

## MATH 450 Seminar in Proof

Prove by induction that  $|P(A)| = 2^n$  if  $|A| = n$ .

*Proof.* By induction.

**Base Case:** Let  $A$  be a set with 0 elements. Then the  $P(A) = \{\emptyset\}$ . Thus  $|P(A)| = 1 = 2^0$ .

**Inductive Hypothesis:** Assume if  $|A| = n$  then  $|P(A)| = 2^n$  is true.

**Inductive Step:**

We will prove that if  $|A| = n + 1$  then  $|P(A)| = 2^{n+1}$ . Now, let  $A$  be a set such that  $|A| = n + 1$ . Let  $B = A - \{a\}$  where  $a \in A$ . Then  $|B| = n$ . Thus  $|P(B)| = 2^n$  from our hypothesis.

Also we can split the subsets of  $A$  into two parts, namely subsets that contain  $a$  and subsets that <sup>do</sup> not *i.e*  $P(B)$ . Note that  $P(B)$  does not have any sets in it that contains  $a$ . Let  $B_1, B_2, B_3, \dots, B_{2^n}$  be the elements of  $P(B)$ . Then,  $B_1 \cup \{a\} \in P(A)$ ,  $B_2 \cup \{a\} \in P(A)$ ,  $B_3 \cup \{a\} \in P(A)$ , ...,  $B_{2^n} \cup \{a\} \in P(A)$  are the subsets of  $A$  that contain the element  $a$ . Since the union of each  $B_i$  and  $\{a\}$  produces  $2^n$  subsets and  $|P(B)| = 2^n$ , then  $|P(A)| = 2^n + 2^n = 2^n(1 + 1) = 2^{n+1}$ .  $\square$