

MATH 450 Seminar in Proof

Prove that the set of all infinite binary strings is uncountable.

Proof. Let us proceed by contradiction. Let A be the set that represents the set of all possible infinite binary strings. Then we assume that A is countable, i.e. $|A| = |\mathbb{N}|$ then we can say that there is a bijection $f : \mathbb{N} \rightarrow A$. Since A is countable we can list the elements of A which are mapped from \mathbb{N} through f .

$$\begin{aligned}f(1) &= a_{11}a_{12}a_{13}a_{14}\dots \\f(2) &= a_{21}a_{22}a_{23}a_{24}\dots \\f(3) &= a_{31}a_{32}a_{33}a_{34}\dots \\&\dots \\&\dots \\&\dots\end{aligned}$$

where a_{ij} is the j^{th} digit of $f(i)$ where $i, j \in \mathbb{N}$. Since f is a bijection, f is onto. Thus we will prove by contradiction that there exists a binary string that is not in $f(A)$ therefore proving that f is not onto and thus A is not countable. Let us define digit d_i as

$$d_i = \begin{cases} 0 & \text{if } a_{ii} = 1 \\ 1 & \text{if } a_{ii} \neq 1 \end{cases}$$

Thus the infinite binary string d whose i^{th} digit is d_i is different from all $f(i)$. By our assumption $d \in A$, but there is no pre-image in \mathbb{N} such that $f(i) = d$ for any i . Thus there is contradiction. Thus the set of all infinite binary strings is not countable. \square