MATH325: Discrete Mathematics II
Assignment a04

Open `main.tex` and for each question you'll see `input` followed by a filename such as `02-05-18.tex`. (which contains your work on chapter 2, section 5, question 18 from our textbook). Enter your answer in the file `02-05-18.tex`. To rebuild and view pdf, in bash shell execute `make`. To build a gzip-tar file, in bash shell execute `make s` and you'll get `submit.tar.gz`.

Objectives

1. Compute recurrence relation.

The following are practice problems for self-study:

- Rosen 8th edition, section 8.1: Odd numbered problems 1-31 except 15, 17, 23, 29.

Some solutions are provided. For this assignment the problems you need to solve are

- Rosen 8th edition, section 8.1: questions 4, 8, 12, 24.

Explain your work completely using the solutions provided as guide and examples on how to write math properly.

In LaTeX math notation is enclosed by the $ symbols. For instance `$x = a_{1} + b^{2}$` gives you $x = a_1 + b^2$. For emphasis you can write also write it as `\[ x = a_{1} + b^{2} \]` to center your math:

$$x = a_1 + b^2$$

For binomial coefficients, `$\binom{5}{2}$` will give you $\binom{5}{2}$. You can also look at the solutions, find something that you can use, copy-and-paste, and modify.

For more information about LaTeX go to my website http://bit.ly/yliow0, click on `Yes` you are one of my students, then look for latex.pdf.) Even easier: ask questions in CCCS discord.

Draw a box around your final answer if the question required an explicit answer (i.e., if it's not a proof question).

Q1. Rosen 8th edition, section 8.1, question 1.

SOLUTION.

Let T_n be the number of moves to solve the n–disks tower of Hanoi problem. The recurrence relation is

$$T_n = 2T_{n-1} + 1 \tag{*}$$

with base case of $T_1 = 1$.

Let $P(n)$ be the statement

$$P(n) = \Big(T_n = 2^n - 1\Big)$$

for $n \geq 1$. We will prove that $P(n)$ holds for all $n \geq 1$ using induction.

Base case. For $n = 1$, $T_1 = 1$ and $2^1 - 1 = 2 - 1 = 1$. Hence $T_1 = 2^1 - 1$. Therefore $P(1)$ holds.

Inductive case. Suppose $P(n)$ holds, i.e.,

$$T^n = 2^n - 1$$

We want to show that $P(n + 1)$ holds as well, i.e., we want to show

$$T^{n+1} = 2^{n+1} - 1$$

From the recurrence relation $(*)$,

$$T_{n+1} = 2T_n + 1$$

Therefore

$$T_{n+1} = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$$

and hence $P(n + 1)$ holds.

Therefore by induction, $P(n)$ holds for all $n \geq 1$, i.e.,

$$T_n = 2^n - 1$$

for all $n \geq 1$. □

Q2. Rosen 8th edition, section 8.1, question 2.

SOLUTION.

Let a_n be the number of permutations on n distinct symbols.

The task T of forming a permutation is the same as the task T_1 of writing down the first symbol following by the task T_2 of writing down $n-1$ symbols distinct from the symbol written down by task T_1. By the multiplication principle, the number of ways to perform T is the number of ways to perform T_1 multiplied by the number of ways to perform T_2, i.e.,

$$a_n = n \cdot a_{n-1}$$

Clearly $a_1 = 1$. To compute a closed form for a_n, note that

$$\begin{aligned}
a_n &= na_{n-1} \\
&= n(n-1)a_{n-2} \\
&= n(n-2)a_{n-3} \\
&= ... \\
&= n(n-2)(n-3)\cdots(n-(k-1))a_{n-k}
\end{aligned}$$

When $n - k = 1$, we have $k = n - 1$ and

$$\begin{aligned}
a_n &= n(n-2)(n-3)\cdots 2 \cdot a_1 \\
&= n(n-2)(n-3)\cdots 2 \cdot 1
\end{aligned}$$

Hence

$$a_n = n!$$

for $n \geq 1$. $\square$

Q3. Rosen 8th edition, section 8.1, question 3.

SOLUTION.

(a) Let the one dollor coin, one dollar bill, five dollar fill be denoted by $a = 1, b = 1, c = 1$. A deposit is a sequence of a, b, c values such that the sum is n. For instance the deposit

$$abac$$

is a deposit for $a + b + a + c = 1 + 1 + 1 + 5 = 8$.

A deposit for n dollar is a sequence of the form

$$\begin{cases} ap \\ bp' \\ cp'' \end{cases}$$

where p is a pattern for the deposit of $n-1$ dollars, p' is a pattern for the deposit of $n-1$ dollars, p'' is a pattern for the deposit of $n-5$ dollars, Hence if a_n denotes the number of ways to deposit n dollars, we have the recurrence relation

$$a_n = a_{n-1} + a_{n-1} + a_{n-5} = 2a_{n-1} + a_{n-5}$$

Clearly

- $a_1 = 2$: the patterns are a and b
- $a_2 = 4$: the patterns are aa, ab, ba, bb
- $a_3 = 8$: the patterns are $aaa, aab, ..., bbb$, i.e., sequence of length 3 using two symbols
- $a_4 = 16$: the patterns are $aaaa, aaab, ..., bbbb$ i.e., sequence of length 4 using two symbols
- $a_5 = 33$: the patterns are $aaaaa, aaaab, ..., bbbbb, c$, i.e., sequence of length 5 using two symbols and c

Hence we have

$$a_n = \begin{cases} 2a_{n-1} + a_{n-5} & \text{if } n \geq 6 \\ 2^n & \text{if } 1 \leq n \geq 4 \\ 33 & \text{if } n = 5 \end{cases}$$

(b)

$$
\begin{aligned}
a_6 &= 2a_5 + a_1 = 2 \cdot 33 + 2 = 68 \\
a_7 &= 2a_6 + a_2 = 2 \cdot 68 + 4 = 72 \\
a_8 &= 2a_7 + a_3 = 2 \cdot 72 + 8 = 152 \\
a_9 &= 2a_8 + a_4 = 2 \cdot 152 + 16 = 320 \\
a_{10} &= 2a_9 + a_5 = 2 \cdot 320 + 33 = 673
\end{aligned}
$$

Answer: $\boxed{673}$

$\square$

Q4. Rosen 8th edition, section 8.1, question 4. A country uses as currency coins with values of 1 peso, 2 pesos, 5 pesos, and 10 pesos and bills with values of 5 pesos, 10 pesos, 20 pesos, 50 pesos, and 100 pesos. Find a recurrence relation for the number of ways to pay a bill of n pesos if the order in which the coins and bills are paid matters.

First 4 initial conditions

$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5$

if a_n denotes the number of ways to deposit n pesos, we have the recurrence relation

for $n \geqslant 100$

$\{a_n = a_{n-1} + a_{n-2} + 2a_{n-5} + 2a_{n-10} + a_{n-20} + a_{n-50} + a_{n-100}\}$

SOLUTION.

Q5. Rosen 8th edition, section 8.1, question 8. a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s. b) What are the initial conditions? c) How many bit strings of length seven contain three consecutive 0s?

(a) let a_n be the number of bit strings of length n containing three consecutive 0s.

we have the recurrence relation

for $n \geqslant 3$

$\{a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}\}$

$\boxed{\{a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}\}}$

(b) First 3 initial conditions are ,

$\boxed{a_1 = 0, a_2 = 0, a_3 = 1}$

(c)

$a_4 = 1 + 0 + 0 + 2^1 = 3$

$a_5 = 3 + 1 + 0 + 2^2 = 8$

$a_6 = 8 + 3 + 1 + 2^3 = 20$

$\{a_7 = a_6 + a_5 + a_4 + 2^4\}$

$\{a_7 = 20 + 8 + 3 + 2^4\} = 47$

$\boxed{47}$

SOLUTION.

Q6. Rosen 8th edition, section 8.1, question 12. a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time. b) What are the initial conditions? c) In how many ways can this person climb a flight of eight stairs?

(a)let a_n be the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

we have the recurrence relation

$\boxed{\text{for } n \geqslant 3 \ \{a_n = a_{n-1} + a_{n-2}a_{n-3}\}}$

(b) The initial conditions are

$\boxed{a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 4}$

(c)

$a_4 = a_3 + a_2 + a_1 = 4 + 2 + 1 = 7$

$a_5 = a_4 + a_3 + a_2 = 7 + 4 + 2 = 13$

$a_6 = a_5 + a_4 + a_3 = 13 + 7 + 4 = 24$

$a_7 = a_6 + a_5 + a_4 = 24 + 13 + 7 = 44$

Now, $a_8 = a_7 + a_6 + a_5 = 44 + 24 + 13 = 81$

$\boxed{81}$

SOLUTION.

Q7. Rosen 8th edition, section 8.1, question 24. Find a recurrence relation for the number of bit sequences of length n with an even number of 0s.

let a_n be the number of bit sequences of length n with an even number of 0s.

we have the recurrence relation

$\boxed{\text{for } n \geqslant 2 \ \{a_n = a_{n-1} + 2^{n-1} - a_{n-1}\}}$

SOLUTION.

Instructions

In `main.tex` change the email address in

```
\renewcommand\AUTHOR{jdoe5@cougars.ccis.edu}
```

yours. In the bash shell, execute "`make`" to recompile `main.pdf`. Execute "`make v`" to view `main.pdf`. Execute "`make s`" to create `submit.tar.gz` for submission.

For each question, you'll see boxes for you to fill. You write your answers in `main.tex` file. For small boxes, if you see

```
1 + 1 = \answerbox{}.
```

you do this:

```
1 + 1 = \answerbox{2}.
```

`answerbox` will also appear in "true/false" and "multiple-choice" questions.

For longer answers that needs typewriter font, if you see

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
\end{answercode}
```

you do this:

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
int x;
\end{answercode}
```

`answercode` will appear in questions asking for code, algorithm, and program output. In this case, indentation and spacing is significant. For program output, I do look at spaces and newlines.

For long answers (not in typewriter font) if you see

```
What is the color of the sky?
\begin{answerlong}
\end{answerlong}
```

you can write

```
What is the color of the sky?
\begin{answerlong}
The color of the sky is blue.
\end{answerlong}
```

For students beyond 245: You can put LaTeX commands in `answerbox` and `answerlong`.

A question that begins with "T or F or M" requires you to identify whether it is true or false, or meaningless. "Meaningless" means something's wrong with the statement and it is not well-defined. Something like "$1 +_2$" or "$\{2\}^{\{3\}}$" is not well-defined. Therefore a question such as "Is $42 = 1 +_2$ true or false?" or "Is $42 = \{2\}^{\{3\}}$ true or false?" does not make sense. "Is $P(42) = \{42\}$ true or false?" is meaningless because $P(X)$ is only defined if X is a set. For "Is $1 + 2 + 3$ true or false?", "$1 + 2 + 3$" is well–defined but as a "numerical expression", not as a "proposition", i.e., it cannot be true or false. Therefore "Is $1 + 2 + 3$ true or false?" is also not a well-defined question.

When writing results of computations, make sure it's simplified. For instance write 2 instead of $1 + 1$. When you write down sets, if the answer is $\{1\}$, I do not want to see $\{1, 1\}$.

When writing a counterexample, always write the simplest.

Here are some examples (see `instructions.tex` for details):

1. T or F or M: $1 + 1 = 2$ .... $\boxed{\text{T}}$

2. T or F or M: $1 + 1 = 3$ .... $\boxed{\text{F}}$

3. T or F or M: $1+^2 =$ .... $\boxed{\text{M}}$

4. $1 + 2 = \boxed{3}$

5. Write a C++ statement to declare an integer variable named `x`.

```
int x;
```

6. Solve $x^2 - 1 = 0$.

   Since $x^2 - 1 = (x - 1)(x + 1)$, $x^2 - 1 = 0$ implies $(x - 1)(x + 1) = 0$. Therefore $x - 1 = 0$ or $x = -1$. Hence $x = 1$ or $x = -1$.

7. Which is true? .... $\boxed{\text{C}}$

   (A) $1 + 1 = 0$
   (B) $1 + 1 = 1$
   (C) $1 + 1 = 2$
   (D) $1 + 1 = 3$
   (E) $1 + 1 = 4$