1 ZMP LQR Riccati Equation

Using $z(t)$ as the 2D position of the ZMP, we formulate:

$$
\begin{aligned}
\underset{u(t)}{\text{minimize}} \quad & \int_0^\infty \|z(t) - z_d(t)\|_2^2 + \|u(t)\|_R^2 \, dt, \\
\text{subject to} \quad & R = R' > 0, \\
& z_d(t) = z_d(t_f), \quad \forall t \geq t_f \\
& \dot{x}(t) = Ax(t) + Bu(t), \quad z(t) = Cx(t) + Du(t) \\
& A = \begin{bmatrix} 0_{2\times 2} & I_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{2\times 2} \\ I_{2\times 2} \end{bmatrix} \\
& C = \begin{bmatrix} I_{2\times 2} & 0_{2\times 2} \end{bmatrix}, \quad D = -\frac{h}{g} I_{2\times 2}
\end{aligned}
$$

This can be rewritten as a cost on state, *in coordinates relative to the final conditions*, $\bar{x} = x - \begin{bmatrix} z_d^T(t_f) & 0 & 0 \end{bmatrix}^T$, $\bar{z}_d(t) = z_d(t) - z_d(t_f)$:

$$
\begin{aligned}
\underset{u(t)}{\text{minimize}} \quad & \int_0^\infty \bar{x}^T Q_1 \bar{x} + \bar{x}^T q_2(t) + q_3(t) + u^T R_1 y + u^T r_2(t) + 2\bar{x}^T N u \\
\text{subject to} \quad & Q_1 = \text{diag}(\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}), \quad q_2(t) = \begin{bmatrix} -2\bar{z}_d(t) \\ 0_{2\times 1} \end{bmatrix}, \quad q_3(t) = \|\bar{z}_d(t)\|_2^2 \\
& R_1 = R + \left(\frac{h}{g}\right)^2 I_{2\times 2}, \quad r_2(t) = 2\bar{z}_d(t)\frac{h}{g}, \quad N = -\frac{h}{g}\begin{bmatrix} I_{2\times 2} \\ 0_{2\times 2} \end{bmatrix} \\
& \dot{\bar{x}}(t) = A\bar{x}(t) + Bu(t) \\
& A = \begin{bmatrix} 0_{2\times 2} & I_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{2\times 2} \\ I_{2\times 2} \end{bmatrix}
\end{aligned}
$$

Note that this implies that $\bar{x}(\infty) = 0$ in order for the cost to be finite.

The resulting cost-to-go is given by

$$J = \bar{x}^T S_1(t)\bar{x} + \bar{x}^T s_2(t) + s_3(t),$$

with the corresponding Riccati differential equation given by

$$
\begin{aligned}
-\dot{S}_1 &= Q_1 - (N + S_1 B)R_1^{-1}(B^T S_1 + N^T) + S_1 A + A^T S_1 \\
-\dot{s}_2 &= q_2(t) - 2(N + S_1 B)R^{-1} r_s(t) + A^T s_2, \quad r_s(t) = \frac{1}{2}(r_2(t) + B^T s_2(t)) \\
-\dot{s}_3 &= q_3(t) - r_s(t)^T R^{-1} r_s(t)
\end{aligned}
$$

Note that S_1 has no time-dependent terms, and therefore $S_1(t)$ is a constant, given by the steady-state solution of the algebraic Riccati equation (e.g. from time-invariant LQR). Similarly, the feedback controller is given by

$$u(t) = -K_1(t)\bar{x} + k_2(t),$$

and again the feedback $K_1(t)$ is a constant (derived from the infinite horizon LQR with Q, R, and N set as above).

1.1 Solving for $s_2(t)$

Given this, the affine terms in the Riccati differential equation are given by the linear differential equations:

$$\dot{s}_2(t) = A_2 s_2(t) + B_2 z_d(t); \quad s_2(t_f) = 0$$

with

$$A_2 = (N + S_1 B)R^{-1}B^T - A^T; \quad B_2 = \begin{bmatrix} 2I_{2\times 2} \\ 0_{2\times 2} \end{bmatrix} + 2\frac{h}{g}(N + S_1 B)R^{-1}$$

Assuming $z_d(t)$ is described by a *continuous* piecewise polynomial of degree k with $n+1$ breaks at t_j (with $t_0 = 0$ and $t_n = t_f$):

$$z_d(t) = \sum_{i=0}^{k} c_{j,i}(t - t_j)^i; \quad \text{for } j = 0, \ldots, n-1, \text{ and } \forall t \in [t_j, t_{j+1}),$$

this system has a closed-form solution given by:

$$s_2(t) = e^{A_2(t-t_j)}\alpha_j + \sum_{i=0}^{k} \beta_{j,i}(t - t_j)^i; \quad \forall t \in [t_j, t_{j+1}),$$

with α_j and $\beta_{j,i}$ vector parameters to be solved for. Taking

$$\begin{aligned}
\dot{s}_2(t) &= A_2 e^{A_2(t-t_j)}\alpha_j + \sum_{i=0}^{k} A_2\beta_{j,i}(t-t_j)^i + \sum_{i=0}^{k} B_2 c_{j,i}(t-t_j)^i \\
&= A_2 e^{A_2(t-t_j)}\alpha_j + \sum_{i=1}^{k} i\beta_{j,i}(t-t_j)^{i-1}
\end{aligned}$$

forces that

$$\begin{aligned}
A_2\beta_{j,i} + B_2 c_{j,i} &= (i+1)\beta_{j,i+1}, \quad \text{for } i = 0, \ldots, k-1 \\
A_2\beta_{j,k} + B_2 c_{j,k} &= 0.
\end{aligned}$$

Note: need to prove that A_2 is full rank (it appears to be in practice). Solve backwards ($i = k, k-1, \ldots, 0$) for $\beta_{j,i}$. Finally, the continuity and the terminal boundary condition $s(t_f) = 0$ gives

$$e^{A(t_{j+1}-t_j)}\alpha_j + \sum_{i=0}^{k} \beta_{j,i}(t_{j+1} - t_j)^{i+1} = s(t_{j+1}).$$

1.2 Reading out $k_2(t)$

The remaining term for the controller is a simple read-out given the solution to $s_2(t)$:

$$k_2(t) = \quad \frac{h}{g} R \;\; ^1 z_d(t) \quad \frac{1}{2} R \;\; ^1 B^T s_2(t)$$

which can be written as

$$k_2(t) = \quad {}_L e^{A_2(t \quad t_j)} \quad {}_{j;R} + \sum_{i=0}^{k} \quad {}_{j;i}(t \quad t_j)^i$$

with

$$\begin{aligned} {}_L &= \quad \frac{1}{2} R \;\; ^1 B^T \\ {}_R &= \quad {}_{j;R} = \quad {}_j \\ {}_{j;i} &= \quad \frac{h}{g} R \;\; ^1 c_{j;i} \quad \frac{1}{2} R \;\; ^1 B^T \quad {}_{j;i} \end{aligned}$$

1.3 Solving for $x_{com}(t)$

The resulting system is

$$\dot{x} = Ax + B\,(K_1 x + k_2(t)) = (A + BK_1)x + Bk_2(t);$$

where $x = [x_{com}; y_{com}; \dot{x}_{com}; \dot{y}_{com}]^T$: Since the solution $k_2(t)$ is the result of another linear system (cascaded in front of this one), it is easiest for me to solve jointly, using $y = \begin{matrix} x \\ s_2 \end{matrix}$:

$$\dot{y} = A_y y + B_y z_d$$

$$A_y = \quad \begin{matrix} A + BK_1 & \frac{1}{2} BR \;\; ^1 B^T \\ 0 & A_2 \end{matrix} \quad ; \quad B_y = \quad \begin{matrix} \frac{h}{g} BR \;\; ^1 \\ B_2 \end{matrix}$$

$$y(t) = e^{A_y(t \quad t_j)} a_j + \sum_{i=0}^{k} b_{j;i}(t \quad t_j)^i$$

i can solve for b using the same technique as above (and re-using the sol),