MATH325: Discrete Mathematics II
Assignment a01

Open `main.tex` and for each question you'll see `input` followed by a filename such as `2-5-18.tex`. Enter your answer in the file `2-5-18.tex`. To rebuild and view pdf, in bash shell execute `make`. To build a gzip-tar file, in bash shell execute `make s` and you'll get `submit.tar.gz`.

In LaTeX math notation is enclosed by the `$` symbols. For instance `$x = a_{1} + b^{2}$` gives you $x = a_1+b^2$. For emphasis you can write also write it as `\[ x = a_{1} + b^{2} \]` to center your math:

$$x = a_1 + b^2$$

For more information about LaTeX go to my website http://bit.ly/yliow0, click on `Yes` you are one of my students, then look for latex.pdf.)

Example. Prove that

(a) Reflexive: $|X| = |X|$
(b) Symmetric: If $|X| = |Y|$, then $|Y| = |X|$.
(c) Transitive: If $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$.

Recall the following definitions: $|X| \leq |Y|$ means there is a 1–1 function $X \to Y$ and $|X| = |Y|$ means there there is a 1–1 and onto function $X \to Y$.

SOLUTION.

(a)

Let X be a set and a function, s.t $f : X \to X$ $f(x) = x$, where x is an element of X f is a 1-1 and onto function. since x, x' are elements of X $f(x) = f(x')$ implies $x = x'$

Therefore, $|X| = |X|$

(b)

Let $f : X \to Y$ that is 1-1 and onto. Since f is bijective it has an inverse function. $f^-1 : Y \to X$, which is 1-1 and onto.

Therefore, if $|X| = |Y|$ then, $|Y| = |X|$

(c)

Let $f : X \to Y$ that is 1-1 and $g : Y \to Z$ which is 1-1. and $h : X \to Z$. then the composite function $gof = h : X \to Z$

Since, this is a composite funct of two 1-1 functions then the composite is 1-1

Therefore, If $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$.

(to be completed)

Q1. Rosen 8th edition, section 2.5, question 18.

Prove that for sets A and B, if $|A| = |B|$, then $|P(A)| = |P(B)|$.

Warning: A and B need not be finite.

SOLUTION.

Let $f : A \to B$ which is 1-1 and onto.

Let $g : P(A) \to P(B)$

let g be defined as $g(X) = f(X)$

Now, for all $X \subseteq A$, that is 1-1 and onto.

Therefore, if $|A| = |B|$, then $|P(A)| = |P(B)|$

(to be completed)

Q2. Rosen 8th edition, section 2.5, question 27.

Prove that the union of countably many countable sets is countable.

Let me explain what the question is asking. Note that you already know what is a *countable set* – it's a set X such that X is finite, say $|X| = |\{1, 2, 3, ..., n\}|$, or $|X| = |\mathbb{N}|$. And a union *countably many sets* means you are trying to compute the union of of a collection of sets and this *collection* is countable. This means that you can label the sets of the collection in a countable way. So either (a) there is a finite number of sets (say n of them) and the union is

$$A_1 \cup A_2 \cup \cdots A_n = \bigcup_{i \in \{1,2,3,\cdots n\}} A_i$$

i.e., the collection of sets $\{A_1, A_2 ..., A_n\}$ is finite or (b) there is infinitely countable number of sets and the union is

$$A_1 \cup A_2 \cup \cdots = \bigcup_{i \in \mathbb{N}} A_i$$

i.e., the collection of sets $\{A_1, A_2 ..., \}$ is infinitely countable. An example of a uncountable union is when you can label the sets with $\mathbb{R}$. For instance for each $x \in \mathbb{R}$, let $A_x = [x - 1, x + 1)$ and then you take

$$\bigcup_{x \in \mathbb{R}} A_x$$

Something like $\bigcup_{x \in [0,1)} A_x$ is also an uncountable union. Therefore you need to prove this: (a) If $\{A_1, A_2, ..., A_n\}$ is countable and each A_i is countable, then $\bigcup_{i \in \{1,2,3,\cdots n\}} A_i$ is also countable and (b) If $\{A_1, A_2, ...\}$ is countable and each A_i is countable, then $\bigcup_{i \in \mathbb{N}} A_i$ is also countable.

Remember: "countable" means (a) finite or (b) countably infinite.

Note: The solution is provided in the textbook. Study the solution carefully and write it in your own words. The solutions at the back of the textbook is usually very brief and sometimes require clarifications and filling in some details. Even better: See if you can prove it in a different, simpler, and clearer way. Yes, there's a simpler proof that uses a proof that I talked about in class for another theorem. That's why studying and understanding proofs are important.

SOLUTION.

Let there be countably many countable sets. s.t A1, A2, A3,....

$A1 = \{a11, a12, a13, ...\}$

$A2 = \{a21, a22, a23, ...\}$

$A3 = \{a31, a32, a33, ...\}$

You can construst such a set.

That the element can be listed $\{ai1, ai2, ai3, ...\}$

$\bigcup_{i=1} A_i$

The elements can be listed aij with i + j = 2, then all terms aij with i + j = 3, then all terms aij with i + j = 4. . .

Therefore, the union of countably many countable sets is countable.

(to be completed)

Q3. Rosen 8th edition, section 2.5, question 37.

Prove that there are countably many programs that can be written in a particular programming language.

(The proof is exactly the same if you change the questions to "prove that there are countable many novels that can be written in a particular alphabetic language".)

Note: The solution is provided in the textbook. Study the solution carefully and write it in your own words. See if you can find a simpler adn clearer proof.

SOLUTION.

There is number of potential strings for a finate alphabet. The number of strings is length n and n is positive. Now, for a particular programming language there countab many program. We can say that since the set all programs would be a subset of all strings of a finite alphabet, which is countable. Then, the set of all programs must be countable since it is a subset of a countable set.

Therefore, there are countably many programs that can be written in a particular programming language. (to be completed)

Q4. Rosen 8th edition, section 2.5, question 38 (modified).

Prove that there are uncountably many functions from $\mathbb{N}$ to $\{0, 1\}$.

Note: The previous questions says that there are *countably* many C++ programs. This questions says that there are *uncountably* many boolean functions $\mathbb{N} \to \{0, 1\}$ is uncountable. I'll let you think about the (computer science/mathematical/engineering) consequence of these two statements.

SOLUTION.

To Prove that there are uncountably many functions from N to {0, 1}. I will suppose that there are countably many functions from N to {0, 1}. and arrive at a contradiction.

Assume, all the possible functions can be listed f1,f2,f3,...

We can form a new function that adds a rule changes what zero gets mapped to of any function. Which creates a new function. If the element is mapped to 0 then map it to 1. If the element is mapped to 1 then map it 0. This can be applied to any element of the set $\mathbb{N}$. This is a contradiction because you for any functon you can a single elemets map and get a new function.

Therfore, here are uncountably many functions from $\mathbb{N}$ to $\{0, 1\}$.

(to be completed)

Spoilers

Here's an important hint: There are two extremely important proof techniques when it comes to the size of infinite sets: the zigzag argument (I used this to prove $|\mathbb{N}| = |\mathbb{Q}|$) and Cantor's diagonalization argument (I used this to prove $|\mathbb{N}| < |\mathbb{R}|$, i.e., $\mathbb{R}$ is not countable). Both methods can be used in this assignment, i.e., one method can be used in one of the three questions and the other can be used in another question.

Instructions

In `main.tex` change the email address in

```
\renewcommand\AUTHOR{jdoe5@cougars.ccis.edu}
```

yours. In the bash shell, execute "`make`" to recompile `main.pdf`. Execute "`make v`" to view `main.pdf`. Execute "`make s`" to create `submit.tar.gz` for submission.

For each question, you'll see boxes for you to fill. You write your answers in `main.tex` file. For small boxes, if you see

```
1 + 1 = \answerbox{}.
```

you do this:

```
1 + 1 = \answerbox{2}.
```

`answerbox` will also appear in "true/false" and "multiple-choice" questions.

For longer answers that needs typewriter font, if you see

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
\end{answercode}
```

you do this:

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
int x;
\end{answercode}
```

`answercode` will appear in questions asking for code, algorithm, and program output. In this case, indentation and spacing is significant. For program output, I do look at spaces and newlines.

For long answers (not in typewriter font) if you see

```
What is the color of the sky?
\begin{answerlong}
\end{answerlong}
```

you can write

```
What is the color of the sky?
\begin{answerlong}
The color of the sky is blue.
\end{answerlong}
```

For students beyond 245: You can put LaTeX commands in `answerbox` and `answerlong`.

A question that begins with "T or F or M" requires you to identify whether it is true or false, or meaningless. "Meaningless" means something's wrong with the statement and it is not well-defined. Something like "$1+_2$" or "$\{2\}^{\{3\}}$" is not well-defined. Therefore a question such as "Is $42 = 1+_2$ true or false?" or "Is $42 = \{2\}^{\{3\}}$ true or false?" does not make sense. "Is $P(42) = \{42\}$ true or false?" is meaningless because $P(X)$ is only defined if X is a set. For "Is $1 + 2 + 3$ true or false?", "$1 + 2 + 3$" is well–defined but as a "numerical expression", not as a "proposition", i.e., it cannot be true or false. Therefore "Is $1 + 2 + 3$ true or false?" is also not a well-defined question.

When writing results of computations, make sure it's simplified. For instance write 2 instead of $1 + 1$. When you write down sets, if the answer is $\{1\}$, I do not want to see $\{1, 1\}$.

When writing a counterexample, always write the simplest.

Here are some examples (see `instructions.tex` for details):

1. T or F or M: $1 + 1 = 2$ .... $\boxed{\text{T}}$
2. T or F or M: $1 + 1 = 3$ .... $\boxed{\text{F}}$
3. T or F or M: $1+^2 =$ .... $\boxed{\text{M}}$
4. $1 + 2 = \boxed{3}$
5. Write a C++ statement to declare an integer variable named `x`.

```
int x;
```

6. Solve $x^2 - 1 = 0$.

   Since $x^2 - 1 = (x - 1)(x + 1)$, $x^2 - 1 = 0$ implies $(x - 1)(x + 1) = 0$. Therefore $x - 1 = 0$ or $x = -1$. Hence $x = 1$ or $x = -1$.

7. Which is true? .... $\boxed{\text{C}}$
   (A) $1 + 1 = 0$
   (B) $1 + 1 = 1$
   (C) $1 + 1 = 2$
   (D) $1 + 1 = 3$
   (E) $1 + 1 = 4$