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Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: ***Graph:*** A *graph* is a pair $G = (V, E)$. The elements of V are the vertices (or nodes or points) and the elements of E are edges (or lines) connecting the vertices.

Note: The vertex on G are referred to as $V(G)$ and the edges on G are referred to $E(G)$. This is independent of the way we define a graph.

Definition 1.2: ***Degree of a Vertex:***: Let $G = (V, E)$ be a non empty graph is the number of edges $(E(G))$ attached to a vertex $v \in V(G)$ in G.

Definition 1.3: ***Euler Path:*** An *Euler Path* on a graph G is a path that traverses each edge exactly once.

Definition 1.4: ***Euler Circuit/Cycle:*** An Euler circuit on a graph G is a Euler Path which starts and ends on the same node.

Definition 1.5: ***Walk***: A walk on a graph G is a unique path using edges and vertex of G

Lemma: ***Nilay's Lemma (Not really):*** If a graph has a degree of at least two, then G has a cycle/walk.

Proof. Let G be a graph. Let v be a vertex in G such that v has at least two degree. Let us construct a walk starting from $v \to v_1 \to v_2 \to .... \to v_k$, where $v_1, v_2, v_3, ..., v_k$ are adjacent to one another (connected with each other with an edge). We can do this recursively starting from v and going to the next adjacent vertex v_1 and repeat the process for all $k > 1$. $\square$

Results to be proven:

1. Any graph where the degree of every vertex is even iff it has an Euler circuit.
2. If there are exactly two vertices a and b of odd degree, there is an Euler path on the graph.

Proof :

1. $\Rightarrow$ Let G be a graph such that it has an Euler circuit. According to the definition of Euler Circuit on a graph, given a vertex v_1, we can traverse every other node only once and end at the vertex v_1. Thus every node/vertex has one entry point and one exit point. We know this because G has an Euler circuit. Thus every node/vertex has two edges attached to it thus making it a node of even degree.
   $\Leftarrow$ Let every node in a graph G have an even degree. Let us prove this by induction. Assume, that there are only two nodes in G, each with two edges on them. Thus it is clear that you will end on the same node that you started with, thus making a Euler circuit on G.
   Now,let G be graph of even degree with more than two nodes. Lets start with an arbitrary node v_1. Now since every node has an even degree, we can follow a trail from v_1 to the next node and repeat

it for the next node we encounter. We know that this is possible because each time we enter a node v_k we know there is an unused edge adjacent to it for us to use to traverse. Thus using a unique path we will come back on v_1 since there are even number of edges, we will always find an untapped edge to let us continue our path back to v_1. Figure 1. 1Euler Circuit

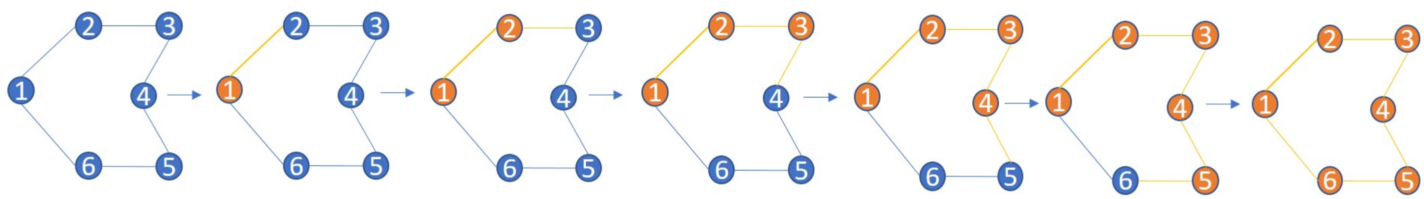

Figure 1: A Euler Circuit.

2. Let G be a graph with Euler circuit. Thus every node/vertex has an even degree. now let us add one node say b and add an edge to a node a in the existing graph G. Note that before adding the edge from b to a, a in G had an even degree. Now if we start drawing our path from b, and since it has only one edge connecting to a we go to a now, if we hypothetically ignore the edge connecting a and b, the remainder of G has nodes with even edges, thus making it a Euler circuit. Therefore, the trail will end at a but since we have already used the edge connecting a and b, we stop at a. Thus we were able to traverse all the edges in G $+b$ exactly once, starting from b and ending at an edge a in G.

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