Banach & Hilbert Spaces: An Introduction to Functional Analysis

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Introduction To The Topic

This presentation is about Hilbert spaces and Banach spaces and some relevant theorems and results. They are the central objects of study in functional analysis.

- Functional analysis itself is a cool mix of:
  - Linear algebra (vector spaces)
  - Topology
  - Real analysis
- It has a ton of applications in physics, particularly in quantum mechanics
- We are primarily interested in sets called function spaces.
  - A vector space where the vectors are functions
  - We also put a topology on the space

Objectives

In this presentation, I hope to accomplish

- High-level review of relevant concepts
    - Metric spaces, completeness
    - Linear algebra (vector spaces, basis, dimension)
- Discussion of continuous linear operators.
    - From vector spaces, we get *linear maps*
    - From topology, we study *continuous maps*
    - In functional analysis, we study both. How do we combine them together?
- Considered one of the "four important theorems" in functional analysis [TODO: decide on a theorem]
    - The Hahn-Banach Theorem
    - The Uniform Boundedness Principle (Banach-Steinhaus Theorem)
    - The Open Mapping Theorem
    - The Closed Graph Theorem

Converging Sequence, Cauchy Sequence

Definition (Converging Sequence)

If X is a metric space, a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to converge to $x_0 \in X$ if for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N_\varepsilon$.

There is a (very similar looking) definition to the notion of a converging sequence: a Cauchy sequence.

Definition (Cauchy Sequence)

If X is a metric space, a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be a Cauchy sequence if for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ when $m, n \geq N_\varepsilon$.

Intuitively, a Cauchy sequence is one where the tail of a sequence is eventually arbitrarily small enough.

Convergent Sequence $\implies$ Cauchy Sequence

Theorem

All convergent sequences are also Cauchy sequences.

Proof.

Suppose that $(x_n) \to x$. Let $\varepsilon > 0$ be arbitrary. Then, there are $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies d(x_n, x) < \frac{\varepsilon}{2}$$

$$n \geq N_2 \implies d(x_m, x) < \frac{\varepsilon}{2}$$

Let $N = \max\{N_1, N_2\}$ and note that $n \geq N$ means

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x_m, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

□

Complete Metric Space

The converse of the previous theorem is the definition of completeness.

Definition (Complete Metric Space)

A metric space X is complete if every Cauchy sequence of entries in X has a limit in X.

This is the most general form of completeness. Equivalent formulations of completeness in $\mathbb{R}$ are dependent on the ordered field axioms of $\mathbb{R}$.

- Least Upper Bound Property
- Nested Intervals Property
- Bolzano Weierstrass Property
- Monotone Convergence Property

Example

The metric space $\mathbb{Q}$ (with the usual metric) is not a complete metric space. Consider the sequence

$$(x_n) = (3, 3.1, 3.14, 3.141, 3.1415, 3.14159, ...)$$

- This is the sequence of rational approximations to π.

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- Note that $x_n \in \mathbb{Q}$ for all n.
- The limit is $\pi \notin \mathbb{Q}$.

Thus, $\mathbb{Q}$ is not a complete metric space.

Another Example

If X is any set under the discrete metric, then X is complete.

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- A sequence which is eventually constant is convergent.

Thus, any Cauchy sequence in X converges, and so X is complete.

Function Space

Definition (Function Space)

We define the function space $C[a, b]$ to be the set of all continuous functions from $[a, b]$ to $\mathbb{R}$.

We can define a metric on the function space as follows

$$d(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|$$

- Note that the maximum exists since $[a, b]$ is compact.

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- Note that the maximum exists since $[a, b]$ is compact.
- Conditions 1, 2, and 3 of a metric space are easy to show. The triangle inequality follows from

$$|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$$

and so

$$\max_{t \in [a,b]} |f(t) - g(t)| \leq \max_{t \in [a,b]} |f(t) - h(t)| + \max_{t \in [a,b]} |h(t) - g(t)|$$

Now, we transition into some results about vector spaces.

Linear Independence, Generating Set

Definition (Linear Independence)

Let V be a vector space over a field $\mathbb{F}$. A set $|V| < \infty$ is linearly independent if

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0$$

where $c_1, \ldots, c_n \in \mathbb{F}$ and $v_1, \ldots, v_n \in V$. An infinite set is linear independent if all of its finite subsets are linearly independent.

Definition (Spanning Set)

Let V be a vector space.

1. If $|S| < \infty \subseteq V$, then the span of S is the set of all linear combinations of elements in S.
2. The span of an infinite set S is the union of the span of all its finite subsets.
3. If $Y \subseteq X$ is a subspace and S is a set such that $\operatorname{span} S = Y$, then S is a generating set (or spanning set) of Y.

Basis & Dimension

Definition (Basis & Dimension)

Let V be a vector space.

1. A basis of V is a linearly independent generating set of V.
2. If β is a basis of V, then the dimension of V is defined to be $\dim V = |\beta|$.
3. If $\dim V < \infty$, then V is finite-dimensional. Otherwise, it is infinite dimensional.

Some things to note

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In functional analysis, we often consider infinite dimensional vector spaces over finite dimensional ones.

Function Space Is A Vector Space

The function space $C[a,b]$ may be turned into a vector space by defining vector addition and scalar multiplication as follows:

$$(f+g)(t) = f(t) + g(t)$$

$$(\alpha f)(t) = \alpha f(t)$$

The additive identity is the zero function, which maps everything in $[a,b]$ to 0.

We remark that the function space $C[a,b]$ is an infinite dimensional vector space.

Linear Map

Fundamental maps between vector spaces are linear maps.

Definition (Linear Map)

Suppose X and Y are vector spaces. A map $T : X \to Y$ is said to be linear if it preserves linear combinations. That is,

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 T v_1 + c_2 T v_2 + \cdots + c_n T v_n$$

Note that we use the notation Tx to mean $T(x)$. This is a common shorthand used in functional analysis.

A consequence of this definition is that linear maps send the identity of X to the identity of Y. This is written as $T0 = 0$.

- Here, we note that the 0 on the left is the identity of X while the 0 on the right is the identity of Y.

Example

For function spaces $C[a,b]$ the differentiation operator given by

$$Tf = f'(t)$$

is a linear operator. This follows from elementary calculus, where we proved

$$(f+g)'(t) = f'(t) + g'(t)$$

$$(\alpha f)'(t) = \alpha f'(t)$$

Norm, Normed Space

Definition (Norm & Normed Space)

Suppose that V is a vector space. A norm on V is a function which maps each $x \in V$ to a scalar in $\mathbb{F}$, denoted $\|x\|$ which satisfies the following properties

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\alpha x\| = |\alpha|\|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

If $\|\cdot\|$ is a norm on a vector space X, then the ordered pair $(V, \|\cdot\|)$ is a normed space. If the norm is inferred from context, we often don't write it explicitly.

Two pointers here

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Two pointers here

- The word "norm" refers to both the *value* of $\|x\|$ and to the *function* which maps each vector to its norm
- The norm essentially serves as a useful way to generalize the notion of vector length

Normed Spaces Are Metric Spaces

Every norm induces a metric. If V is a normed space, we may define a metric on V as

$$d(x, y) = \|x - y\|$$

A lot of the axioms of the metric follow from those of a norm. Symmetry is the only one that requires some work. We note that

$$\begin{aligned} d(y, x) &= \|y - x\| \\ &= \|-(x - y)\| \\ &= \|-1(x - y)\| \\ &= |-1|\|x - y\| \\ &= \|x - y\| \\ &= d(x, y) \end{aligned}$$

In this case, d is the metric induced by the norm. We conclude that every normed space is a metric space.

Examples

1. Consider the space $\mathbb{R}^n$. The L^p-norm is defined as

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

If $p = 2$, this is the usual Euclidean norm on $\mathbb{R}^n$. It induces the usual Euclidean metric on $\mathbb{R}^n$.

- In general, the L^p-norm induces the L^p-distance on $\mathbb{R}^n$

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2. Consider the function space $C[a, b]$. We define a norm on this space by

$$\|f\| = \max_{t\in[a,b]} |x(t)|$$

   This norm induces the same metric on the function space we defined earlier.

Banach Space

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A normed space V is a Banach space if V is a complete metric space under the metric induced by the norm of V.

Examples:

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- The space $\mathbb{R}^n$ is a Banach space.
- The function space $C[a, b]$ is a Banach space (proof omitted in interest of time)

Normed Space That Is Not A Banach Space

Consider the function space $C[a, b]$ with this norm

$$\|f\| = \int_0^1 |f(t)| \, dt$$

This norm induces the metric

$$d(f, g) = \int_0^1 |f(t) - g(t)| \, dt$$

This metric does not make $C[a, b]$ a complete space.

Note that a space can only be complete with respect to some metric. We have seen metrics which make $C[a, b]$ a complete space while other metrics don't make it a complete space. The point is that idea completeness is a *relationship* between a metric and the space.

Inner Product, Inner Product Space

Definition (Inner Product)

Let V be a vector space over a field $\mathbb{F}$. An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that

1. $\langle x + y, z \rangle = \langle x, y \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (complex conjugation)
4. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.

If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space V, then the ordered pair $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Once again, we don't often write the inner product explicitly if it is implied from context.

Note that the third axiom refers to complex conjugation. Hence, the inner product is *conjugate symmetric*. If the underlying scalar field is $\mathbb{R}$, then the axiom becomes

$$\langle x, y \rangle = \langle y, x \rangle$$

Inner Product Space Is A Normed Space

If V is an inner product space, we may define a norm on V as follows: if $x \in V$, then

$$\|x\| = \langle x, x \rangle$$

One can verify that this is a norm on V (only the triangle inequality is non-trivial). This norm is called the norm induced by the inner product.

This result shows us that every inner product space is a normed space. Moreover, since every normed space is a metric space, we have that every inner product space is a metric space too.

Examples

- If $X = \mathbb{R}^n$, the standard inner product on $\mathbb{R}$ is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

This is often called the *dot product*. It induces the L^2-norm on $\mathbb{R}^n$ (and, consequently, induces the L^2 distance on $\mathbb{R}^n$).

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- If $X = \mathbb{C}^n$, then the standard inner product becomes

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i$$

Another Function Space

We may consider the function space $L^2[a,b]$. This function space has a norm which somewhat resembles the L^2-norm for $\mathbb{R}^n$.

$$\|f\| = \left(\int_a^b f(t)^2 \right)^{1/2}$$

This norm can be generated by the following inner product

$$\langle f, g \rangle = \int_a^b f(t)g(t)\ dt$$

The norm on $C[a,b]$ cannot be induced by any inner product. Hence, while all inner products induce norm, not every norm can be induced by an inner product (the L^p-norms for $p \neq 2$ is a common counterexample).

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- Since every inner product space is a normed space, it follows that every Hilbert space is a Banach space.
- The converse need not hold. For instance, the norm we defined on $C[a, b]$ is not induced by any inner product. Hence, it cannot be a Hilbert space (but it is a Banach space).

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- The converse need not hold. For instance, the norm we defined on $C[a, b]$ is not induced by any inner product. Hence, it cannot be a Hilbert space (but it is a Banach space).
- $\mathbb{R}^n$ is a Hilbert space.

Topological Vector Space

So far, we have given results on two seemingly disjoint parts of math

- Linear algebra, which gives us *linear maps*
- Topology, which gives us *continuous operators*

We combine the two concepts in some forthcoming definitions

Definition (Topological Vector Space)

Suppose V is a vector space and let $\mathcal{T}$ be a topology on V. We say $(V, \mathcal{T})$ is a topological vector space if

1. All one-point sets in $\mathcal{T}$ are closed
2. The vector space operations $+$ and $\cdot$ are continuous in $\mathcal{T}$.

Continuous Linear Map

For the rest of these slides, the results will mostly concern normed spaces (and possibly Hilbert spaces). The definition of continuity is the same as that for metrizable spaces (but defined with norms in place of metrics).

Definition (Continuous Linear Map)

Let X, Y be normed spaces and let $T : X \to Y$ be linear. We say T is continuous at a point $x_0 \in X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|x - x_0\| < \delta \implies \|Tx - Tx_0\| < \varepsilon$$

If T is continuous at all points in X, we simply say T is continuous.

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- There is a really nice characterization of continuity for linear operators in terms of a (slightly) more familliar concept. This is discussed in the forthcoming slides.

Bounded Operator

Definition (Bounded Linear Operator)

Let X, Y be normed spaces and let $T : X \to Y$ be linear. The operator T is said to be bounded if there exists $M \in \mathbb{R}$ such that

$$\|Tx\| \leq M\|x\|$$

It is useful to find the *smallest possible* value for which the inequality holds. Excluding the case when $x = 0$ (which is trival and boring), we see that

$$\frac{\|Tx\|}{\|x\|} \leq M$$

To make this inequality hold for all $x \in X$, we simply take the *supremum* over all nonzero values of x. This leads to another useful definition

Norm Of Operator

Definition (Norm Of Operator)

Let X, Y be normed spaces and $T : X \to Y$ linear. Then, the norm of T, denoted $\|T\|$, is defined using two (equivalent) formulations

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} \|Tx\|$$

Some important facts:

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- The norm on an operator does indeed satisfy the norm axioms (proof omitted, in the interest of time)

Norm Of Operator, Examples

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Norm Of Operator, Examples

1. The zero operator $\mathbf{0} : X \to Y$ which sends everything to 0 is bounded and has norm 0.
2. The identity operator on a nontrivial subspace is bounded and has norm 1.
3. Let $\mathcal{P}[0,1]$ be the space of all polynomials on $[0,1]$ with the usual "max" norm. Then, the differentiation operator

$$Tx(t) = x'(t)$$

is unbounded. If $x_n(t) = t^n$, then we have $\|x_n\| = 1$ but

$$\|Tx_n\| = \|nt^{n-1}\| = n$$

which is unbounded.

Bounded Operators On Finite Dimensional Spaces

Theorem

Let X be a normed space with finite dimension. Then, every linear map from X to itself is bounded.

Proof.

SEE THIS PROOF IN THE BOOK AND SEE IF YOU CAN UNDERSTAND IT WELL ENOUGH TO SUCCINTLY PUT IT HERE. □

Characterization Of Continuous Linear Maps

Now, we have this theorem which completely characterizes all continuous linear maps.

Theorem

Let X, Y be normed spaces and let $T : X \to Y$ be linear. Then T is continuous if and only if T is bounded.

We first prove the backwards direction. Assume T is bounded, then $\|T\| < \infty$. Let $\varepsilon > 0$ and $x_0 \in X$ be arbitrary. Let

$$\delta = \frac{\varepsilon}{\|T\|}$$

Assume $\|x - x_0\| < \delta$. Then, by properties of linearity and boundedness, we get

$$\begin{aligned}\|Tx - Tx_0\| &= \|T(x - x_0)\| \\ &= \|T\|\|x - x_0\| \\ &< \|T\| \cdot \frac{\varepsilon}{\|T\|} \\ &= \varepsilon\end{aligned}$$

This shows T is continuous and establishes the backward direction.

Proof (Forward Direction)

Now, assume that T is continuous. Then T is continuous everywhere, so we consider the case when $x = 0$. Then, we know there exists $\delta > 0$ such that

$$\|x\| < \delta \implies \|Tx\| < 1$$