de Rham Cohomology & Stokes' Theorem

Math 412 – Final Project

Nilay Tripathi

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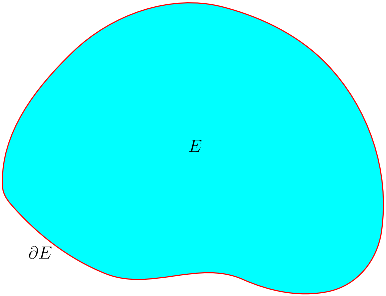

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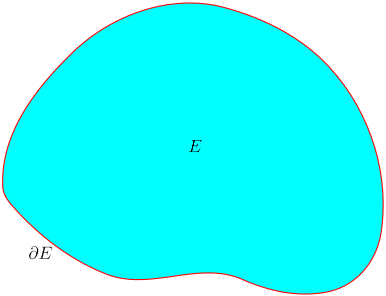

But when might this not work?

Overview & Idea

Idea

Stokes' theorem fails in the presence of *holes*. The de Rham cohomology is a tool to measure how many holes a space has.

> *"de Rham cohomology, which (roughly speaking) measures precisely the extent to which the fundamental theorem of calculus fails in higher dimensions and on general manifolds"*
>
> T. Tao, *Differential Forms and Integration*

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Idea

Stokes' theorem fails in the presence of *holes*. The de Rham cohomology is a tool to measure how many holes a space has.

We will:

- Review some basic terminology to be used in the discussion of Stokes' theorem
- A brief introduction to the idea of homology, which can be used to detect holes.
- Introducing the de Rham cohomology as a tool to measure holes
- Some concrete examples

Cells

Definition (k-Cell)

A k-cell α on a space $U \subseteq \mathbb{R}^n$ is a continuous map $\alpha : [0,1]^k \to U$.

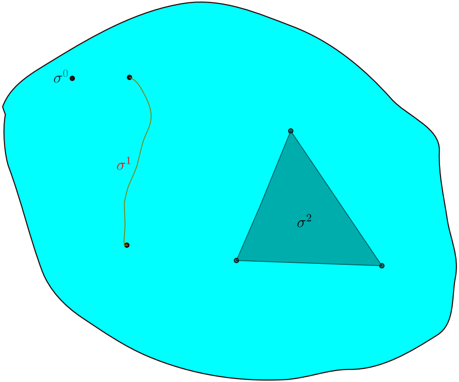

Figure: Some cells: A 0-cell σ^0, 1-cell σ^1, and a 2-cell σ^2.

Closed & Exact Forms

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- is exact if there exists a $(k-1)$-form η with $d\eta = \omega$.

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Corollary

Every exact form is closed.

Indeed, if ω is exact, we have $\omega = d\eta$ for some form η. Then:

$$d\omega = d(d\eta) = d^2\eta = 0$$

So the form is closed.

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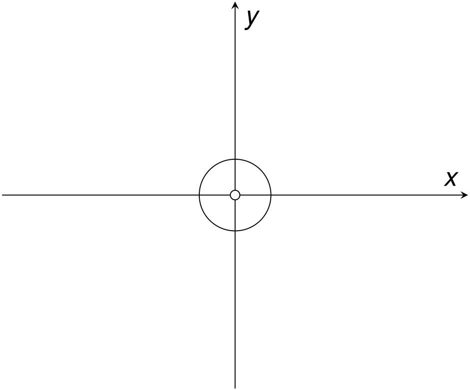
- But ω is not exact. Also on the exam, and can be shown using Stokes' theorem.

Geometric Picture Of ω

Recall that this form is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. This space is called the punctured plane.



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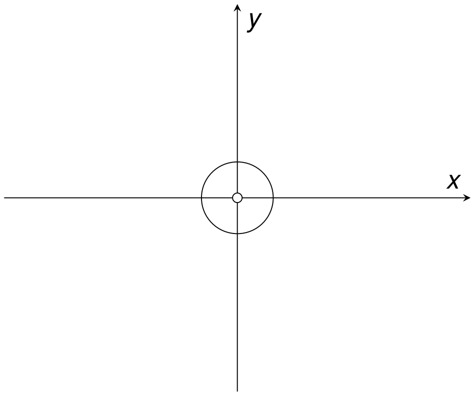

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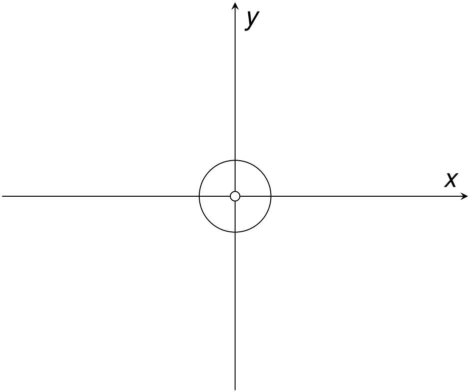

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- The form is a loop, but doesn't bound a disk (due to the hole at the origin).

This form is closed, but not exact. It can exist because of the hole present at the origin.

Poincaré Lemma

Definition (Star Shaped Set)

A set $E \subseteq \mathbb{R}^n$ is star-shaped if there exists a point $\mathbf{x}_0 \in E$ such that for all $\mathbf{x} \in U$ and all $t \in [0, 1]$, we have $t\mathbf{x}_0 + (1 - t)\mathbf{x} \in E$.

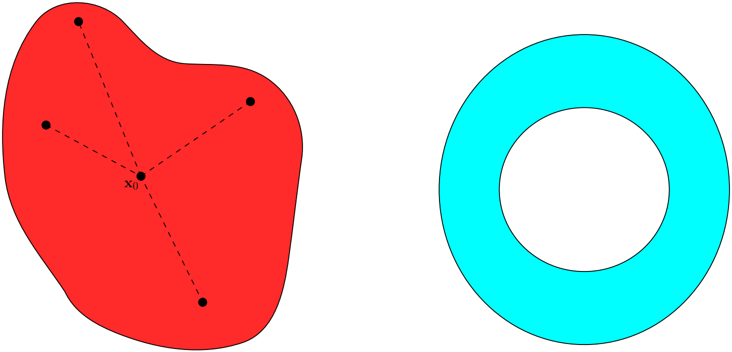

Figure: Left space is star shaped, while the annulus is not.

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- The result holds for a closed ball as well and, consequently, anything homeomorphic to it.

We can use this idea to detect holes in general.

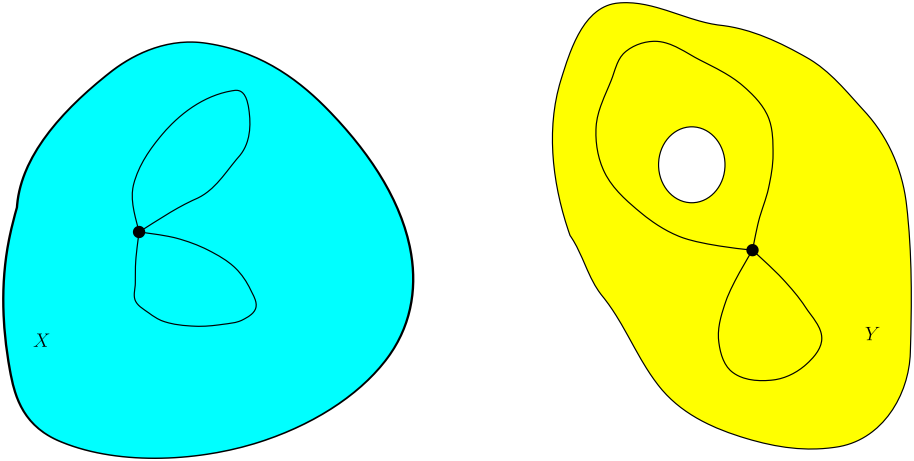

Figure: The space X has no holes while the space Y has a hole.

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A Brief Introduction To Homology

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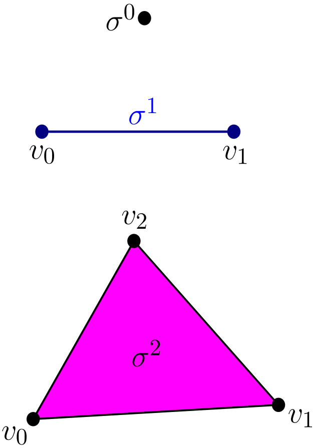
- But in Y (yellow), there exists a closed loop which does not bound a disk because of the hole.

Definition (Chain)

A *k-chain* is a linear combination of k-cells.

Boundary

The boundary of a k-chain is a $(k-1)$-chain. Note: there is a standard orientation, where vertices point from lower index to higher index.



$$\partial(\sigma^0) = 0$$
$$\partial(\sigma^1) = [v_1] - [v_1]$$
$$\partial(\sigma^2) = [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$$

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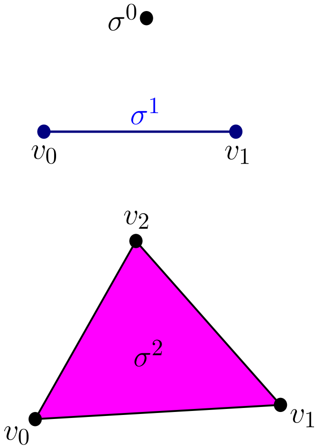

Note that the boundary satisfies

$$\partial^2 = 0$$

- Obvious for $\partial^2(\sigma^0)$.
- $\partial(\sigma^1)$ gives 0-chains, which have boundary 0.
- Also true for σ^2 by keeping track of signs.
- ***Déjà vu maybe??***

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where the boundary maps satisfy $\partial^2 = 0$.

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where the boundary maps satisfy $\partial^2 = 0$. The n-th homology group is then

$$H_n(X) = \ker(\partial_n) / \operatorname{Im}(\partial_{n+1})$$

Let $\Omega^k(X)$ be the set of all smooth differential k-forms on X. Then, we may define a complex:

$$\cdots \xleftarrow{d} \Omega^2(X) \xleftarrow{d} \Omega^1(X) \xleftarrow{d} \Omega^0(X) \leftarrow 0 \leftarrow \cdots$$

Here, d denotes the exterior derivative.

- Note that the arrows are pointing the other direction. Thus, this is a cochain complex.
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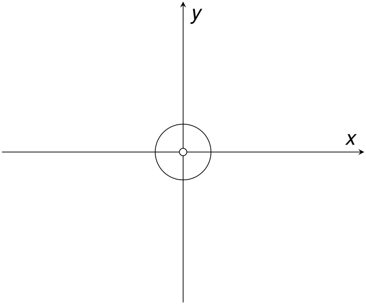
- $\omega \in \operatorname{Im}(d_{n-1})$ iff there is a $(n-1)$-form η with $d\eta = \omega$.
  - So $\operatorname{Im}(d_{n-1})$ consists of the exact k forms.

The Punctured Plane

Return to the punctured plane



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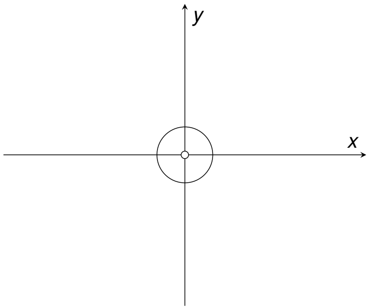

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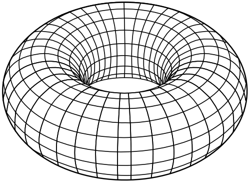
It turns out, the angle form is a generator for the de Rham cohomology group. We see

$$H^1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$$

Another Fun Example: Torus



The torus $T^2 = S^1 \times S^1$ has de Rham cohomology

$$H^k(T^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R}^2 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

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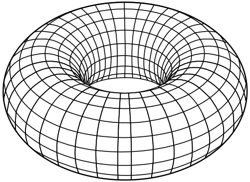

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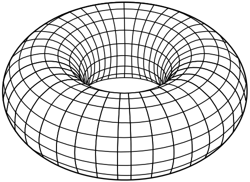

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- One “2-dimensional” hole in the middle.