MATH 450 Seminar in Proof

Prove by induction that $|P(A)| = 2^n$ if $|A| = n$.

Proof. By induction.

Base Case: Let A be a set with 0 elements. Then the $P(A) = \{\emptyset\}$. Thus $|P(A)| = 1 = 2^0$.
Inductive Hypothesis: Assume if $|A| = n$ then $|P(A)| = 2^n$ is true.
Inductive Step:
We will prove that if $|A| = n + 1$ then $|P(A)| = 2^{n+1}$. Now, let A be a set such that $|A| = n + 1$. Let $B = A - \{a\}$ where $a \in A$. Then $|B| = n$. Thus $|P(B)| = 2^n$ from our hypothesis.
Also we can split the subsets of A into two parts, namely subsets that contain a and subsets that does not $i.e$ $P(B)$. Note that $P(B)$ do not have any sets in it that contain a. Let $B_1, B_2, B_3, ..., B_{2^n}$ be the elements of $P(B)$. Then, $B_1 \cup \{a\} \in P(A)$, $B_2 \cup \{a\} \in P(A)$, $B_3 \cup \{a\} \in P(A)$, ..., $B_{2^n} \cup \{a\} \in P(A)$ are the subsets of A that contain the element a. Since the union of each B_i and $\{a\}$ produces 2^n subsets and $|P(B)| = 2^n$, then $|P(A)| = 2^n + 2^n = 2^n(1+1) = 2^{n+1}$. $\square$