Smooth Manifolds & Symplectic Geometry

Nilay Tripathi

August 2024

Manifolds

Definition (Manifold)

A manifold is a topological space M which locally looks like Euclidean space, meaning every point has a neighborhood homeomorphic to $\mathbb{R}^n$. This n is the dimension of the manifold.

Examples:

Manifolds

Definition (Manifold)

A manifold is a topological space M which locally looks like Euclidean space, meaning every point has a neighborhood homeomorphic to $\mathbb{R}^n$. This n is the dimension of the manifold.

Examples:

- $\mathbb{R}^n$ itself is an n-manifold.

Manifolds

Definition (Manifold)

A manifold is a topological space M which locally looks like Euclidean space, meaning every point has a neighborhood homeomorphic to $\mathbb{R}^n$. This n is the dimension of the manifold.

Examples:

- $\mathbb{R}^n$ itself is an n-manifold.
- Spheres S^n are n-manifolds.

Manifold Terminology

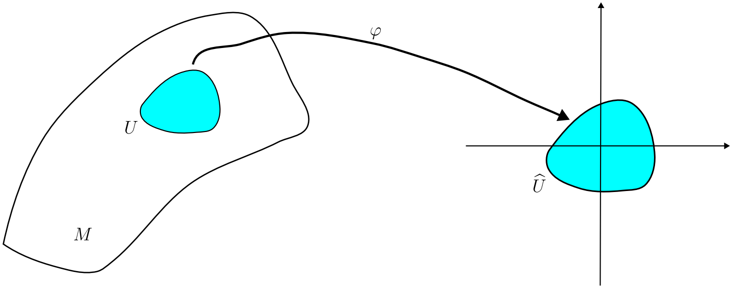
More precisely, the locally Euclidean condition means:

For every point $p \in M$, there exists a homeomorphism $\varphi : U \to \widehat{U}$ where $U \subseteq M$ and $\widehat{U} \subseteq \mathbb{R}^n$ are open sets in their respective domains.

Manifold Terminology

More precisely, the locally Euclidean condition means:

For every point $p \in M$, there exists a homeomorphism $\varphi : U \to \widehat{U}$ where $U \subseteq M$ and $\widehat{U} \subseteq \mathbb{R}^n$ are open sets in their respective domains.



Manifold Terminology

More precisely, the locally Euclidean condition means:

For every point $p \in M$, there exists a homeomorphism $\varphi : U \to \widehat{U}$ where $U \subseteq M$ and $\widehat{U} \subseteq \mathbb{R}^n$ are open sets in their respective domains.

- The pair (U, φ) is a coordinate chart.
- φ can be expressed as $p \mapsto (\varphi^1(p), ..., \varphi^n(p))$. The φ^i are the local coordinates.
- A collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ where the U_α cover M is an atlas on M.

Smooth Manifolds

Roughly: a manifold where calculus can be appropriately extended.

Definition (Transition Maps)

Suppose (U, φ) and (V, ψ) are charts on M where $U \cap V \neq \emptyset$. The transition maps are defined by

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$
$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

These are maps of Euclidean spaces. If they are smooth, then the charts are smoothly compatible.

Definition (Smooth Atlas)

An atlas $\{(U_\alpha, \varphi_\alpha)\}$ on M is a smooth atlas if all possible pairs of transition maps are smooth maps of Euclidean spaces.

Transition Maps

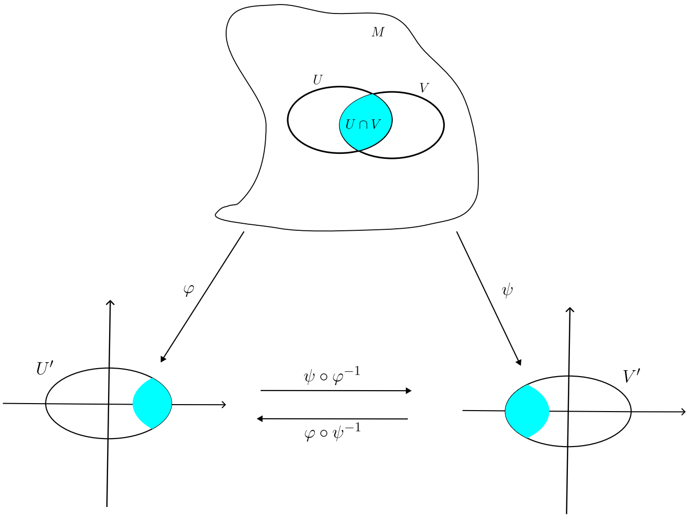


Figure: Transition maps between coordinate charts in a manifold.

Smooth Manifold

Definition (Smooth Manifold)

A smooth manifold is a pair $(M, \mathcal{A})$ where M is a manifold and $\mathcal{A}$ is a maximal smooth atlas i.e. an atlas which contains every possible pair of smoothly compatible coordinate charts.

Smooth Manifold

Definition (Smooth Manifold)

A smooth manifold is a pair $(M, \mathcal{A})$ where M is a manifold and $\mathcal{A}$ is a maximal smooth atlas i.e. an atlas which contains every possible pair of smoothly compatible coordinate charts.

We require the "maximality" condition to ensure we don't get two different manifolds by adding extra maps.

Smooth Manifold

Definition (Smooth Manifold)

A smooth manifold is a pair $(M, \mathcal{A})$ where M is a manifold and $\mathcal{A}$ is a maximal smooth atlas i.e. an atlas which contains every possible pair of smoothly compatible coordinate charts.

We require the "maximality" condition to ensure we don't get two different manifolds by adding extra maps.

Theorem

Given a smooth atlas $\mathcal{A}$, there is a unique maximal smooth atlas $\overline{\mathcal{A}}$ containing $\mathcal{A}$.

Smooth Manifold

Definition (Smooth Manifold)

A smooth manifold is a pair $(M, \mathcal{A})$ where M is a manifold and $\mathcal{A}$ is a maximal smooth atlas i.e. an atlas which contains every possible pair of smoothly compatible coordinate charts.

We require the “maximality” condition to ensure we don’t get two different manifolds by adding extra maps.

Theorem

Given a smooth atlas $\mathcal{A}$, there is a unique maximal smooth atlas $\overline{\mathcal{A}}$ containing $\mathcal{A}$.

So we can specify a smooth manifold by talking about *any* smooth atlas (not necessarily maximal).

Smooth Map

Definition (Smooth Map)

A map $f : M \to N$ between smooth manifolds is smooth at a point $p \in M$ if there exist charts (U, φ) of p in M and (V, ψ) of $f(p)$ in N such that the map

$$f' := \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is a smooth map of Euclidean spaces.

Smooth Map

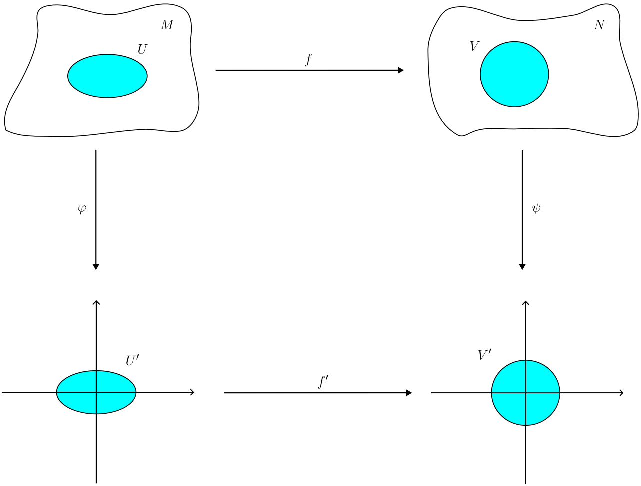


Figure: Smooth map between manifolds M and N.

Tangent Vector & Tangent Space

The purpose of a tangent vector is to generalize directional derivatvies in $\mathbb{R}^n$.

- Suppose $\mathbf{v} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is smooth at a point p. The directional derivative in the direction of $\mathbf{v}$ is

$$D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f\big|_p$$

Tangent Vector & Tangent Space

The purpose of a tangent vector is to generalize directional derivatvies in $\mathbb{R}^n$.

- Suppose $\mathbf{v} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is smooth at a point p. The directional derivative in the direction of $\mathbf{v}$ is

$$D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f\big|_p$$

- So one can view $\mathbf{v}$ as an *operator*: it takes in a smooth function and outputs the change in that smooth function in the direction of $\mathbf{v}$ at the point p.

Tangent Vector & Tangent Space

The purpose of a tangent vector is to generalize directional derivatvies in $\mathbb{R}^n$.

Definition (Smooth Curve)

If M is a smooth manifold, a curve $\lambda : I \to M$ is a smooth curve if λ is a smooth function.

Definition (Tangent Vector)

If $p \in M$, a smooth manifold, a tangent vector to M at p is the function $v : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ defined by

$$v(f) := \frac{d}{dt}(f \circ \lambda)(t)\Big|_{t=0}$$

where $\lambda : I \to M$ is a smooth curve with $\lambda(0) = p$.

Tangent Vector & Tangent Space

The purpose of a tangent vector is to generalize directional derivatvies in $\mathbb{R}^n$.

Definition (Tangent Space)

The tangent space to a manifold M at the point p is the set of all tangent vectors to M at p.

Tangent Vector & Tangent Space

Definition (Tangent Space)

The tangent space to a manifold M at the point p is the set of all tangent vectors to M at p.

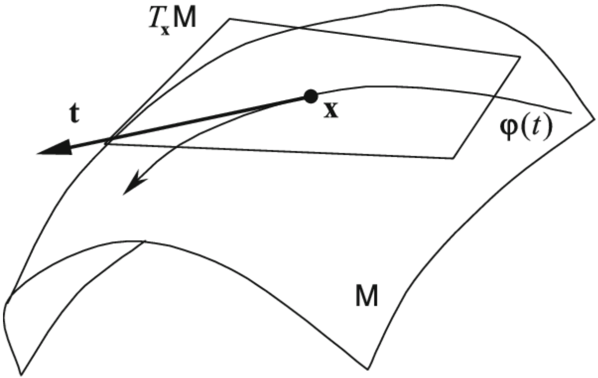


Figure: Tangent space of a manifold

Tangent Vector & Tangent Space

Definition (Tangent Space)

The tangent space to a manifold M at the point p is the set of all tangent vectors to M at p.

Definition (Tangent Bundle)

Given a smooth manifold M, the tangent bundle of M, denoted TM, is the disjoint union of all tangent spaces to M.

$$TM = \bigsqcup_{p \in M} T_p M$$

Tangent Vector & Tangent Space

Definition (Tangent Space)

The tangent space to a manifold M at the point p is the set of all tangent vectors to M at p.

Definition (Tangent Bundle)

Given a smooth manifold M, the tangent bundle of M, denoted TM, is the disjoint union of all tangent spaces to M.

$$TM = \bigsqcup_{p \in M} T_p M$$

- All tangent spaces $T_p M$ are vector spaces.

Tangent Vector & Tangent Space

Definition (Tangent Space)

The tangent space to a manifold M at the point p is the set of all tangent vectors to M at p.

Definition (Tangent Bundle)

Given a smooth manifold M, the tangent bundle of M, denoted TM, is the disjoint union of all tangent spaces to M.

$$TM = \bigsqcup_{p \in M} T_p M$$

- All tangent spaces $T_p M$ are vector spaces.
- The tangent bundle TM has a smooth structure determined by M.

Dual Spaces, Cotangent Space

Definition (Dual Space)

Given a vector space V, its dual space $V^\star$ is the set of linear forms $\alpha : V \to \mathbb{F}$.

For a manifold M and point $p \in M$, the dual space of its tangent space T_pM is its cotangent space T_p^*M.

- Elements of T_pM are vectors on M.
- Elements of T_p^*M are covectors on M.

Tensors

Definition (Tensors)

A *k-tensor* is a k-times linear map $\alpha : V^k \to \mathbb{F}$. The space of k-tensors is a vector space denoted by $\mathcal{T}^k(V)$.

Examples:

- The dual space is the space of 1-tensors $\mathcal{T}^1(V) = V^\star$.
- An inner product $\langle \cdot, \cdot \rangle$ is a 2-tensor.
- The determinant is an n-tensor on $\mathbb{R}^n$.

In manifolds, the inputs are vectors (elements of TM; for covariant tensors) or covectors (elements of T^*M; for contravariant tensors).

Tensors

Definition (Alternating Tensor)

A k-tensor is alternating if its sign changes when two of its arguments are switched.

$$\alpha(v_1, ..., v_i, ..., v_j, ..., v_k) = -\alpha(v_1, ..., v_j, ..., v_i, ..., v_k)$$

The space of alternating tensors is $\Lambda^k(V)$ and is a vector subsapce of $\mathcal{T}^k(V)$.

Examples:

- Every 1-tensor (covector) is vacuously alternating.
- The determinant is an n-tensor that is alternating.

Tensor Product, Wedge Product

Definition (Tensor Product)

If $\alpha \in \mathcal{T}^k(V)$ and $\beta \in \mathcal{T}^\ell(V)$, their tensor product is $\alpha \otimes \beta \in \mathcal{T}^{k+\ell}(V)$ given by

$$(\alpha \otimes \beta)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = \alpha(v_1, \ldots, v_k) \cdot \beta(v_{k+1}, \ldots, v_{k+\ell})$$

Alternating tensors aren't closed under the tensor product.

Tensor Product, Wedge Product

Definition (Tensor Product)

If $\alpha \in \mathcal{T}^k(V)$ and $\beta \in \mathcal{T}^\ell(V)$, their tensor product is $\alpha \otimes \beta \in \mathcal{T}^{k+\ell}(V)$ given by

$$(\alpha \otimes \beta)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = \alpha(v_1, \ldots, v_k) \cdot \beta(v_{k+1}, \ldots, v_{k+\ell})$$

Alternating tensors aren't closed under the tensor product. This means the tensor product of alternating tensors isn't necessarily alternating.

Definition (Wedge Product)

If $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^\ell(V)$, their wedge product $\alpha \wedge \beta \in \Lambda^{k+\ell}(V)$.

Rather than give an explicit formula, we define it as the operation satisfying these properties.

$$\alpha \wedge \alpha = 0$$
$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$$
$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$$
$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$$

Differential Form

Definition (Differential Form)

A differential k-form is a map $\omega : M \to \Lambda^k(T_pM)$ i.e. it assigns an alternating k-tensor $\omega_p \in \Lambda^k(T_pM)$ to every point in M.

The set of k-forms on M is a vector space denoted by $\Omega^k(V)$.

- A 0-form is just a smooth function $f : M \to \mathbb{R}$.

Differential Form

Definition (Differential Form)

A differential k-form is a map $\omega : M \to \Lambda^k(T_pM)$ i.e. it assigns an alternating k-tensor $\omega_p \in \Lambda^k(T_pM)$ to every point in M.

The set of k-forms on M is a vector space denoted by $\Omega^k(V)$.

- A 0-form is just a smooth function $f : M \to \mathbb{R}$.
- Suppose $x = (x^1, ..., x^n)$ is a chart. We let dx^i $(1 \leq i \leq n)$ be a 1-form:

Differential Form

Definition (Differential Form)

A differential k-form is a map $\omega : M \to \Lambda^k(T_pM)$ i.e. it assigns an alternating k-tensor $\omega_p \in \Lambda^k(T_pM)$ to every point in M.

The set of k-forms on M is a vector space denoted by $\Omega^k(V)$.

- A 0-form is just a smooth function $f : M \to \mathbb{R}$.
- Suppose $x = (x^1, ..., x^n)$ is a chart. We let dx^i $(1 \leq i \leq n)$ be a 1-form: For every point $p \in U$, dx^i outputs a function taking in a vector and giving the component of that vector in the i-th direction.

Differential Form

Definition (Differential Form)

A differential k-form is a map $\omega : M \to \Lambda^k(T_pM)$ i.e. it assigns an alternating k-tensor $\omega_p \in \Lambda^k(T_pM)$ to every point in M.

The set of k-forms on M is a vector space denoted by $\Omega^k(V)$.

- A 0-form is just a smooth function $f : M \to \mathbb{R}$.
- Suppose $x = (x^1, ..., x^n)$ is a chart. We let dx^i $(1 \leq i \leq n)$ be a 1-form: For every point $p \in U$, dx^i outputs a function taking in a vector and giving the component of that vector in the i-th direction.
- The set $\{dx^i : 1 \leq i \leq n\}$ is a basis for the set of 1-forms. This means that any $\omega \in \Omega^1(M)$ may be represented as

$$\omega = f_1\ dx^1 + f_2\ dx^2 + \cdots + f_n\ dx^n$$

where the f_i's are smooth functions from $M \to \mathbb{R}$.

Differential Form

Definition (Differential Form)

A differential k-form is a map $\omega : M \to \Lambda^k(T_pM)$ i.e. it assigns an alternating k-tensor $\omega_p \in \Lambda^k(T_pM)$ to every point in M.

The set of k-forms on M is a vector space denoted by $\Omega^k(V)$.

- Generalizing that, let $I = (i_1, ..., i_k)$ be an increasing sequence of the numbers $\{1, ..., n\}$. We define

$$dx^I := dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_1}$$

Definition (Differential Form)

A differential k-form is a map $\omega : M \to \Lambda^k(T_pM)$ i.e. it assigns an alternating k-tensor $\omega_p \in \Lambda^k(T_pM)$ to every point in M.

The set of k-forms on M is a vector space denoted by $\Omega^k(V)$.

- Generalizing that, let $I = (i_1, ..., i_k)$ be an increasing sequence of the numbers $\{1, ..., n\}$. We define

$$dx^I := dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_1}$$

- The set $\{dx^I : I \text{ is an increasing sequence}\}$ is a basis for the set of all k forms $\Omega^k(M)$.

Differential Form Examples

Consider $\mathbb{R}^3$ for a concrete examples:

- Any one form can be written as

$$\omega = f_1(x, y, z)\ dx + f_2(x, y, z)\ dy + f_3(x, y, z)\ dz$$

Differential Form Examples

Consider $\mathbb{R}^3$ for a concrete examples:

- Any one form can be written as

$$\omega = f_1(x, y, z)\ dx + f_2(x, y, z)\ dy + f_3(x, y, z)\ dz$$

- A two form can be written as

$$\omega^2 = f_{x,y}\ dx \wedge dy + f_{y,z}\ dy \wedge dz + f_{x,z}\ dx \wedge dz$$

Differential Form Examples

Consider $\mathbb{R}^3$ for a concrete examples:

- Any one form can be written as

$$\omega = f_1(x,y,z)\ dx + f_2(x,y,z)\ dy + f_3(x,y,z)\ dz$$

- A two form can be written as

$$\omega^2 = f_{x,y}\ dx \wedge dy + f_{y,z}\ dy \wedge dz + f_{x,z}\ dx \wedge dz$$

- A three form is expressed as

$$\omega^3 = f(x,y,z)\ dx \wedge dy \wedge dz$$

Exterior Derivative

Definition (Exterior Derivative)

Given a differential k-form ω, the exterior derivative gives a $(k+1)$-form, denoted $d\omega$, given by

$$d\omega = \sum_I d\omega_I \wedge dx^I$$

Examples: if $\omega = x^2 y \ dx - y \ dy$, then

$$\begin{aligned} d\omega &= (2xy \ dx + x^2 \ dy) \wedge dx - dy \wedge dy \\ &= 2xy \ dx \wedge dx + x^2 \ dy \wedge dx \\ &= -x^2 \ dx \wedge dy \end{aligned}$$

Doing it again:

$$\begin{aligned} d(d\omega) = d^2\omega &= -d(x^2) \wedge dx \wedge dy \\ &= -2x \ dx \wedge dx \wedge dy \\ &= 0 \end{aligned}$$

Exterior Derivative

Definition (Exterior Derivative)

Given a differential k-form ω, the exterior derivative gives a $(k+1)$-form, denoted $d\omega$, given by

$$d\omega = \sum_I d\omega_I \wedge dx^I$$

Examples: if $\omega = x^2 y \ dx - y \ dy$, then

$$\begin{aligned} d\omega &= (2xy \ dx + x^2 \ dy) \wedge dx - dy \wedge dy \\ &= 2xy \ dx \wedge dx + x^2 \ dy \wedge dx \\ &= -x^2 \ dx \wedge dy \end{aligned}$$

Theorem

For any differential form ω, we have $d^2\omega = 0$.

Vector Fields

Definition (Vector Field)

A vector field on M is a map $X : M \to TM$ where $p \mapsto X_p \in T_pM$. It is a smooth vector field if the mapping X is smooth.

- If $x = (x^1, ..., x^n)$ is a chart containing $p \in M$, we define the tangent vector

$$\left(\frac{\partial}{\partial x^i}\right)_p : C^\infty(M) \to \mathbb{R}$$

$$f \mapsto \left.\frac{\partial(f \circ x^{-1})}{\partial x^i}\right|_{x(p)}$$

Then the vectors $(\partial/\partial x^i)_p$ form a basis for the set of smooth vector fields. I.e. any vector field X can be written as

$$X = X^1 \left(\frac{\partial}{\partial x^1}\right)_p + \cdots + X^n \left(\frac{\partial}{\partial x^n}\right)_p$$

Interior Product on Forms

Definition (Interior Product)

Suppose X is a vector field on a manifold M. The interior product with respect to X is the map

$$\iota_X : \Omega^k(M) \to \Omega^{k-1}$$

sending a k-form ω to the $(k-1)$-form defined by

$$(\iota_X \omega)(X_1, ..., X_{k-1}) = \omega(X, X_1, ..., X_{k-1})$$

The interior product obeys this graded version of the Leibniz rule:

$$\iota_X(\alpha \wedge \beta) = \iota_X \alpha \wedge \beta + (-1)^k \alpha \wedge \iota_X \beta$$

Symplectic Manifold

We may consider a special two form on certain manifolds.

Definition (Symplectic Manifold)

A symplectic manifold is a pair (M, ω) where M is a topological manifold and ω is a differential 2-form satisfying

1. ω is closed i.e. $d\omega = 0$.
2. ω is non-degenerate i.e. if for all v, we have $\omega(v, w) = 0$, then it must be that $w = 0$.

The differential form ω is called the symplectic form.

Basic Properties

A symplectic manifold must have even dimension.

- On a local chart, the map ω is alternating and so $A = -A^\top$. This must satisfy

$$\det A = \det(-A^\top) = (-1)^n \det(A^\top) = (-1)^n \det A$$

  But non-degeneracy implies the map A must be invertible, which means $\det A \neq 0$. This only happens if n is even.

A symplectic manifold has a non-vanishing volume form (a differential k-form where k equals the dimension of the manifold).

- Let $\dim M = 2k$, for some k. Then, $\omega^k = \underbrace{\omega \wedge \cdots \wedge \omega}_{k \text{ times}}$ is a volume form that is nonzero.
- This also implies that every symplectic manifold is orientable.

Examples

Examples of symplectic manifolds:

- $\mathbb{R}^{2n}$. For a basis given by $\{v_1, ..., v_n, w_1, ..., w_n\}$, define a 2-form as follows
$$\omega = \sum_{i=1}^{n} dv_i \wedge dw_i$$
- Cotangent bundles of smooth manifolds.
- Certain even-dimensional spheres (S^2 is a symplectic manifold, but S^4 is not).

Symplectic Gradient

Suppose M is a symplectic manifold with ω as the symplectic form.

- Suppose $f : M \to \mathbb{R}$ is a smooth function.

Symplectic Gradient

Suppose M is a symplectic manifold with ω as the symplectic form.

- Suppose $f : M \to \mathbb{R}$ is a smooth function.
- There is a unique vector field X_f such that

$$\iota_{X_f}\omega = df.$$

Symplectic Gradient

Suppose M is a symplectic manifold with ω as the symplectic form.

- Suppose $f : M \to \mathbb{R}$ is a smooth function.
- There is a unique vector field X_f such that

$$\iota_{X_f}\omega = df.$$

- The vector field X_f is the symplectic gradient of f.

Symplectic Gradient

Suppose M is a symplectic manifold with ω as the symplectic form.

- Suppose $f : M \to \mathbb{R}$ is a smooth function.
- There is a unique vector field X_f such that

$$\iota_{X_f}\omega = df.$$

- The vector field X_f is the symplectic gradient of f.

We can also use a different convention and say $\iota_{X_f}\omega = -df$. This will be relevant later on.

Lie Derivative

The Lie derivative of a differential form along a vector field measures the change of the form along the vector field. It is denoted by $\mathcal{L}_X\omega$.

Lie Derivative

The Lie derivative of a differential form along a vector field measures the change of the form along the vector field. It is denoted by $\mathcal{L}_X\omega$.

We can compute the Lie derivative using Cartan's magic formula:

$$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega)$$

Lie Derivative Along Symplectic Gradient

Theorem

The flow along the symplectic gradient preserves ω.

Using Cartan's formula, we compute the Lie derivative as

$$\begin{aligned}
\mathcal{L}_{X_f}\omega &= \iota_{X_f}(d\omega) + d(\iota_{X_f}\omega) \\
&= 0 + d(\iota_{X_f}\omega) \\
&= d(-df) \\
&= 0
\end{aligned}$$

Relation to Hamiltonian Mechanics

Consider a particle moving in $\mathbb{R}^2$. Its state is determined by four numbers:

Relation to Hamiltonian Mechanics

Consider a particle moving in $\mathbb{R}^2$. Its state is determined by four numbers:

- Its two position coordinates (q_1, q_2).
- Its two momentum coordinates (p_1, p_2).

Relation to Hamiltonian Mechanics

Consider a particle moving in $\mathbb{R}^2$. Its state is determined by four numbers:

- Its two position coordinates (q_1, q_2).
- Its two momentum coordinates (p_1, p_2).

The coordinates $(q_1, q_2; p_1, p_2) \in \mathbb{R}^4$ form the phase space of the particle.

Relation to Hamiltonian Mechanics

Consider a particle moving in $\mathbb{R}^2$. Its state is determined by four numbers:

- Its two position coordinates (q_1, q_2).
- Its two momentum coordinates (p_1, p_2).

The coordinates $(q_1, q_2; p_1, p_2) \in \mathbb{R}^4$ form the phase space of the particle.

In general, for a particle moving in $\mathbb{R}^n$, its phase space will be $\mathbb{R}^{2n}$; an even-dimensional manifold, so it admits a symplectic structure.

Hamiltonan Mechanics

Given a particle and its phase space in $\mathbb{R}^{2n}$, the Hamiltonian function is often the sum of the kinetic and potential energy

$$H = K + U$$

The motion is then governed by Hamilton's equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \qquad\qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

We can derive these equations using the tools from symplectic geometry.

Hamiltonian Mechanics

For simplicity, suppose our particle has mass 1. Then, the Hamiltonian is

$$H = K + U$$

Taking the exterior derivative, we get

$$dH = \frac{\partial H}{\partial p}\ dp + \frac{\partial H}{\partial q}\ dq = p\ dp + U_q\ dq$$

Hamtiltonian Mechanics

We can use this to explicitly solve for the symplectic gradient. It will take the form

$$X_H = a\,\frac{\partial}{\partial p} + b\,\frac{\partial}{\partial q}$$

Then, we see

$$\begin{aligned}\omega(X_H, -) &= (dp \wedge dq)\left(a\,\frac{\partial}{\partial p} + b\,\frac{\partial}{\partial q}, -\right)\\ &= a\ dq - b\ dp\end{aligned}$$

This should equal $-dH$, which gives us

$$-dH = -\frac{\partial H}{\partial p}\ dp - \frac{\partial H}{\partial q}\ dq = a\ dq - b\ dp$$

Hamiltonian Mechanics

This implies

$$X_H = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

Hamiltonian Mechanics

This implies

$$X_H = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

Hamilton's equation are then recovered from the components.

This implies

$$X_H = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

Hamilton's equation are then recovered from the components. Flowing around X_H, $\partial/\partial q$ is the change per unit time of q, so

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

Hamiltonian Mechanics

This implies

$$X_H = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

Hamilton's equation are then recovered from the components. Flowing around X_H, $\partial/\partial q$ is the change per unit time of q, so

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

The other equation is recovered similarly to get

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

Sources

- "Quantum Field Theory for Mathematicians: Hamiltonian Mechanics and Symplectic Geometry", Link.
- "An Introduction to Riemannian Geometry with Applications to Mechanics and Relativity", Link.