8.03SC Problem Set 1

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Introduction

Problem 1.1

Problem 1.1 (20 pts)

For the mass and spring discussed during the lecture (Howard Georgei Eq.(1.1)-(1.8)), suppose that the system is hung vertically in the earth's gravitational field, with the top of the spring held fixed. Show that the frequency for vertical oscillations is given by Eq.(1.5). Explain why gravity has no effect on the angular frequency.

The required Diagram is given below:

There is no Horizontal forces acting on the Block. We will equate the Vertical forces.

$kx_o = mg$ where x_o is the extension in the Spring from the natural length. This Force equation gives the equilibrium position $x_o = \frac{mg}{k}$. All oscillations occur about the Equilibrium position.
By the Newton's Second Law, $m\ddot{x} = -k\Delta x + mg$ where Δx is the change in length from the natural position of the spring.
But the oscillations takes place about the equilibrium position of the Spring. So, if we consider the x_o as the mean position, and x is displacement from the mean position. We get, $\Delta x = x + x_o$.
So we get, $m\ddot{x} = -k(x + x_o) + mg = -kx - k\frac{mg}{k} + mg = -kx$

Hence the equation of motion is : $m\ddot{x} = -kx$ or $\ddot{x} = -\omega^2 x$ where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency.

g does not affect the oscillations because it is the source of a constant force. A constant force affect only the shift in equilibrium position and not the motion of S.H.M. itself.

Problem 1.4

Problem 1.4 (20 pts)

A particle of mass m moves on the x axis with potential energy

$$V(x) = \frac{E_0}{a^4}\ x^4 + 4ax^3 - 8a^2x^2$$

a. Find the positions at which the particle is in stable equilibrium.
b. Find the angular frequency of small oscillations about each stable equilibrium position.
c. What do you mean by small oscillations? Be quantitative and give a separate answer for each point of stable equlibrium.

a.
The potential is

$$V(x) = E_0\left(\frac{x^4}{a^4} + 4\frac{x^3}{a^3} - 8\frac{x}{a}\right).$$

Differentiating with respect to x:

$$V'(x) = E_0\left(\frac{4x^3}{a^4} + \frac{12x^2}{a^3} - \frac{8}{a}\right) = \frac{E_0}{a^4}\left(4x^3 + 12ax^2 - 8a^3\right),$$

$$V''(x) = \frac{dV'}{dx} = E_0\left(\frac{12x^2}{a^4} + \frac{24x}{a^3}\right) = \frac{12E_0}{a^4}\left(x^2 + 2ax\right).$$

Equilibrium points satisfy $V'(x) = 0$:

$$4x^3 + 12ax^2 - 8a^3 = 0 \implies x^3 + 3ax^2 - 2a^3 = 0.$$

Introducing the dimensionless variable $y = \frac{x}{a}$ gives

$$y^3 + 3y^2 - 2 = 0.$$

Factoring the cubic:

$$y^3 + 3y^2 - 2 = (y+1)(y^2 + 2y - 2) = 0.$$

Thus the three equilibrium points are

$$x_1 = -a, \qquad x_2 = a(\sqrt{3} - 1), \qquad x_3 = -a(1 + \sqrt{3}).$$

To determine stability, we evaluate the sign of $V''(x)$.

$$V''(x) = \frac{12E_0}{a^4}(x^2 + 2ax).$$

$$V''(-a) = \frac{12E_0}{a^4}(-a^2) < 0 \quad \Rightarrow \quad \text{unstable.}$$

For $x = a(\sqrt{3} - 1)$, let $y = \sqrt{3} - 1$:

$$x^2 + 2ax = a^2(y^2 + 2y) = 2a^2 > 0 \quad \Rightarrow \quad \text{stable.}$$

For $x = -a(1 + \sqrt{3})$, let $z = -(1 + \sqrt{3})$:

$$x^2 + 2ax = a^2(z^2 + 2z) = 2a^2 > 0 \quad \Rightarrow \quad \text{stable.}$$

Stable equilibria: $x = a(\sqrt{3} - 1),\ x = -a(1 + \sqrt{3})$

Unstable equilibrium: $x = -a$

b.

For small oscillations about a point of stable equilibrium x_0, the motion is approximately simple harmonic with angular frequency

$$\omega = \sqrt{\frac{V''(x_0)}{m}}.$$

From part (a), we have

$$V''(x) = \frac{12E_0}{a^4}\left(x^2 + 2ax\right),$$

and the two stable equilibrium positions are

$$x_2 = a(\sqrt{3} - 1), \qquad x_3 = -a(1 + \sqrt{3}).$$

At $x_2 = a(\sqrt{3} - 1)$:
Let $y = \sqrt{3} - 1$, so $x_2 = ay$ and

$$x_2^2 + 2ax_2 = a^2(y^2 + 2y).$$

A short calculation gives

$$y^2 + 2y = (\sqrt{3} - 1)^2 + 2(\sqrt{3} - 1) = 2,$$

so

$$x_2^2 + 2ax_2 = 2a^2.$$

Then

$$V''(x_2) = \frac{12E_0}{a^4}\left(2a^2\right) = \frac{24E_0}{a^2},$$

and therefore

$$\omega_2 = \sqrt{\frac{V''(x_2)}{m}} = \sqrt{\frac{24E_0}{ma^2}}.$$

At $x_3 = -a(1 + \sqrt{3})$:
Let $z = -(1 + \sqrt{3})$, so $x_3 = az$ and

$$x_3^2 + 2ax_3 = a^2(z^2 + 2z).$$

One finds similarly

$$z^2 + 2z = 2,$$

hence

$$x_3^2 + 2ax_3 = 2a^2 \quad \Rightarrow \quad V''(x_3) = \frac{24E_0}{a^2}.$$

Thus

$$\omega_3 = \sqrt{\frac{V''(x_3)}{m}} = \sqrt{\frac{24E_0}{ma^2}}.$$

$$\omega_2 = \omega_3 = \sqrt{\frac{24E_0}{ma^2}}$$

So the angular frequency of small oscillations is the same about both stable equilibrium positions.

c.
By "small oscillations about a stable equilibrium x_0" we mean that the displacement $\xi = x - x_0$ is small enough that the potential $V(x)$ can be approximated by retaining only the quadratic term in its Taylor expansion about x_0:

$$V(x_0 + \xi) \approx V(x_0) + \frac{1}{2}V''(x_0)\,\xi^2,$$

and the higher–order terms,

$$\frac{1}{6}V'''(x_0)\,\xi^3, \qquad \frac{1}{24}V''''(x_0)\,\xi^4, \quad \dots$$

are negligible in comparison with the quadratic term.

From part (b) we already have

$$V''(x) = \frac{12E_0}{a^4}\big(x^2 + 2ax\big).$$

Differentiating once more,

$$V'''(x) = \frac{24E_0}{a^4}(x+a), \qquad V''''(x) = \frac{24E_0}{a^4}.$$

For each minimum x_0 we require

$$\left|\frac{\frac{1}{6}V'''(x_0)\,\xi^3}{\frac{1}{2}V''(x_0)\,\xi^2}\right| = \left|\frac{V'''(x_0)}{3V''(x_0)}\right|\,|\xi| \ll 1,$$

and

$$\left|\frac{\frac{1}{24}V''''(x_0)\,\xi^4}{\frac{1}{2}V''(x_0)\,\xi^2}\right| = \left|\frac{V''''(x_0)}{12V''(x_0)}\right|\,\xi^2 \ll 1.$$

Minimum at $x_2 = a(\sqrt{3}-1)$:

At this point,

$$V''(x_2) = \frac{24E_0}{a^2}, \qquad |V'''(x_2)| = \frac{24\sqrt{3}E_0}{a^3}, \qquad V''''(x_2) = \frac{24E_0}{a^4}.$$

The condition from the cubic term becomes

$$|\xi| \ll \frac{3|V''(x_2)|}{|V'''(x_2)|} = \frac{3\cdot\frac{24E_0}{a^2}}{\frac{24\sqrt{3}E_0}{a^3}} = \sqrt{3}\,a.$$

The quartic term gives a slightly weaker condition,

$$|\xi| \ll \sqrt{\frac{12|V''(x_2)|}{|V''''(x_2)|}} = \sqrt{12}\,a = 2\sqrt{3}\,a,$$

so the dominant restriction comes from the cubic term. Thus for the minimum at x_2 we require

$$|x - a(\sqrt{3}-1)| \ll \sqrt{3}\,a$$

Minimum at $x_3 = -a(1+\sqrt{3})$:

At this point $V''(x_3)$ and $|V'''(x_3)|$ have the same magnitudes as at x_2 (the sign of V''' changes, but only its magnitude enters the inequalities), so we obtain the same numerical bounds:

$$|x + a(1+\sqrt{3})| \ll \sqrt{3}\, a$$

In words: for each stable equilibrium, "small oscillations" means that the amplitude of motion is much smaller than the distance $\sqrt{3}\, a$ from that minimum to the nearby unstable equilibrium at $x = -a$. In this regime the motion is well described by the simple harmonic approximation.

Problem 1.5

Problem 1.5 (20 pts)

Consider a simple pendulum consisting of a point-like mass m attached to a massless string of length L hanging from a fixed support and constrained to move in a vertical plane (see Figure 2). Assume gravitational acceleration to be g.

a. Parametrize the motion of the pendulum in terms of the angle θ, its deviation from the vertical. Find the exact equation of motion ($\vec{\tau} = I\vec{\alpha}$) for the pendulum as a function of θ.

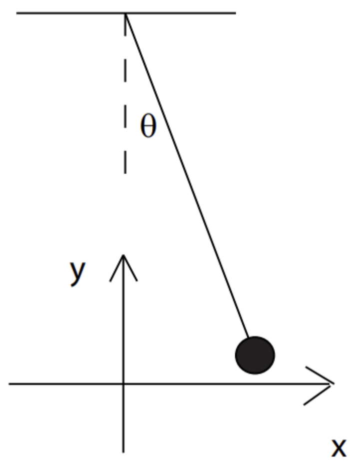


Figure 2: Pendulum

b. Assume that the angle θ is small and find the approximate simple harmonic equation of motion.

c. Justify your approximations. Find the range of θ that the pendulum can be considered a SHM. What is the period of oscillations of this SHM?

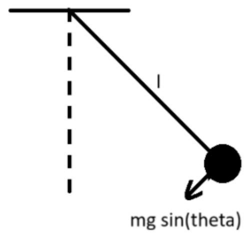
d. Calculate the exact potential energy of the pendulum as a function of θ. Then, show that the Taylor expansion leads to the same result as in part (b).

e. Parametrize the motion of the pendulum in terms of the cartesian coordinate x in the coordinate system with origin at the pendulum equilibrium position and x-axis horizontal in the plane of pendulum. Find the exact equation of motion ($\vec{F} = m\vec{a}$) of the pendulum in terms of x.

f. Assume that x is small and find the approximate simple harmonic equation of motion.

g. Justify your approximations. Find the range of x such that the pendulum can be considered a SHM.

a.



From the Torque equation we get,

$$\tau = I\ddot{\theta} = -mg\sin(\theta)l$$

$$ml^2\ddot{\theta} = -mg\sin(\theta)l$$

$$\ddot{\theta} = -\frac{g\sin(\theta)}{l}$$

b.

If θ is very small $\sin(\theta) = \theta$
Therefore,

$$\ddot{\theta} = -\frac{g}{l}\theta$$

c.

The exact equation of motion of the pendulum is

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

To obtain simple harmonic motion we approximate

$$\sin\theta \approx \theta.$$

This is justified only when the nonlinear terms in the Taylor expansion of $\sin\theta$ are much smaller than the linear term:

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots .$$

For the motion to remain approximately harmonic, we require the cubic term to be negligible compared to the linear term:

$$\left| \frac{\theta^3}{6} \right| \ll |\theta|.$$

Canceling θ (for $\theta \neq 0$), this becomes

$$\frac{\theta^2}{6} \ll 1.$$

If we demand that the cubic correction be less than 5% of the linear term, we impose

$$\frac{\theta^2}{6} < 0.05, \qquad \Rightarrow \qquad \theta^2 < 0.3.$$

Thus

$$|\theta| \lesssim \sqrt{0.3} \approx 0.55 \text{ rad} \approx 30^\circ.$$

The pendulum executes simple harmonic motion to good accuracy for $|\theta| \lesssim 0.5$ rad $(\approx 30^\circ)$

Within this range the approximation $\sin\theta \approx \theta$ is valid and the equation of motion reduces to

$$\ddot{\theta} + \frac{g}{L}\theta = 0,$$

the standard S.H.M. form.

d.

Let the Potential Energy at the Lowest Position be U_0
At an Angular Displacement of θ, the height of the pendulum bob from its Lowest Position is

$$l(1 - \cos(\theta))$$

This gives the Potential Energy as

$$U(\theta) = U_0 + mgl(1 - \cos(\theta))$$

Using Taylor expansion we get,

$$U(\theta) = U_0 + mgl(1 - (1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots))$$

$$U(\theta) = U_0 + mgl\left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \cdots\right)$$

Approximating upto the Quadratic term we get,

$$U(\theta) \approx U_0 + \frac{1}{2}(mgl)\,\theta^2$$

This is the potential of a simple harmonic oscillator with effective "spring constant" $k_{\text{eff}} = mg\ell$ in the angular coordinate. Thus the small–angle equation of motion follows from

$$ml^2\ddot{\theta} = -\frac{dU}{d\theta} \approx -(mgl)\,\theta,$$

or equivalently

$$\ddot{\theta} + \frac{g}{l}\,\theta = 0.$$

Comparing with $\ddot{\theta} + \omega^2\theta = 0$ gives the familiar result

$$\omega = \sqrt{\frac{g}{l}}$$

(Notes: the additive constant U_0 is irrelevant. The neglected terms such as $-\frac{mgl}{24}\theta^4$ quantify the error of the harmonic approximation; requiring those to be small reproduces the small-angle condition used in part (c).)

e.

From the Force equation along the x axis we get,

$$F = m\ddot{x} = -mg\sin(\theta)\cos(\theta)$$

$$\ddot{x} = -g\sin(\theta)\cos(\theta)$$

Note that,

$$\sin(\theta) = \frac{x}{l} \text{ and } \cos(\theta) = \frac{\sqrt{l^2 - x^2}}{l}$$

Therefore we get,

$$\ddot{x} = -\frac{g}{l}\frac{\sqrt{l^2 - x^2}}{l}x$$

f.

If x is very small $x << l$ Then we get

$$\frac{\sqrt{l^2 - x^2}}{l} \approx \frac{l}{l} = 1$$

Then our equation becomes

$$\ddot{x} = -\frac{g}{l}x$$

g.

From part (e), the exact equation of motion in the horizontal coordinate x is

$$\ddot{x} = -\frac{g}{l}\,\frac{\sqrt{l^2 - x^2}}{l}\,x = -\frac{g}{l}\sqrt{1 - \left(\frac{x}{l}\right)^2}\,x.$$

To obtain SHM we replaced

$$\sqrt{1 - \left(\frac{x}{l}\right)^2} \approx 1.$$

Expanding this term,

$$\sqrt{1-u} = 1 - \frac{u}{2} - \frac{u^2}{8} - \cdots, \qquad u = \left(\frac{x}{l}\right)^2,$$

gives

$$\sqrt{1 - \left(\frac{x}{l}\right)^2} = 1 - \frac{x^2}{2l^2} + \cdots.$$

Thus the nonlinear correction is of order

$$\frac{x^2}{2l^2}.$$

For the SHM approximation to be valid, this correction must be much smaller than 1:

$$\frac{x^2}{2l^2} \ll 1.$$

Hence the condition for simple harmonic motion is

$$\boxed{|x| \ll l.}$$

If we require the correction to be less than 5%, then

$$\frac{x^2}{2l^2} < 0.05 \quad \Longrightarrow \quad |x| < 0.316\,l.$$

$|x| \lesssim 0.3\,l$ **for SHM with good accuracy.**

This agrees with part (c), since for small oscillations $x \approx l\theta$.