MATH325: Discrete Mathematics II
Assignment a05

Open `main.tex` and for each question you'll see `input` followed by a filename such as `02-05-18.tex`. (which contains your work on chapter 2, section 5, question 18 from our textbook). Enter your answer in the file `02-05-18.tex`. To rebuild and view pdf, in bash shell execute `make`. To build a gzip-tar file, in bash shell execute `make s` and you'll get `submit.tar.gz`.

Objectives

1. Use generating functions to compute closed forms for linear recurrences.
2. Use generating functions to solve counting problems.
3. Use generating functions to compute/prove identities.

The following are practice problems for self-study:

- Rosen 8th edition, section 8.4: Odd numbered problems 1-57 except 43, 47, 49, 51. 41 was mentioned in my notes (and in class). 46 is in my notes. 53-57 are in my notes.

Some solutions are provided. For this assignment the problems you need to solve are

- Rosen 8th edition, section 8.4: questions 8, 10, 14, 18, 24, 34, 38.

Explain your work completely using the solutions provided as guide and examples on how to write math properly.

In LaTeX math notation is enclosed by the `$` symbols. For instance `$x = a_{1} + b^{2}$` gives you $x = a_1 + b^2$. For emphasis you can write also write it as `\[ x = a_{1} + b^{2} \]` to center your math:

$$x = a_1 + b^2$$

For binomial coefficients, `$\binom{5}{2}$` will give you $\binom{5}{2}$. You can also look at the solutions, find something that you can use, copy-and-paste, and modify.

For more information about LaTeX go to my website http://bit.ly/yliow0, click on `Yes` you are one of my students, then look for latex.pdf.) Even easier: ask questions in CCCS discord.

Draw a box around your final answer if the question required an explicit answer (i.e., if it's not a proof question).

Q1. Rosen 8th edition, section 8.4, question 1.

SOLUTION.

The generating function for 2,2,2,2,2,2 is

$$\begin{aligned} f(x) &= 2 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 \\ &= 2(1 + x + x^2 + x^3 + x^4 + x^5) \\ &= 2\frac{1 - x^6}{1 - x} \end{aligned}$$

□

Q2. Rosen 8th edition, section 8.4, question 2.

SOLUTION.

The generating function for 1, 4, 16, 64, 256 is

$$\begin{aligned} f(x) &= 1 + 4x + 16x^2 + 64x^3 + 256x^4 \\ &= 1 + 4x + 4^2x^2 + 4^3x^3 + 4^4x^4 \\ &= 1 + 4x + (4x)^2 + (4x)^3 + (4x)^4 \\ &= \frac{1-(4x)^5}{1-x} \\ &= \frac{1-1024x^5}{1-x} \end{aligned}$$

$\square$

Q3. Rosen 8th edition, section 8.4, question 3.

SOLUTION.

(a) The generating function for $0, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0,...$ is

$$\begin{aligned} f(x) &= 2x + 2x^2 + \cdots + 2x^6 \\ &= 2x(1 + x + \cdots + x^5) \\ &= 2x\frac{1 - x^6}{1 - x} \end{aligned}$$

(b) The generating function for $0, 0, 0, 1, 1, 1, 1, 1, 1, ...$ is

$$\begin{aligned} f(x) &= x^3 + x^4 + x^5 + \cdots \\ &= x^3(1 + x + x^2 + \cdots) \\ &= x^3\frac{1}{1 - x} \\ &= \frac{x^3}{1 - x} \end{aligned}$$

(c) The generating function for $0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, ...$ is

$$\begin{aligned} f(x) &= x + x^4 + x^7 + x^{10} + \cdots \\ &= x(1 + x^3 + x^6 + x^9 + \cdots) \\ &= x(1 + (x^3) + (x^3)^2 + (x^3)^3 + \cdots) \\ &= x\frac{1}{1 - (x^3)} \\ &= \frac{x}{1 - x^3} \end{aligned}$$

(d) The generating function for $2, 4, 8, 16, 32, 64, 128, 256, ...$ is

$$\begin{aligned} f(x) &= 2 + 4x + 8x^2 + 16x^3 + 32x^4 + \cdots \\ &= 2(1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots) \\ &= 2(1 + (2x) + (2x)^2 + (2x)^3 + \cdots) \\ &= 2\frac{1}{1 - (2x)} \\ &= \frac{2}{1 - 2x} \end{aligned}$$

(e) The generating function for $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, ..., \binom{7}{7}, 0, 0, 0, ...$ is

$$\begin{aligned}
f(x) &= \binom{7}{0} + \binom{7}{1}x + \binom{7}{2}x^2 + \cdots + \binom{7}{7}x^7 \\
&= \sum_{i=0}^{7} \binom{7}{i} x^i \\
&= (1+x)^7
\end{aligned}$$

(f) The generating function for $2, -2, 2, -2, 2, -2, 2, -2, ...$ is

$$\begin{aligned}
f(x) &= 2 + (-2)x + 2x^2 + (-2)x^3 + \cdots \\
&= 2(1 + (-1)x + 1x^2 + (-1)x^3 + \cdots) \\
&= 2(1 + (-x) + (-x)^2 + (-x)^3 + \cdots) \\
&= 2\frac{1}{1-(-x)} \\
&= \frac{2}{1+x}
\end{aligned}$$

(g) The generating function for $1, 1, 0, 1, 1, 1, 1, 1, 1, 1, ...$is

$$\begin{aligned}
f(x) &= 1 + 1x + 0x^2 + 1x^3 + 1x^4 + 1x^5 + \cdots \\
&= 1 + x + (x^3 + x^4 + x^5 + \cdots) \\
&= 1 + x + x^3(1 + x + x^2 + \cdots) \\
&= 1 + x + x^3\frac{1}{1-x} \\
&= 1 + x + \frac{x^3}{1-x} \\
&= \frac{(1+x)(1-x) + x^3}{1-x} \\
&= \frac{1 - x^2 + x^3}{1-x}
\end{aligned}$$

(g) The generating function for $0, 0, 0, 1, 2, 3, 4, ...$ is

$$\begin{aligned}
f(x) &= 1x^3 + 2x^4 + 3x^5 + 4x^6 + \cdots \\
&= x^3(1 + 2x + 3x^2 + 4x^3 + \cdots) \\
&= x^3 \frac{d}{dx}(x + x^2 + x^3 + x^4 + \cdots) \\
&= x^3 \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots) \\
&= x^3 \frac{d}{dx} \frac{1}{1 - x} \\
&= x^3 \frac{-1}{(1 - x)^2} \\
&= -\frac{x^3}{(1 - x)^2}
\end{aligned}$$

Q4. Rosen 8th edition, section 8.4, question 4.

SOLUTION.

(a) The generating function is

$$f(x) = \sum_{i=0}^{6}(-1)x^i = -\sum_{i=0}^{6} x^i = -\frac{1-x^7}{1-x} = \frac{x^7-1}{1-x}$$

(b) The generating function is

$$\begin{aligned} f(x) &= 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + 729x^6 + \cdots \\ &= 1 + (3x) + (3x)^2 + (3x)^3 + (3x)^4 + (3x)^5 + (3x)^6 + \cdots \\ &= \frac{1}{1-3x} \end{aligned}$$

(c) The generating function is

$$\begin{aligned} f(x) &= 0 + 0x + 3x^2 + (-3)x^3 + 3x^4 + (-3)x^5 + 3x^6 + (-3)x^7 + \cdots \\ &= 3x^2(1 - x + x^2 - x^3 + x^4 - x^5 \cdots) \\ &= 3x^2(1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + (-x)^5 \cdots) \\ &= 3x^2 \frac{1}{1-(-x)} \\ &= \frac{3x^2}{1+x} \end{aligned}$$

(d) The generating function is

$$\begin{aligned} f(x) &= x + \sum_{i=0}^{\infty} x^i \\ &= x + \frac{1}{1-x} \\ &= \frac{1-x+1}{1-x} \\ &= \frac{2-x}{1-x} \end{aligned}$$

(e) The generating function is

$$\begin{aligned} f(x) &= \sum_{i=0}^{7} 2^i \binom{7}{i} x^i \\ &= \sum_{i=0}^{7} \binom{7}{i} (2x)^i \\ &= (1+2x)^7 \end{aligned}$$

(f) The generating function is

$$\begin{aligned} f(x) &= (-3) + 3x + (-3)x^2 + 3x^3 + (-3)x^4 + 3x^5 + \cdots \\ &= (-3)\sum_{i=0}^{\infty} (-x)^i \\ &= (-3)\frac{1}{1-(-x)} \\ &= -\frac{3}{1+x} \end{aligned}$$

(g) The generating function is

$$\begin{aligned} f(x) &= 0 + 1x + (-2)x^2 + 4x^3 + (-8)x^4 + 16x^5 + (-32)x^6 + 64x^7 + \cdots \\ &= \sum_{i=0}^{\infty} (-2)^{i-1} x^i \\ &= \sum_{i=0}^{\infty} \frac{1}{-2}(-2)^i x^i \\ &= -\frac{1}{2}\sum_{i=0}^{\infty} (-2x)^i \\ &= -\frac{1}{2}\frac{1}{1-(-2x)} \\ &= -\frac{1}{2(1+2x)} \end{aligned}$$

(h) The generating function is

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} x^{2i} \\ &= \sum_{i=0}^{\infty} (x^2)^i \\ &= \frac{1}{1-x^2} \end{aligned}$$

Q5. Rosen 8th edition, section 8.4, question 5.

SOLUTION.

(a) The generating function is

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} 5x^i \\ &= 5\sum_{i=0}^{\infty} x^i \\ &= 5\frac{1}{1-x} \\ &= \frac{5}{1-x} \end{aligned}$$

(b) The generating function is

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} 3^n x^i \\ &= 5\sum_{i=0}^{\infty} (3x)^i \\ &= \frac{1}{1-3x} \end{aligned}$$

(c) The generating function is

$$\begin{aligned}
f(x) &= \sum_{i=3}^{\infty} 2x^i \\
&= 2\sum_{i=3}^{\infty} x^i \\
&= 2\sum_{k=0}^{\infty} x^{k+3} && \text{(let } k = i - 3\text{)} \\
&= 2\sum_{k=0}^{\infty} x^3 x^k \\
&= 2x^3 \sum_{k=0}^{\infty} x^k \\
&= 2x^3 \frac{1}{1-x} \\
&= \frac{2x^3}{1-x}
\end{aligned}$$

(d) The generating function is

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}(2n+3)x^n \\
&= \sum_{n=0}^{\infty} 2nx^n + \sum_{n=0}^{\infty} 3x^n \\
&= 2\sum_{n=1}^{\infty} nx^n + 3\sum_{n=0}^{\infty} x^n \\
&= 2x\sum_{n=1}^{\infty} nx^{n-1} + 3\frac{1}{1-x} \\
&= 2x\sum_{n=1}^{\infty} \frac{d}{dx}x^n + 3\frac{1}{1-x} \\
&= 2x\frac{d}{dx}\sum_{n=0}^{\infty} x^n + \frac{3}{1-x} \\
&= 2x\frac{d}{dx}\frac{1}{1-x} + \frac{3}{1-x} \\
&= 2x\frac{-1}{(1-x)^2} + \frac{3}{1-x} \\
&= \frac{-2x}{(1-x)^2} + \frac{3}{1-x} \\
&= \frac{-2x}{(1-x)^2} + \frac{3(1+x)}{(1-x)(1+x)} \\
&= \frac{-2x+3(1+x)}{(1-x)^2} \\
&= \frac{3-x}{(1-x)^2}
\end{aligned}
$$

(e) The generating function is

$$
f(x) = \sum_{n=0}^{\infty} \binom{8}{n} x^n
$$

Q6. Rosen 8th edition, section 8.4, question 6.

SOLUTION.

(a) The generating function is

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty}(-1)x^i \\ &= -\sum_{i=0}^{\infty} x^i \\ &= -\frac{1}{1-x} \\ &= \frac{1}{x-1} \end{aligned}$$

(b) The generating function is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} 2^n x^n \\ &= \sum_{n=0}^{\infty}(2x)^n \\ &= \frac{1}{1-2x} \end{aligned}$$

(c) The generating function is

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}(n-1)x^n \\
&= \sum_{n=0}^{\infty} nx^n - \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=1}^{\infty} nx^n - \frac{1}{1-x} \\
&= x\sum_{n=1}^{\infty} nx^{n-1} - \frac{1}{1-x} \\
&= x\sum_{n=1}^{\infty} \frac{d}{dx}x^n - \frac{1}{1-x} \\
&= x\frac{d}{dx}\sum_{n=1}^{\infty} x^n - \frac{1}{1-x} \\
&= x\frac{d}{dx}\frac{1}{1-x} - \frac{1}{1-x} \\
&= x\frac{-1}{(1-x)^2} - \frac{1}{1-x} \\
&= \frac{-x-(1-x)}{(1-x)^2} \\
&= -\frac{1}{(1-x)^2}
\end{aligned}$$

(d) The generating function is

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n \\
&= \frac{1}{x} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^{n+1} \\
&= \frac{1}{x} \sum_{k=1}^{\infty} \frac{1}{k!} x^k && \text{(let } k = n+1\text{)} \\
&= \frac{1}{x} \left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k - 1 \right) \\
&= \frac{1}{x} \left(e^x - 1 \right) \\
&= \frac{e^x - 1}{x}
\end{aligned}
$$

Q7. Rosen 8th edition, section 8.4, question 7.

SOLUTION.

(a) Using binomial theorem, we have

$$\begin{aligned}(3x-4)^3 &= \sum_{n=0}^{3}\binom{3}{n}(3x)^n(-4)^{3-n}\\ &= \sum_{n=0}^{3}\binom{3}{n}3^n(-1)^{3-n}4^{3-n}x^n\\ &= \sum_{n=0}^{3}(-1)^{n+1}\binom{3}{n}4^3\left(\frac{3}{4}\right)^n x^n\end{aligned}$$

Therefore

$$a_n = \begin{cases}\binom{3}{n}(-1)^{n-1}4^3\left(\frac{3}{4}\right)^n & \text{if } n = 0,1,2,3\\ 0 & \text{otherwise}\end{cases}$$

(b) Using binomial theorem, we have

$$\begin{aligned}(x^3+1)^3 &= \sum_{n=0}^{3}\binom{3}{n}(x^3)^n\\ &= \sum_{n=0}^{3}\binom{3}{n}x^{3n}\end{aligned}$$

Therefore

$$a_n = \begin{cases}\binom{3}{n/3} & \text{if } n = 0,3,6,9\\ 0 & \text{otherwise}\end{cases}$$

(c) From

$$\frac{1}{1-5x} = \sum_{n=0}^{\infty}(5x)^n = \sum_{n=0}^{\infty}5^n x^n$$

we get

$$a_n = 5^n$$

for $n \geq 0$.

(d) From

$$\begin{aligned}
\frac{x^3}{1+3x} &= x^3 \frac{1}{1-(-3x)} \\
&= x^3 \sum_{n=0}^{\infty} (-3x)^n \\
&= x^3 \sum_{n=0}^{\infty} (-3)^n x^n \\
&= \sum_{n=0}^{\infty} (-3)^n x^{n+3} \\
&= \sum_{k=3}^{\infty} (-3)^{k-3} x^k && (\text{let } k = n+3)
\end{aligned}$$

we get

$$a_n = \begin{cases} 0 & \text{if } n = 0, 1, 2 \\ (-3)^{n-3} & \text{if } n \geq 3 \end{cases}$$

(e) From

$$\begin{aligned}
x^2 + 3x + 7 + \frac{1}{1-x^2} &= x^2 + 3x + 7 + \sum_{n=0}^{\infty} (x^2)^n \\
&= x^2 + 3x + 7 + \sum_{n=0}^{\infty} x^{2n} \\
&= x^2 + 3x + 7 + \left(1 + x^2 + \sum_{n=2}^{\infty} x^{2n}\right) \\
&= 2x^2 + 3x + 8 + \sum_{n=2}^{\infty} x^{2n}
\end{aligned}$$

we get

$$a_n = \begin{cases} 8 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 1 & \text{if } n \geq 4 \text{ is even} \\ 0 & \text{if } n \geq 3 \text{ is odd} \end{cases}$$

(f) From

$$\begin{aligned}
\frac{x^4}{1-x^4} - x^3 - x^2 - x - 1 &= x^4 \frac{1}{1-x^4} - x^3 - x^2 - x - 1 \\
&= x^4 \sum_{n=0}^{\infty} (x^4)^n - x^3 - x^2 - x - 1 \\
&= x^4 \sum_{n=0}^{\infty} x^{4n} - x^3 - x^2 - x - 1 \\
&= \sum_{n=0}^{\infty} x^{4(n+1)} - x^3 - x^2 - x - 1
\end{aligned}$$

we get

$$a_n = \begin{cases} -1 & \text{if } n = 0, 1, 2, 3 \\ 1 & \text{if } n = 4k \text{ for } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(g) From

$$\begin{aligned}
\frac{x^2}{(1-x)^2} &= x^2 \frac{1}{(1-x)^2} \\
&= x^2 \sum_{n=0}^{\infty} \binom{2+n-1}{n} x^n \\
&= x^2 \sum_{n=0}^{\infty} \binom{n+1}{n} x^n \\
&= x^2 \sum_{n=0}^{\infty} \binom{n+1}{(n+1)-n} x^n \\
&= x^2 \sum_{n=0}^{\infty} \binom{n+1}{1} x^n \\
&= x^2 \sum_{n=0}^{\infty} (n+1) x^n \\
&= \sum_{n=0}^{\infty} (n+1) x^{n+2} \\
&= \sum_{k=2}^{\infty} (k-1) x^k & \text{(let } k = n+2\text{)}
\end{aligned}$$

we get

$$a_n \begin{cases} 0 & \text{if } n = 0, 1 \\ n - 1 & \text{if } n \geq 2 \end{cases}$$

(g) From

$$\begin{aligned} 2e^{2x} &= 2\sum_{n=0}^{\infty} \frac{1}{n!}x^n \\ &= \sum_{n=0}^{\infty} \frac{2}{n!}x^n \end{aligned}$$

we get

$$a_n = \frac{2}{n!}$$

for $n \geq 0$.

Q8. Rosen 8th edition, section 8.4, question 8.

SOLUTION.

For each of these generating functions, provide a closed formula for the sequence it determines.

(a) $= (x^2+1)^3 =$ by multiplying out, we get $x^6+3x^4+3x2+1$ $\boxed{a_0 = 1, a_2 = 3, a_4 = 3, a_6 = 1}$

(b) $= (3x-1)^3 =$ by multiplying out, we get $27x^3-27x^2+9x-1$ $\boxed{a_0 = -1, a_1 = 9, a_2 = -27, a_3 = 27}$

(c) $= 1/(1-2x^2) =$ $\boxed{x^{2n}, \text{ ODD coef's are 0.}}$

(d) $= x^2/(1-x)^3 = x^2 \cdot \binom{n+2}{2}x^n = \binom{n+2}{2}x^{n+2} = \binom{n}{2}x^n$ $\boxed{\binom{n}{2} \text{ if } n \geqslant 2, \text{ 0 if } n = 1, 2}$

(e) $= x - 1 + (1/(1-3x)) = a_n =$ $\boxed{3^n, a_0 = -1 = 3^0 = 0, a_1 = 1 + 3^1 = 4}$

(f) $=$ $\boxed{a_n = (-1)^n 3n \text{ for } n \geqslant 3 \ a_n = (-1)^n \binom{n+2}{2} \text{ for } n < 3}$

(g) $= x/(1+x+x^2) =$ $\boxed{a_n = 0 \text{ for n is a multiple of 3, } a_n = 1\text{, for n} = 1 \text{ ¿ multiple of 3, } a_n = -1\text{, for n} =}$

(h) $= e^{3x^2} - 1 =$ $\boxed{a_0 = 0,\ a_n = 0 \text{ for n is ODD, } a_{2n} = 3^n/m! \text{ for n is EVEN}}$

Q9. Rosen 8th edition, section 8.4, question 9.

SOLUTION.

(a) We have

$$\begin{aligned}
(1 + x^5 + x^{10} + x^{15} + \cdots)^3 &= (1 + (x^5) + (x^5)^2 + (x^5)^3 + \cdots))^3 \\
&= \left(\sum_{n=0} (x^5)^n\right)^3 \\
&= \left(\frac{1}{1 - x^5}\right)^3 \\
&= \sum_{n=0}^{\infty} \binom{3 + n - 1}{n} (x^5)^n \\
&= \sum_{n=0}^{\infty} \binom{n + 2}{n} x^{5n} \\
&= \sum_{n=0}^{\infty} \binom{n + 2}{(n + 2) - n} x^{5n} \\
&= \sum_{n=0}^{\infty} \binom{n + 2}{2} x^{5n}
\end{aligned}$$

In the above power series, the coefficient of x^{10} is the coefficient of x^{5n} when $n = 2$, which is $\binom{2+2}{2} = \binom{4}{2} = 4 \cdot 3/2 = 6$.

(NOTE. Alternatively, this is also the number of solutions to $a + b + c = 10$ where $a = 0, 5, 10, 15, ...$, $b = 0, 5, 10, 15, ...$, $c = 0, 5, 10, 15, ...$)

(b) We have

$$\begin{aligned}
(x^3+x^4+x^5+x^6+x^7+\cdots)^3 &= (x^3(1+x+x^2+x^3+x^4+\cdots)^3 \\
&= \left(x^3\frac{1}{1-x}\right)^3 \\
&= x^9\left(\frac{1}{1-x}\right)^3 \\
&= x^9\sum_{n=0}^{\infty}\binom{3+n-1}{n}x^n \\
&= x^9\sum_{n=0}^{\infty}\binom{n+2}{n}x^n \\
&= x^9\sum_{n=0}^{\infty}\binom{n+2}{(n+2)-n}x^n \\
&= x^9\sum_{n=0}^{\infty}\binom{n+2}{2}x^n \\
&= \sum_{n=0}^{\infty}\binom{n+2}{2}x^{n+9} \\
&= \sum_{k=9}^{\infty}\binom{k-7}{2}x^k && \text{(let } k=n+9\text{)}
\end{aligned}$$

In the above power series, the coefficient of x^{10} is the coefficient of x^k when $k=10$, which is $\binom{3}{2}=3$.

(Note. Alternatively, this is also the number of solutions to $a+b+c=10$ where a,b,c are all ≥ 3.)

(c) We have

$$\begin{aligned}
&(x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)(1 + x + x^2 + x^3 + x^4 + \cdots) \\
&= x^4(1 + x + x^2)x^3(1 + x + x^2 + x^3 + x^4)\frac{1}{1-x} \\
&= x^7 \frac{1-x^3}{1-x}\frac{1-x^5}{1-x}\frac{1}{1-x} \\
&= x^7(1-x^3)(1-x^5)\left(\frac{1}{1-x}\right)^3 \\
&= x^7(1 - x^3 - x^5 + x^8)\sum_{n=0}^{\infty}\binom{3+n-1}{n}x^n \\
&= (x^7 - x^{10} - x^{12} + x^{16})\sum_{n=0}^{\infty}\binom{n+2}{n}x^n \\
&= (x^7 - x^{10} - x^{12} + x^{16})\sum_{n=0}^{\infty}\binom{n+2}{2}x^n
\end{aligned}$$

The coefficient of x^{10} is

$$1 \cdot \binom{3+2}{2} - 1 \cdot \binom{0+2}{2} = \binom{5}{2} - \binom{2}{2} = \frac{5 \cdot 4}{2} - 1 = 9$$

(Note. Alternatively, this is also the number of solutions to $a + b + c = 10$ where $4 \le a \le 6$, $3 \le b \le 7$, $0 \le c$.)

(d) We have

$$\begin{aligned}
&(x^2 + x^4 + x^6 + x^8 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x^4 + x^8 + x^{12} + \cdots) \\
&= x^2(1 + x^2 + x^4 + x^6 + \cdots)x^3(1 + x^3 + x^6 + \cdots)x^4(1 + x^4 + x^8 + \cdots) \\
&= x^{2+3+4}(1 + x^2 + \cdots)(1 + x^3 + \cdots)(1 + x^4 + \cdots) \\
&= x^9(1 + x^2 + \cdots)(1 + x^3 + \cdots)(1 + x^4 + \cdots)
\end{aligned}$$

The term $a_{10}x^{10}$ in the above is the product x^9 and the term of x^1 in $(1+x^2+\cdots)(1+x^3+\cdots)(1+x^4+\cdots)$. Clearly the coefficient of x^1 in $(1+x^2+\cdots)(1+x^3+\cdots)(1+x^4+\cdots)$ is 0. Hence the coefficient of x^{10} in $(x^2+x^4+x^6+x^8+\cdots)(x^3+x^6+x^9+\cdots)(x^4+x^8+x^{12}+\cdots)$ is 0.

(Note. Alternatively, this is also the number of solutions to $a + b + c = 10$ where $a = 2, 4, 6, ...$, $b = 3, 6, 9, ...$, $c = 4, 8, 12, ...$)

(e) The term with x^{10} in

$$(1 + x^2 + x^4 + x^6 + x^8 + \cdots)(1 + x^4 + x^8 + x^{12} + \cdots)(1 + x^6 + x^{12} + x^{18} + \cdots)$$

is given by

$$1 \cdot x^4 \cdot x^6 + x^2 \cdot x^8 \cdot x^0 + x^4 \cdot x^0 \cdot x^6 + x^6 \cdot x^4 \cdot x^0 + x^{10} \cdot x^0 \cdot x^0 = 5x^{10}$$

Therefore the required coefficient is 5. This is also the number of solutions to

$$a + b + c = 10$$

where $a = 0, 2, 4, 6, ...$ and $b = 0, 4, 8, 12, ...$ and $c = 0, 6, 12, 18, ...$. The following are the solution:

1. $(a, b, c) = (0, 4, 6)$
2. $(a, b, c) = (2, 8, 0)$
3. $(a, b, c) = (4, 0, 6)$
4. $(a, b, c) = (6, 4, 0)$
5. $(a, b, c) = (10, 0, 0)$

Note. Whether you should manipulate the power series to get the required coefficient or convert to a counting problem to get the coefficient depends on the scenario. Sometimes you don't know which is better until you try. For instance for (e), you can try to manipulate the power series and see if you can get the answer.

Q10. Rosen 8th edition, section 8.4, question 10.

SOLUTION.

(a) $= 1/(1 - x^3) = \binom{n+2}{2}x^n$ $\boxed{\begin{pmatrix}5\\2\end{pmatrix} = 10}$

(b) $= \binom{n+2}{2}x^n$ $\boxed{\begin{pmatrix}5\\2\end{pmatrix} = 10}$

(c) $=$

(d) $= \boxed{2}$

(e) $= \boxed{0}$

Q11. Rosen 8th edition, section 8.4, question 14.

SOLUTION.

x^n for $1/(1-x)^5 = \binom{n+4}{4}$

$\binom{16}{4} - 5 \cdot \binom{12}{4} + 10 \cdot \binom{8}{4} - 10 \cdot \binom{4}{4}$

$1820 - 2475 + 700 - 10$

$\boxed{35}$

Q12. Rosen 8th edition, section 8.4, question 18.

SOLUTION.

x^n for $1/(1-x)^3 = \binom{n+2}{2}$

$\binom{13}{2} - \binom{5}{2}$

$\boxed{68}$

Q13. Rosen 8th edition, section 8.4, question 24.

SOLUTION.

x^n for $1/(1-x)^2 = n+1$

$1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1$

$\boxed{10}$

Q14. Rosen 8th edition, section 8.4, question 34.

SOLUTION.

$G(x) = 1/(1 - 4x)$

$\boxed{a_k = 4^k}$

Q15. Rosen 8th edition, section 8.4, question 38.

SOLUTION.

Q16. Find a closed form for $0^3 + 1^3 + 2^3 + \cdots + n^3$.

SOLUTION.

Instructions

In `main.tex` change the email address in

```
\renewcommand\AUTHOR{jdoe5@cougars.ccis.edu}
```

yours. In the bash shell, execute "`make`" to recompile `main.pdf`. Execute "`make v`" to view `main.pdf`. Execute "`make s`" to create `submit.tar.gz` for submission.

For each question, you'll see boxes for you to fill. You write your answers in `main.tex` file. For small boxes, if you see

```
1 + 1 = \answerbox{}.
```

you do this:

```
1 + 1 = \answerbox{2}.
```

`answerbox` will also appear in "true/false" and "multiple-choice" questions.

For longer answers that needs typewriter font, if you see

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
\end{answercode}
```

you do this:

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
int x;
\end{answercode}
```

`answercode` will appear in questions asking for code, algorithm, and program output. In this case, indentation and spacing is significant. For program output, I do look at spaces and newlines.

For long answers (not in typewriter font) if you see

```
What is the color of the sky?
\begin{answerlong}
\end{answerlong}
```

you can write

```
What is the color of the sky?
\begin{answerlong}
The color of the sky is blue.
\end{answerlong}
```

For students beyond 245: You can put LaTeX commands in `answerbox` and `answerlong`.

A question that begins with "T or F or M" requires you to identify whether it is true or false, or meaningless. "Meaningless" means something's wrong with the statement and it is not well-defined. Something like "$1+_2$" or "$\{2\}^{\{3\}}$" is not well-defined. Therefore a question such as "Is $42 = 1+_2$ true or false?" or "Is $42 = \{2\}^{\{3\}}$ true or false?" does not make sense. "Is $P(42) = \{42\}$ true or false?" is meaningless because $P(X)$ is only defined if X is a set. For "Is $1 + 2 + 3$ true or false?", "$1 + 2 + 3$" is well–defined but as a "numerical expression", not as a "proposition", i.e., it cannot be true or false. Therefore "Is $1 + 2 + 3$ true or false?" is also not a well-defined question.

When writing results of computations, make sure it's simplified. For instance write 2 instead of $1 + 1$. When you write down sets, if the answer is $\{1\}$, I do not want to see $\{1, 1\}$.

When writing a counterexample, always write the simplest.

Here are some examples (see `instructions.tex` for details):

1. T or F or M: $1 + 1 = 2$ .... $\boxed{\text{T}}$
2. T or F or M: $1 + 1 = 3$ .... $\boxed{\text{F}}$
3. T or F or M: $1+^2=$ .... $\boxed{\text{M}}$
4. $1 + 2 = \boxed{3}$
5. Write a C++ statement to declare an integer variable named `x`.

```
int x;
```

6. Solve $x^2 - 1 = 0$.

Since $x^2 - 1 = (x - 1)(x + 1)$, $x^2 - 1 = 0$ implies $(x - 1)(x + 1) = 0$. Therefore $x - 1 = 0$ or $x = -1$. Hence $x = 1$ or $x = -1$.

7. Which is true? .... $\boxed{\text{C}}$
   (A) $1 + 1 = 0$
   (B) $1 + 1 = 1$
   (C) $1 + 1 = 2$
   (D) $1 + 1 = 3$
   (E) $1 + 1 = 4$