Nilay Bhatt Feb. 19 2017

Divisibility of Three

MATH 450 Seminar in Proof

A number is divisible by three if and only if the sum of the digits is divisible by 3.

Lemma: We are using the following properties of modular arithmetic:

1. $(A + B) \bmod C = ((A \bmod C) + (B \bmod C)) \bmod C$
   and
2. $(A * B) \bmod C = ((A \bmod C) * (B \bmod C)) \bmod C$

where A, B, C are integers and $C > 0$.

Proof. Let β be an integer. We can then write β as:

$$\beta = \beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i$$

where β_i are digits of β.
If we divide β by 3 then, we can write the expansion as:

$$\begin{aligned}
\beta \bmod 3 &= (\beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i) \bmod 3 \\
\beta \bmod 3 &= (\beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i) \bmod 3 \\
&= [(\beta_0 * 10^0) \bmod 3 + (\beta_1 * 10^1) \bmod 3 + (\beta_2 * 10^2) \bmod 3 + ... + (\beta_i * 10^i) \bmod 3] \bmod 3 \\
&= [(\beta_0 \bmod 3 * 10^0 \bmod 3) \bmod 3 + (\beta_1 \bmod 3 * 10^1 \bmod 3) \bmod 3 \\
&+ (\beta_2 \bmod 3 * 10^2 \bmod 3) \bmod 3 + ... + (\beta_i \bmod 3 * 10^i \bmod 3) \bmod 3] \bmod 3 \\
&= [(\beta_0 \bmod 3 * 1) \bmod 3 + (\beta_1 \bmod 3 * 1) \bmod 3 + (\beta_2 \bmod 3 * 1) \bmod 3 \\
&+ ... + (\beta_i \bmod 3 * 1) \bmod 3] \bmod 3 \\
&= [\beta_0 \bmod 3 + \beta_1 \bmod 3 + \beta_2 \bmod 3 + ... + \beta_i \bmod 3] \bmod 3 \\
\beta \bmod 3 &= [\beta_0 + \beta_1 + \beta_2 + ... + \beta_i] \bmod 3
\end{aligned}$$

Thus if β is divisible by three then $\beta \bmod 3 = 0$ and thus the sum if its digits is also divisible by three that is: $[\beta_0 + \beta_1 + \beta_2 + ... + \beta_i] \bmod 3 = 0$.
Also if sum the digits of β written as $\beta_0 + \beta_1 + \beta_2 + ... + \beta_i$ is divisible by 3 then from the algebra done in the equations above, and by the definition of equality, we easily deduce that beta is also divisible by three

□