de Rham Cohomology & Stokes' Theorem

Math 412 – Final Project

Nilay Tripathi

May 3rd, 2024

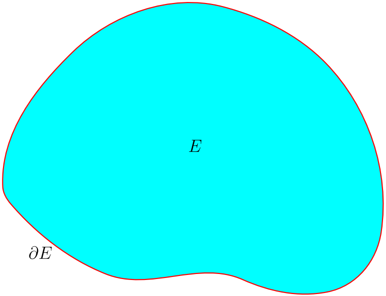
Differential forms provide a unifed approach to integrate in higher dimensions (and also over arbitrary manifolds).

Overview

Differential forms provide a unifed approach to integrate in higher dimensions (and also over arbitrary manifolds).

We have Stokes' theorem:

$$\int_{\partial E} \omega = \int_E d\omega$$

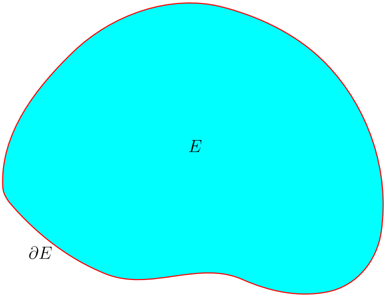


Overview

Differential forms provide a unifed approach to integrate in higher dimensions (and also over arbitrary manifolds).

We have Stokes' theorem:

$$\int_{\partial E} \omega = \int_E d\omega$$



But when might this not work?

Overview & Idea

Idea

Stokes' theorem fails in the presence of *holes*. The de Rham cohomology is a tool to measure how many holes a space has.

> *"de Rham cohomology, which (roughly speaking) measures precisely the extent to which the fundamental theorem of calculus fails in higher dimensions and on general manifolds"*
>
> T. Tao, *Differential Forms and Integration*

Overview & Idea

Idea

Stokes' theorem fails in the presence of *holes*. The de Rham cohomology is a tool to measure how many holes a space has.

We will:

- Review some basic terminology to be used in the discussion of Stokes' theorem
- A brief introduction to the idea of homology, which can be used to detect holes.
- Introducing the de Rham cohomology as a tool to measure holes
- Some concrete examples

Cells

Definition (k-Cell)

A *k-cell* α on a space $U \subseteq \mathbb{R}^n$ is a continuous map $\alpha : [0,1]^k \to U$.

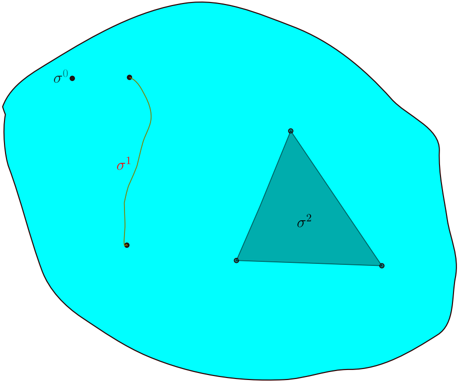


Figure: Some cells: A 0-cell σ^0, 1-cell σ^1, and a 2-cell σ^2.

Closed & Exact Forms

The exterior derivative d takes a differential k-form and outputs a $(k+1)$-form. A differential k-form ω

- is closed if $d\omega = 0$.

Closed & Exact Forms

The exterior derivative d takes a differential k-form and outputs a $(k+1)$-form. A differential k-form ω

- is closed if $d\omega = 0$.
- is exact if there exists a $(k-1)$-form η with $d\eta = \omega$.

Closed & Exact Forms

Theorem

The exterior derivative satisfies $d^2 = 0$.

Closed & Exact Forms

Theorem

The exterior derivative satisfies $d^2 = 0$.

A special case of this was asked on the exam.

Closed & Exact Forms

Theorem

The exterior derivative satisfies $d^2 = 0$.

A special case of this was asked on the exam.

Corollary

Every exact form is closed.

Closed & Exact Forms

Theorem

The exterior derivative satisfies $d^2 = 0$.

A special case of this was asked on the exam.

Corollary

Every exact form is closed.

Indeed, if ω is exact, we have $\omega = d\eta$ for some form η. Then:

$$d\omega = d(d\eta) = d^2\eta = 0$$

So the form is closed.

Closed & Exact Forms

Q: is the converse true (i.e. is every closed form necessarily exact)?

Closed & Exact Forms

Q: is the converse true (i.e. is every closed form necessarily exact)?
Ans: no. Consider the 1-form

$$\omega = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

Closed & Exact Forms

Q: is the converse true (i.e. is every closed form necessarily exact)?
Ans: no. Consider the 1-form

$$\omega = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

- On the exam, we are asked to verify $d\omega = 0$, meaning ω is closed.

Closed & Exact Forms

Q: is the converse true (i.e. is every closed form necessarily exact)?
Ans: no. Consider the 1-form

$$\omega = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

- On the exam, we are asked to verify $d\omega = 0$, meaning ω is closed.
- But ω is not exact.

Closed & Exact Forms

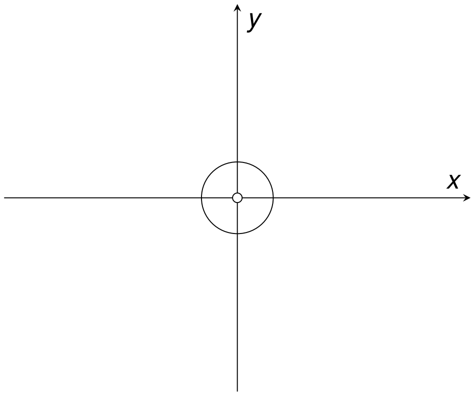
Q: is the converse true (i.e. is every closed form necessarily exact)?
Ans: no. Consider the 1-form

$$\omega = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

- On the exam, we are asked to verify $d\omega = 0$, meaning ω is closed.
- But ω is not exact. Also on the exam, and can be shown using Stokes' theorem.

Geometric Picture Of ω

Recall that this form is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. This space is called the punctured plane.

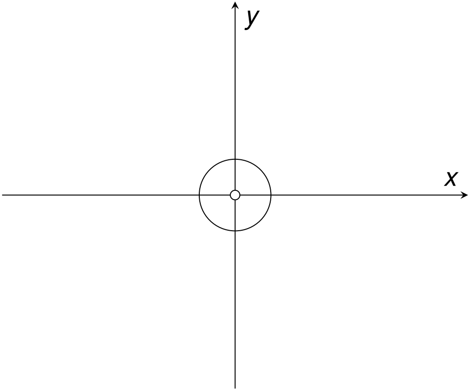


- This form is often called the angle form.

.

Geometric Picture Of ω

Recall that this form is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. This space is called the punctured plane.

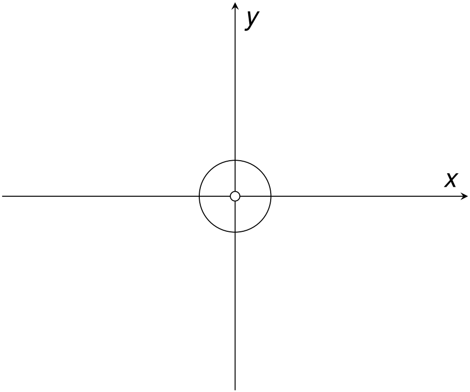


- This form is often called the angle form.
- The form is a loop, but doesn't bound a disk (due to the hole at the origin).

.

Geometric Picture Of ω

Recall that this form is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. This space is called the punctured plane.



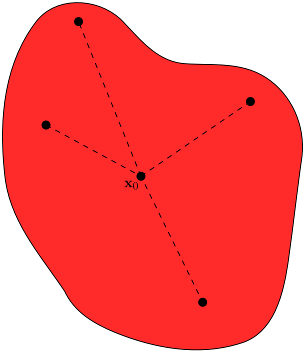
- This form is often called the angle form.
- The form is a loop, but doesn't bound a disk (due to the hole at the origin).

This form is closed, but not exact. It can exist because of the hole present at the origin.

Poincaré Lemma

Definition (Star Shaped Set)

A set $E \subseteq \mathbb{R}^n$ is star-shaped if there exists a point $\mathbf{x}_0 \in E$ such that for all $\mathbf{x} \in U$ and all $t \in [0, 1]$, we have $t\mathbf{x}_0 + (1 - t)\mathbf{x} \in E$.



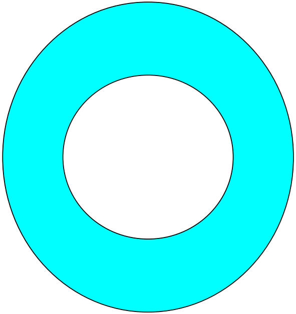


Figure: Left space is star shaped, while the annulus is not.

Definition (Star Shaped Set)

A set $E \subseteq \mathbb{R}^n$ is star-shaped if there exists a point $\mathbf{x}_0 \in E$ such that for all $\mathbf{x} \in U$ and all $t \in [0, 1]$, we have $t\mathbf{x}_0 + (1 - t)\mathbf{x} \in E$.

Theorem (Poincaré Lemma)

If U is a star-shaped domain in $\mathbb{R}^n$, then every closed form on U is exact.

In particular:

Poincaré Lemma

Definition (Star Shaped Set)

A set $E \subseteq \mathbb{R}^n$ is star-shaped if there exists a point $\mathbf{x}_0 \in E$ such that for all $\mathbf{x} \in U$ and all $t \in [0, 1]$, we have $t\mathbf{x}_0 + (1 - t)\mathbf{x} \in E$.

Theorem (Poincaré Lemma)

If U is a star-shaped domain in $\mathbb{R}^n$, then every closed form on U is exact.

In particular:

- The conclusion holds for any open ball and thus, for $\mathbb{R}^n$ itself.

Poincaré Lemma

Definition (Star Shaped Set)

A set $E \subseteq \mathbb{R}^n$ is star-shaped if there exists a point $\mathbf{x}_0 \in E$ such that for all $\mathbf{x} \in U$ and all $t \in [0, 1]$, we have $t\mathbf{x}_0 + (1 - t)\mathbf{x} \in E$.

Theorem (Poincaré Lemma)

If U is a star-shaped domain in $\mathbb{R}^n$, then every closed form on U is exact.

In particular:

- The conclusion holds for any open ball and thus, for $\mathbb{R}^n$ itself.
- The result holds for a closed ball as well and, consequently, anything homeomorphic to it.

A Brief Introduction To Homology

We can use this idea to detect holes in general.

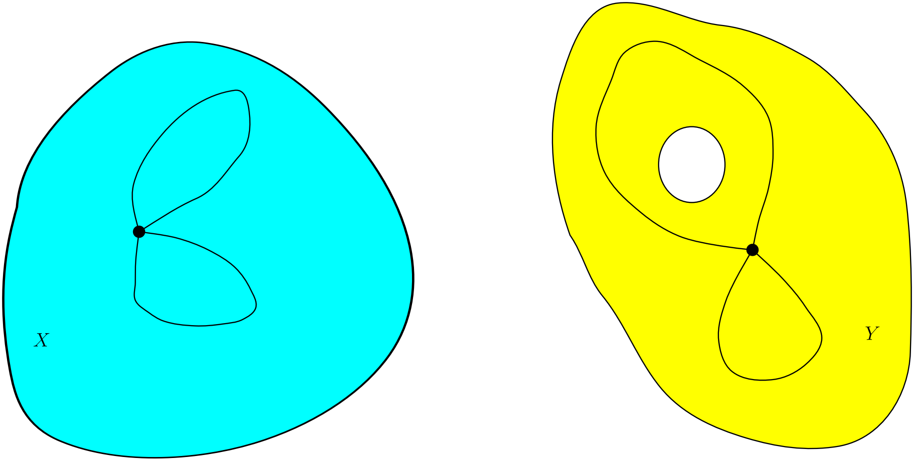


Figure: The space X has no holes while the space Y has a hole.

A Brief Introduction To Homology

- In X (blue), every closed loop completely bounds a disk, so no holes.

A Brief Introduction To Homology

- In X (blue), every closed loop completely bounds a disk, so no holes.
- But in Y (yellow), there exists a closed loop which does not bound a disk because of the hole.

A Brief Introduction To Homology

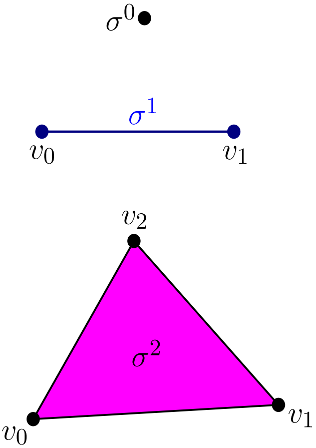
- In X (blue), every closed loop completely bounds a disk, so no holes.
- But in Y (yellow), there exists a closed loop which does not bound a disk because of the hole.

Definition (Chain)

A *k-chain* is a linear combination of k-cells.

Boundary

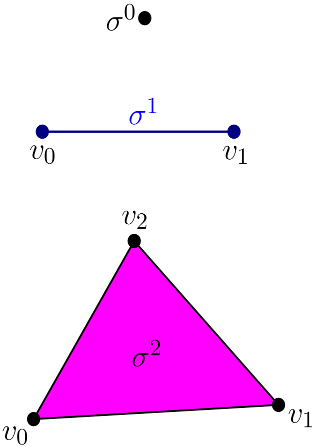
The boundary of a k-chain is a $(k-1)$-chain. Note: there is a standard orientation, where vertices point from lower index to higher index.



$$\partial(\sigma^0) = 0$$
$$\partial(\sigma^1) = [v_1] - [v_1]$$
$$\partial(\sigma^2) = [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$$

Boundary

The boundary of a k-chain is a $(k-1)$-chain. Note: there is a standard orientation, where vertices point from lower index to higher index.



Note that the boundary satisfies

$$\partial^2 = 0$$

- Obvious for $\partial^2(\sigma^0)$.
- $\partial(\sigma^1)$ gives 0-chains, which have boundary 0.
- Also true for σ^2 by keeping track of signs.
- ***Déjà vu maybe??***

Chain Complex & Homology

Definition (Chain Group)

Let $C_n(X)$ be the free abelian group generated by all the n-chains.

Chain Complex & Homology

Definition (Chain Group)

Let $C_n(X)$ be the free abelian group generated by all the n-chains.

A chain complex is a collection of chain groups for all $n \in \mathbb{Z}$

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \to \cdots$$

where the boundary maps satisfy $\partial^2 = 0$.

Chain Complex & Homology

Definition (Chain Group)

Let $C_n(X)$ be the free abelian group generated by all the n-chains.

A chain complex is a collection of chain groups for all $n \in \mathbb{Z}$

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \to \cdots$$

where the boundary maps satisfy $\partial^2 = 0$. The n-th homology group is then

$$H_n(X) = \ker(\partial_n) / \operatorname{Im}(\partial_{n+1})$$

de Rham Cohomology

Let $\Omega^k(X)$ be the set of all smooth differential k-forms on X. Then, we may define a complex:

$$\cdots \xleftarrow{d} \Omega^2(X) \xleftarrow{d} \Omega^1(X) \xleftarrow{d} \Omega^0(X) \leftarrow 0 \leftarrow \cdots$$

Here, d denotes the exterior derivative.

- Note that the arrows are pointing the other direction. Thus, this is a cochain complex.
- The exterior derivative satisfies $d^2 = 0$, so this is a valid cochain complex.

de Rham Cohomology

Let $\Omega^k(X)$ be the set of all smooth differential k-forms on X. Then, we may define a complex:

$$\cdots \xleftarrow{d} \Omega^2(X) \xleftarrow{d} \Omega^1(X) \xleftarrow{d} \Omega^0(X) \leftarrow 0 \leftarrow \cdots$$

Here, d denotes the exterior derivative.

- Note that the arrows are pointing the other direction. Thus, this is a cochain complex.
- The exterior derivative satisfies $d^2 = 0$, so this is a valid cochain complex.

The de Rham cohomology group is then defined as

$$H^n(X) = \ker(d_n)/\operatorname{Im}(d_{n-1})$$

- Note that $\omega \in \ker d$ iff $d\omega = 0$.

de Rham Cohomology

The de Rham cohomology group is then defined as

$$H^n(X) = \ker(d_n) / \operatorname{Im}(d_{n-1})$$

- Note that $\omega \in \ker d$ iff $d\omega = 0$.
  - So $\ker d$ consists of closed k forms.

de Rham Cohomology

The de Rham cohomology group is then defined as

$$H^n(X) = \ker(d_n)/\operatorname{Im}(d_{n-1})$$

- Note that $\omega \in \ker d$ iff $d\omega = 0$.
  - So $\ker d$ consists of closed k forms.
- $\omega \in \operatorname{Im}(d_{n-1})$ iff there is a $(n-1)$-form η with $d\eta = \omega$.

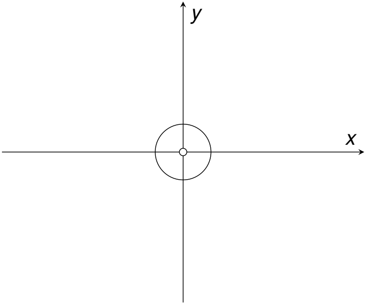
The de Rham cohomology group is then defined as

$$H^n(X) = \ker(d_n)/\operatorname{Im}(d_{n-1})$$

- Note that $\omega \in \ker d$ iff $d\omega = 0$.
  - So $\ker d$ consists of closed k forms.
- $\omega \in \operatorname{Im}(d_{n-1})$ iff there is a $(n-1)$-form η with $d\eta = \omega$.
  - So $\operatorname{Im}(d_{n-1})$ consists of the exact k forms.

The Punctured Plane

Return to the punctured plane

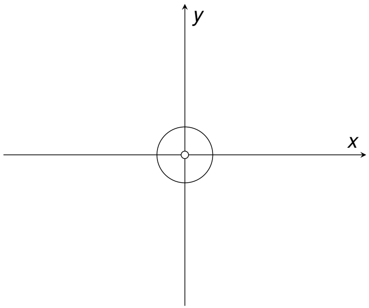


- The angle form ω is a closed, but not exact one form on $\mathbb{R}^2 \setminus \{0\}$. Hence, we have that

$$H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$$

The Punctured Plane

Return to the punctured plane



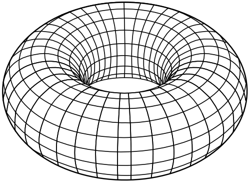
- The angle form ω is a closed, but not exact one form on $\mathbb{R}^2 \setminus \{0\}$. Hence, we have that

$$H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$$

It turns out, the angle form is a generator for the de Rham cohomology group. We see

$$H^1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$$

Another Fun Example: Torus

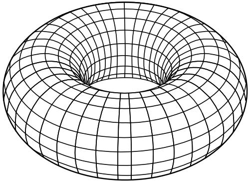


The torus $T^2 = S^1 \times S^1$ has de Rham cohomology

$$H^k(T^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R}^2 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus we have

Another Fun Example: Torus



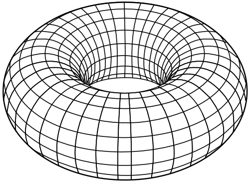
The torus $T^2 = S^1 \times S^1$ has de Rham cohomology

$$H^k(T^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R}^2 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus we have

- Two “1-dimensional” holes (longitude and meridian of torus).

Another Fun Example: Torus



The torus $T^2 = S^1 \times S^1$ has de Rham cohomology

$$H^k(T^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R}^2 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus we have

- Two “1-dimensional” holes (longitude and meridian of torus).
- One “2-dimensional” hole in the middle.