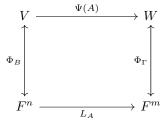
2/23 PSS1

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February 2023

Let us construct a commutativity diagram.



Let us fix $v, v' \in V$ such that

$$\Phi_B(v) = \Phi_B(v') = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$
 (1)

Then, if $B=(\beta_1,\beta_2,...\beta_n)$, then $v=a_1\beta_1+a_2\beta_2+...+a_n\beta_n$, which is also equal to v'. A similar argument for vectors in W follows for Φ_{Γ} , so both Φ_B and Φ_{Γ} are one-to-one. Evidently, multiplying (dot product) any n-dimensional vector by B results in a vector in V (because it is a linear combination of the basis). (A similar argument follows with Φ_{Γ}). Hence, both Φ_B and Φ_{Γ} are also onto, so Φ_B and Φ_{Γ} are isomorphisms. It follows that Φ_{Γ}^{-1} is also a linear transformation. As shown in the commutativity diagram, $\Psi(A)$ is a composition of the transformations Φ_B , L_A (which, by definition, is a linear transformation), and Φ_{Γ}^{-1} . Since the composition of linear transformations is also linear, $\Psi(A)$ is linear.