

## MATH 450 Seminar in Proof

Prove that the set of all infinite binary strings is uncountable.

*Proof.* Let us proceed by contradiction. Let  $A$  be the set that represents the set of all possible infinite binary strings. Then we assume that  $A$  is countable, i.e.  $|A| = |\mathbb{N}|$  then we can say that there is a bijection  $f : \mathbb{N} \rightarrow A$ . Since  $A$  is countable we can list the elements of  $A$  which are mapped from  $\mathbb{N}$  through  $f$ .

$$\begin{aligned}f(1) &= a_{11}a_{12}a_{13}a_{14}\dots \\f(2) &= a_{21}a_{22}a_{23}a_{24}\dots \\f(3) &= a_{31}a_{32}a_{33}a_{34}\dots \\&\dots \\&\dots \\&\dots\end{aligned}$$

where  $a_{ij}$  is the  $j^{th}$  digit of  $f(i)$  where  $i, j \in \mathbb{N}$ . Since  $f$  is a bijection,  $f$  is onto. We will show by construction that there exists a binary string that is not in  $f(A)$  therefore proving by contradiction that  $f$  is not onto and thus  $A$  is not countable.

Let us define digit  $d_i$  as

$$d_i = \begin{cases} 0 & \text{if } a_{ii} = 1 \\ 1 & \text{if } a_{ii} \neq 1 \end{cases}$$

Thus the infinite binary string  $d$  whose  $i^{th}$  digit is  $d_i$  is different from all  $f(i)$ . By our assumption  $d \in A$ , but there is no pre-image in  $\mathbb{N}$ . Thus there is a contradiction. Thus the set of all infinite binary strings is not countable.  $\square$