Lecture 1

More On Rank

We will prove some more theorems regarding the rank of a matrix today. The ultimate goal is to prove that $\operatorname{rank} A = \operatorname{rank} A^T$. Before we get there, here is a theorem

Theorem 1.1
Let A be an $m \times n$ matrix with rank r. Then $r \leq m$ and $r \leq n$ and, by means of a finite number of elementary row and column operations, A can be transformed into a matrix of the form below, denoted as D.

$$D = \begin{bmatrix} I_r & O_1 \\ O_2 & O_3 \end{bmatrix}$$

Where I_r is the $r \times r$ identity matrix, O_1 is the $r \times (n-r)$ zero matrix, O_2 is the $(m-r) \times r$ zero matrix, and O_3 is the $(m-r) \times (n-r)$ zero matrix.

Proof. □

1.1 Elementary Row & Column Operations

There are three kinds of elementary row operations. They are

1. Switch two rows, denoted as $R_i \longleftrightarrow R_j$
2. Multiply a row by a non-zero scalar, denoted as $R_i \to dR_i$, for $d \neq 0$
3. Add a scalar multiple of a row to another row, denoted as $dR_i + R_j \to R_j$

Additionally, each of the elementary row operations above has a corresponding *elementary column operation*, which is performed on the columns instead of the rows.

1.1.1 Elementary Matrices

Each elementary row operation can be defined using an elementary matrix. The elementary matrix is obtained by performing that row operation on the identity. Namely, define the following

$$\begin{aligned} E_{i \longleftrightarrow j} &= \text{matrix obtained after applying type 1 row operation to the identity} \\ E_{di} &= \text{matrix obtained after applying type 2 row operation to the identity} \\ E_{i+dj} &= \text{matrix obtained after applying type 3 row operation to the identity} \end{aligned}$$

Elementary matrices for column operations are defined similarly. The next fact shows how the elementary matrices can be used to apply row/column operations.

> Let A' be the matrix obtained from A by a row (column) operation. Then $A' = EA$ ($A' = AG$) where E (G) is the matrix corresponding to the row (column) operation as seen above.

Essentially the fact says that multiplying an elementary matrix to a matrix on the correct side has the same effect as applying that elementary row/column operation to that matrix. Using this fact, we can prove this corollary

Corollary 1.1

Let A be an $m \times n$ matrix such that $\operatorname{rank} A = r$. Then there exist invertible matrices B and C of sizes $m \times m$ and $n \times n$ respectively such that $D = BAC$ (where D is the matrix defined in theorem 13.1)

Proof. Theorem 13.1 says that there is a finite set of elementary row/column operations to transform any matrix into that form. Combining this with the last fact, let $E_1, E_2, ..., E_k$ be the elementary matrices for the row operations required and let $G_1, G_2, ..., G_\ell$ be the elementary matrices for the column operations. Define

$$\begin{aligned} E &= E_k \cdots E_3 E_2 E_1 \\ G &= G_1 G_2 \cdots G_\ell \end{aligned}$$

Then these matrices are the ones that will satisfy the theorem. These matrices are invertible since the product of invertible matrices is also invertible. □

The next corollary will give us an important result. Before, we will define some important terms related to the columns and rows of a matrix.

Definition 1.1: Row & Column Rank

Let A be an $m \times n$ matrix. Then we have

1. The **row rank** of A is the maximum number of linearly independent rows in A
2. The **column rank** of A is the maximum number of linearly independent columns in A.

With the terms defined, we can present the corollary.

Corollary 1.2

Let A be $m \times n$ matrix. Then we have

1. $\operatorname{rank} A = \operatorname{rank} A^T$
2. $\operatorname{rank} A =$ row rank of A
3. $\operatorname{rank} A =$ column rank of A

Proof.
1. PSS 1.
2. Follows from (1) and (3)
3. This is by definition of $\operatorname{rank} A$.

□

The next theorem gives us some extra ways to determine whether a matrix is invertible.

Corollary 1.3

A matrix A is inveritble if and only if A is a product of elementary matrices.

Proof. Since A is inveritble, it follows that $\operatorname{rank} A = n$ (from rank-nullity theorem). So $D = I_n$. Then $I_n = D = BAC$ and so we have

$$\begin{aligned} A &= B^{-1} C^{-1} \\ &= E_1^{-1} E_2^{-1} \cdots E_k^{-1} G_\ell^{-1} \cdots G_1^{-1} \end{aligned}$$

And the inverse of an elementary matrix is also elementary. Thus, we have A is a product of elementary matrices as desired.

The other direction is easy since all elementary matrices are invertible, so their product is also invertible. □

The next theorem relates the rank of a matrix to the rank of a product of two matrices, which can be viewed as a composition of linear transformations.

Theorem 1.2
Let A and B be matrices such that the product AB is defined. Then, we have

1. $\operatorname{rank} AB \leq \operatorname{rank} A$
2. $\operatorname{rank} AB \leq \operatorname{rank} B$

Proof. Observe that by the definition of rank of a matrix, we have

$$\begin{aligned} \operatorname{rank} AB &= \operatorname{rank}(L_{AB}) \\ &= \dim(\operatorname{Range} L_{AB}) \end{aligned}$$

Now observe that $\operatorname{Range} L_{AB} \subseteq \operatorname{Range} L_A$. Now, we have that $\operatorname{Range} L_{AB} = L_A(\operatorname{Range} L_B)$, which is a subspace of the domain of A. Then, this implies that the result.

For the second part, observe that

$$\begin{aligned} \operatorname{rank} AB &= \operatorname{rank}(AB)^T \\ &= \operatorname{rank}(B^T A^T) \\ &\leq \operatorname{rank} B^T \\ &= \operatorname{rank} B \end{aligned}$$

□