Banach & Hilbert Spaces: An Introduction to Functional Analysis

Nilay Tripathi

December 2023

Introduction To The Topic

Banach spaces and Hilbert spaces are studied in functional analysis.

- Functional analysis itself is a cool mix of:
  - Linear algebra (vector spaces)
  - Topology
  - Real analysis
- It has a ton of applications in physics, particularly in quantum mechanics
- We are primarily interested in sets called function spaces.
  - A vector space where the vectors are functions
  - We also put a topology on the space

Objectives

In this presentation, I hope to accomplish

- High-level review of relevant concepts
  - Metric spaces, completeness
  - Linear algebra (vector spaces, basis, dimension)
- Discussion of continuous linear operators.
  - From vector spaces, we get *linear maps*
  - From topology, we study *continuous maps*
  - In functional analysis, we study both. How do we combine them together?
- One of the "four important theorems" in functional analysis
  - The Hahn-Banach Theorem
  - The Uniform Boundedness Principle (Banach-Steinhaus Theorem)
  - The Open Mapping Theorem
  - **The Closed Graph Theorem**

Converging Sequence, Cauchy Sequence

Definition (Converging Sequence)

If X is a metric space, a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to converge to $x_0 \in X$ if for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N_\varepsilon$.

Definition (Cauchy Sequence)

If X is a metric space, a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be a Cauchy sequence if for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ when $m, n \geq N_\varepsilon$.

Convergent Sequence $\implies$ Cauchy Sequence

Theorem

All convergent sequences are also Cauchy sequences.

Proof.

Suppose that $(x_n) \to x$. Let $\varepsilon > 0$ be arbitrary. Then, there are $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies d(x_n, x) < \frac{\varepsilon}{2}$$
$$n \geq N_2 \implies d(x_m, x) < \frac{\varepsilon}{2}$$

Let $N = \max\{N_1, N_2\}$ and note that $n \geq N$ means

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x_m, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

$\square$

Complete Metric Space

The converse of the previous theorem is the definition of completeness.

Definition (Complete Metric Space)

A metric space X is complete if every Cauchy sequence of entries in X has a limit in X.

This characterization of completeness works in the most general of metric spaces.

Example

The metric space $\mathbb{Q}$ (with the usual metric) is not a complete metric space. Consider the sequence

$$(x_n) = (3, 3.1, 3.14, 3.141, 3.1415, 3.14159, ...)$$

- This is the sequence of rational approximations to π.
- Note that $x_n \in \mathbb{Q}$ for all n.
- The limit is $\pi \notin \mathbb{Q}$.

Thus, $\mathbb{Q}$ is not a complete metric space.

Another Example

If X is any set under the discrete metric, then X is complete.

- Let (x_n) be a Cauchy sequence in X.
- We claim that (x_n) is eventually constant. This happens say for $\varepsilon = \frac{1}{2}$, for example.
- A sequence which is eventually constant is convergent.

Thus, any Cauchy sequence in X converges, and so X is complete.

Function Space

Definition (Function Space)

We define the function space $C[a,b]$ to be the set of all continuous functions from $[a,b]$ to $\mathbb{R}$.

We can define a metric on the function space as follows

$$d(f,g) = \max_{t\in[a,b]} |f(t) - g(t)|$$

- Note that the maximum exists since $[a,b]$ is compact.
- Conditions 1, 2, and 3 of a metric space are easy to show. The triangle inequality follows from

$$|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$$

and so

$$\max_{t\in[a,b]} |f(t) - g(t)| \leq \max_{t\in[a,b]} |f(t) - h(t)| + \max_{t\in[a,b]} |h(t) - g(t)|$$

Linear Independence, Generating Set

Definition (Linear Independence)

Let V be a vector space over a field $\mathbb{F}$. A set $|V| < \infty$ is linearly independent if

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0$$

where $c_1, ..., c_n \in \mathbb{F}$ and $v_1, ..., v_n \in V$. An infinite set is linear independent if all of its finite subsets are linearly independent.

Definition (Spanning Set)

Let V be a vector space.

1. If $|S| < \infty \subseteq V$, then the span of S is the set of all linear combinations of elements in S.
2. The span of an infinite set S is the union of the span of all its finite subsets.
3. If $Y \subseteq X$ is a subspace and S is a set such that $\operatorname{span} S = Y$, then S is a generating set (or spanning set) of Y.

Basis & Dimension

Definition (Basis & Dimension)

Let V be a vector space.

1. A basis of V is a linearly independent generating set of V.
2. If β is a basis of V, then the dimension of V is defined to be $\dim V = |\beta|$.
3. If $\dim V < \infty$, then V is finite-dimensional. Otherwise, it is infinite dimensional.

In functional analysis, we often consider infinite dimensional vector spaces over finite dimensional ones.

Function Space Is A Vector Space

The function space $C[a, b]$ may be turned into a vector space by defining vector addition and scalar multiplication as follows:

$$(f + g)(t) = f(t) + g(t)$$

$$(\alpha f)(t) = \alpha f(t)$$

The additive identity is the zero function, which maps everything in $[a, b]$ to 0.

We remark that the function space $C[a, b]$ is an infinite dimensional vector space.

Linear Map

Fundamental maps between vector spaces are linear maps.

Definition (Linear Map)

Suppose X and Y are vector spaces. A map $T : X \to Y$ is said to be linear if it preserves linear combinations. That is,

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 T v_1 + c_2 T v_2 + \cdots + c_n T v_n$$

Note that we use the notation Tx to mean $T(x)$. This is a common shorthand used in functional analysis.

Example

For function spaces $C[a, b]$ the differentiation operator given by

$$Tf = f'(t)$$

is a linear operator. This follows from elementary calculus, where we proved

$$(f + g)'(t) = f'(t) + g'(t)$$

$$(\alpha f)'(t) = \alpha f'(t)$$

Norm, Normed Space

Definition (Norm & Normed Space)

Suppose that V is a vector space. A norm on V is a function which maps each $x \in V$ to a scalar in $\mathbb{F}$, denoted $\|x\|$ which satisfies the following properties

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\alpha x\| = |\alpha|\|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

If $\| \cdot \|$ is a norm on a vector space X, then the ordered pair $(V, \| \cdot \|)$ is a normed space. If the norm is inferred from context, we often don't write it explicitly.

Normed Spaces Are Metric Spaces

Every norm induces a metric. If V is a normed space, we may define a metric on V as

$$d(x, y) = \|x - y\|$$

A lot of the axioms of the metric follow from those of a norm. Symmetry is the only one that requires some work. We note that

$$\begin{aligned}
d(y, x) &= \|y - x\| \\
&= \| - (x - y)\| \\
&= \| - 1(x - y)\| \\
&= | - 1|\|x - y\| \\
&= \|x - y\| \\
&= d(x, y)
\end{aligned}$$

In this case, d is the metric induced by the norm. We conclude that every normed space is a metric space.

Examples

1. Consider the space $\mathbb{R}^n$. The L^p-norm is defined as

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

If $p = 2$, this is the usual Euclidean norm on $\mathbb{R}^n$. It induces the usual Euclidean metric on $\mathbb{R}^n$.

- In general, the L^p-norm induces the L^p-distance on $\mathbb{R}^n$

Examples

1. Consider the space $\mathbb{R}^n$. The L^p-norm is defined as

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

   If $p = 2$, this is the usual Euclidean norm on $\mathbb{R}^n$. It induces the usual Euclidean metric on $\mathbb{R}^n$.

   - In general, the L^p-norm induces the L^p-distance on $\mathbb{R}^n$

2. Consider the function space $C[a, b]$. We define a norm on this space by

$$\|f\| = \max_{t \in [a,b]} |f(t)|$$

   This norm induces the same metric on the function space we defined earlier.

Examples

1. Consider the space $\mathbb{R}^n$. The L^p-norm is defined as

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

   If $p = 2$, this is the usual Euclidean norm on $\mathbb{R}^n$. It induces the usual Euclidean metric on $\mathbb{R}^n$.

   - In general, the L^p-norm induces the L^p-distance on $\mathbb{R}^n$

2. Consider the function space $C[a, b]$. We define a norm on this space by

$$\|f\| = \max_{t \in [a,b]} |f(t)|$$

   This norm induces the same metric on the function space we defined earlier.

3. A metric space need not be a normed space. The discrete metric on $\mathbb{R}$ is not induced by any norm.

Banach Space

Definition (Banach Space)

A normed space V is a Banach space if V is a complete metric space under the metric induced by the norm of V.

Examples:

- The space $\mathbb{R}^n$ is a Banach space.
- The function space $C[a, b]$ is a Banach space (proof omitted in interest of time)

Normed Space That Is Not A Banach Space

Consider the function space $C[0, 1]$ with this norm

$$\|f\| = \int_0^1 |f(t)| \, dt$$

This norm induces the metric

$$d(f, g) = \int_0^1 |f(t) - g(t)| \, dt$$

This metric does not make $C[a, b]$ a complete space.

Inner Product, Inner Product Space

Definition (Inner Product)

Let V be a vector space over a field $\mathbb{F}$. An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (complex conjugation)
4. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.

If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space V, then the ordered pair $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Once again, we don't often write the inner product explicitly if it is implied from context.

Inner Product Space Is A Normed Space

If V is an inner product space, we may define a norm on V as follows: if $x \in V$, then

$$\|x\| = \sqrt{\langle x, x \rangle}$$

One can verify that this is a norm on V (only the triangle inequality is non-trivial). This norm is called the norm induced by the inner product.

From the observations and examples stated previously, we see

$$\text{inner product space} \implies \text{normed space} \implies \text{metric space}$$

But the reverse implications need not be true.

Examples

- If $X = \mathbb{R}^n$, the standard inner product on $\mathbb{R}^n$ is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

This is often called the *dot product*. It induces the L^2-norm on $\mathbb{R}^n$ (and, consequently, induces the L^2 distance on $\mathbb{R}^n$).

Examples

- If $X = \mathbb{R}^n$, the standard inner product on $\mathbb{R}^n$ is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

  This is often called the *dot product*. It induces the L^2-norm on $\mathbb{R}^n$ (and, consequently, induces the L^2 distance on $\mathbb{R}^n$).

- If $X = \mathbb{C}^n$, then the standard inner product becomes

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i$$

Inner Product Space Need Not Be Normed Space

Not every norm can be induced by an inner product

- The L^p norm for $p = 2$ is the usual Eucidean norm
  - It can be induced by the standard inner product
- However, for $p \neq 2$, the L^p-norm cannot be induced by any inner product

Hilbert Space

Definition (Hilbert Space)

An inner product space V is a Hilbert space if V is a complete metric space under the metric induced by the inner product of V.

- Since every inner product space is a normed space, it follows that every Hilbert space is a Banach space.
- The converse need not hold. For instance, the norm we defined on $C[a, b]$ is not induced by any inner product. Hence, it cannot be a Hilbert space (but it is a Banach space).
- $\mathbb{R}^n$ is a Hilbert space.

Topological Vector Space

So far, we have given results on two seemingly disjoint parts of math

- Linear algebra, which gives us *linear maps*
- Topology, which gives us *continuous operators*

We combine the two concepts in some forthcoming definitions

Definition (Topological Vector Space)

Suppose V is a vector space and let $\mathcal{T}$ be a topology on V. We say $(V, \mathcal{T})$ is a topological vector space if

1. All one-point sets in $\mathcal{T}$ are closed
2. The vector space operations $+$ and $\cdot$ are continuous in $\mathcal{T}$.

Continuous Linear Map

Definition (Continuous Linear Map)

Let X, Y be normed spaces and let $T : X \to Y$ be linear. We say T is continuous at a point $x_0 \in X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|x - x_0\| < \delta \implies \|Tx - Tx_0\| < \varepsilon$$

If T is continuous at all points in X, we simply say T is continuous.

Bounded Operator

Definition (Bounded Linear Operator)

Let X, Y be normed spaces and let $T : X \to Y$ be linear. The operator T is said to be bounded if there exists $M \in \mathbb{R}$ such that

$$\|Tx\| \leq M\|x\|$$

It is useful to find the *smallest possible* value for which the inequality holds. Excluding the case when $x = 0$ (which is trival and boring), we see that

$$\frac{\|Tx\|}{\|x\|} \leq M$$

To make this inequality hold for all $x \in X$, we simply take the *supremum* over all nonzero values of x. This leads to another useful definition

Norm Of Operator

Definition (Norm Of Operator)

Let X, Y be normed spaces and $T : X \to Y$ linear. Then, the norm of T, denoted $\|T\|$, is defined using two (equivalent) formulations

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\| = 1}} \|Tx\|$$

Some important facts:

- The operator T is bounded if $\|T\| < \infty$
- Using the definition of norm of an operator, we may write the boundedness condition as
$$\|Tx\| \leq \|T\|\|x\|$$
- The norm on an operator does indeed satisfy the norm axioms (proof omitted, in the interest of time)

Norm Of Operator, Examples

1. The zero operator $\mathbf{0} : X \to Y$ which sends everything to 0 is bounded and has norm 0.

Norm Of Operator, Examples

1. The zero operator $\mathbf{0} : X \to Y$ which sends everything to 0 is bounded and has norm 0.
2. The identity operator on a nontrivial subspace is bounded and has norm 1.

Norm Of Operator, Examples

1. The zero operator $\mathbf{0} : X \to Y$ which sends everything to 0 is bounded and has norm 0.
2. The identity operator on a nontrivial subspace is bounded and has norm 1.
3. Let $\mathcal{P}[0,1]$ be the space of all polynomials on $[0,1]$ with the usual "max" norm. Then, the differentiation operator

$$Tx(t) = x'(t)$$

is unbounded. If $x_n(t) = t^n$, then we have $\|x_n\| = 1$ but

$$\|Tx_n\| = \|nt^{n-1}\| = n$$

which is unbounded.

Characterization Of Continuous Linear Maps

Now, we have this theorem which completely characterizes all continuous linear maps.

Theorem

Let X, Y be normed spaces and let $T : X \to Y$ be linear. Then T is continuous if and only if T is bounded.

We first prove the backwards direction. Assume T is bounded, then $\|T\| < \infty$. Let $\varepsilon > 0$ and $x_0 \in X$ be arbitrary. Let

$$\delta = \frac{\varepsilon}{\|T\|}$$

Assume $\|x - x_0\| < \delta$. Then, by properties of linearity and boundedness, we get

$$\begin{aligned} \|Tx - Tx_0\| &= \|T(x - x_0)\| \\ &\leq \|T\|\|x - x_0\| \\ &< \|T\| \cdot \frac{\varepsilon}{\|T\|} \\ &= \varepsilon \end{aligned}$$

This shows T is continuous and establishes the backward direction.

Proof (Forward Direction)

Now, assume that T is continuous. Then T is continuous everywhere, so we consider the case when $x = 0$. Then, we know there exists $\delta > 0$ such that

$$\|x\| < \delta \implies \|Tx\| < 1$$

Note that we may write x as follows

$$x = \frac{2x}{\|x\|}\delta \cdot \frac{\|x\|}{2\delta}$$

So we conclude

$$\begin{aligned} Tx &= T\left(\frac{2x\delta}{\|x\|} \cdot \frac{\|x\|}{2\delta}\right) \\ &= \frac{2\|x\|}{\delta} T\left(\underbrace{\frac{x}{2\|x\|}\delta}_{v}\right) \end{aligned}$$

Proof, Forward Direction (Cont.)

Note that $\|v\| = \frac{\delta}{2} < \delta$. Hence, by the continuity condition, we have

$$\|Tv\| < 1$$

And thus, we conclude

$$\|Tx\| < \frac{2}{\delta}\|x\|$$

Hence, we see T is bounded, completing the proof.

Graphs & Closed Operators

Definition (Graph & Closed Operator)

Let X, Y be normed spaces and $T : X \to Y$ be linear. The graph of T is the set

$$G_T := \{(x, Tx) : x \in X\}$$

If the set G_T is closed in the product space $X \times Y$, the linear operator T is a closed linear operator.

Let X and Y be normed spaces. We define a norm on the product space $X \times Y$ as follows:

$$\|(x, y)\| = \|x\|_X + \|y\|_Y$$

Product Of Banach Spaces Is Banach Space

Theorem

Let X and Y be Banach spaces. Then, $X \times Y$ is a Banach space.

Let $z_n = (x_n, y_n)$ be an arbitrary Cauchy sequence in $X \times Y$. Then, for some $N \in \mathbb{N}$ we note that $m, n \geq N$ implies

$$\|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\| < \varepsilon$$

Thus, this shows that $(x_n), (y_n)$ are also Cauchy sequences. Since X, Y are complete, these Cauchy sequences converge. Say $(x_n) \to x$ and $(y_n) \to y$. We claim that $z_n \to (x, y) = z$.

Product Of Banach Spaces (cont.)

Since (x_n), (y_n) converge, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies \|x_n - x\| < \frac{\varepsilon}{2}$$
$$n \geq N_2 \implies \|y_n - y\| < \frac{\varepsilon}{2}$$

So we get

$$\begin{aligned}\|z_n - z\| &= \|x_n - x\| + \|y_n - y\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Thus, we have $(z_n) \to z$. So, all Cauchy sequences in $X \times Y$ converge; so $X \times Y$ is a complete space.

Open Mapping Theorem, Bounded Inverse Theorem

Definition (Open Mapping)

If X and Y are metric spaces, then $T : X \to Y$ is an open mapping if the image of open sets in X is open in Y.

Theorem (Open Mapping Theorem)

Every surjective bounded linear operator between two Banach spaces is an open mapping.

Theorem (Bounded Inverse Theorem)

If T is a bijective linear operator satisfying the conditions of the open mapping theorem, then T^{-1} is bounded.

Theorem (Closed Graph Theorem)

Let X and Y be Banach spaces and $D \subseteq X$. Let $T : D \to Y$ be a closed linear operator. Then T is bounded if D is closed in X.

We consider a "natural" mapping $P : G_T \to D$ defined by

$$P(x, Tx) = x$$

The map P is linear since

$$P(x + cy, T(x + cy)) = P(x + cy, Tx + cTy) = P[(x, Tx) + c(y, Ty)]$$

The map P is bounded since

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|$$

Closed Graph Theorem, Proof (cont.)

Note that P is bijective. The inverse mapping is

$$P^{-1}(x) = (x, Tx)$$

Note that G_T and D are both complete spaces (G_T and D are closed subsets of complete metric spaces). The **bounded inverse theorem** holds and we get that P^{-1} is bounded i.e. there exists M such that $\|P^{-1}(x)\| \leq M\|x\|$.

Closed Graph Theorem, Proof (cont.)

We finish the proof by noting that

$$\begin{aligned} \|Tx\| &\leq \|x\| + \|Tx\| \\ &= \|(x, Tx)\| \\ &= \|P^{-1}(x)\| \\ &\leq M\|x\| \end{aligned}$$

Hence $\|Tx\| \leq M\|x\|$, and so T is bounded.

References

- Kreyszig, *Introductory Functional Analysis with Applications*
- Rudin, *Functional Analysis*