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Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: ***Graph:*** A simple graph $G = (V, E)$ consists of a non-empty finite set $V(G)$ of elements called vertices (or nodes), and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called edges. We call $V(G)$ the vertex set and $E(G)$ the edge set of G. An edge v, w is said to join the vertices v and w, and is usually abbreviated to vw.

Note: The vertex on G are referred to as $V(G)$ and the edges on G are referred to $E(G)$. This is independent of the way we define a graph. Meaning, if we define a graph $H = (W, Q)$ the set of vertices in H is referred as $V(H)$ and the set of edges in H is referred as $E(H$ and not $W(H)$ and $Q(H)$ respectively.

Definition 1.2: ***Adjacency:***: We say that two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then incident with such an edge. Similarly, two distinct edges e and f are adjacent if they have a vertex in common.

Definition 1.3: ***Degree of a Vertex:***: The degree of a vertex v of G is the number of edges incident with v, and is written $deg(v)$; in calculating the degree of v, we usually make the convention that a loop at v contributes 2 (rather than 1) to the degree of v. A vertex of degree 0 is an isolated vertex and a vertex of degree 1 is an end-vertex.

Note: A graph is ***connected*** if it cannot be expressed as the union of two graphs, and disconnected otherwise. Or, in simpler words, if a graph does not have any isolated or end-vertex, then the graph is ***connected***.

Definition 1.4: ***Subgraph:***:A *subgraph* of a graph G is a graph, each of whose vertices belongs to $V(G)$ and each of whose edges belongs to $E(G)$.

Definition 1.5: ***Walk***: Given a graph G, a walk in G is a finite sequence of distinct edges of the form v_0v_1, v_1v_2,...,$v_{m-1}v_m$, also denoted by $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow .... \rightarrow v_m$, in which any two consecutive edges are adjacent or identical. If $v_0 = v_m$ then we call the walk a cycle.

Definition 1.6: ***Euler Path:*** An *Euler Path* on a graph G is a special *walk* that uses each edge exactly once.

Definition 1.7: ***Euler Circuit/Cycle:*** An Euler circuit on a graph G is a Euler Path which starts and ends on the same vertex.

Definition 1.8: ***Traversing:*** The process of passing through each vertex using the edges joining them in a walk or a path or a cycle or a trail.

Lemma: ***Nilay's Lemma (Not really):*** If a connected finite graph has every vertex of degree of at least two, then G has a ***cycle***.

Proof. Let G be a connected finite graph. Let v be a vertex in G such that v has at least degree two. Let us construct a walk $v_0 \to v_1 \to v_2 \to ....$ such that v_1 be any adjacent vertex to v_0, and for each v_i $i > 1$, we choose v_{i+1} to be any adjacent vertex to v_i, except v_{i-1}(already chosen). We know that such a vertex exists because of our hypothesis that every vertex is of at least degree two. Since G is finite graph, the number of vertices it has is finite. Thus, while constructing our walk we will eventually choose a vertex v_k which has already been chosen and included in the walk. If v_k is first such vertex that we encounter,then the path that was created from the first occurrence of v_k to the second one is a *cycle* from v_k to v_k. $\square$

Results to be proven:

1. ***(EULER (1736), HIERHOLZER (1873))*** Any connected graph where the degree of every vertex is even iff it has an Euler circuit.

   Proof. $\implies$ Let G be a connected graph which has Euler circuit E. When traversing E, when we come across any vertex v through an edge $e_{v(1)}$, we know by definition there is another edge $e_{v(2)}$ that is connected to v. Thus making every vertex in G at least degree two. Thus making every vertex in G of even degree.

   $\impliedby$ Let us proceed by induction. Let every vertex in a connected graph G have an even degree. In a most basic graph of no vertex, the proposition is vacuously true ($P(0)$). If there is one vertex $v \in V(G)$, then the number of edges in $|E(G)| = 0$, thus we start and end our Euler Circuit at v ($P(1)$). Now, let there be only two vertices in G and each vertex is of degree two (making them even degree). Then since G is connected, there are no isolated vertex in G. Furthermore, those two vertices share the two edges between them. Therefore, if we construct a walk at either of the vertex in $V(G)$ we will end at the same vertex where we started, and not repeating the edge that we passed through. Thus making the walk a Euler Circuit ($P(2)$).
   Now by our strong induction hypothesis, we say, if G is a graph, $\forall v \in V(G)$ having an even degree and where $|E(G)| \leq k, k \in \mathbb{Z}$, there is an Euler Circuit in G.

   Now, let G be connected graph with $k+1$ vertices, where each vertex in G is of even degree. From the lemma we know that there exists a cycle in G. If a cycle includes all the edges in G then we are done. Let's say it does not. Then there exists a cycle C in G which does not include all the vertices of G . Now, let us remove all the edges from G that are in C and obtain newly made sub-graph $H = (V(G) - V(C), E(G) - E(C))$, made by the remaining edges in G, by our hypothesis all the vertices in H are still even. We know this because when we removed $E(C)$ from $E(G)$, we removed an even number of edges from the cycle C formed in G.

   If H is still connected, *i.e.* there are no isolated vetices in H then we get a graph with less than k edges, and thus H has a Euler Circuit from our hypothesis because $|E(H)| < k$. Furthermore, since H is connected, there must be a common vertex $m \in V(C) \cap V(H)$. So, now we have a Euler circuit in G, where we start from any vertex $v \in V(C)$ and while traversing C, when we reach m, we traverse the Euler Circuit in H, starting and ending at $m \in V(C) \cap V(H)$ and ending our cycle at v, thus traversing along all the edges in $E(G) = E(C) \cup E(H)$ once and all the vertex in $V(G) = V(C) \cup V(H)$.

When we removed C from G, the other possibility was that we may have H disconnected. Thus $H = H'_1 \cup H'_2 \cup H'_3 ... \cup H'_i$. H is formed from a disjoint union of such even degree connected sub-graph H_i. Note that, for each such H_i, $\exists v_i \in V(C) \cap V(H_i)$. Since $\forall i, |E(H'_i)| < k$, from our hypothesis, each such H_i has a Euler Circuit, made by one C'_i, or multiple cycles C_{i_j}.

We can now build an Euler circuit for G. Pick an arbitrary vertex $a \in V(C) \subset V(G)$ from C. Traverse along C starting from a until we reach a vertex $v = V(C) \cap V(H')$. Then, traverse along its Euler circuit starting from v made by the cycle C' in H'. Now we are back to v, and so we continue along C, and do the same for each such $v_i \in V(C) \cap V(H'_i)$ we encounter and after traversing each edge in $E(G) = E(C) \cup E(H'_1) \cup E(H'_2) \cup E(H'_3) ... \cup E(H'_i)$ exactly once and all the vertices in $V(G) = V(C) \cup V(H'_1) \cup V(H'_2) \cup V(H'_3) ... \cup V(H'_i)$, we get our desired Euler path in G starting and ending at $a \in V(G)$.
They say a picture speaks a thousand words, below we try to illustrate what an Euler Circuit will look like on a graph where all the vertex have an even degree. $\square$

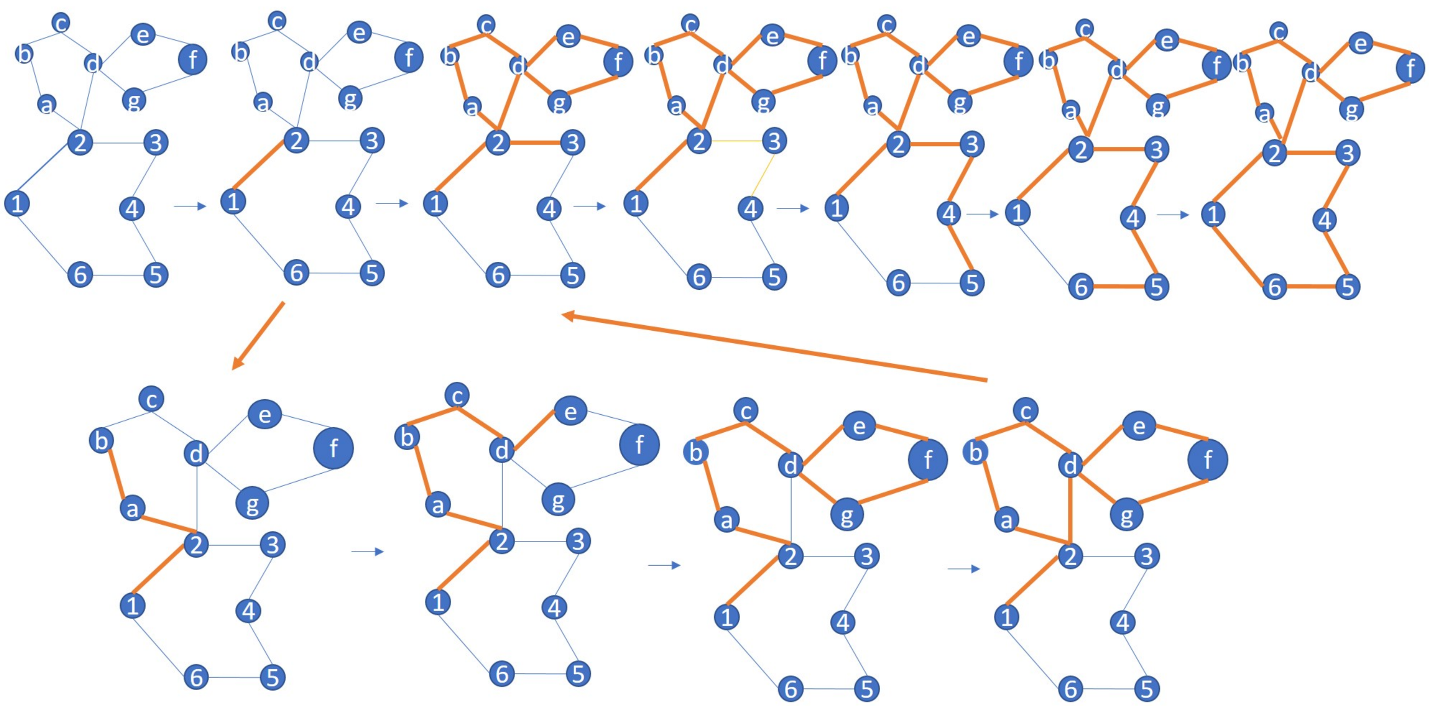

Figure 1: A Euler Circuit.

2. If there are exactly two vertices a and b of odd degree, there is an Euler path on the graph from a to b. (Existence Proof)

Proof. Let G be a graph with Euler circuit. Thus, we know that every vertex in G has an even degree from the theorem stated above. Now let us add one vertex say $b \notin V(G)$ and an edge $e_{ba} \notin E(G)$ to a vertex $a \in V(G)$. Note that before adding the edge from e_{ba} to a, $a \in V(G)$ had an even degree. We start our path from b, and since it has only one edge e_{ba} connecting to a. We know that $a \in V(G)$ and since G has a Euler Circuit, we know that we can construct a cycle that starts and ends at $a \in G$. We cannot use the edge e_{ba} to go back to b as we have already included in our path. Therefore, the path will end at a. Thus, we know have a graph $G' = (V(G) + b, E(G) + e_{ba})$, where a and b are two vertices in G' that are of odd degree and an Euler Path starting from b and ending at a.

$\square$