8.022 Problem Set 1

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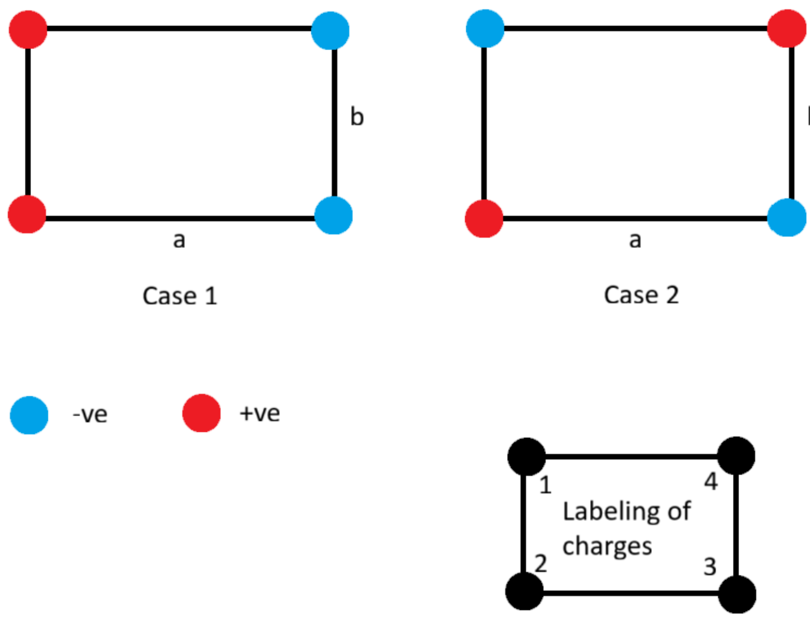
Problem 1

(Purcell 1.4)

1.4 *Work for a rectangle* $**$
Two protons and two electrons are located at the corners of a rectangle with side lengths a and b. There are two essentially different arrangements. Consider the work required to assemble the system, starting with the particles very far apart. Is it possible for the work to be positive for either of the arrangements? If so, how must a and b be related? You will need to solve something numerically.

The 2 cases are showed below in the diagram.



We will use help of the following steps:
Step 1: Fix charge 1 at its place.
Step 2: Calculate the Energy U_1 required to bring charge 2 to its place.
Step 3: Calculate the Energy U_2 required to bring charge 3 in its place.
Step 4: Calculate the Energy U_3 required to bring charge 4 in its place.
Step 5: The total energy required is $U = U_1 + U_2 + U_3$.
This method works owing to the Law of Conservation of Energy, Law of superposition of Charges and their fields and the fact that Electric Field is a Conservative Force Field.

We will consider, $k = \frac{1}{4\pi\epsilon_o}$ which is always positive. Magnitude of each charge is q
Case 1:
$U_1 = \frac{kq^2}{b}$
$U_2 = -\frac{kq^2}{\sqrt{a^2+b^2}} - \frac{kq^2}{a}$
$U_3 = -\frac{kq^2}{a} - \frac{kq^2}{\sqrt{a^2+b^2}} + \frac{kq^2}{b}$
$U = \frac{kq^2}{b} - \frac{kq^2}{\sqrt{a^2+b^2}} - \frac{kq^2}{a} - \frac{kq^2}{a} - \frac{kq^2}{\sqrt{a^2+b^2}} + \frac{kq^2}{b} = kq^2 2(\frac{1}{b} - \frac{1}{a} - \frac{1}{\sqrt{a^2+b^2}})$
Case 2:
$U_1 = -\frac{kq^2}{b}$
$U_2 = \frac{kq^2}{\sqrt{a^2+b^2}} - \frac{kq^2}{a}$
$U_3 = -\frac{kq^2}{a} + \frac{kq^2}{\sqrt{a^2+b^2}} - \frac{kq^2}{b}$

$U = -\frac{kq^2}{b} + \frac{kq^2}{\sqrt{a^2+b^2}} - \frac{kq^2}{a} - \frac{kq^2}{a} + \frac{kq^2}{\sqrt{a^2+b^2}} - \frac{kq^2}{b} = kq^2 2(-\frac{1}{b} - \frac{1}{a} + \frac{1}{\sqrt{a^2+b^2}})$

Case 1 can be negative or positive. positive when b is sufficiently smaller than a. Case 2 is always negative.

Problem 2

(Purcell 1.9)

1.61 *Potential energy of a sphere* **

A spherical volume of radius R is filled with charge of uniform density ρ. We want to know the potential energy U of this sphere of charge, that is, the work done in assembling it. In the example in Section 1.15, we calculated U by integrating the energy density of the electric field; the result was $U = (3/5)Q^2/4\pi\epsilon_0 R$. Derive U here by building up the sphere layer by layer, making use of the fact that the field outside a spherical distribution of charge is the same as if all the charge were at the center.

Let the radius of the sphere at a specific moment be r. The charge stored in the sphere at that instant is $\frac{4}{3}\pi r^3 \rho$.
The question tells us to build the sphere layer by layer. So if we add a layer of elementary thickness dr over the existing sphere, the charge added to the existing sphere is $4\pi r^2 dr\rho$.
The potential energy stored for adding a layer of elementary thickness is $dU = \frac{(\frac{4}{3}\pi r^3 \rho)(4\pi r^2 dr\rho)}{4\pi\epsilon_o r} = \frac{4\pi\rho^2 r^4}{3\epsilon_o} dr$
So $U = \frac{4\pi\rho^2}{3\epsilon_o} \int_0^R r^4 dr = \frac{4\pi\rho^2}{3\epsilon_o} \frac{R^5}{5}$
Replacing $\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$ we get,
$U = \frac{4\pi}{3\epsilon_o} \frac{R^5}{5} \frac{9Q^2}{(4\pi)^2 R^6} = \frac{3}{5} \frac{Q^2}{4\pi\epsilon_o R}$

Hence, $U = \frac{3}{5} \frac{Q^2}{4\pi\epsilon_o R}$

Problem 7

7 Coulomb force between line charges: a rod of length l1 with line charge density λ_1 and a rod of length l_2 with line charge density λ_2 lie on the x axis. Their ends are separated by a distance D as shown in the figure.

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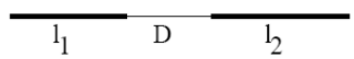


(a) What is the force $\vec{F}$ between these charges?

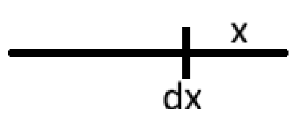
(b) Show that for $D >> l_1$ and $D >> l_2$, this force reduces to the Coulomb forces between a pair of point charges, $q_1 = l_1\lambda_1$, $q_2 = l_2\lambda_2$.

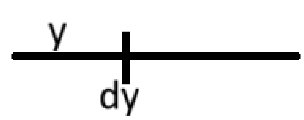
We will consider

$$k = \frac{1}{4\pi\epsilon_o}$$

a.

Consider the diagram below to understand the notations of x and y.





Force between dx and dy is given by

$$dF = \frac{k\,\lambda_1\lambda_2\,dx\,dy}{(D+x+y)^2}$$

So the net force is

$$F = k\,\lambda_1\lambda_2 \int_0^{l_1}\int_0^{l_2} \frac{dy\,dx}{(D+x+y)^2}$$

$$\int_0^{l_2} \frac{dy}{(D+x+y)^2} = \left[-\frac{1}{D+x+y}\right]_0^{l_2} = \frac{1}{D+x} - \frac{1}{D+x+l_2}$$

$$F = k\,\lambda_1\lambda_2 \int_0^{l_1} \left(\frac{1}{D+x} - \frac{1}{D+x+l_2}\right) dx$$

$$\int_0^{l_1} \frac{dx}{D+x} = \ln(D+l_1) - \ln D$$

$$\int_0^{l_1} \frac{dx}{D+x+l_2} = \ln(D+l_1+l_2) - \ln(D+l_2)$$

$$F = k\,\lambda_1\lambda_2 \Big[\ln(D+l_1) - \ln D - \ln(D+l_1+l_2) + \ln(D+l_2)\Big]$$

$$F = k\,\lambda_1\lambda_2\, \ln\!\left(\frac{(D+l_1)(D+l_2)}{D\,(D+l_1+l_2)}\right)$$

b.

$$\ln\!\left(\frac{(D+l_1)(D+l_2)}{D(D+l_1+l_2)}\right) = \ln\left(1+\frac{l_1}{D}\right) + \ln\left(1+\frac{l_2}{D}\right) - \ln\left(1+\frac{l_1+l_2}{D}\right)$$

$$\ln(1+u) \approx u - \frac{u^2}{2} \quad \text{for } |u| \ll 1$$

Let

$$x = \frac{l_1}{D}, \qquad y = \frac{l_2}{D},$$

Since $D >> l_1$ and $D >> l_2$

$$x, y \ll 1$$

Then

$$\ln(1+x) \approx x - \frac{x^2}{2}, \qquad \ln(1+y) \approx y - \frac{y^2}{2},$$

$$\ln\left(1+x+y\right) \approx (x+y) - \frac{(x+y)^2}{2}$$

$$\ln\left(1+x\right) + \ln\left(1+y\right) - \ln\left(1+x+y\right) \approx x + y - (x+y) - \frac{x^2}{2} - \frac{y^2}{2} + \frac{(x+y)^2}{2} = xy$$

$$\ln\left(\frac{(D+l_1)(D+l_2)}{D(D+l_1+l_2)}\right) \approx \frac{l_1 l_2}{D^2}$$

$$F \approx k\,\lambda_1\lambda_2\,\frac{l_1 l_2}{D^2} = k\,\frac{(\lambda_1 l_1)(\lambda_2 l_2)}{D^2} = k\,\frac{q_1 q_2}{D^2}$$