MATH 450 Seminar in Proof

Prove that the set of integers are countable.

Proof. We know that a set X is countable if there exists a bijection in $f : \mathbb{N} \to X$. We know that $f : \mathbb{N} \to \mathbb{Z}$ is a well defined function from the class lecture. So we will show that it is a bijection. Let

$$f(n) = \begin{cases} -\left(\dfrac{n-1}{2}\right) & \text{if n is odd} \\ \dfrac{n}{2} & \text{if n is even} \end{cases}$$

One-to-One:

Let $f(a) = f(b)$ where $a, b \in \mathbb{N}$. Then since, $f(a) = f(b)$ we know that either $f(a) = -\left(\dfrac{a-1}{2}\right)$ and $f(b) = -\left(\dfrac{b-1}{2}\right)$ or $f(a) = \dfrac{a}{2}$ and $f(b) = \dfrac{b}{2}$. In both the cases we get $a = b$ if $f(a) = f(b)$. thus the function is one-to-one.

Onto:

Let $y \in \mathbb{Z}$. If $y > 0$ then we have $f(n) = y = \dfrac{n}{2} = y \to n = 2y$. Since $y > 0 \to 2y > 0 \in \mathbb{N}$. Thus $2y = n \in \mathbb{N}$. If $y < 0$ then we have $y = -\left(\dfrac{n-1}{2}\right)$. Solving for n we get $n = -2y + 1$. Since $y < 0$, $-2y > 0$ and thus $-2y + 1 > 0$. Thus $-2y + 1 = n \in \mathbb{N}$. Hence, the function is onto.

Thus the $f : \mathbb{N} \to \mathbb{Z}$ with

$$f(n) = \begin{cases} -\left(\dfrac{n-1}{2}\right) & \text{if n is odd} \\ \dfrac{n}{2} & \text{if n is even} \end{cases}$$

is well defined and is a bijection. Thus the set of integers are countable. $\square$