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Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: ***Graph:*** A *graph* is a pair $G = (V, E)$. The elements of V are the vertices (or nodes or points) and the elements of E are edges (or lines) connecting the vertices.

So, I have to say, I don't see that these definitions offer much more precision than what you started with. There is a more precise way to define "connecting the vertices".

Note: The vertex on G are referred to as $V(G)$ and the edges on G are referred to $E(G)$. This is independent of the way we define a graph.

The plural of "vertex" is "vertices"! :)

I"m not sure what you mean by this.

reread this - the wording is off

Definition 1.2: ***Degree of a Vertex:***: Let $G = (V, E)$ be a non empty graph is the number of edges $(E(G))$ attached to a vertex $v \in V(G)$ in G.

There should be rigor in this too.

Definition 1.3: ***Euler Path:*** An *Euler Path* on a graph G is a path that traverses each edge exactly once.

there is also a rigorous way to define this

Definition 1.4: ***Euler Circuit/Cycle:*** An Euler circuit on a graph G is a Euler Path which starts and ends on the same node.

Not quite sure what you mean by this.

Definition 1.5: ***Walk***: A walk on a graph G is a unique path using edges and vertex of G.

Lemma: ***Nilay's Lemma (Not really):*** If a connected graph has every vertex of degree of at least two, then G has a *cycle*.

You haven't defined this

degree two

Proof. Let G be a finite graph. Let v be a vertex in G such that v has at least two degree. Let us construct a walk starting from v. Let v_1 be an adjacent vertex to v, v_2 be an adjacent vertex to v_1 and so on. So the walk we create will look like $v \to v_1 \to v_2 \to .... \to v_k$. We can do this recursively all $k > 1$ because of our hypothesis that each vertex has at least two degrees. Since G is finite graph, the number of vertices it has is limited. Thus, while constructing our walk we will encounter a vertex v_i which has already been traversed (already included in the walk). The path that was created from the first occurrence of v_i to the second one is a *cycle* from v_i to v_i. □

This can be defined rigorously too.

Still avoiding the rigor....

Results to be proven:

1. ***(EULER (1736), HIERHOLZER (1873))*** Any connected graph where the degree of every vertex is even iff it has an Euler circuit.

   Did you prove the other direction? I can't find it anywhere.

2. If there are exactly two vertices a and b of odd degree, there is an Euler path on the graph.

Proof :

1. $\Rightarrow$ Let G be a connected graph which has Euler circuit E. When traversing E, when we come across any vertex v through an edge $e_v(1)$, we know by definition there is another edge $e_v(2)$ that is connected to v. Thus making every vertex in G at least degree two. Thus making every vertex in G of even degree.

I actually don't see this. Do you use the same edge twice? Are there two edges between the two vertices?

Not a sentence….

⇐ Let us proceed by induction. Let every vertex in a connected graph G have an even degree. If there are only two vertex in G. Thus it is clear that you will end on the same vertex that you started with, thus making a Euler circuit on G.

C'mon, you know we don't say this in a proof.

Now, let G be connected graph with more than two vertices. From the lemma we know that there exists a cycle in G. If a cycle covers all the vertices in G then we are done. Let's say it does not. Then there exists a cycle C in G which does not include all the vertices. Now, let us remove all the

Which one? What's left?

edges from G that are in C. Call this new sub-graph H, by our hypothesis all the vertices in H are still even and thus H contains a cycle. Let us choose a common vertex v in C and H. We know

What does this mean? And why?

this is possible because there are no isolated vertices in G. Now since v still has an even degree we

Not quite sure what you mean

produce a cycle C' in H that starts and ends at v. If C' has all the remaining edges in H then we are done, and thus G has a Euler circuit. If not we continue the above process recursively until all

This is still lacking so much rigor.

the vertices in G are traversed. The final tour would be the union of all the cycles that we created recursively in G and return to the initial vertex in C where we started, thus making an Euler circuit in G.

This is not induction though. Where is your inductive hypothesis? What are you assuming holds for k and showing holds for k+1?

They say a picture speaks a thousand words, below we try to illustrate what an Euler Circuit will look like on a graph where all the vertex have an even degree.

The picture is nice for your presentation, but as you know, the proof needs to stand on its own without the picture.

I don't think this is obvious. How do you know that removing the edges in C does affect the degrees of the remaining vertices/

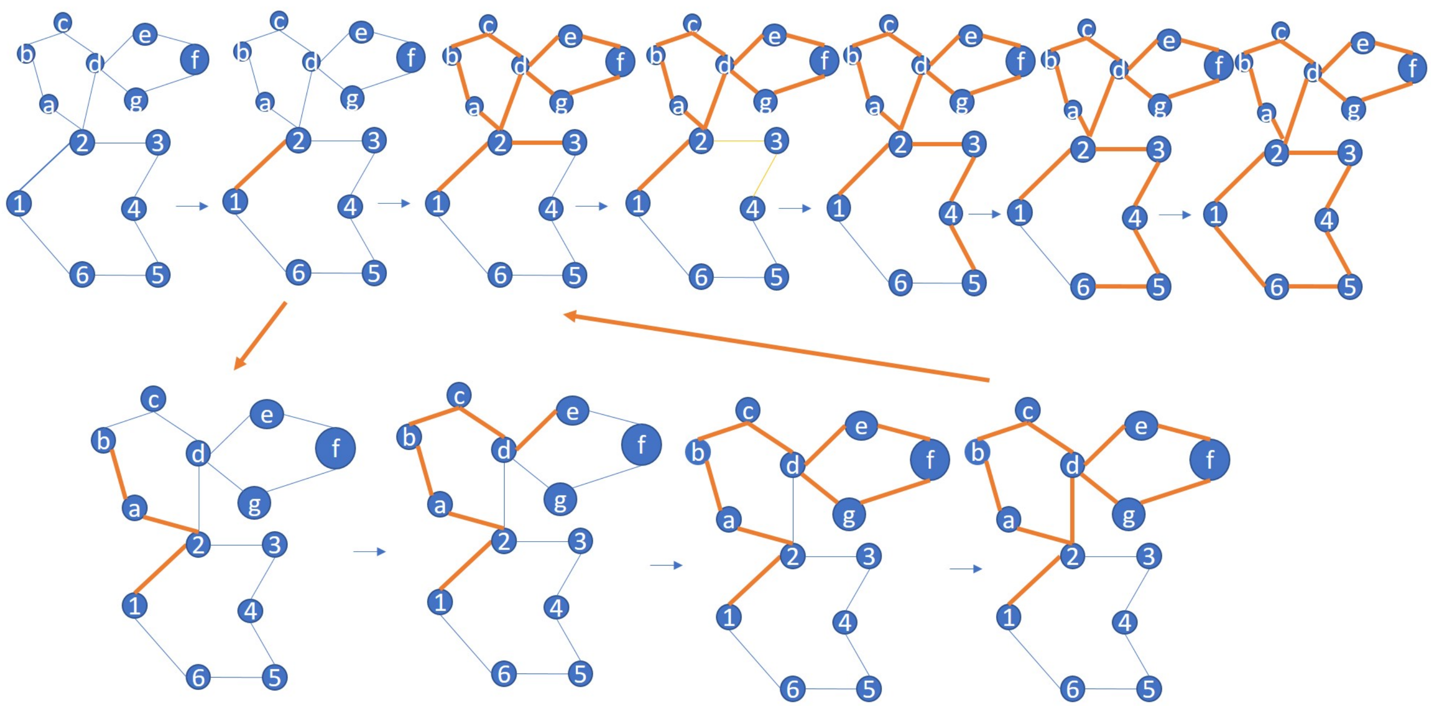

Figure 1: A Euler Circuit.

I can't find where you actually proved this.

Is a node the same as a vertex? (If so, need to define that.)

2. Let G be a graph with Euler circuit. Thus every node/vertex has an even degree. now let us add one node say b and add an edge to a node a in the existing graph G. Note that before adding the edge from b to a, a in G had an even degree. Now if we start drawing our path from b, and since it has only one edge connecting to a we go to a now, if we hypothetically ignore the edge connecting a and b, the remainder of G has nodes with even edges, thus making it a Euler circuit. Therefore, the trail will end at a but since we have already used the edge connecting a and b, we stop at a. Thus we were able to traverse all the edges in $G + b$ exactly once, starting from b and ending at an edge a in G.

incorrect notation

■

What other vertex does it connect to?