

Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: *Euler Path*: An *Euler Path* on a graph G is a path that traverses each edge exactly once.

Definition 1.2: *Euler Circuit/Cycle*: An Euler circuit on a graph G is a Euler Path which starts and ends on the same node.

Results to be proven:

1. Show that any graph where the degree of every vertex is even iff it has an Eulerian circuit.
2. Show that if there are exactly two vertices a and b of odd degree, there is an Eulerian path on the graph.

Proof :

1. \Rightarrow Let G be a graph such that it has an Euler circuit. According to the definition of Euler Circuit on a graph, given a vertex v_1 , we can traverse every other node only once and end at the vertex v_1 . Thus every node/vertex has one entry point and one exit point. We know this because G has an Euler circuit. Thus every node/vertex has two edges attached to it thus making it a node of even degree.

\Leftarrow Let every node in a graph G have an even degree. Let us prove this by induction. Assume, that there are only two nodes in G , each with two edges on them. Thus it is clear that you will end on the same node that you started with, thus making a Euler circuit on G .

Now, let G be graph of even degree with more than two nodes. Let's start with an arbitrary node v_1 . Now since every node has an even degree, we can follow a trail from v_1 to the next node and repeat it for the next node we encounter. We know that this is possible because each time we enter a node v_k we know there is an unused edge adjacent to it for us to use to traverse. Thus using a unique path we will come back on v_1 since there are even number of edges, we will always find an untapped edge to let us continue our path back to v_1 . Figure 1. 1Euler Circuit

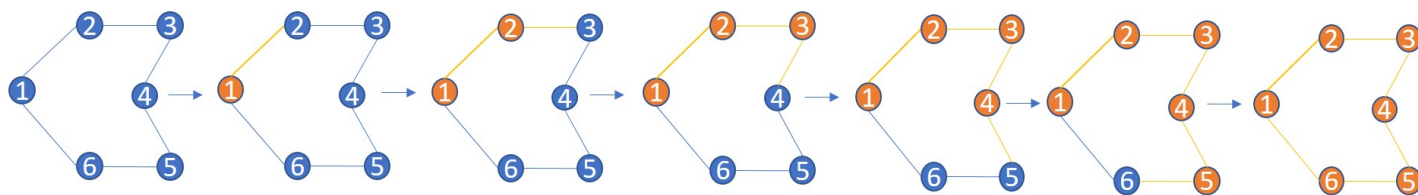


Figure 1: A Euler Circuit.

2. Let G be a graph with Euler circuit. Thus every node/vertex has an even degree. now let us add one node say b and add an edge to a node a in the existing graph G . Note that before adding the edge from b to a , a in G had an even degree. Now if we start drawing our path from b , and since it

has only one edge connecting to a we go to a now, if we hypothetically ignore the edge connecting a and b , the remainder of G has nodes with even edges, thus making it a Euler circuit. Therefore, the trail will end at a but since we have already used the edge connecting a and b , we stop at a . Thus we were able to traverse all the edges in $G + b$ exactly once, starting from b and ending at an edge a in G .

