Nilay Bhatt Feb. 19 2017

Divisibility of Three

MATH 450 Seminar in Proof

A number is divisible by three if and only if the sum of the digits is divisible by 3.

Proof. $\Rightarrow$ Let β be a whole number divisible by 3. We can then define β as:

$$
\begin{aligned}
\beta &= \beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i \\
(\beta) \mod 3 &= (\beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i) \mod 3 \\
0 &= (\beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i) \mod 3 \\
&= ((\beta_0 * 10^0) \mod 3 + (\beta_1 * 10^1) \mod 3 + (\beta_2 * 10^2) \mod 3 + ... + (\beta_i * 10^i) \mod 3) \mod 3 \\
&= (((\beta_0) \mod 3 * (10^0) \mod 3) \mod 3 + ((\beta_1) \mod 3 * (10^1) \mod 3) \mod 3 \\
&+ ((\beta_2) \mod 3 * (10^2)) \mod 3) \mod 3 + ... + ((\beta_i) \mod 3 * (10^i) \mod 3) \mod 3) \mod 3 \\
&= (((\beta_0) \mod 3 * 1) \mod 3 + ((\beta_1) \mod 3 * 1) \mod 3 + ((\beta_2) \mod 3 * 1) \mod 3) \\
&+ \mod 3 + ... + ((\beta_i) \mod 3 * 1) \mod 3) \mod 3 \\
&= ((\beta_0) \mod 3 + (\beta_1) \mod 3 + (\beta_2) \mod 3 + ... + (\beta_i) \mod 3) \mod 3 \\
0 &= (\beta_0 + \beta_1 + \beta_2 + ... + \beta_i) \mod 3
\end{aligned}
$$

We are using the properties of modular arithmetic over here. We are using the following properties:

1. $(A + B) \mod C = ((A \mod C) + (B \mod C)) \mod C$
   and
2. $(A * B) \mod C = ((A \mod C) * (B \mod C)) \mod C$

Thus if a number is divisible by three then the sum if it's digits are also divisible by three.

$\Leftarrow$ Letβ be a whole number such that the sum of its digits $(\beta_0 + \beta_1 + \beta_2 + ... + \beta_i)$ is divisible by 3. Then we can say:

$$
\begin{aligned}
(\beta_0 + \beta_1 + \beta_2 + ... + \beta_i) \mod 3 = 0 \\
((\beta_0) \mod 3 + (\beta_1) \mod 3 + (\beta_2) \mod 3 + ... + (\beta_i) \mod 3) \mod 3 = \\
(((\beta_0) \mod 3 * 1) \mod 3 + ((\beta_1) \mod 3 * 1) \mod 3 + ((\beta_2) \mod 3 * 1) \mod 3) + \\
\mod 3 + ... + ((\beta_i) \mod 3 * 1) \mod 3) \mod 3 = \\
(((\beta_0) \mod 3 * (10^0) \mod 3) \mod 3 + ((\beta_1) \mod 3 * (10^1) \mod 3) \mod 3 + \\
((\beta_2) \mod 3 * (10^2)) \mod 3) \mod 3 + ... + ((\beta_i) \mod 3 * (10^i) \mod 3) \mod 3) \mod 3 = \\
((\beta_0 * 10^0) \mod 3 + (\beta_1 * 10^1) \mod 3 + (\beta_2 * 10^2) \mod 3 + ... + (\beta_i * 10^i) \mod 3) \mod 3 = \\
(\beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + ... + \beta_i * 10^i) \mod 3 = 0 \\
(\beta) \mod 3 = 0
\end{aligned}
$$

Thus if the sum of the digits of a number are divisible by 3 then then number itself is divisible by 3.

$\square$