

Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: Graph: A graph G consists of a non-empty finite set $V(G)$ of elements called **vertices**, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called **edges**; the use of the word 'family' permits the existence of multiple edges. We call $V(G)$ the vertex set and $E(G)$ the edge family of G . An edge e_{vw} is said to join the vertices v and w , and is usually abbreviated to vw .

Definition 1.2: Simple Graph: A simple graph is an **undirected, unweighted** graph such that there is at most 1 edge between any two vertices, and no loops, where a loop is an edge from a vertex x to itself.

Note: The union of two graphs $G = (V_1, E_1), H = (V_2, E_2)$ as $G \cup H = (V_1 \cup V_2, E_1 \cup E_2)$. If we have the union of a graph and a single vertex v or edge e we may write $G \cup v$ or $G \cup e$.

Note: The vertices in G are referred to as $V(G)$ and the edges on G are referred to $E(G)$. This is independent of the way we define a graph. Meaning, if we define a graph $H = (W, Q)$ the set of vertices in H is referred as $V(H) = W$ and the set of edges in H is referred as $E(H) = Q$ and not $W(H)$ and $Q(H)$ respectively.

Definition 1.3: Adjacency: We say that two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and f are **adjacent** if they have a vertex in common.

Note: A graph G is **complete** if any two vertices in G are adjacent.

Definition 1.4: Degree of a Vertex: The degree of a vertex v of G is the number of edges connected with v , and is written $\deg(v)$; in calculating the degree of v , we usually make the convention that a loop at v contributes 2 (rather than 1) to the degree of v . A vertex of degree 0 is an **isolated** vertex and a vertex of degree 1 is an **end-vertex**.

Note: A graph is **connected** if it cannot be expressed as the union of two distinct graphs, and **disconnected** otherwise.

Definition 1.5: Subgraph: A subgraph H of a graph G is a graph, such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Note: if $G = (V, E)$ be a graph then $G \cup G = G$ where we preserve the cardinality and mapping in G for $V(G)$ and $E(G)$.

Definition 1.6: Walk: Given a graph G , a **walk** in G is a finite sequence of distinct edges of the form $v_0v_1, v_1v_2, \dots, v_{m-1}v_m$, also denoted by $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_m$, in which any two consecutive edges are adjacent. If $v_0 = v_m$ then we call the walk a **cycle**.

Note: As a convention we will denote any cycle or path P as a sequenced n -tuple of the vertices P visits in a given order. For example if P is a path that visits vertices a, b, c in the respective order we say $P = (a, b, c)$. If P is a cycle, we simply acknowledge that P also contains the edge from the last vertex in the tuple back to the first.

Definition 1.7: Euler Path: An **Euler Path** on a graph G is a special walk that uses each edge exactly once.

Definition 1.8: (Euler Circuit/Cycle: An **Euler circuit** on a graph G is an Euler Path with a cycle.

Definition 1.9: Traversing: The process of passing through each vertex of a walk or cycle in a graph G using the edges joining them in a walk or a path or a cycle.

Definition 2.0: Hamiltonian Path: A **Hamiltonian path** is a path that visits every vertex of the graph once. A **Hamiltonian cycle** is a Hamiltonian path that is a cycle. If graph G contains a Hamiltonian cycle, we say G is **Hamiltonian**.

Lemma: *Nilay's Lemma (Not really)*: If a connected graph has every vertex of degree of at least two, then G has a *cycle*.

Proof. Let G be a connected graph. Let v_0 be a vertex in G such that v_0 has at least degree two. Let us construct a walk $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots$ such that v_1 be any adjacent vertex to v_0 , and for each v_i where $i > 1$, we choose v_{i+1} to be any adjacent vertex to v_i , except v_{i-1} (already chosen). We know that such a vertex exists because of our hypothesis that every vertex is of at least degree two. Since G is finite graph, the number of vertices it has is finite. Thus, while constructing our walk we will eventually choose a vertex v_k which has already been chosen and included in the walk. If v_k is the first such vertex that we encounter, then the path that was created from the first occurrence of v_k to the second one is a cycle from v_k to v_k . \square

Results to be proven:

1. (***EULER (1736), HIERHOLZER (1873)***) A connected graph G has an Euler Circuit if and only if the degree of each vertex of G is even.

Proof. \implies Let G be a connected graph which has Euler circuit C . Whenever C passes through a vertex in $V(G)$ through an edge in $E(G)$, there is a contribution of 2 edges which are adjacent to the vertex, towards the degree of that vertex. From our hypothesis we know that edges in C are distinct, thus each vertex must have even degree.

\Leftarrow Let us proceed by induction. In a most basic connected graph G of no edges and one vertex, the proposition is vacuously true. If a connected graph G has one vertex $v \in V(G)$, then the number of edges in $E(G) = 1$, thus we start and end our Euler Circuit at v (loop, contribution of 2 towards the degree).

Now, let there be only two vertices in a connected graph G and each vertex is of degree two (making them even degree). Then since G is connected, there are no isolated vertices in G . Furthermore, those two vertices share the two edges between them. Therefore, if we construct a walk at either of the vertices in $V(G)$ we will end at the same vertex

where we started, and not repeating the edge that we passed through. Thus making the walk an Euler Circuit in G .

Now, for our strong induction hypothesis, we say, if G is a graph where $|E(G)| \leq k, k \in \mathbb{Z}$, and all vertices in G have an even degree then there exists is an Euler Circuit in G .

Now, let G be a connected graph with $k+1$ vertices, where each vertex in G is of even degree. From the lemma we know that there exists a cycle in G . If that cycle includes all the edges in G then we are done. Let's say it does not. Then there exists a cycle C in G which does not include all the vertices of G .

Now, let us remove all the edges from G that are in C and obtain newly made sub-graph $H = (V(G), E(G) - E(C))$, made by the remaining edges in G ; by our hypothesis degree all the vertices in H are still even. We know this because when we removed $E(C)$ from $E(G)$, we removed an even number of edges from each vertex from the cycle C formed in G .

Suppose H is still connected, *i.e.* there are no isolated vertices in H then H is a graph with less than k edges, and thus H has a Euler Circuit from our hypothesis because $|E(H)| < k$. Also, since G is connected, there must be a common vertex $m \in V(C) \cap V(H)$. If there weren't any common vertex in C and H then, $C \cup H$ will form a disjoint union, thus making G disconnected. So, now we have an Euler circuit in G , where we start from any vertex $v \in V(C)$ and while traversing C , when we reach m , we traverse the Euler circuit in H , starting and ending at $m \in V(C) \cap V(H)$ and ending our cycle at v , thus traversing along all the edges in $E(G) = E(C) \cup E(H)$ once and all the vertex in $V(G) = V(C) \cup V(H)$.

When we removed C from G , the other possibility was that we may have H disconnected. Thus $H = H'_1 \sqcup H'_2 \sqcup H'_3 \sqcup \dots \sqcup H'_i$. H is formed from a disjoint union of even degree connected sub-graphs H'_i . Note that, for each such H'_i , $\exists v_i \in V(C) \cap V(H_i)$. Since $|E(H'_i)| < k$, from our hypothesis, each H'_i has an Euler Circuit, made by one C'_i , or multiple cycles C_{ij} .

We can now build an Euler circuit for G . Pick an arbitrary vertex

$a \in V(C) \subset V(G)$ from C . Traverse along C starting from a until we reach a vertex $v = V(C) \cap V(H'_i)$. Then, traverse along H'_i 's Euler circuit starting from v made by the cycle(s) $C'_{i,j}$ in H'_i .

Now we are back to v , and so we continue along C , and do the same for each such $v_i \in V(C) \cap V(H'_i)$ we encounter and after traversing each edge in $E(G) = E(C) \cup E(H'_1) \cup E(H'_2) \cup E(H'_3) \cup \dots \cup E(H'_i)$ exactly once and all the vertices in $V(G) = V(C) \cup V(H'_1) \cup V(H'_2) \cup V(H'_3) \cup \dots \cup V(H'_i)$, we obtain our desired Euler path in G starting and ending at $a \in V(G)$. \square

They say a picture speaks a thousand words, below we try to illustrate what an Euler Circuit will look like on a graph where all the vertex have an even degree.

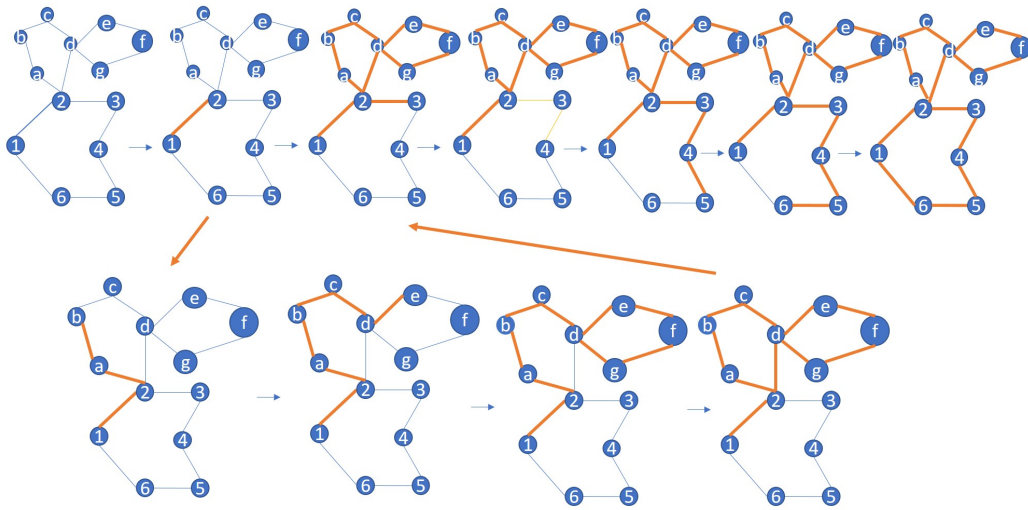


Figure 1: A Euler Circuit.