2/23 PSS1

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Let us construct a commutativity diagram.

$$\begin{tikzcd}
V \arrow[r, "\Psi(A)"] \arrow[d, leftrightarrow, "\Phi_B"'] & W \arrow[d, leftrightarrow, "\Phi_\Gamma"] \\
F^n \arrow[r, "L_A"'] & F^m
\end{tikzcd}$$

Let us fix $v, v' \in V$ such that

$$\Phi_B(v) = \Phi_B(v') = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \tag{1}$$

Then, if $B = (\beta_1, \beta_2, ...\beta_n)$, then $v = a_1\beta_1 + a_2\beta_2 + ... + a_n\beta_n$, which is also equal to v'. A similar argument for vectors in W follows for Φ_Γ, so both Φ_B and Φ_Γ are one-to-one. Evidently, multiplying (dot product) any n-dimensional vector by B results in a vector in V (because it is a linear combination of the basis). (A similar argument follows with Φ_Γ). Hence, both Φ_B and Φ_Γ are also onto, so Φ_B and Φ_Γ are isomorphisms. It follows that Φ_Γ^{-1} is also a linear transformation. As shown in the commutativity diagram, $\Psi(A)$ is a composition of the transformations Φ_B, L_A (which, by definition, is a linear transformation), and Φ_Γ^{-1}. Since the composition of linear transformations is also linear, $\Psi(A)$ is linear.