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Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: ***Graph:*** A *graph* is a pair $G = (V, E)$. The elements of V are the vertices (or nodes or points) and the elements of E are edges (or lines) connecting the vertices.

Note: The vertex on G are referred to as $V(G)$ and the edges on G are referred to $E(G)$. This is independent of the way we define a graph.

Definition 1.2: ***Degree of a Vertex:***: Let $G = (V, E)$ be a non empty graph is the number of edges $(E(G))$ attached to a vertex $v \in V(G)$ in G.

Definition 1.3: ***Euler Path:*** An *Euler Path* on a graph G is a path that traverses each edge exactly once.

Definition 1.4: ***Euler Circuit/Cycle:*** An Euler circuit on a graph G is a Euler Path which starts and ends on the same node.

Definition 1.5: ***Walk***: A walk on a graph G is a unique path using edges and vertex of G.

Lemma: ***Nilay's Lemma (Not really):*** If a graph has a degree of at least two, then G has a *cycle*.

Proof. Let G be a finite graph. Let v be a vertex in G such that v has at least two degree. Let us construct a walk starting from v. Let v_1 be an adjacent vertex to v, v_2 be an adjacent vertex to v_1 and so on. So the walk we create will look like $v \to v_1 \to v_2 \to .... \to v_k$. We can do this recursively all $k > 1$ because of our hypothesis that each vertex has at least two degrees. Since G is finite graph, the number of vertices it has is limited. Thus, while constructing our walk we will encounter a vertex v_i which has already been traversed (already included in the walk). The path that was created from the first occurrence of v_i to the second one is a *cycle* from v_i to v_i. □

Results to be proven:

1. Any graph where the degree of every vertex is even iff it has an Euler circuit.

2. If there are exactly two vertices a and b of odd degree, there is an Euler path on the graph.

Proof :

1. $\Rightarrow$ Let G be a graph which has Euler circuit E. When traversing E, when we come across any vertex v through an edge $e_v(1)$, we know by definition there is another edge $e_v(2)$ that is connected to v. Thus making every vertex in G at least degree two. Thus making every vertex in G of even degree.

   $\Leftarrow$ Let us proceed by induction. Let every vertex in a graph G have an even degree. Assume, that there are only two nodes in G, each with two edges on them. Thus it is clear that you will end on the same node that you started with, thus making a Euler circuit on G.

Now,let G be graph of even degree with more than two nodes. Lets start with an arbitrary node v_1. Now since every node has an even degree, we can follow a trail from v_1 to the next node and repeat it for the next node we encounter. We know that this is possible because each time we enter a node v_k we know there is an unused edge adjacent to it for us to use to traverse. Thus using a unique path we will come back on v_1 since there are even number of edges, we will always find an untapped edge to let us continue our path back to v_1.

Figure 1. 1Euler Circuit

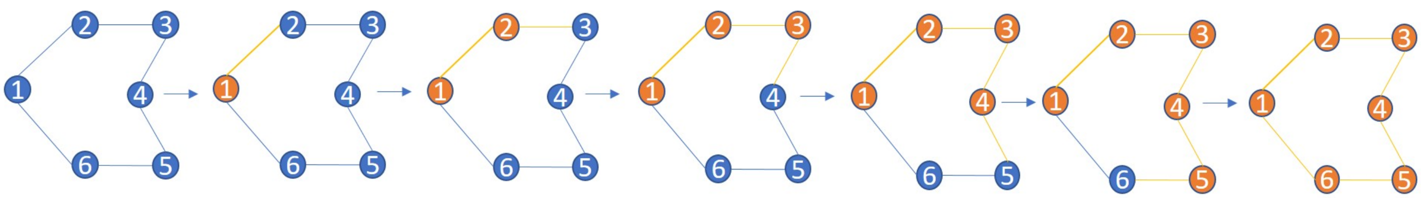

Figure 1: A Euler Circuit.

2. Let G be a graph with Euler circuit. Thus every node/vertex has an even degree. now let us add one node say b and add an edge to a node a in the existing graph G. Note that before adding the edge from b to a, a in G had an even degree. Now if we start drawing our path from b, and since it has only one edge connecting to a we go to a now, if we hypothetically ignore the edge connecting a and b, the remainder of G has nodes with even edges, thus making it a Euler circuit. Therefore, the trail will end at a but since we have already used the edge connecting a and b, we stop at a. Thus we were able to traverse all the edges in G $+b$ exactly once, starting from b and ending at an edge a in G.

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