MATH325: Discrete Mathematics II
Assignment a03

Open `main.tex` and for each question you'll see `input` followed by a filename such as `02-05-18.tex`. (which contains your work on chapter 2, section 5, question 18 from our textbook). Enter your answer in the file `02-05-18.tex`. To rebuild and view pdf, in bash shell execute `make`. To build a gzip-tar file, in bash shell execute `make s` and you'll get `submit.tar.gz`.

Objectives

1. Solve counting problems using principle of inclusion-exclution.

The following are practice problems for self-study:

- Rosen 8th edition, section 8.6: Odd numbered problems 1-25 except 15. Solutions to many of the questions are included. 23 was covered in class. Some questions (example 19, 24) are small theorems – you are encourage to try them and talk to me if you have questions.

For this assignment the problems you need to solve are

- Rosen 8th edition, section 8.6: questions 4, 6, 10, 17.

Explain your work completely using the solutions provided as guide and examples on how to write math properly.

In LaTeX math notation is enclosed by the $ symbols. For instance `$x = a_{1} + b^{2}$` gives you $x = a_1 + b^2$. For emphasis you can write also write it as `\[ x = a_{1} + b^{2} \]` to center your math:

$$x = a_1 + b^2$$

For binomial coefficients, `$\binom{5}{2}$` will give you $\binom{5}{2}$. You can also look at the solutions, find something that you can use, copy-and-paste, and modify.

For more information about LaTeX go to my website http://bit.ly/yliow0, click on `Yes` you are one of my students, then look for latex.pdf.) Even easier: ask questions in CCCS discord.

For each question in section 8.6, you must define your universe U and the propositional formulas P_1, P_2, P_3. (Of course different problems have different numbers of propositional formulas; you might need more or less.) You can use other subscripts such as $P_A, \ldots$ if you find it easier to read the solution that way.

After stating the above, you have to argue that the required number is $N(P_1' P_2' P_3')$. In most cases, it should be obvious. If that's the case you just say "The required number is clearly $N(P_1' P_2' P_3' \ldots)$". In some cases it might not be clear. You h ave to judge for yourself.

After stating the above, you must state the version of P.I.E. you are using. For instance you might want to say "By the principle of inclusion-exclusion,

$$N(P_1' P_2' P_3') = |U| - \ldots$$

Do NOT use U or P_i's if it's not defined.

After that you should compute the values $|U|$, $N(P_1)$, ..., $N(P_1 P_2)$, ... If the computation of $N(P_2)$ is similar to $N(P_1)$, then you may say "Similarly $N(P_2) = \ldots$". For easy and straightforward cases (example: the number is something from Discrete I), you simply state the number. For instance: "We have $|U| = 2^5$." or "Clearly $|U| = 2^5$." Learn to write *full* sentences: writing "$|U| = 2^5$" out of nowhere is very confusing and misleading.

Again you do not need to show trivial computations. The general rule is this: If the result is from Discrete I (or previous classes), then you can use the relevant formula right away without too much explanation. If a result you want to use is not from previous classes, then you need to explain how to change the computation to one involving previous results.

Once all the quantities are stated/computated, you substitute them into the P.I.E. that you have stated.

Finally state what you have found clearly. For instance you might want to say "Therefore the number of students studying Biology and Physics is 115".

Draw a box around your final answer if the question required an explicit answer (i.e., if it's not a proof question).

Q1. Rosen 8th edition, section 8.6, question 1.

SOLUTION.

Let U be the apples in the bushel. Define

$$P_w(x) = (\text{there are worms in } x)$$
$$P_b(x) = (\text{there are bruises on } x)$$

The required number is the number of apples that can be sold which is the number of apples without worms and without bruises, i.e., the required number is $N(P'_w P'_b)$.

By the principle of inclusion-exclusion

$$N(P'_w P'_b) = |U| - (N(P_w) + N(P_b)) + N(P_w P_b)$$

We are given

$$|U| = 100$$
$$N(P_w) = 20$$
$$N(P_b) = 15$$
$$N(P_w P_b) = 10$$

Therefore

$$N(P'_w P'_b) = 100 - (20 + 15) + 10 = 75$$

Hence the number of apples that can be sold is 75.

ANSWER: $\boxed{75}$ □

Q2. Rosen 8th edition, section 8.6, question 2.

SOLUTION.

Let

$$\begin{aligned} U &= \text{the set of applicants} \\ P_1(x) &= (x \text{ has altitude sickness}) \\ P_2(x) &= (x \text{ is not in good enough shape}) \\ P_3(x) &= (x \text{ has allergies}) \end{aligned}$$

We are given the following information

$$\begin{aligned} |U| &= 1000 \\ N(P_1) &= 450 \\ N(P_2) &= 622 \\ N(P_3) &= 30 \\ N(P_1P_2) &= 111 \\ N(P_1P_3) &= 14 \\ N(P_2P_3) &= 18 \\ N(P_1N_2P_3) &= 9 \end{aligned}$$

The required number is $N(P_1'P_2'P_3')$. By the principle of inclusion-exclusion,

$$\begin{aligned} N(P_1'P_2'P_3') = |U| & \\ & - (N(P_1) + N(P_2) + N(P_3)) \\ & + (N(P_1P_2) + N(P_1P_3) + N(P_2P_3)) \\ & - (N(P_1P_2P_3)) \\ = 1000 & - (450 + 622 + 30) + (111 + 14 + 18) - 9 \\ = 32 & \end{aligned}$$

Therefore the number of applicants that qualify is 32.

ANSWER: $\boxed{32}$ $\square$

Q3. Rosen 8th edition, section 8.6, question 3.

SOLUTION.

Let
$$U = \{(x_1, x_2, x_3) \in \mathbb{N}^3 \mid x_1 + x_2 + x_3 = 13\}$$
where $\mathbb{N} = \{0, 1, 2, 3, ...\}$. Define

$$\begin{aligned} P_1(x_1, x_2, x_3) &= (x_1 \geq 6) \\ P_2(x_1, x_2, x_3) &= (x_2 \geq 6) \\ P_3(x_1, x_2, x_3) &= (x_3 \geq 6) \end{aligned}$$

The required number is the number of (x_1, x_2, x_3) satisfying

$$x_1 + x_2 + x_3 = 13, \quad 0 \leq x_1 < 6, \quad 0 \leq x_2 < 6, \quad 0 \leq x_3 < 6$$

i.e., $N(P_1'P_2'P_3')$. By the inclusion-exclusion principle,

$$\begin{aligned} N(P_1'P_2'P_3') = |U| & \\ & - (N(P_1) + N(P_2) + N(P_3)) \\ & + (N(P_1P_2) + N(P_1P_3) + N(P_2P_3)) \\ & - (N(P_1P_2P_3)) \end{aligned}$$

We have
$$|U| = \frac{(13+2)!}{13!2!} = \binom{15}{2}$$
(This is from Discrete I, i.e., it is the number of distribution of 13 0's into 3 boxes, which is the same as the number of permutations of 13 0's and 2 1's).

$N(P_1)$ is the number of solutions to

$$x_1 + x_2 + x_3 = 13, \;\; x_1 \geq 6, \;\; x_2 \geq 0, \;\; x_3 \geq 0$$

which is the number of solutions to

$$(x_1 - 6) + x_2 + x_3 = 13 - 6, \;\; x_1 - 7 \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0$$

which is the number of solutions to

$$x_1' + x_2 + x_3 = 7, \;\; x_1' \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0$$

which is $\frac{(7+2)!}{7!2!} = \binom{9}{2}$. By symmetry, clearly $N(P_2) = N(P_3) = \binom{9}{2}$.

$N(P_1P_2)$ is the number of solutions to

$$x_1 + x_2 + x_3 = 13, \;\; x_1 \geq 6, \;\; x_2 \geq 6, \;\; x_3 \geq 0$$

which is the number of solutions to

$$(x_1 - 6) + (x_2 - 6) + x_3 = 13 - 6 - 6, \;\; x_1 - 6 \geq 0, \;\; x_2 - 6 \geq 0, \;\; x_3 \geq 0$$

which is the number of solutions to

$$x_1' + x_2' + x_3 = 1, \;\; x_1' \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0$$

which is $\binom{1+2}{1!2!} = \binom{3}{1}$. By symmetry, clearly $N(P_1P_3) = N(P_2P_3) = \binom{3}{1}$.

$N(P_1P_2P_3)$ is the number of solutions to

$$x_1 + x_2 + x_3 = 13, \;\; x_1 \geq 6, \;\; x_2 \geq 6, \;\; x_3 \geq 6$$

which is the number of solutions to

$$(x_1 - 6) + (x_2 - 6) + (x_3 - 6) = 13 - 6 - 6 - 6, \;\; x_1 - 6 \geq 0, \;\; x_2 - 6 \geq 0, \;\; x_3 - 6 \geq 0$$

which is the number of solutions to

$$x_1' + x_2' + x_3' = -5, \;\; x_1' \geq 0, \;\; x_2' \geq 0, \;\; x_3' \geq 0$$

which is clearly 0.

Therefore

$$\begin{aligned} N(P_1'P_2'P_3') &= \binom{15}{2} - 3\binom{9}{2} + 3\binom{3}{1} - 0 \\ &= \frac{15 \cdot 14}{2} - 3 \cdot \frac{8 \cdot 7}{2} + 3 \cdot 3 - 0 \\ &= 6 \end{aligned}$$

The number of solutions is 6.

ANSWER: $\boxed{6}$ $\square$

Q4. Rosen 8th edition, section 8.6, question 4.

SOLUTION. $N = \binom{17+3}{17} = 1140$

$N(P1) = numberofsolutionswithx1 >= 4 = \binom{16}{13} = 560$

$N(P2) = numberofsolutionswithx2 >= 5 = \binom{15}{12} = 455$

$N(P3) = numberofsolutionswithx3 >= 6 = \binom{14}{11} = 364$

$N(P4) = numberofsolutionswithx4 >= 9 = \binom{11}{8} = 165$

$N(P1P2) = \binom{11}{8} = 165$

$N(P1P3) = \binom{10}{7} = 120$

$N(P1P4) = \binom{7}{4} = 35$

$N(P2P3) = \binom{9}{6} = 84$

$N(P2P4) = \binom{6}{3} = 20$

$N(P3P4) = \binom{4}{2} = 6$

$N(P1P2P3) = \binom{4}{2} = 6$

$N(P1P2P4) = 0$

$N(P2P3P4) = 0$

$N(P1P2P3P4) = 6$

$1140 - (560 + 455 + 354 + 165) + (165 + 120 + 35 + 84 + 20 + 6) - 6$

$\boxed{20}$

Q5. Rosen 8th edition, section 8.6, question 5.

SOLUTION.

Let $U = \{1, 2, 3, ..., 200\}$. Note that $\lfloor\sqrt{200}\rfloor = \lfloor 14.14...\rfloor = 14$. The primes less than or equal to 14 are 2, 3, 5, 7, 11, 13.

First we count the number of integers in U which are not divisible by 2, 3, 5, 7, 11, 13. Define

$$P_p(x) = (x \text{ divisible by } p)$$

for $p = 2, 3, 5, 7, 11$, and 13. We want to compute $N(P_2'P_3'P_5'P_7'P_{11}'P_{13}')$. The required number is then

$$N(P_2'P_3'P_5'P_7'P_{11}'P_{13}') + |\{2, 3, 5, 7, 11, 13\}| - 1 = N(P_2'P_3'P_5'P_7'P_{11}'P_{13}') + 5$$

By the principle of inclusion-exclusion,

$$\begin{aligned}
&N(P_2'P_3'P_5'P_7'P_{11}'P_{13}') \\
&= 200 \\
&\quad - \left(\left\lfloor\frac{200}{2}\right\rfloor + \left\lfloor\frac{200}{3}\right\rfloor + \left\lfloor\frac{200}{5}\right\rfloor + \left\lfloor\frac{200}{7}\right\rfloor + \left\lfloor\frac{200}{11}\right\rfloor + \left\lfloor\frac{200}{13}\right\rfloor\right) \\
&\quad + \left(\left\lfloor\frac{200}{2\cdot 3}\right\rfloor + \left\lfloor\frac{200}{2\cdot 5}\right\rfloor + \left\lfloor\frac{200}{2\cdot 7}\right\rfloor + \left\lfloor\frac{200}{2\cdot 11}\right\rfloor + \left\lfloor\frac{200}{2\cdot 13}\right\rfloor + \left\lfloor\frac{200}{3\cdot 5}\right\rfloor + \left\lfloor\frac{200}{3\cdot 7}\right\rfloor + \left\lfloor\frac{200}{3\cdot 11}\right\rfloor\right. \\
&\qquad \left. + \left\lfloor\frac{200}{3\cdot 13}\right\rfloor + \left\lfloor\frac{200}{5\cdot 7}\right\rfloor + \left\lfloor\frac{200}{5\cdot 11}\right\rfloor + \left\lfloor\frac{200}{5\cdot 13}\right\rfloor + \left\lfloor\frac{200}{7\cdot 11}\right\rfloor + \left\lfloor\frac{200}{7\cdot 13}\right\rfloor + \left\lfloor\frac{200}{11\cdot 13}\right\rfloor\right) \\
&\quad - \left(\left\lfloor\frac{200}{2\cdot 3\cdot 5}\right\rfloor + \left\lfloor\frac{200}{2\cdot 3\cdot 7}\right\rfloor + \left\lfloor\frac{200}{2\cdot 3\cdot 11}\right\rfloor + \left\lfloor\frac{200}{2\cdot 3\cdot 13}\right\rfloor + \left\lfloor\frac{200}{2\cdot 5\cdot 7}\right\rfloor + \left\lfloor\frac{200}{2\cdot 5\cdot 11}\right\rfloor\right. \\
&\qquad + \left\lfloor\frac{200}{2\cdot 5\cdot 13}\right\rfloor + \left\lfloor\frac{200}{3\cdot 5\cdot 7}\right\rfloor + \left\lfloor\frac{200}{3\cdot 5\cdot 11}\right\rfloor + \left\lfloor\frac{200}{3\cdot 5\cdot 13}\right\rfloor + \left\lfloor\frac{200}{3\cdot 7\cdot 11}\right\rfloor + \left\lfloor\frac{200}{3\cdot 7\cdot 13}\right\rfloor \\
&\qquad \left. + \left\lfloor\frac{200}{3\cdot 11\cdot 13}\right\rfloor + \left\lfloor\frac{200}{5\cdot 7\cdot 11}\right\rfloor + \left\lfloor\frac{200}{5\cdot 11\cdot 13}\right\rfloor\right)
\end{aligned}$$

Note that the other terms are 0 since

$$\left\lfloor\frac{200}{2\cdot 3\cdot 5\cdot 7}\right\rfloor = \left\lfloor\frac{200}{210}\right\rfloor = 0$$

Hence we have

$$\begin{aligned}
N(P_2'P_3'P_5'P_7'P_{11}'P_{13}') &= 200 - 267 + 132 - 24 \\
&= 41
\end{aligned}$$

Therefore the number of primes less than or equal to 200 is $41 + 5 = 46$.

ANSWER: $\boxed{46}$ $\square$

Note. You can (and should) of course write a simple function to count primes up to 200:

```
>>> def isprime(n):
...     for d in range(2, n):
...         if n % d == 0: return False
...     return True
...
>>> count = 0
>>> for x in range(2, 201):
...     if isprime(x): count += 1
...
>>> print(count)
46
```

The computation of the terms in the inclusion-exclusion can of course be done with a program too.

```
>>> p = [2,3,5,7,11,13]
>>> for x in p:
...     s += 200/x
...
>>> print(s)
267
>>> s = 0
>>> for i in range(6):
...     for j in range(i+1,6):
...         s += 200/(p[i] * p[j])
...
>>> print(s)
132
>>> s = 0
>>> for i in range(6):
...     for j in range(i+1,6):
...         for k in range(j+1,6):
...             s += 200/(p[i] * p[j] * p[k])
...
>>> print(s)
24
>>> s = 0
>>> for i in range(6):
...     for j in range(i+1,6):
...         for k in range(j+1,6):
```

```
...             for l in range(l+1,6):
...                 s += 200/(p[i] * p[j] * p[k] * p[l])
...
>>> print(s)
0
>>> 200 - 267 + 132 - 24
41
```

If you want to checkout the primes you can do this:

```
>>> ps = [x for x in range(2,201) if isprime(x)]
>>> print(len(ps))
46
>>> for x in ps:
...     print(x)
...
2
3
5
7
11
13
17
19
23
29
31
37
41
43
47
53
59
61
67
71
73
79
83
89
97
101
103
107
109
113
127
131
137
139
```

```
149
151
157
163
167
173
179
181
191
193
197
199
>>>
```

Q6. Rosen 8th edition, section 8.6, question 6.

An integer is called squarefree if it is not divisible by the square of a positive integer greater than 1. Find the number of squarefree positive integers less than 100.

SOLUTION.

99

$-(\left\lfloor \frac{99}{2^2} \right\rfloor + \left\lfloor \frac{99}{3^2} \right\rfloor + \left\lfloor \frac{99}{5^2} \right\rfloor + \left\lfloor \frac{99}{7^2} \right\rfloor)$

$+\left\lfloor \frac{99}{2^2 \cdot 3^2} \right\rfloor$

$99 - (24 + 11 + 3 + 2) + (2)$

$\boxed{61}$

Q7. Rosen 8th edition, section 8.6, question 7.

SOLUTION.

[HINT: Let $U = \{x \in \mathbb{Z} \mid 2 \leq x \leq 10000\}$; the integer 1 is excluded from U since 1 is a second power, third power, fourther power, etc. Define $A_1 = \{x \in U \mid x \text{ is a second power}\}$, $A_2 = \{x \in U \mid x \text{ is a third power}\}$, $A_3 = \{x \in U \mid x \text{ is a fourth power}\}$, etc. You hope that these sets and their intersections are easier to count. If this is the case, then the inclusion-exclusion principle can be used. For instance in the case of A_1, we have $A_1 = \{2^2, 3^2, ..., 100^2\}$. Therefore $|A_1| = 99$.]

Note that we only need to consider prime powers. For instance the integer 16 is a fourth-power and also a second-power.

Let $U = \{2, 3, \ldots, 10000\}$ and

$$\begin{aligned}
P_2(x) &= (x \text{ is a second power}) \\
P_3(x) &= (x \text{ is a third power}) \\
P_5(x) &= (x \text{ is a 5-th power}) \\
P_7(x) &= (x \text{ is a 7-th power}) \\
P_{11}(x) &= (x \text{ is a 11-th power})
\end{aligned}$$

(Note: I'm excluding 1 because 1 is a d–th power for all positive integer d which means that if 1 is included in U, then all the P_i will be non-empty and the there will be many non-zero terms in the inclusion-exclusion computation. To force many terms to be zero, I prefer to remove 1 first from U right at the beginning. If you don't, you will still get the same answer.)

Note that the d–th powers are

$$1^d, 2^d, 3^d, ...$$

The number of integer d–th power ≤ 10000, is

$$\lfloor 10000^{1/d} \rfloor$$

Note that

$$\begin{aligned}
10000^{1/d} < 2 &\iff 10000 < 2^d \\
&\iff \log_2 10000 < d \\
&\iff \log_2 10000 < d \\
&\iff 13.287712... < d
\end{aligned}$$

Therefore for $d \geq 14$, the only integer which is a d–th power is 1. Hence, on excluding 1, the number of integers in U which are d–th powers is

$$\lfloor 10000^{1/d} \rfloor - 1$$

and if 0 if $d > 13$.

By the inclusion-exclusion principle

$$
\begin{aligned}
&N(P_2P_3P_5P_7P_{11}) \\
&= |U| \\
&\quad - (\lfloor 10000^{1/2} \rfloor - 1 + \lfloor 10000^{1/3} \rfloor - 1 + \lfloor 10000^{1/5} \rfloor - 1 + \lfloor 10000^{1/7} \rfloor - 1 + \lfloor 10000^{1/11} \rfloor - 1+ \\
&\qquad + \lfloor 10000^{1/13} \rfloor - 1) \\
&\quad + (\lfloor 10000^{1/(2\cdot 3)} \rfloor - 1 + \lfloor 10000^{1/(2\cdot 5)} \rfloor - 1) \\
&= 9999 - (99 + 20 + 5 + 2 + 1 + 1) + (3 + 1) \\
&= 9875
\end{aligned}
$$

Hence the number of integers from 1 to 10000 which are d–powers for some $d > 1$ is 9875.

ANSWER: $\boxed{9875}$ □

Note. Here's a quick-and-dirty check (i.e. it's not efficient). The program computes the d–th powers which are at most 10000 for $d > 1$

```
xs = []
for x in range(1, 10001):
    for d in range(2, 15):
        y = x**d
        if y <= 10000 and y not in xs:
            xs.append(y)
xs.sort()
print(len(xs))
print(xs)
```

The number of integers in `xs` is 125. Therefore the non-powers are $10000 - 125 = 9875$.

Q8. Rosen 8th edition, section 8.6, question 8.

SOLUTION.

Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$. A function $f : X \to Y$ is onto if

$$\begin{aligned}
&y_1 \text{ is in the image of } f\\
&y_2 \text{ is in the image of } f\\
&y_3 \text{ is in the image of } f\\
&y_4 \text{ is in the image of } f\\
&y_5 \text{ is in the image of } f
\end{aligned}$$

where the image of f is the set of values attained by f, i.e. the image of f is $\{f(x_1), \ldots, f(x_7)\}$.

Define U to be the set of all functions from X to Y and

$$\begin{aligned}
P_1(f) &= (y_1 \text{ is not in the image of } f)\\
P_2(f) &= (y_2 \text{ is not in the image of } f)\\
P_3(f) &= (y_3 \text{ is not in the image of } f)\\
P_4(f) &= (y_4 \text{ is not in the image of } f)\\
P_5(f) &= (y_5 \text{ is not in the image of } f)
\end{aligned}$$

Hence the required number is

$$N(P_1' P_2' P_3' P_4' P_5')$$

Note that $|U| = 5^7$.

$N(P_i)$ is the set of functions from X to Y where y_i is not in the image of the functions. The number of such functions is 4^7 since, excluding y_i, there are 4 possible values for each of the 7 values of X.

$N(P_i P_j)$ (for $i < j$) is the number of functions from X to Y except that y_i and y_j are not in the image of the functions. This means that for each value of X, there are 3 possible values. Hence there are 3^7 such functions.

Arguing in the same manner,

$$\begin{aligned}
N(P_i P_j P_k) &= 2^7\\
N(P_i P_j P_k P_l) &= 1^7\\
N(P_1 P_2 P_3 P_4 P_5) &= 0^7
\end{aligned}$$

By the inclusion-exclusion principle,

$$\begin{aligned} N(P_1'P_2'P_3'P_4'P_5') &= 5^7 - \binom{5}{1}4^7 + \binom{5}{2}3^7 - \binom{5}{3}2^7 + \binom{5}{4}1^7 - \binom{5}{5}0^7 \\ &= 16800 \end{aligned}$$

The required number is 16800.

ANSWER: $\boxed{16800}$ □

Note. Here's a quick check:

```
>>> x1,x2,x3,x4,x5,x6,x7 = 1,2,3,4,5,6,7
>>> y1,y2,y3,y4,y5 = 1,2,3,4,5
>>> X = [x1,x2,x3,x4,x5]
>>> X = [x1,x2,x3,x4,x5,x6,x7]
>>> Y = [y1,y2,y3,y4,y5]
>>> 5**7
78125
>>> fs = []
>>> for a in Y:
...     for b in Y:
...         for c in Y:
...             for d in Y:
...                 for e in Y:
...                     for f in Y:
...                         for g in Y:
...                             fn = [(x1,a),(x2,b),(x3,c),
                                      (x4,d),(x5,e),(x6,f),(x7,g)]
...                             fs.append(fn)
...
>>> len(fs)
78125
>>> count = 0
>>> for fn in fs:
...     image = []
...     for x,y in fn:
...         if y not in image: image.append(y)
...     image.sort()
...     if image == Y: count += 1
...
>>> print(count)
16800
>>>
```

Q9. Rosen 8th edition, section 8.6, question 9.

SOLUTION.

[HINT: Let the 6 toys be $T = \{t_1, t_2, \ldots, t_6\}$ and the three children be $C = \{c_1, c_2, c_3\}$. Each distribution is a function $f : T \to C$. The fact that each child gets at least one toy is the same as saying the function f is onto.]

The total number of ways is the number of onto functions from a set of 6 elements to a set of 3. Let T be a set of 6 elements and $C = \{c_1, c_2, c_3\}$. let $U = \{f : T \to C\}$. Define

$$\begin{aligned} P_1(f) &= (\text{the image of } f \text{ does not contain } c_1) \\ P_2(f) &= (\text{the image of } f \text{ does not contain } c_2) \\ P_3(f) &= (\text{the image of } f \text{ does not contain } c_3) \end{aligned}$$

We have $|U| = 3^6$.

$N(P_1)$ is the number of function from T to $C - \{c_1\} = \{c_2, c_3\}$. Therefore $N(P_1) = 2^6$. Likewise $N(P_2) = N(P_3) = 2^6$.

$N(P_1P_2)$ is the number of function from T to $C - \{c_1, c_2\} = \{c_3\}$. Therefore $N(P_1P_2) = 1^6$. Likewise $N(P_1P_3) = N(P_2P_3) = 1^6$.

$N(P_1P_2P_3)$ is the number of functions from T to $C - \{c_1, c_2, c_3\} = \emptyset$. There are no such functions. Therefore $N(P_1P_2P_3) = 0$.

Therefore, by the inclusion-exclusion principle, the total number of ways is

$$3^6 - \binom{3}{1}2^6 + \binom{3}{2}1^6 = 729 - 192 + 3 = 540$$

The required number is 540.

ANSWER: $\boxed{540}$ $\square$

Q10. Rosen 8th edition, section 8.6, question 10. In how many ways can eight distinct balls be distributed into three distinct urns if each urn must contain at least one ball?

$3^8 - (C\binom{3}{1} \cdot 2^8) + (C\binom{3}{2} \cdot 1^8)$

SOLUTION.

Q11. Rosen 8th edition, section 8.6, question 11.

SOLUTION.

????

Q12. Rosen 8th edition, section 8.6, question 12.

SOLUTION.

The following lists all permutations of 1234. The derangements are marked with *

```
1234
1243
1324
1342
1423
1432
2134
2143 *
2314
2341 *
2413 *
2431
3124
3142 *
3214
3241
3412 *
3421 *
4123 *
4132
4213
4231
4312 *
4321 *
```

Note that there are 9 derangements.

We can verify that the above exhaustive enumeration is correct with a computation. Let

$$U = \{(x_1, x_2, x_3, x_4) \mid (x_1, x_2, x_3, x_4) \text{ is a permutation of } 1, 2, 3, 4\}$$

and define the following propositional formulas:

$$\begin{aligned}
P_1((x_1, x_2, x_3, x_4)) &= (x_1 = 1) \\
P_2((x_1, x_2, x_3, x_4)) &= (x_2 = 2) \\
P_3((x_1, x_2, x_3, x_4)) &= (x_3 = 3) \\
P_4((x_1, x_2, x_3, x_4)) &= (x_4 = 4)
\end{aligned}$$

The number of derangements is $N(P_1'P_2'P_3'P_4')$. By the inclusion-exclusion principle

$$\begin{aligned}
N(P_1'P_2'P_3'P_4') &= 4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3}1! + \binom{4}{4}0! \\
&= 4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3}1! + \binom{4}{4}0! \\
&= 24 - 4 \cdot 6 + 6 \cdot 2 - 4 \cdot 1 + 1 \\
&= 9
\end{aligned}$$

Q13. Rosen 8th edition, section 8.6, question 13.

SOLUTION.

The required number is

$$\begin{aligned} D_7 &= 7!\sum_{k=0}^{7}(-1)^k\frac{1}{k!} \\ &= 5040\left(1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}-\frac{1}{5040}\right) \\ &= 1854 \end{aligned}$$

Answer: $\boxed{1854}$ □

Q14. Rosen 8th edition, section 8.6, question 14.

SOLUTION.

The probability is

$$\frac{D_{10}}{10!} = \frac{10!}{10!}\sum_{k=0}^{10}(-1)^k\frac{1}{k!} = \sum_{k=0}^{10}(-1)^k\frac{1}{k!} \simeq 0.3678$$

Answer: $\boxed{\text{approximately } 0.3678}$ □

Q15. Rosen 8th edition, section 8.6, question 15.

SOLUTION.

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Q16. Rosen 8th edition, section 8.6, question 16.

SOLUTION.

This is the number of derangements of n symbols, i.e., the required number is

$$D_n = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

$\square$

Q17. Rosen 8th edition, section 8.6, question 17.

(Show all work clearly.)

How many ways can the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 be arranged so that no even digit is in its original position?

Digits 0,1,2,3,4,5,6,7,8,9 can be arranged 10! ways.

There are 5 even numbers in the list.

$10! - \binom{5}{1}9! + \binom{5}{2}8! - \binom{5}{3}7! + \binom{5}{4}6! - \binom{5}{5}5!$

$3628800 - 181440 + 403200 - 50400 + 3600 - 120$

$\boxed{2170680}$

SOLUTION.

Q18. Rosen 8th edition, section 8.6, question 22.

SOLUTION.

One can prove this using principle of inclusion-exclusion or we can use Euler's theorem on *phi*-function (question 23 or from class notes) to obtain

$$\phi(pq) = \phi(p)\phi(q) = (p-1)(q-1)$$

□

Q19. Rosen 8th edition, section 8.6, question 23.

SOLUTION.

Euler's theorem for the Euler ϕ–function was proven in class. $\square$

Instructions

In `main.tex` change the email address in

```
\renewcommand\AUTHOR{jdoe5@cougars.ccis.edu}
```

yours. In the bash shell, execute "`make`" to recompile `main.pdf`. Execute "`make v`" to view `main.pdf`. Execute "`make s`" to create `submit.tar.gz` for submission.

For each question, you'll see boxes for you to fill. You write your answers in `main.tex` file. For small boxes, if you see

```
1 + 1 = \answerbox{}.
```

you do this:

```
1 + 1 = \answerbox{2}.
```

`answerbox` will also appear in "true/false" and "multiple-choice" questions.

For longer answers that needs typewriter font, if you see

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
\end{answercode}
```

you do this:

```
Write a C++ statement that declares an integer variable name x.
\begin{answercode}
int x;
\end{answercode}
```

`answercode` will appear in questions asking for code, algorithm, and program output. In this case, indentation and spacing is significant. For program output, I do look at spaces and newlines.

For long answers (not in typewriter font) if you see

```
What is the color of the sky?
\begin{answerlong}
\end{answerlong}
```

you can write

```
What is the color of the sky?
\begin{answerlong}
The color of the sky is blue.
\end{answerlong}
```

For students beyond 245: You can put LaTeX commands in `answerbox` and `answerlong`.

A question that begins with "T or F or M" requires you to identify whether it is true or false, or meaningless. "Meaningless" means something's wrong with the statement and it is not well-defined. Something like "$1+_2$" or "$\{2\}^{\{3\}}$" is not well-defined. Therefore a question such as "Is $42 = 1+_2$ true or false?" or "Is $42 = \{2\}^{\{3\}}$ true or false?" does not make sense. "Is $P(42) = \{42\}$ true or false?" is meaningless because $P(X)$ is only defined if X is a set. For "Is $1 + 2 + 3$ true or false?", "$1 + 2 + 3$" is well–defined but as a "numerical expression", not as a "proposition", i.e., it cannot be true or false. Therefore "Is $1 + 2 + 3$ true or false?" is also not a well-defined question.

When writing results of computations, make sure it's simplified. For instance write 2 instead of $1 + 1$. When you write down sets, if the answer is $\{1\}$, I do not want to see $\{1, 1\}$.

When writing a counterexample, always write the simplest.

Here are some examples (see `instructions.tex` for details):

1. T or F or M: $1 + 1 = 2$ .... $\boxed{\text{T}}$

2. T or F or M: $1 + 1 = 3$ .... $\boxed{\text{F}}$

3. T or F or M: $1+^2 =$ .... $\boxed{\text{M}}$

4. $1 + 2 = \boxed{3}$

5. Write a C++ statement to declare an integer variable named `x`.

```
int x;
```

6. Solve $x^2 - 1 = 0$.

   Since $x^2 - 1 = (x - 1)(x + 1)$, $x^2 - 1 = 0$ implies $(x - 1)(x + 1) = 0$. Therefore $x - 1 = 0$ or $x = -1$. Hence $x = 1$ or $x = -1$.

7. Which is true? .... $\boxed{\text{C}}$
   (A) $1 + 1 = 0$
   (B) $1 + 1 = 1$
   (C) $1 + 1 = 2$
   (D) $1 + 1 = 3$
   (E) $1 + 1 = 4$