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Results about Euler's path and circuits

MATH 450 Seminar in Proof

Definition 1.1: ***Euler Path:*** An *Euler Path* on a graph G is a path that traverses each edge exactly once.
Definition 1.2: ***Euler Circuit/Cycle:*** An Euler circuit on a graph G is a Euler Path which starts and ends on the same node.

Results to be proven:

1. Show that any graph where the degree of every vertex is even iff it has an Eulerian circuit.

2. Show that if there are exactly two vertices a and b of odd degree, there is an Eulerian path on the graph.

Proof :

1. $\Rightarrow$ Let G be a graph such that it has an Euler circuit. According to the definition of Euler Circuit on a graph, given a vertex v_1, we can traverse every other node only once and end at the vertex v_1. Thus every node/vertex has one entry point and one exit point. We know this because G has an Euler circuit. Thus every node/vertex has two edges attached to it thus making it a node of even degree.
$\Leftarrow$ Let every node in a graph G have an even degree. Let us prove this by induction. Assume, that there are only two nodes in G, each with two edges on them. Thus it is clear that you will end on the same node that you started with, thus making a Euler circuit on G.
Now,let G be graph of even degree with more than two nodes. Lets start with an arbitrary node v_1. Now since every node has an even degree, we can follow a trail from v_1 to the next node and repeat it for the next node we encounter. We know that this is possible because each time we enter a node v_k we know there is an unused edge adjacent to it for us to use to traverse. Thus using a unique path we will come back on v_1 since there are even number of edges, we will always find an untapped edge to let us continue our path back to v_1.

2. Let G be a graph with Euler circuit. Thus every node/vertex has an even degree. now let us add one node say b and add an edge to a node a in the existing graph G. Note that before adding the edge from b to a, a in G had an even degree. Now if we start drawing our path from b, and since it has only one edge connecting to a we go to a now, if we hypothetically ignore the edge connecting a and b, the remainder of G has nodes with even edges, thus making it a Euler circuit. Therefore, the trail will end at a but since we have already used the edge connecting a and b, we stop at a. Thus we were able to traverse all the edges in $G + b$ exactly once, starting from b and ending at an edge a in G.

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