Eigenvalues & Diagonalization

NT

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In these notes, we look at a certain class of vectors associated to a matrix: **eigenvectors** and their associated **eigenvalues**. In addition to several applications in other disciplines, they will reveal important information regarding the structure of these matrices and their associated linear operators.

1 Eigen-things

Throughout these notes, we will study square matrices.

Definition 1.1: Eigenvalues & Eigenvectors

Let A be an $n \times n$ matrix of real numbers. A real number $\lambda \in \mathbb{R}$ is an **eigenvalue** of A if there exists a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} = \lambda\mathbf{v}$. The vector $\mathbf{v}$ is then an **eigenvector** of A corresponding to the eigenvalue λ.

Example 1.2. Consider the matrix A given below. Observe that this matrix is the standard matrix for a reflection about the y-axis.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using geometry, we see that the y-axis is fixed by the transformation. We verify this by computing

$$A\mathbf{e}_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{e}_2$$

So $\mathbf{e}_2$ is an eigenvector of A with corresponding eigenvalue 1.

In the reflection, the x-axis isn't fixed but it is negated. Thus, we may compute

$$A\mathbf{e}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\mathbf{e}_1$$

So $\mathbf{e}_1$ is also an eigenvector of A with corresponding eigenvalue -1.

Example 1.3. Not all matrices have eigenvalues. For an angle $\theta \in [0, 2\pi)$, the rotation matrix R_θ is defined as

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Multiplying a vector $\mathbf{v} \in \mathbb{R}^2$ by R_θ rotates $\mathbf{v}$ counterclockwise by an angle θ. We thus notice that for a nonzero vector $\mathbf{v}$, $R_\theta\mathbf{v}$ can never be a scalar multiple of $\mathbf{v}$ since rotation will change the direction of the vector. Thus, R_θ does not have eigenvectors and eigenvalues.

As we will see later, if we permit eigenvalues to be *complex numbers* as well as real numbers, then every matrix *will* have complex eigenvalues.

1.1 Finding the Eigenvalues & Eigenvectors

We now turn to the question of actually determining the eigenvalues and eigenvectors of a matrix $A \in \mathbb{R}^{n\times n}$. Fortunately, this isn't a difficult task. For $\lambda \in \mathbb{R}$ to be an eigenvalue of A, it must satisfy the equation

$$A\mathbf{v} = \lambda\mathbf{v}$$

for a nonzero vector $\mathbf{v} \in \mathbb{R}^n$. The equation above is equivalent to

$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} &\iff A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \\ &\iff A\mathbf{v} - \lambda I_n\mathbf{v} = \mathbf{0} \\ &\iff (A - \lambda I_n)\mathbf{v} = \mathbf{0} \end{aligned}$$

In particular, since eigenvectors are taken to be nonzero, for λ to be an eigenvalue of A, the last equation above must have nontrivial solutions. Said differently, this means that $\mathrm{Null}(A - \lambda I_n)$ must be nontrivial. From the invertible matrix theorem, the matrix $A - \lambda I_n$ cannot be invertible which is equivalent to $\det(A - \lambda I_n) = 0$. This gives us the general method for computing eigenvalues. In fact, this polynomial is so important, it is given a special name.

Definition 1.4: Characteristic Polynomial

For an $n \times n$ matrix A, its **characteristic polynomial** is the polynomial in t given by $\det(A - tI_n)$. The **characteristic equation** is the polynomial equation $\det(A - tI_n) = 0$.

We then see that the eigenvalues will be the roots of the characteristic polynomial. Once an eigenvalue λ has been identified, we can then work on finding the eigenvectors. The eigenvectors are the nonzero vectors $\mathbf{v} \in \mathbb{R}^n$ satisfying $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$. The set of all eigenvectors of λ is the **eigenspace** corresponding to λ. Since it is the nullspace of a matrix, we see that the eigenspace is a subspace of $\mathbb{R}^n$.

Summarizing the discussion above, we now know how to determine the eigenvalues of a matrix and the corresponding basis of eigenvectors.

Fact 1.5: Finding Eigenvalues & Eigenvectors

Suppose A is an $n\times n$ matrix. To find the eigenvalues of A and the corresponding basis of eigenvectors for each eigenvalue:

1. Solve the **characteristic equation** $\det(A - tI_n) = 0$. The solutions in t are the eigenvalues of A.

2. For each eigenvalue of t, the basis for the eigenspace is just a basis for $\mathrm{Null}(A - tI_n)$.

Example 1.6. Let us return to the matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ from the first example. We already know what the eigenvalues and eigenvectors are by inspection; let us confirm it with our newly found procedure. We first compute the characteristic polynomial as

$$\det \begin{bmatrix} -1-t & 0 \\ 0 & 1-t \end{bmatrix} = -(1+t)(1-t)$$

The eigenvalues are thus found by solving the equation $-(1+t)(1-t) = 0$ which has solutions $t = \pm 1$. These are the eigenvalues of A (as we've seen).

Let us find a basis for the eigenspace corresponding to the eigenvalue $t = 1$. We see that

$$A - 1\cdot I_n = A - I_n = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

How convenient, this matrix happens to be in row echelon form already! We see that the general form of the solution is $\mathbf{v} = \begin{bmatrix} 0 & x_2 \end{bmatrix}^T = x_2 \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Thus, a basis for this nullspace is the set $\left\{ \begin{bmatrix} 0 & 1 \end{bmatrix}^T \right\}$, which is thus a basis for the eigenspace for eigenvalue 1. Note that this confirms our previous belief that $\mathbf{e}_2$ was an eigenvector corresponding to eigenvalue 1.

For the other eigenvalue $t = -1$, we see

$$A - (-1)I_n = A + I_n = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Similar logic as the previous eigenvalue shows us that a basis for the null space here is $\left\{\begin{bmatrix} 1 & 0 \end{bmatrix}\right\}^T$. We therefore see that $\mathbf{e}_1$ is an eigenvector corresponding to eigenvalue -1, again confirming our geometric reasoning from earlier.