

## Divisibility of Three

### MATH 450 Seminar in Proof

**A number is divisible by three if and only if the sum of the digits is divisible by 3.**

**Lemma:** We are using the following properties of modular arithmetic:

1.  $(A + B) \bmod C = ((A \bmod C) + (B \bmod C)) \bmod C$   
and
2.  $(A * B) \bmod C = ((A \bmod C) * (B \bmod C)) \bmod C$

where  $A, B, C$  are integers.

*Proof.* Let  $\beta$  be an integer. We can then write  $\beta$  as:

$$\beta = \beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + \dots + \beta_i * 10^i$$

where  $\beta_i$  are digits of  $\beta$ .

$\Rightarrow$  Let  $\beta$  be divisible by 3. Then  $\beta \bmod 3 = 0$ , thus we can write the expansion as:

$$\begin{aligned}\beta \bmod 3 &= (\beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + \dots + \beta_i * 10^i) \bmod 3 \\ 0 &= (\beta_0 * 10^0 + \beta_1 * 10^1 + \beta_2 * 10^2 + \dots + \beta_i * 10^i) \bmod 3 \\ &= [(\beta_0 * 10^0) \bmod 3 + (\beta_1 * 10^1) \bmod 3 + (\beta_2 * 10^2) \bmod 3 + \dots + (\beta_i * 10^i) \bmod 3] \bmod 3 \\ &= [(\beta_0 \bmod 3 * 10^0 \bmod 3) \bmod 3 + (\beta_1 \bmod 3 * 10^1 \bmod 3) \bmod 3 \\ &\quad + (\beta_2 \bmod 3 * 10^2 \bmod 3) \bmod 3 + \dots + (\beta_i \bmod 3 * 10^i \bmod 3) \bmod 3] \bmod 3 \\ &= [(\beta_0 \bmod 3 * 1) \bmod 3 + (\beta_1 \bmod 3 * 1) \bmod 3 + (\beta_2 \bmod 3 * 1) \bmod 3 \\ &\quad + \dots + (\beta_i \bmod 3 * 1) \bmod 3] \bmod 3 \\ &= [\beta_0 \bmod 3 + \beta_1 \bmod 3 + \beta_2 \bmod 3 + \dots + \beta_i \bmod 3] \bmod 3 \\ 0 &= [\beta_0 + \beta_1 + \beta_2 + \dots + \beta_i] \bmod 3\end{aligned}$$

Thus if a number is divisible by three then the sum of its digits is also divisible by three.

$\Leftarrow$  Let  $\beta$  be a number such that the sum of its digits  $\beta_0 + \beta_1 + \beta_2 + \dots + \beta_i$  is divisible by 3. From the algebra done in the equations above, and by the definition of equality, we can follow the last equation

$$[\beta_0 + \beta_1 + \beta_2 + \dots + \beta_i] \bmod 3 = 0$$

from bottom up and thus  $\beta$  is also divisible by 3.

Thus if the sum of the digits of a number are divisible by 3 then the number itself is divisible by 3.

□