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Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: ***Graph:*** A *graph* is a pair $G = (V, E)$. The elements of V are the vertices (or nodes or points) and the elements of E are edges (or lines) connecting the vertices.

Note: The vertex on G are referred to as $V(G)$ and the edges on G are referred to $E(G)$. This is independent of the way we define a graph.

Definition 1.2: ***Degree of a Vertex:***: Let $G = (V, E)$ be a non empty graph is the number of edges $(E(G))$ attached to a vertex $v \in V(G)$ in G.

Definition 1.3: ***Euler Path:*** An *Euler Path* on a graph G is a path that traverses each edge exactly once.

Definition 1.4: ***Euler Circuit/Cycle:*** An Euler circuit on a graph G is a Euler Path which starts and ends on the same node.

Definition 1.5: ***Walk***: A walk on a graph G is a unique path using edges and vertex of G.

Lemma: ***Nilay's Lemma (Not really):*** If a connected graph has every vertex of degree of at least two, then G has a *cycle*.

Proof. Let G be a finite graph. Let v be a vertex in G such that v has at least two degree. Let us construct a walk starting from v. Let v_1 be an adjacent vertex to v, v_2 be an adjacent vertex to v_1 and so on. So the walk we create will look like $v \to v_1 \to v_2 \to .... \to v_k$. We can do this recursively all $k > 1$ because of our hypothesis that each vertex has at least two degrees. Since G is finite graph, the number of vertices it has is limited. Thus, while constructing our walk we will encounter a vertex v_i which has already been traversed (already included in the walk). The path that was created from the first occurrence of v_i to the second one is a *cycle* from v_i to v_i. □

Results to be proven:

1. ***(EULER (1736), HIERHOLZER (1873))*** Any connected graph where the degree of every vertex is even iff it has an Euler circuit.

2. If there are exactly two vertices a and b of odd degree, there is an Euler path on the graph.

Proof :

1. $\Rightarrow$ Let G be a connected graph which has Euler circuit E. When traversing E, when we come across any vertex v through an edge $e_v(1)$, we know by definition there is another edge $e_v(2)$ that is connected to v. Thus making every vertex in G at least degree two. Thus making every vertex in G of even degree.

$\Leftarrow$ Let us proceed by induction. Let every vertex in a connected graph G have an even degree. If there are only two vertex in G. Thus it is clear that you will end on the same vertex that you started with, thus making a Euler circuit on G.
Now, let G be connected graph with more than two vertices. From the lemma we know that there exists a cycle in G. If a cycle covers all the vertices in G then we are done. Let's say it does not. Then there exists a cycle C in G which does not include all the vertices. Now, let us remove all the edges from G that are in C. Call this new sub-graph H, by our hypothesis all the vertices in H are still even and thus H contains a cycle. Let us choose a common vertex v in C and H. We know this is possible because there are no isolated vertices in G. Now since v still has an even degree we produce a cycle C' in H that starts and ends at v. If C' has all the remaining edges in H then we are done, and thus G has a Euler circuit. If not we continue the above process recursively until all the vertices in G are traversed. The final tour would be the union of all the cycles that we created recursively in G and return to the initial vertex in C where we started, thus making an Euler circuit in G.
They say a picture speaks a thousand words, below we try to illustrate what an Euler Circuit will look like on a graph where all the vertex have an even degree.

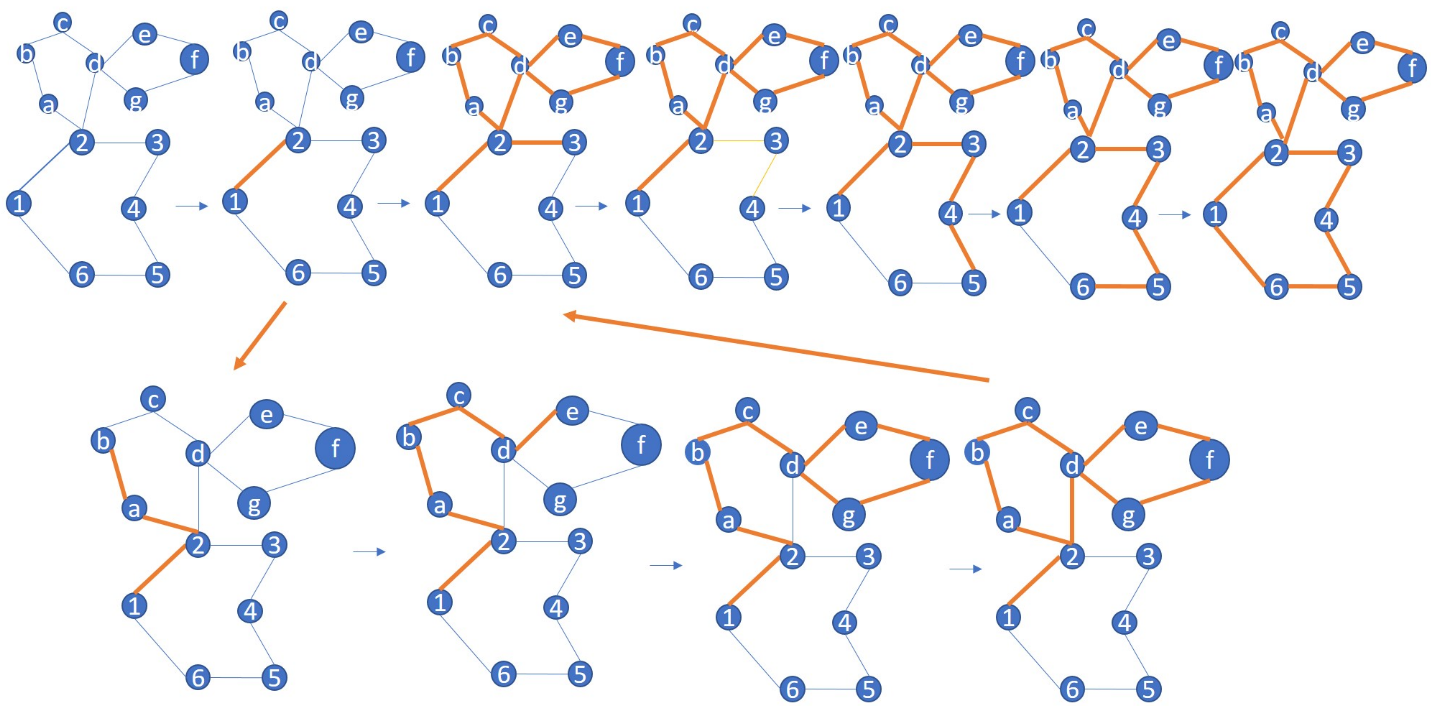

Figure 1: A Euler Circuit.

2. Let G be a graph with Euler circuit. Thus every node/vertex has an even degree. now let us add one node say b and add an edge to a node a in the existing graph G. Note that before adding the edge from b to a, a in G had an even degree. Now if we start drawing our path from b, and since it has only one edge connecting to a we go to a now, if we hypothetically ignore the edge connecting a and b, the remainder of G has nodes with even edges, thus making it a Euler circuit. Therefore, the trail will end at a but since we have already used the edge connecting a and b, we stop at a. Thus we were able to traverse all the edges in $G + b$ exactly once, starting from b and ending at an edge a in G.

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