Topology Presentation Notes

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1 Metric Space Preliminaries

- Sequences and Cauchy sequences
  - A sequence converges to a point $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} : n \geq N_\varepsilon \implies d(x_n, x) < \varepsilon$$

  - A sequence is Cauchy if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} : m, n \geq N_\varepsilon \implies d(x_n, x_m) < \varepsilon$$

  - Every convergent sequence is Cauchy. PROOF: use triangle inequality with $\varepsilon/2$ argument.
- Complete metric space: every Cauchy sequence in X converges to a limit in X.
  - The metric space $\mathbb{Q}$ is not complete. PROOF: consider the sequence of rational approximations to any irrational number.
  - Discrete spaces are complete. PROOF: every Cauchy sequence is eventually constant.

1.1 Function Space

We define the **function space**, $C[a, b]$, to be all continuous functions from $[a, b]$ to $\mathbb{R}$. That is

$$C[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous}\}$$

We define a metric on the function space as follows: for $f, g \in C[a, b]$

$$d(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|$$

- The metric is well defined (i.e. is finite). PROOF: $f - g$ is continuous and $[a, b]$ is compact. EVT implies existence of a maximum (and minimum)

2 Vector Space Preliminaries

- A vector space V over a field $\mathbb{F}$ has two operations: vector addition and scalar multiplication where
  - Vector addition is an abelian group
  - Scalar multiplicatin satisfies: $1v = v$, $a(bv) = (ab)v$, and two distributive laws: scalar multiplication distributes over vector addition and field addition.
- Linearly independent sets: a (finite) set V is linearly independent if for scalars c_i and vectors v_i

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0$$

  Infinite sets are linearly independent if all of its finite subsets are L.I.

- If V is a V.S. over $\mathbb{F}$ and $S \subseteq V$ is finite, then span S is defined as

$$\text{span } S = \{c_1v_1 + \cdots + c_nv_n : c_i \in \mathbb{F}, v_i \in S\}$$

  The span of an infinite set is the union of the span of all its finite subsets.

- If $E \subseteq X$ is a subspace, then a set S is a spanning set if span $S = E$.
- A basis is a linearly independent generating set.
  - It is the smallest generating set and the largest L.I. set (in a f.d. V.S.)
  - Every vector has a unique representation in a basis
  - The dimension of a V.S. is the size of its basis (either finite or infinite)
  - Every V.S. has a basis. For f.d. spaces, all bases have the same size

2.1 Function Spaces

We turn the function space $C[a, b]$ into a vector space over $\mathbb{R}$. For $f, g \in C[a, b]$ and $\alpha \in \mathbb{R}$, define

$$(f + g)(t) = f(t) + g(t)$$
$$(\alpha f)(t) = \alpha f(t)$$

The additive identity is the zero function. It is also an infinite dimensional V.S.

2.2 Linear Maps

Linear maps preserve linear combos. So $T : X \to Y$ is linear if

$$T(c_1v_1 + c_2v_2 + \cdots c_nv_n) = c_1Tv_1 + c_2Tv_2 + \cdots + c_nTv_n$$

Notably, linear maps send the identity of X to the identity of Y (i.e. $T0 = 0$)

- The differentiation operator on $C[a, b]$, $Df = f'$, is linear. PROOF: derivative rules from calculus.

3 Normed Spaces

- A norm on a V.S. V generalizes the length of a vector. It satisfies these axioms
  1. $\|x\| \geq 0$ with $\|x\| = 0 \iff x = 0$
  2. $\|\alpha x\| = |\alpha|\|x\|$ (norm only depends on direction)
  3. $\|x + y\| \leq \|x\| + \|y\|$ (satisfies triangle inequality)

  We use the word "norm" to mean both the value $\|x\|$ and the function $x \mapsto \|x\|$.

- Normed space $\implies$ metric space
  - Define the metric as $d(x, y) = \|x - y\|$

3.1 Examples

- The L^p-norms on $\mathbb{R}^n$ are defined by

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

  - The L^p-norm induces the L^p-metric on $\mathbb{R}^n$.
  - If $p = 2$, this is the usual notion of length/metric on $\mathbb{R}^2$.