \frac{qR}{2M} B_{\text{av}}(t) + c$$

Choosing the constant of integration such that $v = 0$ when $B_{\text{av}} = 0$, we obtain

$$v(t) = \frac{qR}{2M} B_{\text{av}}(t)$$

c.

The magnetic force provides the centripetal force required to maintain the particle in a circular orbit of radius R . Hence,

$$qvB_0(R) = \frac{Mv^2}{R}$$

Substituting the expression for $v(t)$ from part (b),

$$q \left(\frac{qR}{2M} B_{\text{av}} \right) B_0(R) = \frac{M}{R} \left(\frac{qR}{2M} B_{\text{av}} \right)^2$$

Simplifying,

$$qB_0(R) = \frac{q}{2} B_{\text{av}}$$

Therefore, the required condition for stable circular motion is

$$B_0(R) = \frac{1}{2} B_{\text{av}}$$

Can this work for a uniform B'_0 ?

From the condition for stable circular motion derived earlier, the magnetic field at the particle's orbit must satisfy

$$B_0(R) = \frac{1}{2} B_{\text{av}},$$

where B_{av} is the average magnetic field over the area enclosed by the circular orbit of radius R .

If the magnetic field is spatially uniform, i.e.

$$B_0(r) = \text{constant},$$

then the average magnetic field over the disk is simply

$$B_{\text{av}} = B_0(R).$$

Substituting into the stability condition gives

$$B_0(R) = \frac{1}{2} B_0(R),$$

which implies

$$B_0(R) = 0.$$

This is a contradiction unless the magnetic field vanishes entirely. Therefore, a uniform magnetic field cannot satisfy the condition required for stable circular motion.

Stable circular motion is not possible for a uniform B'_0 .

Physically, the induced electric field that accelerates the particle depends on the *average* magnetic field through the orbit, while the magnetic force that provides centripetal acceleration depends on the *local* magnetic field at the particle's position. These two requirements can only be satisfied if the magnetic field varies with radius.