MATH325: Discrete Mathematics II
Assignment a06

Open `main.tex` and for each question you'll see `input` followed by a filename such as `02-05-18.tex`. (which contains your work on chapter 2, section 5, question 18 from our textbook). Enter your answer in the file `02-05-18.tex`. To rebuild and view pdf, in bash shell execute `make`. To build a gzip-tar file, in bash shell execute `make s` and you'll get `submit.tar.gz`.

Objectives

1. Use generating functions to compute closed forms for linear recurrences.
2. Use generating functions to compute/prove identities.
3. Use characteristic equation to compute closed forms for linear recurrences.

The following are practice problems for self-study:

- Rosen 8th edition, section 8.2: Odd numbered problems 1-53 except 41,49,51.

For this assignment the problems you need to solve are

- Q1 and Q2 and
- Rosen 8th edition, section 8.2: questions 4, 12, 28, 30.

Explain your work completely using the solutions provided as guide and examples on how to write math properly.

In LaTeX math notation is enclosed by the `$` symbols. For instance `$x = a_{1} + b^{2}$` gives you $x = a_1 + b^2$. For emphasis you can write also write it as `\[ x = a_{1} + b^{2} \]` to center your math:

$$x = a_1 + b^2$$

For binomial coefficients, `$\binom{5}{2}$` will give you $\binom{5}{2}$.

If you are writing a series of computations you can align them like this:

```
\begin{align*}
a(x) &= \sum_{n=0}^\infty a_n x^n \\
     &= a_0 + a_1 x + \sum_{n=2}^\infty a_n x^n \\
     &= 123 + 234 x + \sum_{n=2}^\infty (a_{n-1} + 1) x^n \\
```

```
\end{align*}
```

gives you this:

$$
\begin{aligned}
a(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= 123 + 234x + \sum_{n=2}^{\infty} (a_{n-1} + 1) x^n
\end{aligned}
$$

For more information about LaTeX go to my website http://bit.ly/yliow0, click on `Yes` you are one of my students, then look for latex.pdf.) Even easier: ask questions in CCCS discord.

Draw a box around your final answer if the question required an explicit answer (i.e., if it's not a proof question).

Solving linear recurrences

The first method is always by generating functions and is the best.

The second method is to use characteristic equation. For instance if the linear recurrence is

$$a_n = 5a_{n-1} - 6a_{n-2}$$

then the characteristic equation is

$$x^2 - 5x + 6$$

which has roots $r_1 = 2, r_2 = 3$. Therefore the general form for a_n is

$$a_n = C_1 2^n + C_2 3^n$$

If the linear recurrence in

$$b_n = 4b_{n-1} - 4b_{n-2}$$

then the characteristic equation is

$$x^2 - 4x + 4$$

which has roots $r_1 = r_2 = 2$. Therefore the general form for b_n is

$$b_n = C_1' 2^n + C_2' n 2^n$$

In both cases, to find the constants (C_i or C_i'), you need two base conditions.

What about the nonhomogeneous case? For instance

$$a_n = 5a_{n-1} - 6a_{n-2} + \underline{3n + 1}$$

If you use the generating function, you will find the closed form for a_n. (See question below.) But what if you are using the characteristic equation method?

Step 1. You ignore the nonhomogeneous term and solve

$$a_n = 5a_{n-1} - 6a_{n-2}$$

This is sometimes called the homogeneous part of the linear recurrence. Sometimes to differentiate this and the original, you can write

$$a_n^{(h)} = 5a_{n-1}^{(h)} - 6a_{n-2}^{(h)}$$

i.e., the sequence is $a_0^{(h)}, a_1^{(h)}, a_2^{(h)}, \ldots$. So

$$a_n^{(h)} = C_1 2^n + C_2 3^n$$

Step 2. You make an educated guess that involves the nonhomogeneous term. (This is the part I don't like about characteristic equation method.) The nonhomogeneous part of

$$a_n = 5a_{n-1} - 6a_{n-2} + \underline{3n+1}$$

is a polynomial $3n+1$ of degree 1. You try and see if $a_n^{(p)} = An + B$ is a solution of

$$a_n = 5a_{n-1} - 6a_{n-2} + \underline{3n+1}$$

Substituting $a_n^{(p)} = An + B$ into the linear recurrence, you get

$$An + B = 5(A(n-1) + B) - 6(A(n-2) + B) + 3n + 1$$

i.e.,

$$\begin{aligned}
An + B &= 5(A(n-1) + B) - 6(A(n-2) + B) + 3n + 1 \\
&= 5(An - A + B) - 6(An - 2A + B) + 3n + 1 \\
&= (5A - 6A + 3)n + (-5A + 5B + 12A - 6B + 1) \\
&= (3 - A)n + (7A - B + 1)
\end{aligned}$$

which means $A = 3 - A$ and $B = 7A - B + 1$, i.e., $A = 3/2$ and $B = 23/4$. Therefore

$$a_n^{(p)} = An + B = \frac{3}{2}n + \frac{23}{4}$$

Step 3. You add the homogeneous and particular solutions:

$$a_n = a_n^{(h)} + a_n^{(p)} = C_1 2^n + C_2 3^n + \frac{3}{2}n + \frac{23}{4}$$

Suppose the base cases are $a_0 = 2, a_1 = 3$. You can then solve for C_1 and C_2 with the base cases:

$$\begin{aligned}
2 = a_0 &= C_1 + C_2 + \frac{23}{4} \\
3 = a_1 &= 2C_1 + 3C_2 + \frac{3}{2} + \frac{23}{4}
\end{aligned}$$

i.e.,

$$\begin{aligned}
C_1 + C_2 &= 2 - \frac{23}{4} = -\frac{15}{4} \\
2C_1 + 3C_2 &= 3 - \frac{29}{4} = -\frac{17}{4}
\end{aligned}$$

This yields $C_1 = -15/4 - C_2 = -15/4 - 13/4 = -7$ and $C_2 = 13/4$. Hence

$$a_n = -7 \cdot 2^n + \frac{13}{4}3^n + \frac{3}{2}n + \frac{23}{4}$$

Here's a quick check:

```
def an(n):
    if n == 0: return 2
    elif n == 1: return 3
    else: return 5 * an(n - 1) - 6 * an(n - 2) + 3 * n + 1

def bn(n):
    return -7 * 2**n + (13/4.0) * 3**n + (1.5) * n + (23.0/4)

for n in range(10):
    print(n, an(n), bn(n))
```

The output is

```
0 2 2.0
1 3 3.0
2 10 10.0
3 42 42.0
4 163 163.0
5 579 579.0
6 1936 1936.0
7 6228 6228.0
8 19549 19549.0
9 60405 60405.0
```

The problem with the characteristic equation method is that you have to guess the particular solution.

- Suppose the recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2} + 7n^2$. Then you guess $a_n^{(p)} = A + Bn + Cn^2$.
- Suppose the recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2} + 2 + 7n^5$. Then you guess $a_n^{(p)} = A + Bn + Cn^2 + Dn^3 + En^4 + Fn^5$.
- Suppose the recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2} + 7 \cdot 5^n$. Then you guess $a_n^{(p)} = A \cdot 5^n$.
- Suppose the recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2} + (7 + 10n) \cdot 5^n$. Then you guess $a_n^{(p)} = (A + Bn) \cdot 5^n$.
- Suppose the recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2} + 7 \cdot 2^n$. Then you cannot guess $a_n^{(p)} = A \cdot 2^n$. Why? Because the homogeneous solution has a $C_1 2^n$ and this overlaps with your guess. You have to guess $a_n^{(p)} = C \cdot n2^n$
- Suppose the recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2} + (7 + 9n) \cdot 2^n$. Then you cannot guess $a_n^{(p)} = (A + Bn) \cdot 2^n$. Why? Because the homogeneous solution

has a $C_1 2^n$ and your guess contains $A2^n$ with overlaps with $C_1 2^n$. You have to guess $a_n^{(p)} = (A + Bn) \cdot n2^n$

- Suppose the recurrence relation is $a_n = 4a_{n-1} - 4a_{n-2} + (7 + 9n) \cdot 2^n$. The homogeneous solution is $a_n^{(h)} = C_1 2^n + C_2 n 2^n$. Then you cannot guess $a_n^{(p)} = (A + Bn) \cdot 2^n$. Why? Because the homogeneous solution has a $C_1 2^n$ and your guess contains $A2^n$ with overlaps with $C_1 2^n$. Your guess also cannot be $a_n^{(p)} = (A + Bn) \cdot n2^n$. Why? Because the homogeneous solution has a $C_2 n 2^n$ and your guess contains $Bn2^n$ with overlaps with $C_2 n 2^n$. You have to guess $a_n^{(p)} = (A + Bn) \cdot n^2 2^n$.

- So in general if the nonhomogeneous part looks like (polynomial of degree d) $\cdot$ r^n, you write down (polynomial of degree d) $\cdot$ r^n, where the polynomial has constants $A, B, C, ...$ for the coefficients for the polynomial. You check check if you need to change r^n to nr^n, $n^2 r^n$, etc. until your guess does not have any term that overlaps with the any term of the homogeneous solution.

- See theorem 6 in Rosen.

The problem is that you have memorize the above cases or memorize Theorem 6 of Rosen. And theorem 6 goes not even cover all cases. On the other hand, generating functions always work, as long as when a_n's are converted into a power series, the corresponding power series for the the nonhomogenous part becomes a power series that is one of the standard forms or can be manipulated into one of the standard forms. That's why I prefer to simply work with power series.

Q1. Find a closed form for $0^3 + 1^3 + 2^3 + \cdots + n^3$.

SOLUTION.

Q2. Using generating function method, find a closed form for a_n where

$$a_n = 5a_{n-1} - 6a_{n-2} + \underline{3n + 1}$$

where $a_0 = 2$ and $a_1 = 3$.

SOLUTION.

Q3. Rosen 8th edition, section 8.2, question 1.

SOLUTION.

(a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$: Linear. Homogeneous. Degree 3.

(b) $a_n = 2\underline{n}a_{n-1} + a_{n-2}$: Not linear. Homogeneous. Degree 2.

(c) $a_n = a_{n-1} + a_{n-4}$: Linear. Homogeneous. Degree 4.

(d) $a_n = a_{n-1} + \underline{2}$: Linear. Not homogeneous. Degree 1.

(e) $a_n = \underline{a_{n-1}^2} + a_{n-2}$: Not linear. Homogeneous. Degree 2.

(f) $a_n = a_{n-2}$: Linear. Homogeneous. Degree 2.

(g) $a_n = a_{n-2}$: Linear. Homogeneous. Degree 2.

Q4. Rosen 8th edition, section 8.2, question 3.

SOLUTION.

(a) The characteristic equation of

$$a_n = \begin{cases} 2a_{n-1} & \text{for } n \geq 1 \\ 3 & \text{for } n \geq 0 \end{cases}$$

is

$$x - 2$$

The root is $r_1 = 2$. Therefore the general closed form for a_n is

$$a_n = C_1 2^n$$

From the base condition

$$3 = C_1 2^0 = C_1$$

Hence

$$a_n = 3 \cdot 2^n = 3^{n+1}$$

for $n \geq 0$. $\square$

(Check: $a_0 = 3, a_1 = 3 \cdot 2^1 = 6 = 2 \cdot 3 = 2 \cdot a_0, a_2 = 3 \cdot 2^2 = 12 = 2 \cdot 6 = 2 \cdot a_1$, $a_3 = 3 \cdot 2^3 = 24 = 2 \cdot 12 = 2 \cdot a_2$.)

(b) The characteristic equation of

$$a_n = \begin{cases} a_{n-1} & \text{if } n \geq 1 \\ 2 & \text{if } n = 0 \end{cases}$$

is

$$x - 1$$

The root is $r_1 = 1$. Therefore the general closed form for a_n is

$$a_n = C_1 1^n = C_1$$

From the base condition

$$2 = a_0 = C_1$$

Hence

$$a_n = 2$$

for $n \geq 0$. $\square$

(Check: $a_0 = 2, a_1 = 2 = a_0, a_2 = 2 = a_1$, $a_3 = 2 = a_2$.)

(c) The characteristic equation of

$$a_n = \begin{cases} 5a_{n-1} - 6a_{n-2} & \text{if } n \geq 2 \\ 1 & \text{if } n = 0 \\ 0 & \text{if } n = 1 \end{cases}$$

is

$$x^2 - 5x + 6$$

The roots are $r_1 = 2$, $r_2 = 3$. Therefore the general closed form for a_n is

$$a_n = C_1 2^n + C_2 3^n$$

From the base cases

$$\begin{aligned} 1 = a_0 &= C_1 + C_2 \\ 0 = a_1 &= 2C_1 + 3C_2 \end{aligned}$$

Therefore $C_1 = 3, C_2 = -2$ Hence

$$a_n = 3 \cdot 2^n - 2 \cdot 3^n$$

for $n \geq 0$. □

(Check:

```
def an(n):
    if n == 0: return 1
    elif n == 1: return 0
    else: return 5 * an(n - 1) - 6 * an(n - 2)

def bn(n):
    return 3 * 2**n - 2 * 3**n

for n in range(10):
    print(n, an(n), bn(n))
```

has output

```
0 1 1
1 0 0
2 -6 -6
3 -30 -30
4 -114 -114
5 -390 -390
6 -1266 -1266
7 -3990 -3990
8 -12354 -12354
9 -37830 -37830
```

)

(d) The characteristic equation of

$$a_n = \begin{cases} 4a_{n-1} - 4a_{n-2} & \text{if } n \geq 2 \\ 6 & \text{if } n \geq 0 \\ 8 & \text{if } n \geq 1 \end{cases}$$

is

$$x^2 - 4x + 4$$

The roots are $r_1 = 2$, $r_2 = 2$. Therefore the general closed form for a_n is

$$a_n = C_1 2^n + C_2 n 2^n$$

From the base cases

$$\begin{aligned} 6 &= a_0 = C_1 \\ 8 &= a_1 = 2C_1 + 2C_2 \end{aligned}$$

Therefore $C_1 = 6, C_2 = -2$ Hence

$$a_n = 6 \cdot 2^n - 2 \cdot n 2^n = (6 - 2n)2^n$$

for $n \geq 0$. $\square$

Check:

```
def an(n):
    if n == 0: return 6
    elif n == 1: return 8
    else: return 4 * an(n-1) - 4 * an(n - 2)

def bn(n):
    return 6 * 2**n - 2 * n * 2**n

for n in range(10):
    print(n, an(n), bn(n))
```

```
0 6 6
1 8 8
2 8 8
3 0 0
4 -32 -32
5 -128 -128
6 -384 -384
7 -1024 -1024
8 -2560 -2560
9 -6144 -6144
```

(e) The characteristic equation of

$$a_n = \begin{cases} -4a_{n-1} - 4a_{n-2} & \text{if } n \geq 2 \\ 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \end{cases}$$

is

$$x^2 + 4x + 4$$

The roots are $r_1 = -2$, $r_2 = -2$. Therefore the general closed form for a_n is

$$a_n = C_1(-2)^n + C_2 n(-2)^n = C_1(-1)^n 2^n + C_2 n(-1)^n 2^n = (C_1 + C_2 n)(-1)^n 2^n$$

From the base cases

$$0 = a_0 = C_1$$
$$1 = a_1 = -2C_1 - 2C_2$$

Therefore $C_1 = 0, C_2 = -1/2$ Hence

$$a_n = (0 + (-1/2)n)(-1)^n 2^n = (-1)^{n+1} n 2^{n-1}$$

for $n \geq 0$. □

Check:

```
def an(n):
    if n == 0: return 0
    elif n == 1: return 1
    else: return -4 * an(n - 1) - 4 * an(n - 2)

def bn(n):
    return (-1)**(n + 1) * n * 2**(n - 1)

for n in range(10):
    print(n, an(n), bn(n))
```

```
0 0 0.0
1 1 1
2 -4 -4
3 12 12
4 -32 -32
5 80 80
6 -192 -192
7 448 448
8 -1024 -1024
9 2304 2304
```

(f) The characteristic equation of

$$a_n = \begin{cases} 4a_{n-2} & \text{if } n \geq 2 \\ 0 & \text{if } n = 0 \\ 4 & \text{if } n = 1 \end{cases}$$

is

$$x^2 - 4$$

The roots are $r_1 = -2$, $r_2 = 2$. Therefore the general closed form for a_n is

$$a_n = C_1(-2)^n + C_2 2^n = C_1(-1)^n 2^n + C_2 2^n = ((-1)^n C_1 + C_2)2^n$$

From the base cases

$$0 = a_0 = C_1 + C_2$$
$$4 = a_1 = -2C_1 + 2C_2$$

Therefore $C_1 = -1, C_2 = 1$ Hence

$$a_n = ((-1)^n(-1) + 1)2^n = ((-1)^{n+1} + 1)2^n$$

for $n \geq 0$. □

Check:

```
def an(n):
    if n == 0: return 0
    elif n == 1: return 4
    else: return  4 * an(n - 2)

def bn(n):
    return ((-1)**(n+1) + 1) * 2**n

for n in range(10):
    print(n, an(n), bn(n))
```

```
0 0 0
1 4 4
2 0 0
3 16 16
4 0 0
5 64 64
6 0 0
7 256 256
8 0 0
9 1024 1024
```

(g) The characteristic equation of

$$a_n = \begin{cases} a_{n-2}/4 & \text{if } n \geq 2 \\ 1 & \text{if } n = 0 \\ 0 & \text{if } n = 1 \end{cases}$$

is

$$x^2 - 1/4$$

The roots are $r_1 = -1/2$, $r_2 = 1/2$. Therefore the general closed form for a_n is

$$a_n = C_1(-1/2)^n + C_2(1/2)^n = C_1(-1)^n(1/2)^n + C_2(1/2)^n = ((-1)^n C_1 + C_2)(1/2)^n$$

From the base cases

$$\begin{aligned} 1 &= a_0 = C_1 + C_2 \\ 0 &= a_1 = -C_1 + C_2 \end{aligned}$$

Therefore $C_1 = 1/2, C_2 = 1/2$ Hence

$$a_n = ((-1)^n(1/2) + (1/2))(1/2)^n = ((-1)^n + 1)(1/2)^{n+1}$$

for $n \geq 0$. □

Check:

```
def an(n):
    if n == 0: return 1
    elif n == 1: return 0
    else: return  an(n - 2) / 4.0

def bn(n):
    return ((-1)**n + 1) * 0.5**(n + 1)

for n in range(10):
    print(n, an(n), bn(n))
```

```
0 1 1.0
1 0 0.0
2 0.25 0.25
3 0.0 0.0
4 0.0625 0.0625
5 0.0 0.0
6 0.015625 0.015625
7 0.0 0.0
8 0.00390625 0.00390625
9 0.0 0.0
```

Q5. Rosen 8th edition, section 8.2, question 4.

SOLUTION. Solve these recurrence relations together with the initial conditions given.

a)

$r^2 - r - 6 = 0$

$r = -2, 3$

$a_n = a_1(-2)^n + a_2 3^n$

$3 = a_1 + a_2$

$6 = -2a_1 + 3a_2$

$a_1 = \frac{3}{5}$ $a_2 = \frac{12}{5}$

$a_n = (\frac{3}{5})(-2)^n + (\frac{12}{5})3^n$

b)

$r^2 - 6r + 10 = 0$

$r = -2, 5$

$a_n = a_1 2^n + a_2 5^n$

$2 = a_1 + a_2$

$1 = -2a_1 + 5a_2$

$a_1 = 3$ $a_2 = -1$

$a_n = 3 \cdot 2^n - 5^n$

c)

$r^2 - 6r + 8 = 0$

$r = -2, 4$

$a_n = a_1 2^n + a_2 4^n$

$4 = a_1 + a_2$

$10 = -2a_1 + 4a_2$

$a_1 = 3\ a_2 = 1$

$a_n = 3 \cdot 2^n - 4^n$

d)

$r^2 - 2r + 1 = 0$

$r = -1, 1$

$a_n = a_1 1^n + a_2 1^n$

$4 = a_1$

$1 = a_1 + a_2$

$a_1 = 4\ a_2 = -3$

$a_n = 4 - 3^n$

e)

$r^2 - 1 = 0$

$r = -1, 1$

$a_n = a_1(-1)^n + a_2 1^n$

$5 = a_1 + a_2$

$-1 = -a_1 + a_2$

$a_1 = 3\ a_2 = 2$

$a_n = 3 \cdot (-1)^n + 2$

(f)

$r^2 + 6r + 9 = 0$

$r = -3, -3$

$a_n = a_1(-3)^n + a_2(-3)^n$

$3 = a_1$

$-3 = -3a_1 + -3a_2$

$a_1 = 3$ $a_2 = -2$

$a_n = 3(-3)^n - 2n(-3)^n$

(g)

$r^2 + 4r - 5 = 0$

$r = -5, 1$

$a_n = a_1(-5)^n + a_2 1^n$

$2 = a_1 + a_2$

$8 = -5a_1 + a_2$

$a_1 = -1$ $a_2 = 3$

$a_n = -(-5)^n + 3$

Q6. Rosen 8th edition, section 8.2, question 12.

SOLUTION.

$r^3 - 2r^2 - r + 2 = 0$

$r = 1, -1, 2$

$a_n = a_1 + a_2(-1)^n + a_3 2^n$

$3 = a_1 + a_2 + a_3$

$6 = a_1 - a_2 + 2a_3$

$a - 1 = 6, a_2 = -2, a_3 = -1$

$a_n = 6 - 2(-1)^n - 2^n$

Q7. Rosen 8th edition, section 8.2, question 13.

SOLUTION.

The characteristic equation of

$$a_n = \begin{cases} 7a_{n-2} + 6a_{n-3} & \text{if } n \geq 3 \\ 9 & \text{if } n = 0 \\ 10 & \text{if } n = 1 \\ 32 & \text{if } n = 2 \end{cases}$$

is

$$x^3 - 7x - 6$$

By trial and error, one of the roots is -1. By long division, we get $x^3 - 7x - 6 = (x+1)(x^2 - x - 6)$. The roots are $r_1 = -2$, $r_2 = -1$, $r_3 = 3$. Therefore the general closed form for a_n is

$$a_n = C_1(-2)^n + C_2(-1)^n + C_3 3^n = (-1)^n(C_1 2^n + C_2) + C_3 3^n$$

From the base cases

$$9 = a_0 = C_1 + C_2 + C_3 \quad (1)$$
$$10 = a_1 = -2C_1 - C_2 + 3C_3 \quad (2)$$
$$32 = a_2 = 4C_1 + C_2 + 9C_3 \quad (3)$$

(1)+(2) and (2)+(3) give us

$$19 = -C_1 + 4C_3 \quad (4)$$
$$21 = C_1 + 6C_3 \quad (5)$$

(4) + (5) gives us $40 = 10C_3$, i.e., $C_3 = 4$. (4) then gives us $C_1 = -3$. From (1), we get $C_2 = 8$. Hence

$$a_n = (-1)^n((-3)2^n + 8) + 4 \cdot 3^n$$

for $n \geq 0$. □

Check:

```
def an(n):
    if n == 0: return 9
    elif n == 1: return 10
    elif n == 2: return 32
    else: return 7 * an(n - 2) + 6 * an(n - 3)

def bn(n):
    return (-1)**n*((-3)*2**n + 8) + 4 * 3**n

for n in range(10):
    print(n, an(n), bn(n))
```

```
0 9 9
1 10 10
2 32 32
3 124 124
4 284 284
5 1060 1060
6 2732 2732
7 9124 9124
8 25484 25484
9 80260 80260
```

Q8. Rosen 8th edition, section 8.2, question 23.

SOLUTION.

(a) If $a_n = -2^{n+1}$, then

$$\begin{aligned} 3a_{n-1} + 2^n &= 3(-2^{(n-1)+1}) + 2^n \\ &= 3(-2^n) + 2^n \\ &= (3(-1) + 1) \cdot 2^n \\ &= -2 \cdot 2^n \\ &= -2^{n+1} \\ &= a_n \end{aligned}$$

Hence

$$a_n^{(p)} = -2^{n+1}$$

is a (particular) solution of $a_n = 3a_{n-1} + 2^n$.

(b) The characteristic equation of $a_n = 3a_{n-1}$ is

$$x - 3$$

The root of the charactistic equation is $r_1 = 3$. Therefore the general solution to $a_n = 3a_{n-1}$ is $C_1 3^n$, i.e.,

$$a_n^{(h)} = C_1 3^n$$

where C_1 is a constant. Hence the general solution to $a_n = 3a_{n-1} + 2^n$ is

$$a_n = a_n^{(h)} + a_n^{(p)} = C_1 3^n - 2^{n+1}$$

where C_1 is a constant.

(c) Using the base case,

$$1 = a_0 = C_1 3^0 - 2^{0+1} = C_1 - 2$$

Hence $C_1 = 3$. Therefore

$$a_n = 3 \cdot 3^n - 2^{n+1}$$

□

Check:

```
def an(n):
    if n == 0:
        return 1
```

```
    else:
        return 3 * an(n - 1) + 2**n

def bn(n):
    return 3 * 3**n - 2**(n + 1)

for n in range(10):
    print(n, an(n), bn(n))
```

has output

```
0 1 1
1 5 5
2 19 19
3 65 65
4 211 211
5 665 665
6 2059 2059
7 6305 6305
8 19171 19171
9 58025 58025
```

Q9. Rosen 8th edition, section 8.2, question 24.

SOLUTION.

(a) If $a_n = n2^n$, then

$$\begin{aligned} 2a_{n-1} + 2^n &= 2(n-1)2^{n-1} + 2^n \\ &= (n-1)2^n + 2^n \\ &= n2^n \\ &= a_n \end{aligned}$$

Hence

$$a_n^{(p)} = n2^n$$

is a (particular) solution of $a_n = 2a_{n-1} + 2^n$.

(b) The characteristic equation of $a_n = 2a_{n-1}$ is

$$x - 2$$

The root of the charactistic equation is $r_1 = 2$. Therefore the general solution to $a_n = 3a_{n-1}$ is $C_1 2^n$, i.e.,

$$a_n^{(h)} = C_1 2^n$$

where C_1 is a constant. Hence the general solution to $a_n = 3a_{n-1} + 2^n$ is

$$a_n = a_n^{(h)} + a_n^{(p)} = C_1 2^n + n2^n$$

where C_1 is a constant.

(c) Using the base case,

$$2 = a_0 = C_1 2^0 + 0 = C_1$$

Hence $C_1 = 2$. Therefore

$$a_n = 2 \cdot 2^n + n2^n = (2+n)2^n$$

□

Check:

```
def an(n):
    if n == 0:
        return 2
    else:
        return 2 * an(n - 1) + 2**n
```

```
def bn(n):
    return (2 + n) * 2**n

for n in range(10):
    print(n, an(n), bn(n))
```

has output

```
0 2 2
1 6 6
2 16 16
3 40 40
4 96 96
5 224 224
6 512 512
7 1152 1152
8 2560 2560
9 5632 5632
```

Q10. Rosen 8th edition, section 8.2, question 25.

SOLUTION.

(a) If $a_n = An + B$ is a particular solution of $a_n = 2a_{n-1} + n + 5$, then

$$\begin{aligned} An + B &= 2(A(n-1) + B) + n + 5 \\ &= (2A+1)n + (-2A + 2B + 5) \end{aligned}$$

Hence

$$\begin{aligned} A &= 2A + 1 \\ B &= -2A + 2B + 5 \end{aligned}$$

i.e.,

$$\begin{aligned} A &= -1 \\ B &= 2A - 5 \end{aligned}$$

Hence $A = -1$ and $B = -7$. Therefore

$$a^{(p)} = -n - 7$$

is a particular solution of $a_n = 2a_{n-1} + n + 5$.

(b) The characteristic equation of $a_n = 2a_{n-1}$ is

$$x - 2$$

The root of the chracteristic equation is $r_1 = 2$. Therefore the general solution of $a_n = 2a_{n-1}$ is

$$a^{(h)} = C_1 2^n$$

Hence the general solution of $a_n = 2a_{n-1} + n + 5$ is

$$a_n = a^{(h)} + a^{(p)} = C_1 2^n - n - 7$$

(c) From $a_0 = 4$,

$$4 = a_0 = C_1 2^0 - 0 - 7 = C_1 - 7$$

i.e., $C_1 = 11$. Hence the solution is

$$a_n = 11 \cdot 2^n - n - 7$$

Check:

```
def an(n):
    if n == 0:
        return 4
    else:
        return 2 * an(n - 1) + n + 5

def bn(n):
    return 11 * 2**n - n - 7

for n in range(10):
    print(n, an(n), bn(n))
```

has output

```
0 4 4
1 14 14
2 35 35
3 78 78
4 165 165
5 340 340
6 691 691
7 1394 1394
8 2801 2801
9 5616 5616
```

Q11. Rosen 8th edition, section 8.2, question 25.

SOLUTION.

(a) If $a_n = An + B$ is a particular solution of $a_n = 2a_{n-1} + n + 5$, then

$$\begin{aligned} An + B &= 2(A(n-1) + B) + n + 5 \\ &= (2A+1)n + (-2A + 2B + 5) \end{aligned}$$

Hence

$$\begin{aligned} A &= 2A + 1 \\ B &= -2A + 2B + 5 \end{aligned}$$

i.e.,

$$\begin{aligned} A &= -1 \\ B &= 2A - 5 \end{aligned}$$

Hence $A = -1$ and $B = -7$. Therefore

$$a^{(p)} = -n - 7$$

is a particular solution of $a_n = 2a_{n-1} + n + 5$.

(b) The characteristic equation of $a_n = 2a_{n-1}$ is

$$x - 2$$

The root of the chracteristic equation is $r_1 = 2$. Therefore the general solution of $a_n = 2a_{n-1}$ is

$$a^{(h)} = C_1 2^n$$

Hence the general solution of $a_n = 2a_{n-1} + n + 5$ is

$$a_n = a^{(h)} + a^{(p)} = C_1 2^n - n - 7$$

(c) From $a_0 = 4$,

$$4 = a_0 = C_1 2^0 - 0 - 7 = C_1 - 7$$

i.e., $C_1 = 11$. Hence the solution is

$$a_n = 11 \cdot 2^n - n - 7$$

Check:

```
def an(n):
    if n == 0:
        return 4
    else:
        return 2 * an(n - 1) + n + 5

def bn(n):
    return 11 * 2**n - n - 7

for n in range(10):
    print(n, an(n), bn(n))
```

has output

```
0 4 4
1 14 14
2 35 35
3 78 78
4 165 165
5 340 340
6 691 691
7 1394 1394
8 2801 2801
9 5616 5616
```

Q12. Rosen 8th edition, section 8.2, question 26.

SOLUTION.

The characteristic equation of $a_n = 6a_{n-1} - 12a_{n-1} + 8a_{n-3} + F(n)$ is

$$x^3 - 6x^2 + 12x - 8$$

By trial and error, 2 is a root. By long division

$$x^3 - 6x^2 + 12x - 8 = (x-2)(x^2 - 4x + 4)$$

Hence

$$x^3 - 6x^2 + 12x - 8 = (x-2)^3$$

Hence the roots of $x^3 - 6x^2 + 12x - 8$ are $r_1 = r_2 = r_3 = 2$. Therefore the general solution to homogeneous part of the recurrence relation is

$$a_n^{(h)} = C_1 2^n + C_2 2^n + C_1 n^2 2^n$$

(a) The general form of the closed form for $a_n = 6a_{n-1} - 12a_{n-1} + 8a_{n-3} + n^2$ is

$$a_n = C_1 2^n + C_2 2^n + C_1 n^2 2^n + A + Bn + Cn^2$$

(b) The general form of the closed form for $a_n = 6a_{n-1} - 12a_{n-1} + 8a_{n-3} + 2^n$ is

$$a_n = C_1 2^n + C_2 n 2^n + C_1 n^2 2^n + An^3 2^n$$

(c) The general form of the closed form for $a_n = 6a_{n-1} - 12a_{n-1} + 8a_{n-3} + n2^n$ is

$$a_n = C_1 2^n + C_2 n 2^n + C_3 n^2 2^n + (A + Bn) n^3 2^n$$

where C_1, C_2, C_3, A, B are constants.

(d) The general form of the closed form for $a_n = 6a_{n-1} - 12a_{n-1} + 8a_{n-3} + (-2)^n$ is

$$a_n = C_1 2^n + C_2 n 2^n + C_3 n^2 2^n + A(-2)^n$$

where C_1, C_2, C_3, A are constants.

(e) The general form of the closed form for $a_n = 6a_{n-1} - 12a_{n-1} + 8a_{n-3} + n^2 2^n$ is

$$a_n = C_1 2^n + C_2 n 2^n + C_3 n^2 2^n + (A + Bn + Cn^2) n^3 2^n$$

where C_1, C_2, C_3, A, B, C are constants.

(f) The general form of the closed form for $a_n = 6a_{n-1} - 12a_{n-1} + 8a_{n-3} + n^3(-2)^n$ is

$$a_n = C_1 2^n + C_2 n 2^n + C_3 n^2 2^n + (A + Bn + Cn^2 + Dn^3)(-2)^n$$

where $C_1, C_2, C_3, A, B, C, D$ are constants.

(g) The general form of the closed form for $a_n = 6a_{n-1} - 12a_{n-1} + 8a_{n-3} + 3$ is

$$a_n = C_1 2^n + C_2 n 2^n + C_3 n^2 2^n + A$$

where C_1, C_2, C_3, A are constants.

Q13. Rosen 8th edition, section 8.2, question 28.

SOLUTION.

a)

$a_n = 2a_{n-1}$

$a_n^{(h)} = a2^n$

$p_2 = -2, 4p_2 = p_1$

$-2p_2 + 2p_1 - p_0 = 0$

$p = -8 \ p_0 = -12$

$a_n = a2^n - 2n^2 - 8n - 12$

b)

$a = 13$

$a_n = 13 \cdot 2^n - 8n - 12$

Q14. Rosen 8th edition, section 8.2, question 30.

SOLUTION.

a)

$a_n = -5a_{n-1} - 6a_{n-2}$

$r^2 + 5r + 6 = 0$

$r = -2, -3$

$a_n^{(h)} = a(-2)^n + \beta(-3)^n$

$c = 16$

$a_n^{(p)} = 16 \cdot 4^n$

$a_n = a(-2)^n + \beta(-3)^n + 4^{n+2}$

b)

$a_1 = 56$

$a_2 = 4a + 9\beta + 256$

$a_n = (-2)^n + 2(-3) + 4^{n+2}$

Instructions

In `main.tex` change the email address in

```
\renewcommand\AUTHOR{jdoe5@cougars.ccis.edu}
```

yours. In the bash shell, execute "`make`" to recompile `main.pdf`. Execute "`make v`" to view `main.pdf`. Execute "`make s`" to create `submit.tar.gz` for submission.

For each question, you'll see boxes for you to fill. You write your answers in `main.tex` file. For small boxes, if you see

```
1 + 1 = \answerbox{}.
```

you do this:

```
1 + 1 = \answerbox{2}.
```

`answerbox` will also appear in "true/false" and "multiple-choice" questions.

For longer answers that needs typewriter font, if you see

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
\end{answercode}
```

you do this:

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
int x;
\end{answercode}
```

`answercode` will appear in questions asking for code, algorithm, and program output. In this case, indentation and spacing is significant. For program output, I do look at spaces and newlines.

For long answers (not in typewriter font) if you see

```
What is the color of the sky?
\begin{answerlong}
\end{answerlong}
```

you can write

```
What is the color of the sky?
\begin{answerlong}
The color of the sky is blue.
\end{answerlong}
```

For students beyond 245: You can put LaTeX commands in `answerbox` and `answerlong`.

A question that begins with "T or F or M" requires you to identify whether it is true or false, or meaningless. "Meaningless" means something's wrong with the statement and it is not well-defined. Something like "$1+_2$" or "$\{2\}^{\{3\}}$" is not well-defined. Therefore a question such as "Is $42 = 1+_2$ true or false?" or "Is $42 = \{2\}^{\{3\}}$ true or false?" does not make sense. "Is $P(42) = \{42\}$ true or false?" is meaningless because $P(X)$ is only defined if X is a set. For "Is $1 + 2 + 3$ true or false?", "$1 + 2 + 3$" is well–defined but as a "numerical expression", not as a "proposition", i.e., it cannot be true or false. Therefore "Is $1 + 2 + 3$ true or false?" is also not a well-defined question.

When writing results of computations, make sure it's simplified. For instance write 2 instead of $1 + 1$. When you write down sets, if the answer is $\{1\}$, I do not want to see $\{1, 1\}$.

When writing a counterexample, always write the simplest.

Here are some examples (see `instructions.tex` for details):

1. T or F or M: $1 + 1 = 2$ .... $\boxed{\text{T}}$
2. T or F or M: $1 + 1 = 3$ .... $\boxed{\text{F}}$
3. T or F or M: $1+^2 =$ .... $\boxed{\text{M}}$
4. $1 + 2 = \boxed{3}$
5. Write a C++ statement to declare an integer variable named `x`.

```
int x;
```

6. Solve $x^2 - 1 = 0$.

   Since $x^2 - 1 = (x - 1)(x + 1)$, $x^2 - 1 = 0$ implies $(x - 1)(x + 1) = 0$. Therefore $x - 1 = 0$ or $x = -1$. Hence $x = 1$ or $x = -1$.

7. Which is true? .... $\boxed{\text{C}}$

   (A) $1 + 1 = 0$
   (B) $1 + 1 = 1$
   (C) $1 + 1 = 2$
   (D) $1 + 1 = 3$
   (E) $1 + 1 = 4$