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Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: ***Graph:*** A **graph** G consists of a non-empty finite set $V(G)$ of elements called **vertices**, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called **edges**; the use of the word 'family' permits the existence of multiple edges. We call $V(G)$ the vertex set and $E(G)$ the edge family of G. An edge e_{vw} is said to join the vertices v and w, and is usually abbreviated to vw.

Note: The vertices in G are referred to as $V(G)$ and the edges on G are referred to $E(G)$. This is independent of the way we define a graph. Meaning, if we define a graph $H = (W, Q)$ the set of vertices in H is referred as $V(H) = W$ and the set of edges in H is referred as $E(H) = Q$ and not $W(H)$ and $Q(H)$ respectively.

Definition 1.2: ***Adjacency:***: We say that two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and f are **adjacent** if they have a vertex in common.

Definition 1.3: ***Degree of a Vertex:***: The degree of a vertex v of G is the number of edges connected with v, and is written $deg(v)$; in calculating the degree of v, we usually make the convention that a loop at v contributes 2 (rather than 1) to the degree of v. A vertex of degree 0 is an **isolated** vertex and a vertex of degree 1 is an **end-vertex**.

Note: A graph is ***connected*** if it cannot be expressed as the union of two distinct graphs, and **disconnected** otherwise.

Definition 1.4: ***Subgraph:***A **subgraph** H of a graph G is a graph, such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Note: if $G = (V, E)$ be a graph then $G \cup G = G$ where we preserve the cardinality and mapping in G for $V(G)$ and $E(G)$.

Definition 1.5: ***Walk***: Given a graph G, a **walk** in G is a finite sequence of distinct edges of the form v_0v_1, v_1v_2,...,$v_{m-1}v_m$, also denoted by $v_0 \to v_1 \to v_2 \to .... \to v_m$, in which any two consecutive edges are adjacent. If $v_0 = v_m$ then we call the walk a **cycle**.

Definition 1.6: ***Euler Path:*** An **Euler Path** on a graph G is a special walk that uses each edge exactly once.

Definition 1.7: ***(Euler Circuit/Cycle:*** An **Euler circuit** on a graph G is an Euler Path with a cycle.

Definition 1.8: ***Traversing:*** The process of passing through each vertex of a walk or cycle in a graph G using the edges joining them in a walk or a path or a cycle.

Lemma: ***Nilay's Lemma (Not really):*** If a connected graph has every vertex of degree of at least two, then G has a *cycle*.

Proof. Let G be a connected graph. Let v_0 be a vertex in G such that v_0 has at least degree two. Let us construct a walk $v_0 \to v_1 \to v_2 \to ....$ such that v_1 be any adjacent vertex to v_0, and for each v_i where $i > 1$, we choose v_{i+1} to be any adjacent vertex to v_i, except v_{i-1}(already chosen). We know that such a vertex exists because of our hypothesis that every vertex is of at least degree two. Since G is finite graph, the number of vertices it has is finite. Thus, while constructing our walk we will eventually choose a vertex v_k which has already been chosen and included in the walk. If v_k is the first such vertex that we encounter, then the path that was created from the first occurrence of v_k to the second one is a cycle from v_k to v_k. $\square$

Results to be proven:

1. ***(EULER (1736), HIERHOLZER (1873))*** A connected graph G has an Euler Circuit if and only if the degree of each vertex of G is even.

   Proof. $\Longrightarrow$ Let G be a connected graph which has Euler circuit C. Whenever C passes through a vertex in $V(G)$ through an edge in $E(G)$, there is a contribution of 2 edges which are adjacent to the vertex, towards the degree of that vertex. From our hypothesis we know that edges in C are distinct, thus each vertex must have even degree.

$\Longleftarrow$ Let us proceed by induction. In a most basic connected graph G of no edges and one vertex, the proposition is vacuously true. If a connected graph G has one vertex $v \in V(G)$, then the number of edges in $E(G) = 1$, thus we start and end our Euler Circuit at v (loop, contribution of 2 towards the degree).

Now, let there be only two vertices in a connected graph G and each vertex is of degree two (making them even degree). Then since G is connected, there are no isolated vertices in G. Furthermore, those two vertices share the two edges between them. Therefore, if we construct a walk at either of the vertices in $V(G)$ we will end at the same vertex where we started, and not repeating the edge that we passed through. Thus making the walk an Euler Circuit in G.

Now, for our strong induction hypothesis, we say, if G is a graph where $|E(G)| \leq k, k \in \mathbb{Z}$, and all vertices in G have an even degree then there exists is an Euler Circuit in G.

Now, let G be a connected graph with $k+1$ vertices, where each vertex in G is of even degree. From the lemma we know that there exists a cycle in G. If that cycle includes all the edges in G then we are done. Let's say it does not. Then there exists a cycle C in G which does not include all the vertices of G.

Now, let us remove all the edges from G that are in C and obtain newly made sub-graph $H = (V(G), E(G) - E(C))$, made by the remaining edges in G; by our hypothesis degree all the vertices in H are still even. We know this because when we removed $E(C)$ from $E(G)$, we removed an even number of edges from each vertex from the cycle C formed in G.

Suppose H is still connected, *i.e.* there are no isolated vertices in H then H is a graph with less than k edges, and thus H has a Euler Circuit from our hypothesis because $|E(H)| < k$. Also, since G is connected, there must be a common vertex $m \in V(C) \cap V(H)$. If there weren't any common vertex in C and H then, $C \cup H$ will form a disjoint union, thus making G disconnected. So, now we have an Euler circuit in G, where we start from any vertex $v \in V(C)$ and while traversing C, when we reach m, we traverse the Euler circuit in H, starting and ending at $m \in V(C) \cap V(H)$ and ending our cycle at v, thus traversing along all the edges in $E(G) = E(C) \cup E(H)$ once and all the vertex in $V(G) = V(C) \cup V(H)$.

When we removed C from G, the other possibility was that we may have H disconnected. Thus $H = H_1' \sqcup H_2' \sqcup H_3' \sqcup ... \sqcup H_i'$. H is formed from a disjoint union of even degree connected sub-graphs H_i'. Note that, for each such H_i', $\exists v_i \in V(C) \cap V(H_i)$. Since $|E(H_i')| < k$, from our hypothesis, each H_i' has an Euler Circuit, made by one C_i', or multiple cycles C_{i_j}.

We can now build an Euler circuit for G. Pick an arbitrary vertex $a \in V(C) \subset V(G)$ from C. Traverse along C starting from a until we reach a vertex $v = V(C) \cap V(H_i')$. Then, traverse along H_i''s Euler circuit starting from v made by the cycle(s) $C'_{i_{j)}}$ in H_i'.

Now we are back to v, and so we continue along C, and do the same for each such $v_i \in V(C) \cap V(H_i')$ we encounter and after traversing each edge in $E(G) = E(C) \cup E(H_1') \cup E(H_2') \cup E(H_3') \cup ... \cup E(H_i')$ exactly once and all the vertices in $V(G) = V(C) \cup V(H_1') \cup V(H_2') \cup V(H_3') \cup ... \cup V(H_i')$, we obtain our desired Euler path in G starting and ending at $a \in V(G)$. $\square$

They say a picture speaks a thousand words, below we try to illustrate what an Euler Circuit will look like on a graph where all the vertex have an even degree.

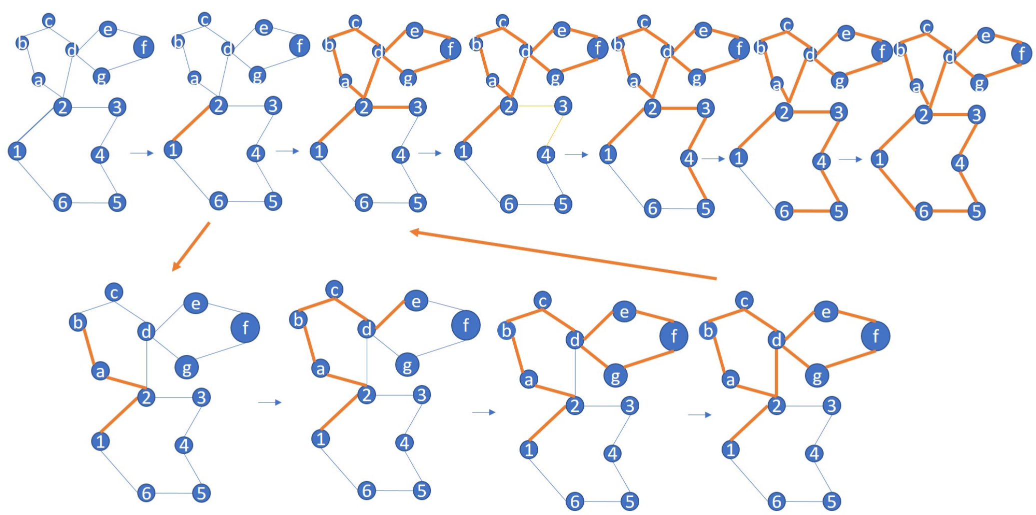

Figure 1: A Euler Circuit.