MATH 450 Seminar in Proof

Prove that the set of odd integers are countable.

Proof. We know that a set X is countable if there exists a bijection in $f : \mathbb{N} \to X$. We will show that $f : \mathbb{N} \to 2\mathbb{Z} - 1$ this is a well defined function and is a bijection. Let

$$f(n) = \begin{cases} n & \text{if n is odd} \\ 1 - n & \text{if n is even} \end{cases}$$

We first show that the function is well defined meaning $f(n) \in 2\mathbb{Z} - 1$ for every $n \in N$. If n is even then $n = 2k$, $k > 0 \in \mathbb{Z}$. Thus $f(n) = 1 - 2k$. Since $2k > 0$ and is even, $1 - 2k < 0$ and is odd, thus $f(n) \in 2\mathbb{Z} - 1$. If n is odd then $n = 2k - 1$, where $k > 0 \in \mathbb{Z}$. Then $f(n) = n = 2k - 1 \in \mathbb{Z}$. Thus our function is well defined. Now we will prove that it is bijective.

One-to-One:

Let $f(a) = f(b)$ where $a, b \in \mathbb{N}$. Then since, $f(a) = f(b)$ we know that either $f(a) = a$ and $f(b) = b$ or $f(a) = 1 - a$ and $f(b) = 1 - b$. In both the cases we get $a = b$ if $f(a) = f(b)$. Thus the function is one-to-one.

Onto:

Let $y \in \mathbb{Z}$. If $y > 0$ then we have $f(n) = y \to n = y$. Since $y > 0 \to n > 0 \in \mathbb{N}$. Thus $y = n \in \mathbb{N}$. If $y < 0$ then we have $y = 1 - n$. Solving for n we get $n = 1 - y$. Since $y < 0$, $-y > 0$ and thus $-y + 1 > 0$. Thus $-y + 1 = n \in \mathbb{N}$. Hence, the function is onto.

Thus the $f : \mathbb{N} \to 2\mathbb{Z} - 1$ with

$$f(n) = \begin{cases} n & \text{if n is odd} \\ 1 - n & \text{if n is even} \end{cases}$$

is well defined and is bijective. Thus the set of odd integers are countable. $\square$