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MATH 450 Seminar in Proof

Prove: If $f : A \to B$ be a function such that A and B are finite $|A| = |B|$, then f is one-to-one if only if it is onto.

Proof.
$\Longleftarrow$ Let $f : A \to B$ be a function such that $|A| = |B|$ and f is onto. Then from the definition of onto, for all $b \in B$ there exists $a \in A$ such that, $f(a) = b$. Let f be not one-to-one. Then there exists $a_1, a_2 \in A, a_1 \neq a_2$ such that $f(a_1) = f(a_2) = b \in B$. Since f is onto, every $b \in B$ has a pre-image in A. Also, since f is well defined, each $a \in A$ has only one image in B. But by our assumption f is not one-to-one and so there exists $a_1, a_2 \in A$ where $a_1 \neq a_2$ such that they both have the same image. Also, by our assumption since $|A| = |B|$ we will have an element in B which does not have a pre-image in A making it not onto. Hence there is a contradiction $\rightarrow\leftarrow$; and so, f is one-to-one.

$\Longrightarrow$ Let $f : A \to B$ be a function such that $|A| = |B|$ and f is one-to-one. Then from the definition of one-to-one, if $f(a_1) = f(a_2)$ then $a_1 = a_2$. Also note that since f is one-to-one $|A| = |f[A]|$. Let f be not onto. Then there exist a $b \in B$ such that there does not exist any $a \in A$ where $f(a) = b$. This is a contradiction because, according to our assumption $|A| = |B|$ but then if f is not onto it implies that there exists more elements in B than there are in A. Also, since the cardinality of A and B is same then two elements in A map to one element in B thus making it not one-to-one. This is where our contradiction lies. Therefore f has to be onto. $\square$