MATH 450 Seminar in Proof

Prove that $f : A \to P(A)$ is not onto. Note that A is an infinite set and $P(A)$ is the power set of A. Hint: consider the set $\{a \in A : a \notin f(a)\}$

Proof. Let us proceed by contradiction. Let $f : A \to P(A)$ be onto. Then $f[A] = P(A)$. Let $B = \{a \in A : a \notin f(a)\}$. Thus $B \subseteq A$ and also $B \in f[A]$. Since f is onto, we know that B has a pre-image in A. Let $f(a) = B$, where $a \in A$. Thus, there are now two possibilities, $a \in B$ or $a \notin B$. If $a \in B$ then by definition $a \notin f(a)$ but then since $f(a) = B$ there is a contradiction. If $a \notin B$ then by definition $a \in f(a)$ but then since $f(a) = B$ then again there is a contradiction. Thus f is not onto. $\square$