8.03SC Problem Set 2

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Problem 2.2

Problem 2.2 (20 pts)

In a lab close to the Large Hadron Collider, a delicate crystal of mass M is supported by four massless springs in parallel, each with spring constant k. The whole setup is put on a table. When the graduate students in the lab move the table across the floor, the tabletop vibrates, producing an effective vertical force $F = MA_0 \cos(\omega_d t)$ on the mass M in the tabletop reference frame.

a. Let x be the vertical displacement of the crystal from its equilibrium position. Write down the equation of motion of the instrument. (You can assume that there is no drag force in this part of the question)

b. Find the vibration amplitude of the crystal (the mass M) in the steady state. (Assuming that there are very small energy losses in this system such that the homogeneous solution die out as $t \to \infty$.)

c. To reduce this vibration amplitude of the crystal in (a) by a factor of ten, how would you propose to modify the four springs, i.e. how much longer (shorter) should the new springs be? Assume $k/M \gg \omega_d^2$. (Hint: the spring constant of a spring is proportional to its area and inversely proportional to its length.)

d. A better way to reduce this vibration amplitude by a factor of ten from what it originally was is to insert some kind of a soft massless cushion between the crystal and the table, in parallel to the springs. Assuming that the cushion produces a resistive force $-b$ times the velocity of M, derive an equation that allows you to determine the value of b in terms of k, M, ω_d and solve for b, for $k/M \gg \omega_d^2$.

Net Spring constant

$$k_{eff.} = 4k$$

a.

$$M\ddot{x} + 4kx = F = MA_0 \cos(\omega_d t)$$

b.

$$\ddot{x} + \frac{4k}{M}x = A_0 \cos(\omega_d t)$$

$$\implies x = (c_1 \cos\left(\sqrt{\frac{4k}{M}}t\right) + c_2 \sin\left(\sqrt{\frac{4k}{M}}t\right)) + \frac{A_0 \cos(\omega_d t)}{(-\omega_d^2 - \frac{4k}{M})}$$

Since the Homogeneous solution dies

$$x = \frac{A_0 \cos(\omega_d t)}{(-\omega_d^2 + \frac{4k}{M})}$$

$$\implies x = \frac{MA_0}{-M\omega_d^2 + 4k}\cos(\omega_d t)$$

Therefore, **Amplitude is given by**

$$x_{max} = \frac{MA_0}{4k - M\omega_d^2}$$

c.

If $k/M >> \omega_d^2$ then,

$$x_{max} = \frac{MA_0}{4k}$$

So to reduce the Amplitude by a factor of a ten, the k needs to be made 10 times.

This can be achieved by making each Springs one-tenth the current length.

d.

Our force equation becomes

$$M\ddot{x} + b\dot{x} + 4kx = MA_0\cos(\omega_d t)$$

$$\implies \ddot{x} + \frac{b}{M}\dot{x} + \frac{4k}{M}x = A_0\cos(\omega_d t)$$

Solving this by assuming that the Homogeneous solution dies, we get

$$x = \frac{A_0}{\frac{b^2\omega_d^2}{M^2} + (\frac{4k}{M} - \omega_d^2)^2}\Big(\frac{b\omega_d}{M}\sin(\omega_d t) + (\frac{4k}{M} - \omega_d^2)\cos(\omega_d t)\Big)$$

$$\implies x = \frac{MA_0}{\sqrt{(4k - M\omega_d^2)^2 + (b\omega_d)^2}}\cos(\omega_d t - \arctan(\frac{b\omega_d}{4k - M\omega_d^2}))$$

Considering, $k/M >> \omega_d^2$

$$x_{max} = \frac{MA_0}{\sqrt{16k^2 + b^2\omega_d^2}}$$

By comparing,

$$\frac{MA_0}{\sqrt{16k^2 + b^2\omega_d^2}} = \frac{1}{10}\frac{MA_0}{4k}$$

$$\implies 16k^2 + b^2\omega_d^2 = 1600k^2$$

$$\implies b = \frac{\sqrt{1584}k}{\omega_d}$$

Problem 2.4

Problem 2.4 (20 pts)

Consider a mass m moving on a horizontal air track. The mass is attached on both sides to two identical springs each with spring constant k and relaxed length ℓ_0 (see Figure 2). The end of the first spring is fixed. The end of the second spring is attached to an electrical motor that causes it to undergo harmonic motion with amplitude Δ and angular frequency ω. At $t = 0$ the springs are relaxed at $\ell = \ell_0$ and the mass is not moving $\dot{x}(0) = 0$. Define the origin of the coordinate system to be at the position of the mass at $t = 0$ such that $x(0) = 0$

At $t = 0$ the motor has been switched on such that the end of the spring starts to move according to the following equation:

$$x_{end} = \Delta \sin(\omega t) + \ell_0$$

Assume that the motion of the mass is affected by a small velocity dependent air friction, $-b\dot{x}$.

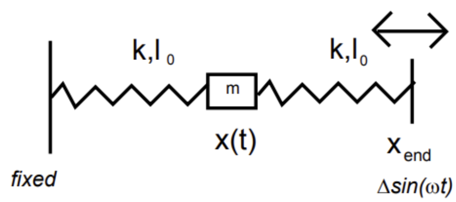

Figure 2: Classroom demonstration: mass on the air track

a. Set up carefully a one dimensional equation of motion for mass m including all the forces. Organize your equation neatly to clearly indicate the oscillator terms and the terms related to the external force.
b. Postulate the complete solution for the motion of the mass $x(t) = x_{free}(t) + x_{driven}(t)$ without solving. Indicate which constants in the solution depend only on oscillator properties, which on the properties of external force and which need to be determined using the initial conditions.
c. Find the amplitude and phase of the steady state motion of the mass. Sketch their dependence on the frequency of the driving force ω and the given parameters.
d. Find the frequency ω_{max} for which the amplitude is maximum.
e. Use the initial conditions to find the specific solution including both the free oscillator and the steady state solution with the appropriate choice of all parameters.

a.

Equation of Motion comes out to be

$$m\ddot{x} = -2kx - b\dot{x} + k\Delta \sin(\omega t)$$

$$\implies \ddot{x} + \frac{b}{m}\dot{x} + \frac{2k}{m}x = \frac{k\Delta}{m}\sin(\omega t)$$

b.

Since the equation of motion is linear, the displacement can be written as the sum of a free (homogeneous) solution and a driven (particular) solution,

$$x(t) = x_{\text{free}}(t) + x_{\text{driven}}(t)$$

The free solution depends on the initial conditions and represents the natural motion of the system; due to the presence of damping, this contribution decays with time. The driven solution is determined solely by the external driving and oscillates at the driving frequency.

At sufficiently long times, the free motion dies out and the observed motion is entirely given by the driven (steady–state) solution.

c.

The Steady State Motion corresponds to the Particular Solution of the Equation of Motion.
It is given by

$$x_{driven} = \left\{ \frac{1}{D^2 + \frac{b}{m}D + \frac{2k}{m}} \right\} \left[\frac{k\Delta}{m} \sin(\omega t) \right] \text{ here } D \equiv \frac{d}{dt}$$

$$\implies x_{driven} = \frac{k\Delta}{\sqrt{b^2\omega^2 + (2k - m\omega^2)^2}} \cos\left(\omega t - \arctan\left(\frac{b\omega}{2k - m\omega^2} \right) \right)$$

Therefore,

$$\textbf{Amplitude}(A_P) = \frac{k\Delta}{\sqrt{b^2\omega^2 + (2k - m\omega^2)^2}}$$

$$\textbf{Phase}(\delta_P) = \arctan\left(\frac{b\omega}{2k - m\omega^2} \right)$$

d.

For $A_P = \frac{k\Delta}{\sqrt{b^2\omega^2+(2k-m\omega^2)^2}}$ to be maximum, $b^2\omega^2 + (2k - m\omega^2)^2$ should be minimum.
Let

$$f(\omega) = b^2\omega^2 + (2k - m\omega^2)^2$$

Solving $\frac{df}{d\omega} = 0$, we get,

$$\frac{df}{d\omega} = 2b^2\omega + 2(2k - m\omega^2)(-2m\omega) = 0$$

$$\implies \omega = \sqrt{\frac{2k}{m} - \frac{b^2}{2m^2}}$$

Therefore for maximum amplitude, we need

$$\omega_{max} = \sqrt{\frac{2k}{m} - \frac{b^2}{2m^2}}$$

e.

For finding x_{free}, we need to solve the Complementary Equation of the equation of motion.

$$r^2 + \frac{b}{m}r + \frac{2k}{m} = 0$$

$$r = \frac{-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - (4)(1)(\frac{2k}{m})}}{2}$$

$$\implies r = \frac{-b}{2m} \pm \frac{\sqrt{b^2 - 8km}}{2m}$$

In the question it is mentioned that the Velocity dependent Air Friction is small. So

$$b^2 < 8km$$

Let

$$\gamma = \frac{b}{2m} \textbf{ and } \omega_o = \frac{\sqrt{8km - b^2}}{2m}$$

Therefore, the Solution given by the Complementary Equation corresponds to

$$x_{free} = e^{-\gamma t}(c_1 \cos(\omega_o t) + c_2 \sin(\omega_o t))$$

c_1 **and** c_2 **are arbitrary constants**

Hence equation of motion comes out to be

$$x(t) = x_{\textbf{free}}(t) + x_{\textbf{driven}}(t)$$

$$\implies x(t) = e^{-\gamma t}(c_1 \cos(\omega_o t) + c_2 \sin(\omega_o t)) + A_P \cos(\omega t - \delta_P)$$

Solving for $x(t) = 0$, we get

$$c_1 = -A_P \cos(\delta_P)$$

Solving for $\dot{x}(t) = 0$, we get

$$c_2 = \frac{\gamma c_1 - \omega A_P \sin(\delta_P)}{\omega_o}$$

$$c_2 = -\frac{A_P(\gamma \cos(\delta_P) + \omega \sin(\delta_P))}{\omega_o}$$

1 Problem 2.5

Problem 2.5 (20 pts)

Consider a system of three springs and two masses as shown in Figure 3, where the masses are constrained to move only in the vertical direction The spring constants are $K_A = 78$, The system is prepared in a lab on Earth and the gravitation force is pointing downward. $K_B = 60$ and $K_C = 24$ measured in $[N/m]$. The two masses are $m_1 = 4$ and $m_2 = 12$ measured in $[kg]$. The unstretched lengths of spring A and B equal the unstretched length C.

a. Define coordinate system(s) and write equations of motion for the two masses.

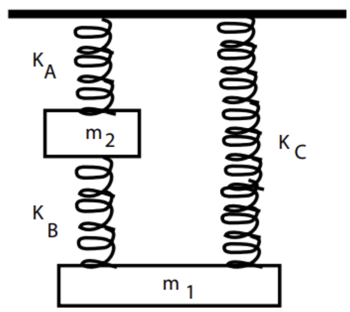

Figure 3: Two masses hanging vertically on springs

b. Write the equation of motion in matrix form. Show clearly the matrix $M^{-1}K$ as defined in the textbook.

c. Find the normal modes of oscillations and their associated angular frequencies.

d. If the mass m_1 is displaced by 1 cm up from its equilibrium position and m_2 is held at its original equlibrium position and both blocks released from rest at $t = 0$, write expressions for the subsequent motion of both blocks.

a.

The Coordinate System is defined as : The Direction towards the ceiling i.e. the Upward Direction is considered $+x$ The Opposite Direction i.e. the Downward Direction is called $-x$

- The Displacement of m_1 is termed as x_1
- The Displacement of m_2 is termed as x_2

Equations of Motion are given by:

$$m_1\ddot{x}_1 = -k_B(x_1 - x_2) - k_C(x_1)$$

$$m_2\ddot{x}_2 = -k_A(x_2) - k_B(x_2 - x_1)$$

b.

The Equations of motion can be re-written as

$$m_1\ddot{x}_1 = -(k_B + k_C)x_1 - (-k_B)x_2$$

$$m_2\ddot{x}_2 = -(-k_B)x_1 - (k_A + k_B)x_2$$

Expressing this in the form of a Matrix Equation where:

- Matrix M denotes the Mass Distribution of the system.
- Matrix X denotes the Displacement Vectors of each mass of the system.
- Matrix K denotes the Distribution of Stiffness in-between the masses of the system.

Hence representing the system in terms of

$$M\ddot{X} = -KX$$

We get,

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; K = \begin{bmatrix} k_B + k_C & -k_B \\ -k_B & k_A + k_B \end{bmatrix}$$

$$\implies M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

Therefore we get,

$$M^{-1}K = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} k_B + k_C & -k_B \\ -k_B & k_A + k_B \end{bmatrix} = \begin{bmatrix} \frac{k_B+k_C}{m_1} & -\frac{k_B}{m_1} \\ -\frac{k_B}{m_2} & \frac{k_A+k_B}{m_2} \end{bmatrix}$$

c.

For finding Normal Modes, we need to solve

$|M^{-1}K - \omega^2 I| = 0$ Here ω represent Angular frequencies of Normal Modes

$$\implies \left| \begin{bmatrix} \frac{k_B+k_C}{m_1} & -\frac{k_B}{m_1} \\ -\frac{k_B}{m_2} & \frac{k_A+k_B}{m_2} \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\implies \begin{vmatrix} \frac{k_B+k_C}{m_1} - \omega^2 & -\frac{k_B}{m_1} \\ -\frac{k_B}{m_2} & \frac{k_A+k_B}{m_2} - \omega^2 \end{vmatrix} = 0$$

This gives the Corresponding Angular Frequencies of the Normal Modes to be

$$\omega_1 = \sqrt{\frac{1}{2}\left[\left(\frac{k_B + k_C}{m_1} + \frac{k_A + k_B}{m_2}\right) - \sqrt{\left(\frac{k_B + k_C}{m_1} - \frac{k_A + k_B}{m_2}\right)^2 + \frac{4k_B^2}{m_1 m_2}}\right]}$$

$$\omega_2 = \sqrt{\frac{1}{2}\left[\left(\frac{k_B+k_C}{m_1}+\frac{k_A+k_B}{m_2}\right)+\sqrt{\left(\frac{k_B+k_C}{m_1}-\frac{k_A+k_B}{m_2}\right)^2+\frac{4k_B^2}{m_1 m_2}}\right]}$$

d.

The General Equation of Motion for the 2 masses are given by

$$x_1(t) = A_1\cos(\omega_1 t+\delta_1) + A_2\cos(\omega_2 t+\delta_2)$$

$$x_2(t) = \alpha_1 A_1\cos(\omega_1 t+\delta_1) + \alpha_2 A_2\cos(\omega_2 t+\delta_2)$$

$x_1(0) = 1$ gives us

$$A_1\cos(\delta_1) + A_2\cos(\delta_2) = 1$$

$x_2(0) = 0$ gives us

$$\alpha_1 A_1\cos(\delta_1) + \alpha_2 A_2\cos(\delta_2) = 0$$

$x_1'(0) = 0$ gives us

$$-A_1\omega_1\sin(\delta_1) - A_2\omega_2\sin(\delta_2) = 0$$

$x_2'(0) = 0$ gives us

$$-\alpha_1 A_1\omega_1\sin(\delta_1) - \alpha_2 A_2\omega_2\sin(\delta_2) = 0$$

From these I.C.s, we get,

$$\delta_1 = \delta_2 = 0 \ ; \ A_1 = \frac{\alpha_2}{\alpha_2-\alpha_1} \ ; \ A_2 = -\frac{\alpha_1}{\alpha_2-\alpha_1} \ ; \ \alpha_j = \frac{k_B+k_C-m_1\omega_j^2}{k_B}, j = 1,2$$