Topology Presentation Notes

Nilay Tripathi

December 4, 2023

1 Metric Space Preliminaries

- Sequences and Cauchy sequences
  - A sequence converges to a point $x_0 \in X$ if
  $$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} : n \geq N_\varepsilon \implies d(x_n, x) < \varepsilon$$
  - A sequence is Cauchy if
  $$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} : m, n \geq N_\varepsilon \implies d(x_n, x_m) < \varepsilon$$
  - Every convergent sequence is Cauchy. PROOF: use triangle inequality with $\varepsilon/2$ argument.
- Complete metric space: every Cauchy sequence in X converges to a limit in X.
  - The metric space $\mathbb{Q}$ is not complete. PROOF: consider the sequence of rational approximations to any irrational number.
  - Discrete spaces are complete. PROOF: every Cauchy sequence is eventually constant.
  - Briefly comment about equivalent formulas of completeness in $\mathbb{R}$ (i.e. least upper bound property, montone convergence property, etc.). Cauchy completeness is the most general definition and works in all metric spaces.

1.1 Function Space

We define the **function space**, $C[a, b]$, to be all continuous functions from $[a, b]$ to $\mathbb{R}$. That is

$$C[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous}\}$$

We define a metric on the function space as follows: for $f, g \in C[a, b]$

$$d(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|$$

- The metric is well defined (i.e. is finite). PROOF: $f - g$ is continuous and $[a, b]$ is compact. EVT implies existence of a maximum (and minimum)

2 Vector Space Preliminaries

- A vector space V over a field $\mathbb{F}$ has two operations: vector addition and scalar multiplication where
  - Vector addition is an abelian group
  - Scalar multiplicatin satisfies: $1v = v$, $a(bv) = (ab)v$, and two distributive laws: scalar multiplication distributes over vector addition and field addition.

- Linearly independent sets: a (finite) set V is linearly independent if for scalars c_i and vectors v_i

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0$$

  Infinite sets are linearly independent if all of its finite subsets are L.I.

- If V is a V.S. over $\mathbb{F}$ and $S \subseteq V$ is finite, then span S is defined as

$$\text{span } S = \{c_1v_1 + \cdots + c_nv_n : c_i \in \mathbb{F}, v_i \in S\}$$

  The span of an infinite set is the union of the span of all its finite subsets.

- If $E \subseteq X$ is a subspace, then a set S is a spanning set if span $S = E$.

- A basis is a linearly independent generating set.
  - It is the smallest generating set and the largest L.I. set (in a f.d. V.S.)
  - Every vector has a unique representation in a basis
  - The dimension of a V.S. is the size of its basis (either finite or infinite)
  - Every V.S. has a basis. For f.d. spaces, all bases have the same size

2.1 Function Spaces

We turn the function space $C[a,b]$ into a vector space over $\mathbb{R}$. For $f, g \in C[a,b]$ and $\alpha \in \mathbb{R}$, define

$$\begin{aligned}(f+g)(t) &= f(t) + g(t)\\ (\alpha f)(t) &= \alpha f(t)\end{aligned}$$

The additive identity is the zero function. It is also an infinite dimensional V.S.

2.2 Linear Maps

Linear maps preserve linear combos. So $T : X \to Y$ is linear if

$$T(c_1v_1 + c_2v_2 + \cdots c_nv_n) = c_1Tv_1 + c_2Tv_2 + \cdots + c_nTv_n$$

Notably, linear maps send the identity of X to the identity of Y (i.e. $T0 = 0$)

- The differentiation operator on $C[a,b]$, $Df = f'$, is linear. PROOF: derivative rules from calculus.

3 Normed Spaces

- A norm on a V.S. V generalizes the length of a vector. It satisfies these axioms
  1. $\|x\| \geq 0$ with $\|x\| = 0 \iff x = 0$
  2. $\|\alpha x\| = |\alpha|\|x\|$ (norm only depends on direction)
  3. $\|x+y\| \leq \|x\| + \|y\|$ (satisfies triangle inequality)

  We use the word "norm" to mean both the value $\|x\|$ and the function $x \mapsto \|x\|$.

- Normed space $\implies$ metric space
  - Define the metric as $d(x,y) = \|x-y\|$

3.1 Examples

- The L^p-norms on $\mathbb{R}^n$ are defined by

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

  - The L^p-norm induces the L^p-metric on $\mathbb{R}^n$.
  - If $p = 2$, this is the usual notion of length/distance on $\mathbb{R}^2$.

- Consider the function space $C[a, b]$. Define a norm on this space as

$$\|f\| = \max_{t \in [a,b]} |f(t)|$$

- Metric space $\not\Longrightarrow$ normed space: the discrete metric on $\mathbb{R}$ is not induced by any norm.

3.2 Banach Spaces

A **Banach space** is any normed space where the norm induces a complete metric space.

- $\mathbb{R}^n$ is a Banach space (when we consider its usual Euclidean norm)
- The function space $C[a, b]$ is a Banach space, under its usual metric.
- Not all normed spaces are Banach spaces.
  - Define a different norm on $C[a, b]$ as:

$$\|f\| = \int_0^1 |f(t)| \; dt$$

    The metric induced by the norm is

$$d(f, g) = \int_0^1 |f(t) - g(t)| \; dt$$

    Under this metric $C[a, b]$ is not a complete space.

4 Inner Product Spaces

An **inner product** satisfies the following axioms

1. $\langle x + y, z\rangle = \langle x, y\rangle + \langle y, z\rangle$
2. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$
3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
4. $\langle x, x\rangle \geq 0$ with $\langle x, x\rangle = 0 \iff x = 0$

If the V.S. is over $\mathbb{R}$, axiom 3 becomes $\langle x, y\rangle = \langle y, x\rangle$.

- We have inner product space $\Longrightarrow$ normed space. A norm may be defined as

$$\|x\| = \sqrt{\langle x, x\rangle}$$

- However, the reverse implication is NOT true. EXAMPLE: L^p-norms when $p \neq 2$.

4.1 Examples

- The standard inner product on $\mathbb{R}^n$ is
$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$
It is better known as the dot product. It induces the Euclidean norm on $\mathbb{R}^n$.
- If $X = \mathbb{C}^n$ instead, then the standard inner product becomes
$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

4.2 Hilbert Space

A **Hilbert space** is an I.P.S. where the norm induced by the inner product induces a complete metric space.

- Hilbert space $\implies$ Banach space
- Converse not true. EXAMPLE: the max norm on $C[a, b]$ is not given by any inner product, so it cannot be a Hilbert space
- $\mathbb{R}^n$, with the Euclidean norm/standard inner product is a Hilbert space.

5 Topological Vector Spaces

We now combine the notions of vector spaces and topological spaces. A **topological vector space** is a vector space with a topology such that

1. All one point sets in closed
2. The vector space operations $+$ and $\cdot$ are continuous

5.1 Continuous & Bounded Linear Map

- We say a linear map $T : X \to Y$ is continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that
$$\|x - x_0\| < \delta \implies \|Tx - Tx_0\| < \varepsilon$$
If T is continuous at all points in X, then T is continuous.
- We say a linear map $T : X \to Y$ is bounded if there exists $M \in \mathbb{R}$ such that
$$\|Tx\| \leq M\|x\|$$
- We define the norm of an operator as follows
$$\|T\| = \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in X \\ \|x\|=1}} \|Tx\|$$
This gives us that for any bounded operator: $\|Tx\| \leq \|T\|\|x\|$.
  - The zero operator has norm 0. PROOF:
$$\frac{\|Tx\|}{\|x\|} = \frac{\|0\|}{\|x\|} = 0$$
  - The identity operator has norm 1. PROOF:
$$\frac{\|Tx\|}{\|x\|} = \frac{\|x\|}{\|x\|} = 1$$
  - The differentiation operator $D(f) = f'$ is unbounded. PROOF: consider polynomials on $[0, 1]$. If $x_n(t) = t^n$, then
$$\|Tx_n\| = \|nt^{n-1}\| = n$$

5.2 Continuous $\iff$ Bounded

We prove the following result.

Theorem 1. *A linear operator $T : X \to Y$, between normed spaces is continuous iff it is bounded.*

Proof. Assume T is bounded. Then, we have $\|T\| < \infty$. Let $\varepsilon > 0$ and $x_0 \in X$ be arbitrary. Let

$$\delta = \frac{\varepsilon}{\|T\|}$$

and assume $\|x - x_0\| < \delta$. Then, we have

$$\begin{aligned} \|Tx - Tx_0\| &= \|T(x - x_0)\| \\ &\leq \|T\|\|x - x_0\| \\ &< \|T\| \cdot \frac{\varepsilon}{\|T\|} \\ &= \varepsilon \end{aligned}$$

Hence, T is continuous as it is continuous at all points.

Conversely, assume T is continuous. Then, T is continuous at $x = 0$. So, there is a $\delta > 0$ such that

$$\|x\| < \delta \implies \|Tx\| < 1$$

Rewrite x as

$$x = \frac{2x\delta}{\|x\|} \cdot \frac{\|x\|}{2\delta}$$

So

$$\begin{aligned} Tx &= T\left(\frac{2x\delta}{\|x\|} \cdot \frac{\|x\|}{2\delta}\right) \\ &= \frac{2\|x\|}{\delta} T\left(\frac{x}{\|x\|} \cdot \frac{\delta}{2}\right) \end{aligned}$$

Put $v = \frac{x}{\|x\|} \cdot \frac{\delta}{2}$ and note that

$$\|v\| = \frac{\delta}{2}\left\|\frac{x}{\|x\|}\right\| = \frac{\delta}{2} < \delta$$

So $\|Tv\| < 1$. Hence, we see that

$$\|Tx\| < \frac{2}{\delta}\|x\|$$

which shows that T is bounded. $\square$

6 Closed Graph Theorem

6.1 Open Mapping Theorem

We state the open mapping theorem, and its corollary, the bounded inverse theorem.

- An open mapping sends open sets in the domain to open sets in the range i.e. the image of open sets is open
- Open Mapping Thrm: every surjective bounded linear operator between Banach spaces is an open mapping
  - IDEA OF PROOF: similar to the "neighborhood trick": show that for an open set U, any point in $T(U)$ contains an open set contained in $T(U)$.

  - However, the details are complicated since we are in an arbitrary space

- Bounded Inverse Thrm: if T is a bijective, continuous linear operator, then T^{-1} is bounded.
  - IDEA OF PROOF: show T^{-1} is continuous. Indeed, $T : X \to Y$ maps open sets to open sets. But this is roughly the same as the preimage of open sets in X being open under T^{-1}. Thus, T^{-1} is continuous and, therefore, bounded.

6.2 Norm On Product Space

If X, Y are Banach spaces, we define a norm on $X \times Y$ as follows

$$\|(x, y)\| = \|x\| + \|y\|$$

We remark: there are other norms on the product space (e.g. the "max" norm), but the closed graph theorem will go through for all of them. I think this is the easiest one.

Theorem 2. *If X, Y are Banach spaces, then so is $X \times Y$.*

Proof Sketch. Let (z_n) be an arbitrary Cauchy sequence. W.T.S. (z_n) converges

- From definition of the product norm, we have (x_n) and (y_n) are Cauchy
- The two Cauchy sequences converge. Then, (z_n) converges to the ordered pair (limit of x_n, limit of y_n)
- To show this, use a $\varepsilon/2$ argument.

□

6.3 Proof of Closed Graph Theorem

A linear operator $T : X \to Y$ if the graph of T

$$G_T = \{(x, Tx) : x \in X\}$$

is a closed set in $X \times Y$.

Theorem 3 (Closed Graph Thrm). *Suppose $T : D \to Y$ is a closed linear operator. Then, T is bounded if D is closed.*

Proof Idea.
- Define a linear map $P : G_T \to D$ as

$$P(x, Tx) = x$$

- P is linear since

$$P(x + cy, T(x + cy)) = P(x + cy, Tx + cTy) = P[(x, Tx) + c(y, Ty)]$$

- P is bounded since

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|$$

- P is bijective.
- G_T is a closed subset of $X \times Y$, a complete space $\implies$ G_T is closed.
- Same argument shows D is a complete space.
- Invoke the bounded inverse theorem to get that P^{-1} is bounded.

- The proof is finished by noting that

$$\begin{aligned}\|Tx\| &\leq \|x\| + \|Tx\| \\ &= \|(x, Tx)\| \\ &= \|P^{-1}(x)\| \\ &\leq M\|x\|\end{aligned}$$

  So T is bounded, as desired.

$\square$