Nilay Bhatt Feb. 8 2017

Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: ***Graph:*** A **graph** G consists of a non-empty finite set $V(G)$ of elements called **vertices**, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called **edges**; the use of the word 'family' permits the existence of multiple edges. We call $V(G)$ the vertex set and $E(G)$ the edge family of G. An edge e_{vw} is said to join the vertices v and w, and is usually abbreviated to vw.

So, I haven't really looked at any of the definitions again because I figured I'd wait until you and Landon come up with a collective set and then I'll look over those. Also, I think you need my feedback on the proof itself sooner so I'd rather get to that right away.

Note: The vertices in G are referred to as $V(G)$ and the edges on G are referred to $E(G)$. This is independent of the way we define a graph. Meaning, if we define a graph $H = (W, Q)$ the set of vertices in H is referred as $V(H) = W$ and the set of edges in H is referred as $E(H) = Q$ and not $W(H)$ and $Q(H)$ respectively.

Definition 1.2: ***Adjacency:***: We say that two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and f are **adjacent** if they have a vertex in common.

Definition 1.3: ***Degree of a Vertex:***: The degree of a vertex v of G is the number of edges connected with v, and is written $deg(v)$; in calculating the degree of v, we usually make the convention that a loop at v contributes 2 (rather than 1) to the degree of v. A vertex of degree 0 is an **isolated** vertex and a vertex of degree 1 is an **end-vertex**.

Note: A graph is ***connected*** if it cannot be expressed as the union of two distinct graphs, and **disconnected** otherwise.

Definition 1.4: ***Subgraph:***:A **subgraph** H of a graph G is a graph, such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 1.5: ***Walk***: Given a graph G, a **walk** in G is a finite sequence of distinct edges of the form v_0v_1, v_1v_2,...,$v_{m-1}v_m$, also denoted by

$v_0 \to v_1 \to v_2 \to .... \to v_m$, in which any two consecutive edges are adjacent. If $v_0 = v_m$ then we call the walk a **cycle**. Yeah you're right you did define this. I didn't notice it before I guess.

Definition 1.6: ***Euler Path:*** An **Euler Path** on a graph G is a special walk that uses each edge exactly once.

Definition 1.7: ***(Euler Circuit/Cycle:*** An **Euler circuit** on a graph G is an Euler Path with a cycle.

Definition 1.8: ***Traversing:*** The process of passing through each vertex of a walk or cycle in a graph G using the edges joining them in a walk or a path or a cycle.

Lemma: ***Nilay's Lemma (Not really):*** If a connected graph has every vertex of degree of at least two, then G has a *cycle*.

I'm not sure where v is in your walk.

Proof. Let G be a connected graph. Let v be a vertex in G such that v has at least degree two. Let us construct a walk $v_0 \to v_1 \to v_2 \to ....$ such that v_1 be any adjacent vertex to v_0, and for each v_i $i > 1$, we choose v_{i+1} to be any adjacent vertex to v_i, except v_{i-1}(already chosen). We know that such a vertex exists because of our hypothesis that every vertex is of at least degree two. Since G is finite graph, the number of vertices it has is finite. Thus, while constructing our walk we will eventually choose a vertex v_k which has already been chosen and included in the walk. If v_k is first such vertex that we encounter,then the path that was created from the first occurrence of v_k to the second one is a cycle from v_k to v_k. ☐

comma or "where" the space

Results to be proven:

1. ***(EULER (1736), HIERHOLZER (1873))*** A connected graph G has an Euler Circuit if and only if the degree of each vertex of G is even.

   Proof. $\implies$ Let G be a connected graph which has Euler circuit C. Whenever C passes through a vertex in $V(G)$ through an edge in $E(G)$,there is a contribution of 2 edges which are adjacent to the vertex, towards the degree of that vertex. Since each edge occurs exactly

   space

I don't this word is well-defined here

once in C, each vertex must have even degree.

$\Longleftarrow$ Let us proceed by induction. . In a most basic connected graph G of no edges and one vertex, the proposition is vacuously true. If a connected graph G has one vertex $v \in V(G)$, then the number of edges in $E(G) = 1$, thus we start and end our Euler Circuit at v (loop, contribution of 2 towards the degree).

Now, let there be only two vertices in a connected graph G and each vertex is of degree two (making them even degree). Then since G is
vertices
connected, there are no isolated vertex in G. Furthermore, those two vertices share the two edges between them. Therefore, if we construct a walk at either of the vertex in $V(G)$ we will end at the same vertex where we started, and not repeating the edge that we passed through. Thus making the walk a Euler Circuit. Still not a sentence. :)

Don't use this word yet - you haven't actually defined your induction hypothesis yet.
This is what you say if you're calling it.

Now by our strong induction hypothesis, we say, if G is a graph,
write out $\forall v \in V(G)$ having an even degree and where $|E(G)| \leq k, k \in \mathbb{Z}$, there is an Euler Circuit in G. read this aloud - it doesn't parse right

Now, let G be a connected graph with $k+1$ vertices, where each vertex in G is of even degree. From the lemma we know that there exists a
that
cycle in G. If a cycle includes all the edges in G then we are done. Let's say it does not. Then there exists a cycle C in G which does not include all the vertices of G .

Careful - I don't think you can take all these away. What about the edges that remain that have those vertices as endpoints?

Now, let us remove all the edges from G that are in C and obtain newly made sub-graph $H = (V(G) - V(C), E(G) - E(C))$, made by
semi-colon degrees of the
the remaining edges in G, by our hypothesis all the vertices in H are still even. We know this because when we removed $E(C)$ from $E(G)$, we removed an even number of edges from the cycle C formed in G.

Have to be a little bit more precise. You could have removed an odd number from one vertex and an odd number from another, still resulting in an even total amount.

H is
If H is still connected, *i.e.* there are no isolated vetices in H then we get a graph with less than k edges, and thus H has a Euler Circuit from our
an
hypothesis because $|E(H)| < k$. Furthermore, since H is connected,
Make this justification more explicit. Why? there must be a common vertex $m \in V(C) \cap V(H)$. So, now we have a Euler circuit in G, where we start from any vertex $v \in V(C)$ and while

Oh my gosh I understand this so much better now. :)

Wait, this was hypothetical. Use the word "suppose" if you want it to be a fact you can call.

traversing C, when we reach m, we traverse the Euler Circuit in H, starting and ending at $m \in V(C) \cap V(H)$ and ending our cycle at v, thus traversing along all the edges in $E(G) = E(C) \cup E(H)$ once and all the vertex in $V(G) = V(C) \cup V(H)$.

I still want you to think about what union means for graphs, when you can have multiple edges between two vertices. If you take the union of a graph with itself do you get double the edges?

When we removed C from G, the other possibility was that we may have H disconnected. Thus $H = H_1' \cup H_2' \cup H_3' ... \cup H_i'$. H is formed from a disjoint union of such even degree connected sub-graph H_i. Note that, for each such H_i, $\exists v_i \in V(C) \cap V(H_i)$. Since $\forall i, |E(H_i')| < k$, from our hypothesis, each ~~such~~ H_i has a Euler Circuit, made by one C_i', or multiple cycles C_{i_j}.

Do you remember the symbol we use for disjoint union from topology? It's the "square" U.

what is this referring to?

why?

Where H'_i is?

We can now build an Euler circuit for G. Pick an arbitrary vertex $a \in V(C) \subset V(G)$ from C. Traverse along C starting from a until we reach a vertex $v = V(C) \cap V(H')$. Then, traverse along its Euler circuit starting from v made by the cycle C' in H'. shouldn't these have subscripts?

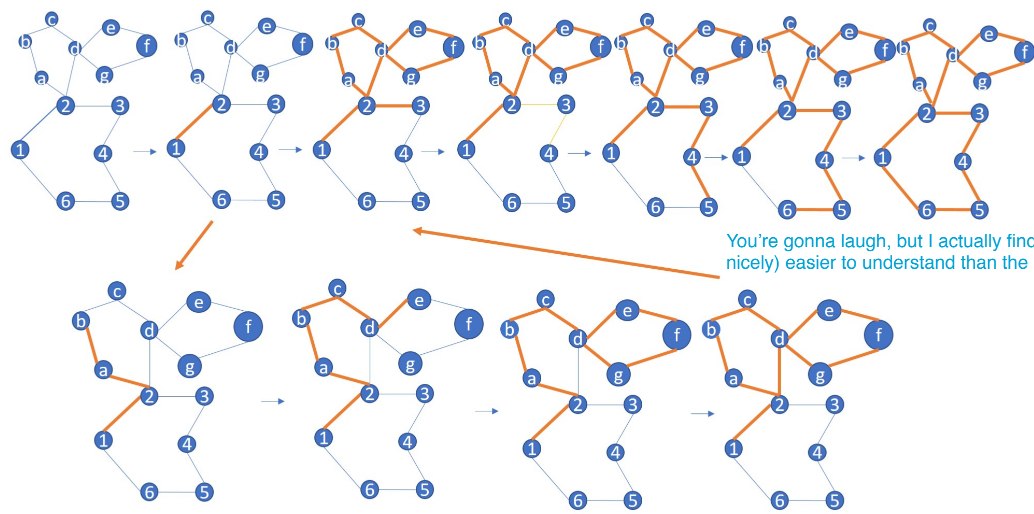
Which one? Not sure what you mean.

Now we are back to v, and so we continue along C, and do the same for each such $v_i \in V(C) \cap V(H_i')$ we encounter and after traversing each edge in $E(G) = E(C) \cup E(H_1') \cup E(H_2') \cup E(H_3') \cup ... \cup E(H_i')$ exactly once and all the vertices in $V(G) = V(C) \cup V(H_1') \cup V(H_2') \cup V(H_3') \cup ... \cup V(H_i')$, we get our desired Euler path in G starting and ending at $a \in V(G)$.

obtain (or have)

They say a picture speaks a thousand words, below we try to illustrate what an Euler Circuit will look like on a graph where all the vertex have an even degree.

☐ This should go before the comment about the picture :)



You're gonna laugh, but I actually find your proof (now that it's written out nicely) easier to understand than the picture!

Figure 1: A Euler Circuit.

2. If there are exactly two vertices a and b of odd degree, there is an Euler path on the graph from a to b. (Existence Proof)

Wait, is this the same one you proved before? I don't remember this one.

Wait, why would you get to assume this? That's not one of your hypotheses.

Proof. Let G be a graph with Euler circuit. Thus, we know that every vertex in G has an even degree from the theorem stated above. Now let us add one vertex say $b \notin V(G)$ and an edge $e_{ba} \notin E(G)$ to a vertex $a \in V(G)$. Note that before adding the edge from e_{ba} to a, $a \in V(G)$ had an even degree. We start our path from b, and since it has only one edge e_{ba} connecting to a. We know that $a \in V(G)$ and since G has a Euler Circuit, we know that we can construct a cycle that starts and ends at $a \in G$. We cannot use the edge e_{ba} to go back to b as we have already included in our path. Therefore, the path will end at a. Thus, we know have a graph $G' = (V(G) + b, E(G) + e_{ba})$, where a and b are two vertices in G' that are of odd degree and an Euler Path starting from b and ending at a.

not a sentence

$\square$

Be consistent with notation. Denote this the way you've been doing throughout.

What does + mean?

To preserve generality, you should really start with an arbitrary graph that has exactly two vertices of odd degree and go from there (as opposed to building one from another graph).