Numerical Computation of Rank Probabilities for the M6 Financial Forecasting Competition

Abstract

A fast method of computing rank probabilities of random variables is presented, making no assumptions about the distributions of the variables.

Keywords: Thurston choice models, finance, forecasting competitions, multivariate time series, Price forecasting, volatility forecasting

1. Motivation

As laid out in the M6 Financial Duathlon Competition guideslines, part of the challenge posed to participants is the creation of rank probabilities for returns of stocks and ETFs arranged in portfolios of size five (See Makridakis, Cirillo, Spiliotic, Assimenos & Michailidis (2021)). As noted therein, the realised total returns of all assets in each asset class over one month are divided into quintiles, ranking from 1 (worst performing) to 5 (best performing). Subsequently, a Ranked Probability Score (RPS) is applied: at heart a Brier score using a five-by-five table whose $(i, k)'th$ entry is the assigned probability that asset i's return is ranked k'th out of five.

There are many approaches to the estimation of rank probabilities. As with previous competitions the M6 encourages diversity in approach. That said, it is not unreasonable to postulate that as an intermediate step, a model for the joint distribution of future returns might be employed in some if not many methodologies. While remaining ambivalent on the matter of the generative model, this work provides a convenient alternative to Monte Carlo, which would be the customary last step of that analysis.

To make the discussion concrete Figure 1 depicts some hypothetical skew-normal return distributions for five assets. These have been generated on a lattice with 100 points representing increments of 0.1. The choice of distribution and scale is arbitrary, and for that matter results are preserved under any strictly monotonic transformation (such as exponentiation). Any set of distributions can be employed, whether analytic forms exist for them or not.

2. Method

Informally speaking we'd like to estimate all five placegetter probabilities, given five performance distributions, for all five contestants in a race.

But due to the possibility of ties, some minor distinctions between state prices, as they shall be termed here, and probabilities can arise. This presentation adopts that financial terminology, in keeping with Cotton (2021) and, as a fine point, it is expectations of functionals of order statistics that shall be computed, not probabilities.

The distinction is a commonplace one and not terribly profound - it arises in the splitting of a lottery payout between all winners, in tournaments were ties are common, and in dividends determined by the parimutuel system, common at racetracks. Some readers may prefer to assume the probability of ties is small, and treat state prices as synonymous with probability.

2.1. Distributions supported on a lattice, approximate translation, and state prices

We consider only return distributions f that are supported on an equally-spaced lattice of values. We use F to denote the (cumulative) distribution function and f the density, though both are, under this assumption, synonymous with vectors. This is not a serious loss of generality, provided the algorithm supports reasonably sized lattices, which it does. Nor is there loss of generality in assuming the distributions are supported on integers - though as a convenience that isn't assumed in the accompanying code.

Following Cotton (2021) we define, for any $f(\cdot) : \mathcal{Z} \to \mathcal{R}$ and any $a \in \mathcal{R}$ a shifted distribution $f^{\to a}(\cdot)$ also supported on the integers $\mathcal{Z}$:

$$f^{\to a}(j) := (1-r)f^{\to \lfloor a \rfloor}(j) + rf^{\to \lfloor a \rfloor + 1}(j) \tag{1}$$

where $r = a - \lfloor a \rfloor$ is the fractional part of the shift a. This extends the obvious right shift operator applicable when a is an integer. Formerly $f^{\to \lfloor a \rfloor}(j) := f(j - \lfloor a \rfloor)$. If $f(\cdot)$ is the distribution for X then $f^{\to a}$ approximates the distribution $X + a$.

Now suppose X_i represents the one month return of stock i, and $X^{(1)}$ the first order statistic (minimum) of the returns. As an example the quantity

$$p_{i,1} = E\left[\frac{\iota_{X_i = X^{(1)}}}{\sum_{j=1}^{5} \iota_{X_j = X^{(1)}}}\right] \tag{2}$$

is, ignoring ties, the probability that the i'th stock is ranked 1'st. We refer to this as a state price. Here ι is the indicator function and the denominator counts ties.

2.2. Functionals of independent densities

Let $\mathcal{F} = \{f_i\}_{i=1}^5$ be a collection of densities supported on a subset of the integers $\{-L, \dots, L\}$ extending symmetrically around zero. Treated as vectors, these atomic densities are converted to cumulative distribution functions via the cumulative sum operation. Conversely they are converted back by prepending zero and then differencing. We will denote the respective cumulative distribution vectors as $\{F_i\}_{i=1}^5$.

Any collection of discrete densities, so represented, implies a vector p_i of state prices which can be interpreted as the winning probabilities in a contest where the *lowest* score wins, and where ties result in split winnings. As noted in Cotton (2021) this can be estimated using the notion of multiplicity: the expected number of ties conditioned on a choice of winning score.

2.3. Review of multiplicity calculus and removal of one asset

To proceed formerly, recall from Cotton (2021) that any group of horses A can be characterized by a three tuple $\Upsilon_A = (f_A(), S_A(), m_A())$ at least as far as the winning performance is concerned. The first two introduce redundancy, but that is convenient in the code. Here f_A is the first order statistic density (winning density) and the winning survival function S_A is derived by $S_A(j) = Prob(X^{(1)} > j) = 1 - F_A(j)$ where F_A is the winning cumulative distribution. The multiplicity $m(j)$ reports the expected number of ties assuming a winning score j.

Combining groups A and B into a single portfolio relies on the following estimate for combined multiplicity.

$$m_{A\cup B}(j) = \frac{m_A(j)f_A(j)S_B(j) + (m_A(j) + m_B(j))f_A(j)f_B(j) + m_B(j)f_B(j)S_A(j)}{f_A(j)S_B(j) + f_A(j)f_B(j) + f_B(j)S_A(j)} \tag{3}$$

The corresponding operation for $S's$ is mere multiplication, whereupon we recover winning density also. As utilized in Cotton (2021) the multiplicity relations can be inverted to enable us to remove the i'th asset. The multiplicity with i left out:

$$m_{\hat{i}}(j) = \frac{m(j)f_i(j)S_{\hat{i}}(j) + m(j)f_i(j)f_{\hat{i}}(j) + m(j)f_{\hat{i}}(j)S_i(j) - m_i(j)f_i(j)S_{\hat{i}}(j)}{f_{\hat{i}}(f_i + S_i)} \tag{4}$$

2.4. Approximate state prices for lowest score

We compute losing probability for asset i by removing it from the race, using formula 4. Then an approximate win state price can be estimated by considering all possible values j taken by asset i and lowest return j' taken by the other assets in the portfolio.

$$\begin{aligned} p_i &= \sum_{j,j'} f_i(j) f_{\hat{i}}(j') E\left[\frac{W}{M} | X_i = j, X_{\hat{i}}^{(1)} = j'\right] \\ &\approx \sum_j f_i(j) \left\{ \frac{f_{\hat{i}}}{1 + m_{\hat{i}}(j)} + S_{\hat{i}}(j) \right\} \end{aligned} \tag{5}$$

as shown in more detail in Cotton (2021).

2.5. Approximate state prices for combinations

Next we outline a method of pricing so-called quinella wagers. These require a patron to predict which two horses will finish first and second, but do not require specification of order. To date there has not existed a fast method of pricing quinellas permitting arbitrary performance distributions in either the racing or choice model literature, yet we can proceed as follows:

1. Remove two horses i and j from the race using repeated application of Equation 4.
2. Compute the distribution of the worst performance of horses i and j, and the multiplicity.
3. Compare the worst performance of the two horses to the rest of the runners, represented by the density and multiplicity achieved in the first step.

This yields excellent approximations of these symmetric, combinatorial prices.

Naturally one makes the elementary observation that by summation, quinella prices imply prices for finishing *either* first or second. And since the winning probabilities are already computed, one can compute second place probabilities by subtraction.

2.6. Rank probabilities in a five horse race

This procedure obviously generalizes to computation of symmetric state prices involving three or more horses. However quinella pricing is sufficient for the M6 Competition.

That's because by symmetry one can compute fourth and fifth place probabilities in a five horse race - simply by reversing the vectors that represent discrete performance distributions and applying the previous algorithm.

Then the state price for third, which is loosely speaking the third place probability, can be computed for each horse by subtracting the first, second, forth and fifth place state prices from unity.

2.7. Dependence

Having established a route to the computation of five-way rank probabilities in the case of independent assets, we now consider the case where they are dependent. Depending on the computational budget and structure assumed, quadrature can be used. We give only the example of a Gaussian copula, and therein the special case where off diagonal entries are all equal to a constant parameter $0 \leq \rho < 1$.

The return X_i of the i'th asset can be assumed to satisfy

$$X_i = F_i^{-1}\left(\Phi(Z_i)\right)$$

where F_i is the cumulative distribution on the lattice (the 1-margin), Φ is the standard cumulative normal distribution, and Z_i is an auxiliary random variable with standard normal distribution. (As written this is merely a restatement of the distributional transform, or the definition of F_i.)

In the independent case all assets are assumed to satisfy this generative model, with corresponding Z_i's that are independent and identically distributed. A simplistic dependence structure now introduces a connection between the Z_i's, such as

$$Z_i = \rho Z + \sqrt{1-\rho^2}\epsilon_i$$

where $Z \sim N(0,1)$ is a common factor, ρ is a correlation parameter and the ϵ_i's are independent N(0,1). All state prices can be viewed as iterated expectations, first conditioning on a choice of Z. If we let $F_i(\cdot; z)$ denote the cumulative distribution of the i'th asset return knowing $Z = z$ then rearranging we have

$$F_i(x; z) = \Phi\left(\frac{\Phi^{-1}(F_i(x)) + \rho z}{\sqrt{1-\rho^2}}\right) \tag{6}$$

Thus given any vector valued linear functional G acting on collections of densities, we can define $g(z)$ as the action of G on the set of densities transformed according to Equation 6 with parameter z. Then, the integral

$$\int_{z=-\infty}^{\infty} g(z)\phi(z)dz$$

can be estimated using standard methods such as Gaussian quadrature.

In our example, this results in an M6 submission given in Table 2 when $\rho = 0.1$ is chosen. However when $\rho = 0.25$ is considered instead, we see slightly different values as reported in Table 3. Because the common factor influences conditional probabilities in monotone fashion, the results are not overly sensitive to choice of correlation. The reader will observe that the rounded answers are almost identical to two significant digits.

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
location	-0.50	-0.25	0.00	1.00	1.50
scale	1.00	1.50	1.20	1.30	2.00

Table 1: Location and scale parameters for skew-normal return distributions used to illustrate the rank probability approach.

3. Uses

As noted in Section 2 the most obvious application of the numerical algorithm presented starts with an assumption of asset return densities (and a gaussian copula dependence parameter ρ). Viewed in this manner, the computational device is merely a tool and not in any way a prescription. However some brief comments are supplied on some ways in which it might be used.

3.1. Imputation of five-way rank probabilities

Figure 4 depicts a partially completed submission. Here an opinion has been expressed as to the probability that a stock will be the worst performing of the five, but no other information is supplied.

	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5
Asset 1	0.37	0.33	0.20	0.08	0.02
Asset 2	0.32	0.24	0.21	0.15	0.08
Asset 3	0.20	0.26	0.28	0.19	0.07
Asset 4	0.04	0.10	0.19	0.37	0.31
Asset 5	0.07	0.08	0.12	0.21	0.52

Table 2: Example submission, $\rho = 0.1$.

	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5
Asset 1	**0.38**	0.33	0.20	0.08	0.02
Asset 2	0.32	0.24	0.21	0.15	**0.07**
Asset 3	0.20	0.26	0.29	**0.18**	0.07
Asset 4	0.04	**0.09**	0.19	0.37	0.31
Asset 5	**0.06**	**0.08**	0.12	0.21	0.53

Table 3: Example submission, $\rho = 0.25$.

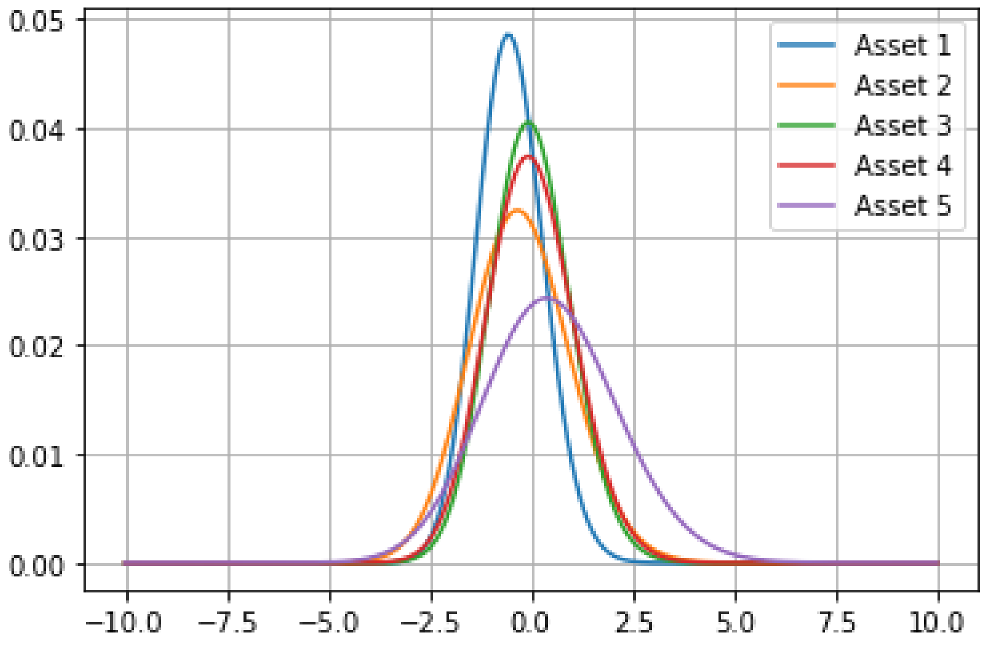

Figure 1: Skew-normal return distributions for five assets.

	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5
Asset 1	0.25	?	?	?	?
Asset 2	0.15	?	?	?	?
Asset 3	0.20	?	?	?	?
Asset 4	0.22	?	?	?	?
Asset 5	0.18	?	?	?	?

Table 4: A partially completed competition entry showing only worst-performer probabilities.

	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5	Assumed 1	Discrepancy
Asset 1	**0.252**	0.218	0.197	0.178	0.156	**0.250**	0.002
Asset 2	**0.149**	0.176	0.198	0.222	0.255	**0.150**	-0.001
Asset 3	**0.200**	0.203	0.202	0.200	0.195	**0.200**	0.000
Asset 4	**0.220**	0.210	0.201	0.191	0.178	**0.220**	0.000
Asset 5	**0.179**	0.193	0.202	0.209	0.216	**0.180**	-0.001

Table 5: Imputed five-way rank probabilities taking Table 4 as a starting point. A gaussian correlation of $\rho = 0.25$ has been assumed, and this leads to a small differential in Rank 1 probabilities.

This information is insufficient to determine the remaining probabilities, of course, but we can illustrate one of the many uses of the provided numerical techniques (using also those in Cotton (2021)) by making an assumption about the underlying generative model.

Suppose, for illustration, that we assume all five assets have similar volatilities and, for that matter, identical return profiles up to a translation. The imputation proceeds by calibrating the relative location parameters of the return distributions. Figure 2 depicts return distributions consistent with Table 4 where, as can be seen, we remark that only very small differences in return densities are required.

Using the procedure provided in Section 2 the corresponding five-way rank probabilities (state prices, to be pedantic) are reported in Table 5. Again, purely for illustration, the parameter $\rho = 0.25$ has been used. The penultimate column repeats the assumed probabilities taken from Table 4 and the last column reports the discrepancy after calibration.

The small discrepancy noted arises from two sources. One is due to numerical issues discussed in Cotton (2021) that, in our example, modify the losing probabilities by roughly one part in a

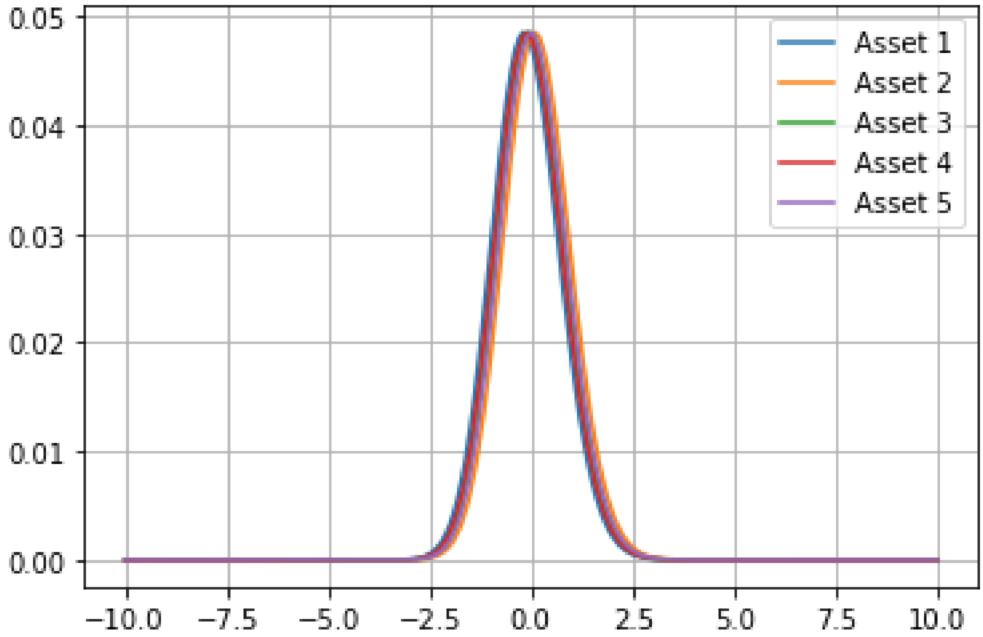

Figure 2: Implied relative return distributions consistent with Table 4.

million.

By far the largest component in the last column is due to inconsistency in the calibration approach. Technically speaking, one should calibrate assuming $\rho = 0.25$ whereas we have calibrated using $\rho = 0$ implicitly. However the errors are still quite small, as can be seen, and so a more complex procedure might not be warranted.

3.2. Ex-ante analysis of submissions

It should be clear that there are many other types of imputation are possible. Certainly, imputation from best-stock probabilities rather than worst is virtually identical.

In the example above we have used the calibration procedure given in Cotton (2021) but this was designed to accommodate very large races - up to hundreds of thousands of participants if not more. In five-horse races we have many other options at our disposal and these lend themselves to ex-ante analysis of submissions.

One class of approach takes submitted rank probabilities, removes k randomly chosen entries from the table, and then calibrates (using a derivative free optimizer) numerous choices of generative model and correlation parameter. By seeking to infer a model that is consistent with the supplied submission one might hope to form a prior opinion as to the reasonableness of the supplied rank probabilities, and also suggest changes to some of those values supplied.

It is plausible that shrinkage towards a calibrated generative model, achieved in this fashion, might improve ex-ante performance. That is a hypothesis, however, that awaits the competition itself.

Another hypothesis is that overly simplistic assumptions about rank-probabilities can be discerned in some entries, and that this alone might help predict their performance in the contest. For instance some entries might, contrary to the approach suggested herein, reinvent the Harville model Harville (1973) (in which Rank 4 probabilities are assumed to be renormalized Rank 5 probabilities. This independence assumption is also known as Luce's Axiom of Choice - see Cotton (2021) for discussion).

3.3. Races with more than five runners

It goes without saying that the approach generalizes beyond five horse races, albeit with an equally obvious rise in computation that is proportional to the number of ways of choosing $\lfloor n/2 \rfloor$ entrants out of n.

3.4. Generalizing beyond one-parameter dependence

The quadrature employed is a commonplace idea from factor models used in finance. Quadrature of this sort was used in the numerical pricing of synthetic collateralized debt obligations, devised by the present author for use by Morgan Stanley some 20 years ago (and independently invented at rival firms).

The dependence structure can be generalized in a rather obvious way by allowing more than one factor to vary. The accuracy and computational convenience is then subject to the efficacy of two, three or higher dimensional quadrature schemes.

4. Conclusion

A fast computation for five-way rank probabilities has been provided, and thereby a tool for creating plausible matrices required in one side of the M6 Financial Forecasting Duathlon.

The intent is to increase participation in the M6 competition by offering participants a streamlined way to convert some types of belief, or modeling outputs, into rank probabilities. This leaves the user in complete control over the generative model for asset returns and, to a lessor extent, the dependence structure between them.

As an intermediate step, a numerical procedure for pricing quinella bets has been presented. It is perhaps of note that this might be the first example of a quinella calculator which makes no assumption about performance distributions and is, therefore, quite general. A critique of prior art used at racetracks was given in Cotton (2021), but not the quinella calculation itself. As this result applies to order statistics in general, there are likely to be applications beyond the racetrack.

As regards participation in the M6 Competition, the use of numerical procedure brings the usual set of advantages over the more obvious Monte Carlo approach. For instance, it is much easier to discern the impact of changing assumptions and more practical to run backtests and estimation procedures.

Also in the context of the M6 competition, it has been shown that this newly developed technique can be wielded in other ways as well. Importantly, it might be be used to solicit opinions and forecasts from those who might not otherwise be willing to complete full five-way rank probability tables.

References

Cotton, P. (2021). Inferring Relative Ability from Winning Probability in Multientrant Contests. *SIAM Journal on Financial Mathematics*, *12*. doi:10.1137/19M1276261.

Harville, D. A. (1973). Assigning probabilities to the outcomes of multi-entry competitions. *Journal of the American Statistical Association*, . doi:10.1080/01621459.1973.10482425.

Makridakis, S., Cirillo, P., Spiliotic, V., Assimenos, N., & Michailidis, M. (2021). The M6 Financial Duathlon Competition (Draft). URL: `https://mofc.unic.ac.cy/wp-content/uploads/2021/10/M6-Guidelines-1.pdf`.