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Divisibility of Three

MATH 450 Seminar in Proof

A number is divisible by three if and only if the sum of the digits is divisible by 3.

Lemma: We are using the following properties of modular arithmetic:

1. $(A + B) \bmod C = ((A \bmod C) + (B \bmod C)) \bmod C$
   and
2. $(A * B) \bmod C = ((A \bmod C) * (B \bmod C)) \bmod C$

where A, B, C are integers. Actually C > 0 (we don't mod by negative numbers)

Proof. Let β be an integer. We can then write β as:

$$\beta = \beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i$$

where β_i are digits of β.
$\Rightarrow$ Let β be divisible by 3. Then $\beta \bmod 3 = 0$, thus we can write the expansion as:

$$\begin{aligned}
\beta \bmod 3 &= (\beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i) \bmod 3 \\
0 &= (\beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i) \bmod 3 \\
&= [(\beta_0 * 10^0) \bmod 3 + (\beta_1 * 10^1) \bmod 3 + (\beta_2 * 10^2) \bmod 3 + ... + (\beta_i * 10^i) \bmod 3] \bmod 3 \\
&= [(\beta_0 \bmod 3 * 10^0 \bmod 3) \bmod 3 + (\beta_1 \bmod 3 * 10^1 \bmod 3) \bmod 3 \\
&+ (\beta_2 \bmod 3 * 10^2 \bmod 3) \bmod 3 + ... + (\beta_i \bmod 3 * 10^i \bmod 3) \bmod 3] \bmod 3 \\
&= [(\beta_0 \bmod 3 * 1) \bmod 3 + (\beta_1 \bmod 3 * 1) \bmod 3 + (\beta_2 \bmod 3 * 1) \bmod 3 \\
&+ ... + (\beta_i \bmod 3 * 1) \bmod 3] \bmod 3 \\
&= [\beta_0 \bmod 3 + \beta_1 \bmod 3 + \beta_2 \bmod 3 + ... + \beta_i \bmod 3] \bmod 3 \\
0 &= [\beta_0 + \beta_1 + \beta_2 + ... + \beta_i] \bmod 3
\end{aligned}$$

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Thus if a number is divisible by three then the sum if its digits is also divisible by three.
You're not actually redefining beta here
$\Leftarrow$ Let β be a number such that the sum of its digits $\beta_0 + \beta_1 + \beta_2 + ... + \beta_i$ is divisible by 3. From the algebra done in the equations above, and by the definition of equality, we can follow the last equation

$$[\beta_0 + \beta_1 + \beta_2 + ... + \beta_i] \bmod 3 = 0$$

from bottom up and thus β is also divisible by 3.
Thus if the sum of the digits of a number are divisible by 3 then then number itself is divisible by 3.

$\square$

I think you can combine the algebra here the way we discussed, i.e. don't substitute 0 on either end and just show beta mod 3 = (sum) mod 3. Then the two directions fall out almost instantly. The algebra is used in both so in some sense it doesn't make sense to associate it with the first direction specifically.