An Overivew Of Hermite Polynomials

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Overview

1. Definition of inner product and orthogonality
2. Inner products on polynomials
3. Definition of Hermite polynomials
4. Generating function of Hermite polynomials; establishing orthogonality
5. A quick application to quantum physics

Inner Products

Definition (Inner Product)

Let V be a vector space over $\mathbb{C}$. An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that

1. $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle = 0$ iff $x = 0$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
4. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

Orthogonality

In a vector space V, we say that $u, v \in V$ are **orthogonal** if $\langle u, v \rangle = 0$.

Inner Products On Polynomials

Continuous functions on a closed interval $[a, b]$ form a vector space. We can consider two polynomials p and q and define an inner product as

$$\langle p(x), q(x) \rangle = \int_a^b p(x)\overline{q(x)}\, dx$$

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What if we want to consider polynomials over $\mathbb{R}$. The integral above would diverge....

Inner Products On Polynomials

For polynomials over $\mathbb{R}$, add a weight function, $w(x)$, which decays rapidly. Thus, the inner product is of the form

$$\langle p, q \rangle = \int_{\mathbb{R}} p(x)\overline{q(x)}w(x)\ dx$$

Exponential weight functions are common. We will consider a Gaussian weight function i.e.

$$w(x) = e^{-x^2}$$

Of course, other weights are possible, but we will consider this one in our presentation.

Orthogonal Bases Of Polynomials

The set $\{1, x, x^2, \ldots\}$ is a basis of the subspace of polynomials.

- The Gram-Schmidt process takes any basis and returns an orthogonal basis which spans the same subset
  - An orthogonal basis is useful as it simplifies many calculations in that space.
- Using the inner product $\langle p, q \rangle = \int_{\mathbb{R}} p(x)\overline{q(x)}w(x)$, with $w(x) = e^{-x^2}$, we can apply the Gram Schmidt process to get a sequence of polynomials called the Hermite polynomials, which we denote as $\{H_n\}$.

Subspace Of Square Integrable Functions

Definition (Square Integrable Function)

We say a function f is square-integrable on $\mathbb{R}$ with respect to a weight function $w(x)$ (a real-valued function) if

$$\int_{\mathbb{R}} f(x)\overline{f(x)}w(x)\ dx \text{ is finite}$$

In our discussion of Hermite polynomials, we take $w(x) = e^{-x^2}$.

- The set of square integrable functions is a subspace of integrable functions. But it is not necessarily a complete subspace
  - Roughly speaking, there are "gaps" in this space.

Completion Of The Space

We can complete the space using Cauchy sequences with the L^2 norm, which is

$$\|f\| = \left(\int_{\mathbb{R}} f(x)\overline{f(x)}w(x)\ dx \right)^{1/2}$$

- The terms of Cauchy sequences eventually stay within a certain neighborhood
- The limits of Cauchy sequences "fill in" the gaps
- Specifically, take two Cauchy sequences to be "equivalent" if they have the same limit.
  - The equivalence classes form the completion of the space
- The Hermite polynomials $\{H_n\}$ are an orthogonal basis for this completed space when $w(x) = e^{-x^2}$.

Hermite Polynomials

An Explicit Formula

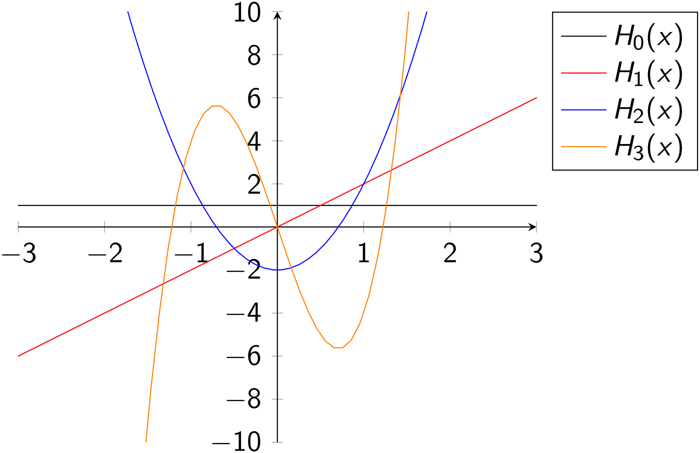
An explicit formula for the Hermite polynomials is given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Note that derivatives of e^{-x^2} are of the form $p(x) \cdot e^{-x^2}$. This cancels out with the e^{x^2} right before and leaves a polynomial.

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
&\vdots
\end{aligned}$$

Graph Of First Four Hermite Polynomials



Generating Function

Generating Function

The Hermite polynomials may equivalently be defined using the **generating function** below

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

From the generating function, one can show that the Hermite polynomials may be recursively defined as the following recurrence relation

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x)$$

Establishing Orthogonality

We will use the generating function to establish orthogonality. Multiplying two Hermite polynomials gives us

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} H_n(x)H_m(x)\frac{t^n}{n!}\frac{s^m}{m!} = \exp(-t^2 - s^2 + 2xt + 2xs)$$

Multiplying by the weight function and integrating, we see

$$\begin{aligned}
\int_{\mathbb{R}} e^{-x^2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} H_n(x)H_m(x)\frac{t^n s^m}{n!m!}\ dx &= \int_{\mathbb{R}} \exp(-x^2 - t^2 - s^2 + 2xt + 2xs)\ dx \\
&= \int_{\mathbb{R}} \exp\left[-(\underbrace{x - t - s}_{u})^2 + 2st\right]\ dx \\
&= \int_{\mathbb{R}} e^{2st} e^{-u^2}\ du \\
&= e^{2st} \cdot \sqrt{\pi}
\end{aligned}$$

Establishing Orthogonality

Using the power series expansion for e^x, we see

$$\int_{\mathbb{R}} e^{-x^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_n(x) H_m(x) \frac{t^n s^m}{n! m!} = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n s^n t^n}{n!}$$

Equating the coefficients of each series shows that the value is 0 when $m \neq n$ and nonzero when $m = n$. This establishes orthogonality of the sequence of polynomials.

Raising And Lowering Operator

Definition (Raising/Lowering Operators)

If f is a differentiable function, the raising operator is given by

$$a^{+} = \left(-\frac{d}{dx} + x\right) f = -\frac{df}{dx} + xf(x)$$

The lowering operator is given by

$$a^{-} = \left(\frac{d}{dx} + x\right) f = \frac{df}{dx} + xf(x)$$

They let us increase or decrease eigenvalues of operators. In quantum physics, this is applied to the Hamiltonian operator (gives the total energy of the system).

Raising And Lowering Operators

We apply these operators to the Gaussian function $f(x) = e^{-x^2/2}$. We have that

$$a^-(f) = -xe^{-x^2/2} + xe^{-x^2/2} = 0$$

Also

$$a^+(f) = xe^{-x^2/2} + xe^{-x^2/2} = 2xe^{-x^2/2}$$

And

$$a^+(2xe^{-x^2/2}) = (4x^2 - 2)e^{-x^2/2}$$

Hermite Functions

Hermite Functions

Functions of the form $H_n(x)e^{-x^2/2}$ (possibly with some appropriate scaling constants) are called **Hermite functions**. The n-th Hermite function is given by

$$\psi_n(x) = H_n(x)e^{-x^2/2}$$

Once again, some additional scaling constants may be added.

Energy Operator, Eigenfunctions

Definition (Energy Operator)

Suppose that f is a twice-differentiable function. The energy operator for a quantum harmonic oscillator (QHO) is given by

$$E(f) = (x^2 - D^2)f$$

where D^2 is the second derivative operator.

Definition (Eigenfunction of Operator)

A function f is said to be an **eigenfunction** of an operator T if

$$T(f) = \lambda f$$

where $\lambda \in \mathbb{C}$.

Energy Operator

Energy operator has some useful applications

- Can be used to determine physical properties of a quantum mechanical system

The Hermite functions are eigenfunctions of the energy operator for the QHO.

We will show it for the simpliest Hermite function $\psi = e^{-x^2/2}$. We have

$$\begin{aligned}
E(\psi) &= -\frac{d^2}{dx^2} e^{-x^2/2} + x^2 e^{-x^2/2} \\
&= -\frac{d}{dx}(-xe^{-x^2/2}) + x^2 e^{-x^2/2} \\
&= e^{-x^2/2}(1 - x^2 + x^2) \\
&= e^{-x^2/2}
\end{aligned}$$

In general, we have $E(\psi_n) = \lambda_n \psi_n$, where $\lambda_n = 2n + 1$.