MATH 450 Seminar in Proof

Prove that the set of odd negative integers is countable.

Proof. We know that a set X is countable if there exists a bijection in $f : \mathbb{N} \to X$. Let

$$f(n) = \begin{cases} -n & \text{if } n \text{ is odd} \\ -1-n & \text{if } n \text{ is even} \end{cases}$$

We will show that $f : \mathbb{N} \to -2\mathbb{Z} - 1$ is well defined and is a bijection. We first show that the function has a valid co-domain $f(n) \in -2\mathbb{Z} - 1$ for every $n \in N$. If n is even then $n = 2k$, $k > 0$, $k \in \mathbb{Z}$. Then $f(n) = -1 - n = -1 - 2k$. Since $2k > 0$ and is even, $-1 - 2k < 0$ and is odd. Thus, $f(n) \in -2\mathbb{Z} - 1$. If n is odd then $n = 2k - 1$, where $k > 0$ and $k \in \mathbb{Z}$. Then $f(n) = -n = -2k - 1$. Therefore $f(n) \in -2\mathbb{Z} - 1$. Thus our function has a valid co-domain. Now we will prove that it is bijective.

One-to-One:

Let $f(a) = f(b)$ where $a, b \in \mathbb{N}$. Then since, $f(a) = f(b)$ we know that either $f(a) = -a$ and $f(b) = -b$ or $f(a) = -1-a$ and $f(b) = -1-b$. In both the cases we get $a = b$ if $f(a) = f(b)$. Thus the function is one-to-one. If compare the expressions in a different way such as, $f(b) = -1-b$ and $f(a) = -a$ without loss of generality and if $f(a) = f(b)$ then we have $b = a - 1$. This is not valid as if $a = 1$ then $b = 0$ thus $b \notin \mathbb{N}$

Onto:

Let $y \in -2\mathbb{Z} - 1$. Then $y < 0$. Thus, $f(-y) = y$ or $f(-1 - y) = -1 - (-1 - y) = y$. Thus for every $y \in -2\mathbb{Z} - 1$ there exits $n \in \mathbb{N}$ such that $n = -y$ if n is odd or $n = -1 - y$ if n is even. Hence, the function is onto.

Thus $f : \mathbb{N} \to 2\mathbb{Z} - 1$ with

$$f(n) = \begin{cases} -n & \text{if } n \text{ is odd} \\ -1 - n & \text{if } n \text{ is even} \end{cases}$$

is bijective. Thus the set of odd integers are countable. $\square$