MATH 450 Seminar in Proof

Prove that the set of integers are countable.

Proof. We know that a set X is countable if there exists a bijection in $f : \mathbb{N} \to X$. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined as

$$f(n) = \begin{cases} -\left(\dfrac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

We know that $f(n)$ has a valid co-domain. We will now show that it is a bijection.

One-to-One:

Let $f(a) = f(b)$ where $a, b \in \mathbb{N}$. Therefore, either $f(a) = -\left(\dfrac{a-1}{2}\right)$ and $f(b) = -\left(\dfrac{b-1}{2}\right)$ or $f(a) = \dfrac{a}{2}$ and $f(b) = \dfrac{b}{2}$. Note that if we have $f(b) = -\left(\dfrac{b-1}{2}\right)$ and $f(a) = \dfrac{a}{2}$ with out loss of generality, we will have $b = 1 - a$, we know that $a \geq 1$ (because $a \in \mathbb{N}$) which makes $b \leq 0$ implies $b \notin \mathbb{N}$. Therefore both $f(a)$ and $f(b)$ will have the same structure.

In both the former cases we get $a = b$ if $f(a) = f(b)$. thus the function is one-to-one.

Onto:

Let $y \in \mathbb{Z}$. If $y > 0$ then we have $f(2y) = \dfrac{2y}{2} = y$. Since $y > 0$, then we know that $2y > 0$ and $2y \in \mathbb{N}$. If $y < 0$ then we have $f(-2y+1) = -\left(\dfrac{-2y+1-1}{2}\right) = y$. Since $y < 0$, which implies that $-2y > 0$ and thus $-2y+1 > 0$. Thus $-2y+1 \in \mathbb{N}$. Hence, the function is onto.

Thus the $f : \mathbb{N} \to \mathbb{Z}$ with

$$f(n) = \begin{cases} -\left(\dfrac{n-1}{2}\right) & \text{if n is odd} \\ \dfrac{n}{2} & \text{if n is even} \end{cases}$$

is well defined and is a bijection. Thus the set of integers are countable.

$\square$