Euler paths and circuits on digraphs and genome sequencing

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Overview

1. Definitions
2. Eulerian Graphs
3. Applications

Definitions

Definition (Graph)

A **graph** G consists of a non-empty finite set $V(G)$ of elements called **vertices**, and a finite 'family' $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called **edges**.

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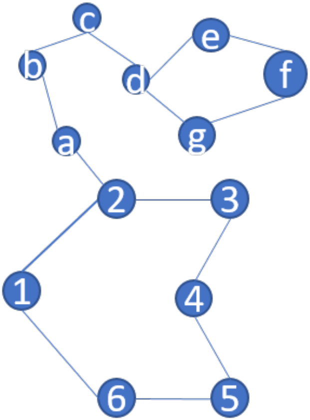
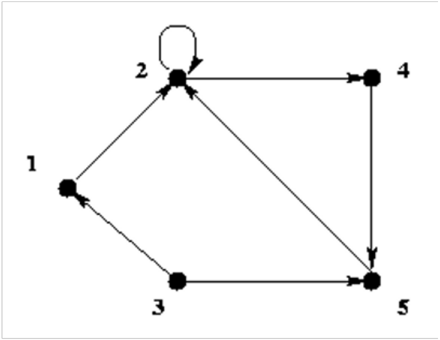
- NYC grid with all the streets being one-way, which then restricts our movement on the graph.

Graphs and Digraphs





Euler Paths vs. Circuits

Definition (Euler Path)

An **Euler path** on a graph G is a special walk that uses each edge exactly once, and it starts and ends at **different** vertices.

Definition (Euler Circuit)

An **Euler circuit** on a graph G is a walk that uses each edge exactly once, and it starts and ends at the **same** vertex.

Criterion for an Euler path or circuit on a graph

Euler Path

A given graph G has an Euler path if and only if the graph is connected and **all but 2 vertices in the graph are of odd degree.**

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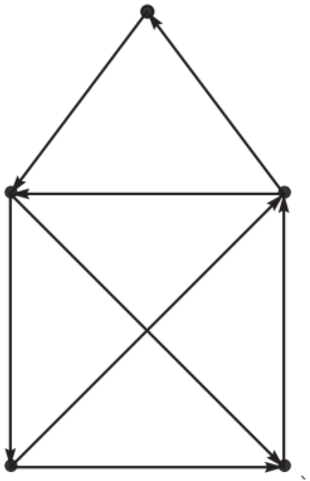
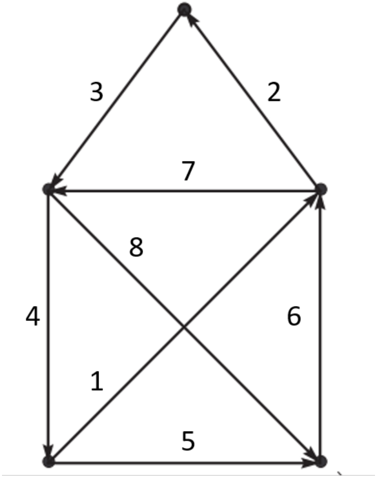
- What extra condition do you think we need to add for a digraph to have an Euler path?

Can we find an Euler Path?



YES WE CAN!



Criterion for an Euler path on a digraph

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Euler Path on a digraph

A digraph D_g has an Euler path iff

- The graph is connected
- All but 2 vertices in the graph are of odd degree
- **the $|\textbf{indegree} - \textbf{outdegree}| = 1$ for those two odd vertices in D_g.**

Criterion for an Euler circuit on a digraph

Euler Circuit

A graph G has an Euler circuit if and only if G is connected and all the vertices are of **even degree** .

Criterion for an Euler circuit on a digraph

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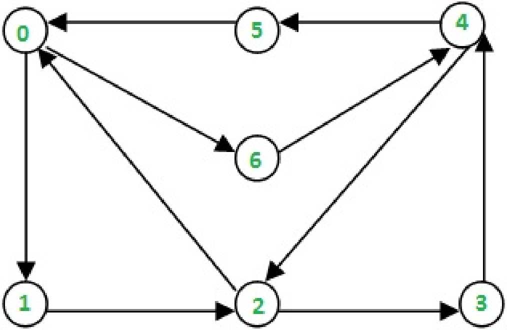
- What extra condition do you think will be needed on a digraph for it to have an Euler circuit?

Euler Circuit on digraph



Euler Circuit : 0 -> 6 -> 4 -> 5 -> 0 -> 1 -> 2 -> 3 -> 4 -> 2 -> 0

Criterion for an Euler circuit on a digraph

Euler Circuit

A graph G has an Euler circuit if and only if G is connected and all the vertices are of **even degree** .

Euler circuit on a digraph

A graph D_g has an Euler circuit if and only if D_g is connected and all the vertices are of even degree and the **indegree** $=$ **outdegree** for all vertices.

- A formal proof done as part of the final project in Senior Seminar (Spring 2017)

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Recall: For an Euler Path all but 2 vertices must be of odd degree and to have an Euler circuit has all vertices must be of even degree.

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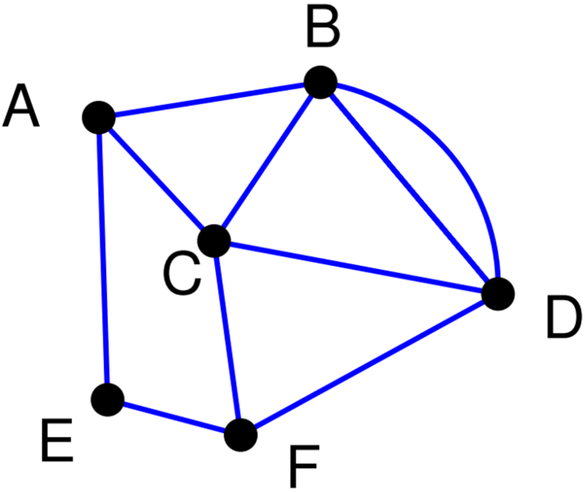
This is called **Fleury's Algorithm**

Find an Euler Path?



What's next?

This is cool, but for all the applied mathematicians in the room and computer engineers like me the question becomes:

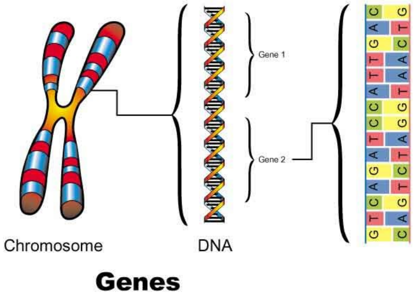
How does that help me?

Applications

1. Mathematical modeling
2. **DNA fragment reconstruction**
3. Postman problem
4. Traveling salesman problem
5. many more ....

Genome Sequencing

DNA



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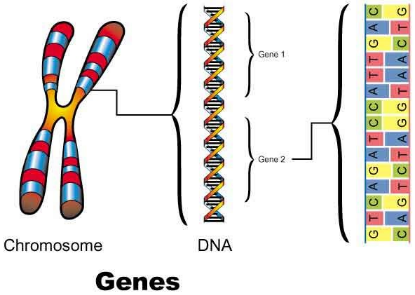

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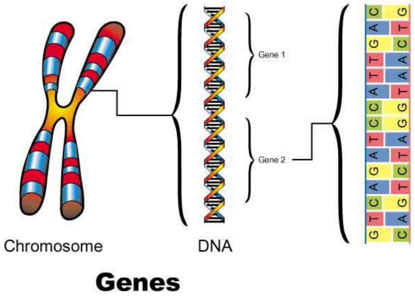


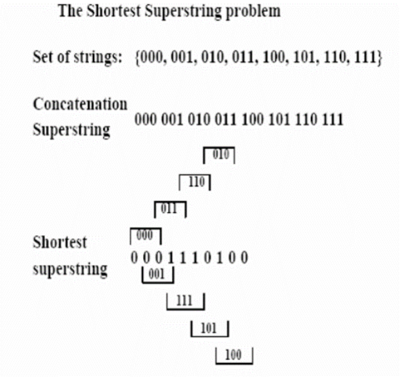
- Protein that stores the genetic information of a living organism
- Four components to it: A,T,G,C. A connects to T and G connects to C
- Fragments of DNA are used in genetic research and discovery
- Strings of such ATGC pairs have this information stored in them

Genome Sequencing

- Reads are taken of a known DNA strand (Sanger, 1977)
- They vary in size from 30-800 nucleotides
- Reads are taken in an overlapping form
- Reconstructing the specific genome with the overlapping to create a superstring is NP-complete

Shortest Super String Problem

Problem: **Given a set of strings, find a shortest string that contains all of them** - NP Complete



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- Consolidate the duplicate edges (collapse them, while conserving the directed edges
- The final directed Euler Path is the super string

What is a $k - mer$ composition of a given genome string

Given a string : $\boldsymbol{TAATGCCATGGGATGTT}$, what is a 3-mer composition?
$= \{TAA, AAT, ATG, TGC, GCC, CCA, CAT,$
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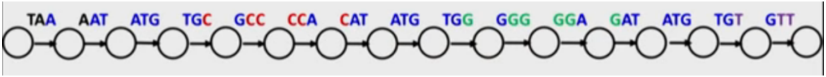

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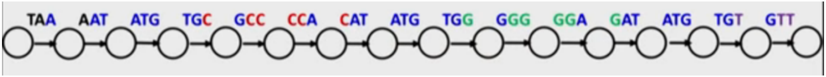
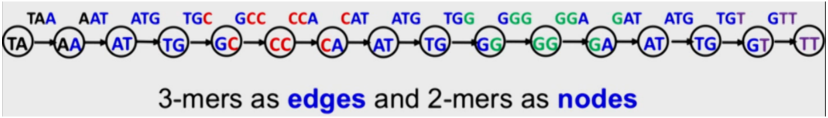
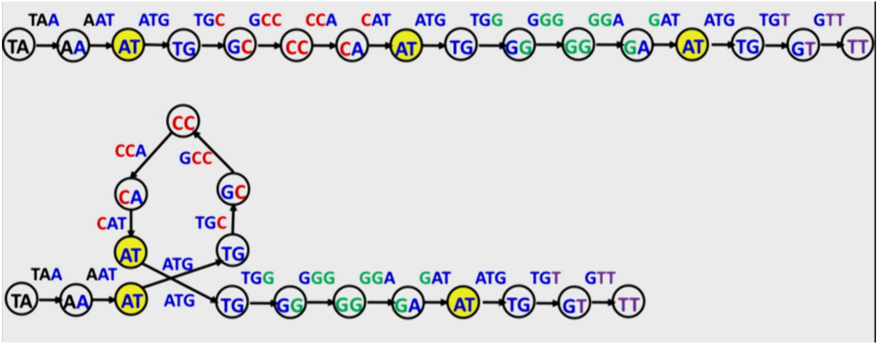




Collapsing like vertices on itself



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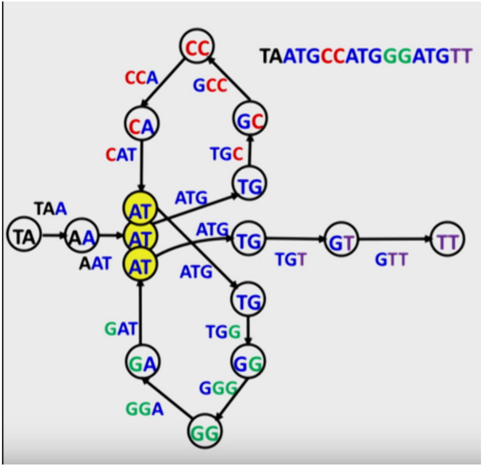

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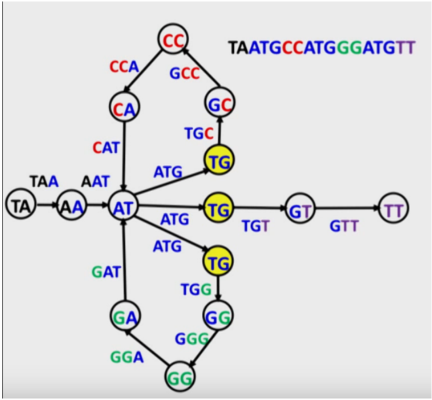
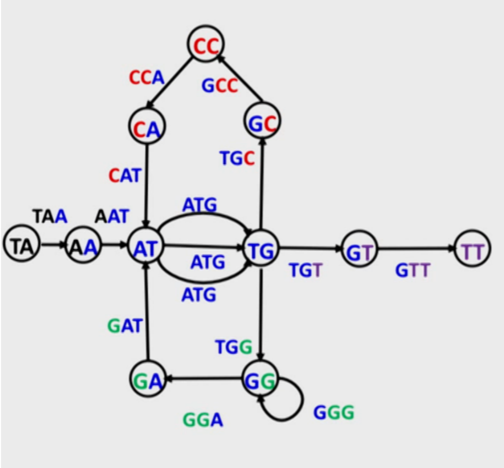


DeBruijn Graph of TAATGCCATGGGATGTT



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- What if there are multiple Euler paths?
- That is where we use paired Debruijn graphs (A special form of 4-mer and 6-mer pairing)

References

John Smith (2012)
Title of the publication
Journal Name 12(3), 45 – 678.

The End