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Results about Euler's path and circuits

MATH 450 Seminar in Proof

So you are not allowing for double edges or loops, just to confirm

Definition 1.1: ***Graph:*** A simple graph $G = (V, E)$ consists of a non-empty finite set $V(G)$ of elements called vertices (or nodes), and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called edges. We call $V(G)$ the vertex set and $E(G)$ the edge set of G. An edge v, w is said to join the vertices v and w, and is usually abbreviated to vw.

needs parentheses (since it's a pair)

vertices! "in" or "of"

Note: The vertex on G are referred to as $V(G)$ and the edges on G are referred to as $E(G)$. This is independent of the way we define a graph. Meaning, if we define a graph $H = (W, Q)$ the set of vertices in H is referred as $V(H)$ and the set of edges in H is referred as $E(H$ and not $W(H)$ and $Q(H)$ respectively.

I am actually a little confused by this. What set is W if it isn't the vertices? (Similarly for Q.)

Definition 1.2: ***Adjacency:***: We say that two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then incident with such an edge. Similarly, two distinct edges e and f are adjacent if they have a vertex in common.

boldface this since you're defining it

boldface (technically a new definition)

bf

A technicality: when you defined "incident with" above, you defined it for vertices, not edges.

Definition 1.3: ***Degree of a Vertex:***: The degree of a vertex v of G is the number of edges incident with v, and is written $deg(v)$; in calculating the degree of v, we usually make the convention that a loop at v contributes 2 (rather than 1) to the degree of v. A vertex of degree 0 is an isolated vertex and a vertex of degree 1 is an end-vertex.

bf

So you do want to allow for these? If so, need to adjust your definition above.

bf

I think you need a condition on these graphs.

Note: A graph is ***connected*** if it cannot be expressed as the union of two graphs, and disconnected otherwise. Or, in simpler words, if a graph does not have any isolated or end-vertex, then the graph is ***connected***. bf

vertices

This would actually require proof to show that these definitions are equivalent. I don't think you need to show it here unless you want to, but it should definitely be acknowledged.

I also think you need to be precise in your meaning of union since you are allowing for loops and double edge. E.g., if you take the union of two graphs that both have vertices v and w, and each one has an edge vw, do you now have two different edges between v and w?

Definition 1.4: ***Subgraph:***:A *subgraph* of a graph G is a graph, each of whose vertices belongs to $V(G)$ and each of whose edges belongs to $E(G)$.

I'm not sure what this is referring to

bf

Definition 1.5: ***Walk***: Given a graph G, a walk in G is a finite sequence of distinct edges of the form v_0v_1, v_1v_2,...,$v_{m-1}v_m$, also denoted by $v_0 \to v_1 \to v_2 \to .... \to v_m$, in which any two consecutive edges are adjacent or identical. If $v_0 = v_m$ then we call the walk a cycle. bf

I think these contradict each other

of G

Definition 1.6: ***Euler Path:*** An *Euler Path* on a graph G is a special *walk* that uses each edge exactly once.

italics instead of boldface is fine, but be consistent throughout

I don't think this should be italicized since you're not defining it here

bf or italics

Definition 1.7: ***Euler Circuit/Cycle:*** An Euler circuit on a graph G is a Euler Path which starts and ends on the same vertex. Can you rephrase this using how you've defined it above?

an

of G?

Definition 1.8: ***Traversing:*** The process of passing through each vertex using the edges joining them in a walk or a path or a cycle or a trail.

You didn't define this - do you need it?

I don't think you need this since you've defined graph as finite

only because the two "of"s makes it confusing to parse

Lemma: ***Nilay's Lemma (Not really):*** If a connected finite graph has every vertex of degree ~~of~~ at least two, then G has a ***cycle***.

An Euler cycle? (That's the only one you defined.)

Proof. Let G be a connected finite graph. Let v be a vertex in G such that v has at least degree two. Let us construct a walk $v_0 \to v_1 \to v_2 \to \dots$. such that v_1 be any adjacent vertex to v_0, and for each v_i with $i > 1$, we choose v_{i+1} to be any adjacent vertex to v_i, except v_{i-1}(already chosen). We know that such a vertex exists because of our hypothesis that every vertex is of at least degree two. Since G is a finite graph, the number of vertices it has is finite. Thus, while constructing our walk we will eventually choose a vertex v_k which has already been chosen and included in the walk. If v_k is the first such vertex that we encounter,then the path that was created from the first occurrence of v_k to the second one is a *cycle* from v_k to v_k. $\square$

no italics

Results to be proven:

Something's off about the way this is written. What proposition is this?

1. ***(EULER (1736), HIERHOLZER (1873))*** Any connected graph where the degree of every vertex is even iff it has an Euler circuit.

I think this word is what's making it be off.

The way this sentence is written, it's not clear the "connected" applies to this part too.

Proof. $\Longrightarrow$ Let G be a connected graph which has Euler circuit E. When traversing E, when we come across any vertex v through an edge $e_{v(1)}$, we know by definition there is another edge $e_{v(2)}$ that is connected to v. Thus making every vertex in G at least degree two. Thus making every vertex in G of even degree.

Can you use the notation you've defined above?

Can't you use a word you've defined?

Not a sentence!

Same

So here's the thing: I don't think you need to involve traversing here. The path already exists, so you can talk about any v_i in the middle of it - doesn't matter when we "come across" it.

Here is where you first define G. Then you redefine it as you go. (It's not the same G throughout)

You can't name G yet because it changes throughout the proof.

$\Longleftarrow$ Let us proceed by induction. Let every vertex in a connected graph G have an even degree. In a most basic graph of no vertices vertex, the proposition is vacuously true $(P(0))$. If there is one vertex $v \in V(G)$, then the number of edges in $|E(G)| = 0$, thus we start and end our Euler Circuit at v $(P(1))$. Now, let there be only two vertices in G and each vertex is of degree two (making them even degree). Then since G is connected, there are no isolated vertex in G. Furthermore, those two vertices share the two edges between them. Therefore, if we construct a walk at either of the vertex in $V(G)$ we will end at the same vertex where we started, and not repeating the edge that we passed through. Thus making the walk a Euler Circuit $(P(2))$.

You haven't defined P so I would just leave this out.

Can't combine these this way. In E(G), not in IE(G)I

Can you skip some lines in here (and/or increase the margins)? It's hard to fit comments in.

Now by our strong induction hypothesis, we say, if G is a graph, $\forall v \in V(G)$ having an even degree and where $|E(G)| \leq k, k \in \mathbb{Z}$, there is an Euler Circuit in G.

Now, let G be connected graph with $k+1$ vertices, where each vertex in G is of even degree. From the lemma we know that there exists a cycle in G. If a cycle includes all the edges in G then we are done. Let's say it does not. Then there exists a cycle C in G which does not include all the vertices of G . Now, let us remove all the edges from G that are in C and obtain newly made sub-graph $H = (V(G) - V(C), E(G) - E(C))$, made by the remaining edges in G, by our hypothesis all the vertices in H are still even. We know this because when we removed $E(C)$ from $E(G)$, we removed an even number of edges from the cycle C formed in G.

If H is still connected, *i.e.* there are no isolated vetices in H then we get a graph with less than k edges, and thus H has a Euler Circuit from our hypothesis because $|E(H)| < k$. Furthermore, since H is connected, there must be a common vertex $m \in V(C) \cap V(H)$. So, now we have a Euler circuit in G, where we start from any vertex $v \in V(C)$ and while traversing C, when we reach m, we traverse the Euler Circuit in H, starting and ending at $m \in V(C) \cap V(H)$ and ending our cycle at v, thus traversing along all the edges in $E(G) = E(C) \cup E(H)$ once and all the vertex in $V(G) = V(C) \cup V(H)$.

When we removed C from G, the other possibility was that we may have H disconnected. Thus $H = H_1' \cup H_2' \cup H_3' ... \cup H_i'$. H is formed from a disjoint union of such even degree connected sub-graph H_i. Note that, for each such H_i, $\exists v_i \in V(C) \cap V(H_i)$. Since $\forall i, |E(H_i')| < k$, from our hypothesis, each such H_i has a Euler Circuit, made by one C_i', or multiple cycles C_{i_j}.

We can now build an Euler circuit for G. Pick an arbitrary vertex $a \in V(C) \subset V(G)$ from C. Traverse along C starting from a until we reach a vertex $v = V(C) \cap V(H')$. Then, traverse along its Euler circuit starting from v made by the cycle C' in H'. Now we are back to v, and so we continue along C, and do the same for each such $v_i \in V(C) \cap V(H_i')$ we encounter and after traversing each edge in $E(G) = E(C) \cup E(H_1') \cup E(H_2') \cup E(H_3') ... \cup E(H_i')$ exactly once and all the vertices in $V(G) = V(C) \cup V(H_1') \cup V(H_2') \cup V(H_3') ... \cup V(H_i')$, we get our desired Euler path in G starting and ending at $a \in V(G)$.

Skip a line here

They say a picture speaks a thousand words, below we try to illustrate what an Euler Circuit will look like on a graph where all the vertex have an even degree. $\square$

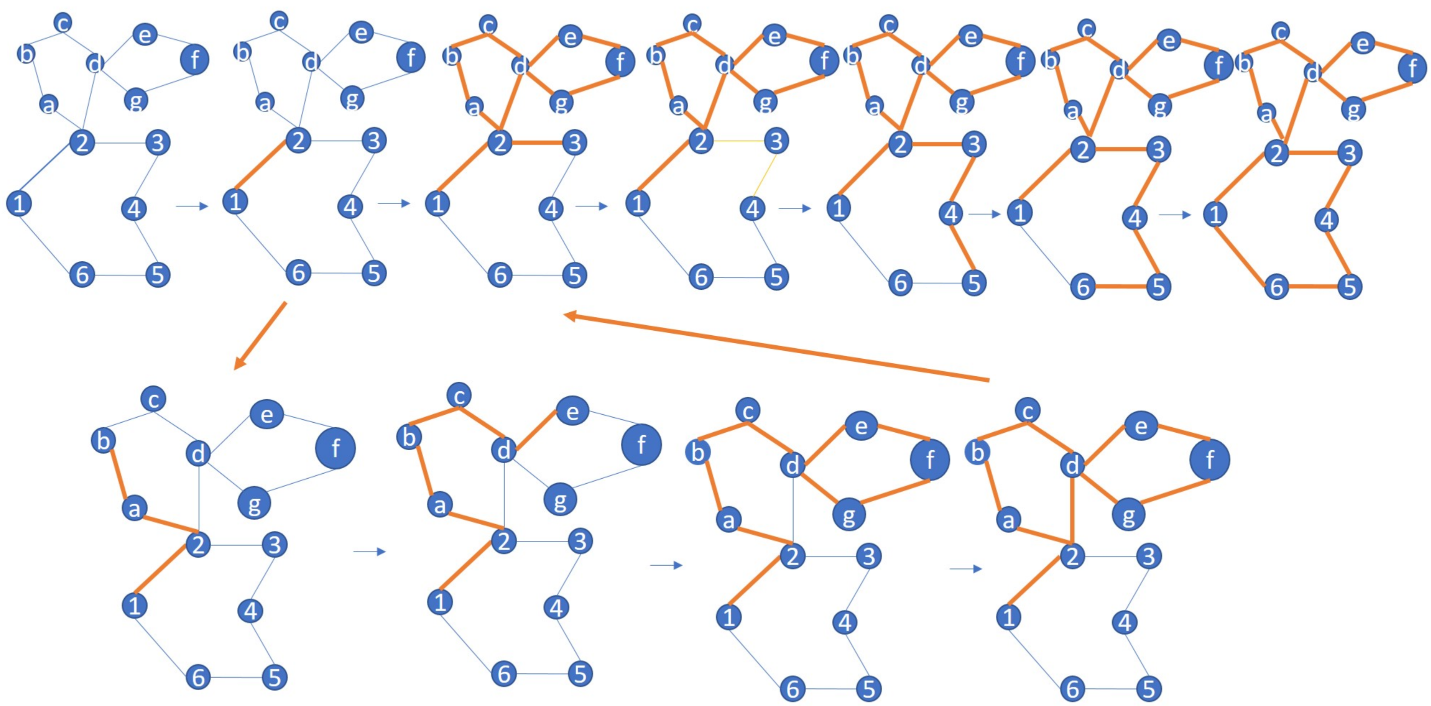

Figure 1: A Euler Circuit.

2. If there are exactly two vertices a and b of odd degree, there is an Euler path on the graph from a to b. (Existence Proof)

Proof. Let G be a graph with Euler circuit. Thus, we know that every vertex in G has an even degree from the theorem stated above. Now let us add one vertex say $b \notin V(G)$ and an edge $e_{ba} \notin E(G)$ to a vertex $a \in V(G)$. Note that before adding the edge from e_{ba} to a, $a \in V(G)$ had an even degree. We start our path from b, and since it has only one edge e_{ba} connecting to a. We know that $a \in V(G)$ and since G has a Euler Circuit, we know that we can construct a cycle that starts and ends at $a \in G$. We cannot use the edge e_{ba} to go back to b as we have already included in our path. Therefore, the path will end at a. Thus, we know have a graph $G' = (V(G) + b, E(G) + e_{ba})$, where a and b are two vertices in G' that are of odd degree and an Euler Path starting from b and ending at a.

$\square$