Presentation

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Outline

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1. Value of an European option
   - Basic concepts
   - Underlying assumptions
2. Calculus of the option value using FFT
   - Expression of the option value
   - Trapezoid rule
   - Simpson's rule
3. Future goals

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What is an European value ?

Definition

European option : In finance, it is a contract which gives the buyer (the owner) the right, to buy or sell a stock at a specified strike price on a specified date. This date T is called the maturity of the option.

We must also give the following definitions :

- Stock price : Current price of the stock.
- Spot price : Expected price of the stock on a given date.

The value of an option

An option can be exchanged on the market like a stock. Therefore, there is an option value. This value is defined by a relation in which the spot price intervenes.

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Assumptions

- We consider the Black-Scholes model.
- The characteristic function of the log terminal sport price is known.
- Initially we have $S_0 = 1$
- The interest rate is constant

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Initial call value

Definition of the call value

$$C_T(k) = \int_k^{\infty} e^{-rT}(e^s - e^k)q_T(s)ds \tag{1}$$

- q_T the risk-neutral density of the log spot price
- k the log stike price of the stock
- r the interest rate

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Transform of the call value

As we want a square-integrable function we consider

$$c_T(k) = exp(\alpha k) C_T(k)$$

That leads to the following equation

More useful expression of call value

$$C_T(k) = \frac{exp(-\alpha k)}{\pi} \int_0^{\infty} e^{-i\nu k} \psi(\nu) d\nu \qquad (2)$$

Where ψ is the Fourier transform of c_T.

The purpose of the FFT

We would like to use the FFT to compute quickly the call value given by the expression (2). In order to do that, we should write $C_T(k)$ like the following sum

$$\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j)$$

To obtain a such expression, we try two different methods. First, we approach the integral in (2) by the trapezoid method. Then we will consider the Simpson's method.

Trapezoid rule

Final result for $C_T(k)$

$$C_T(k) \approx \frac{exp(-\alpha k)}{\pi} \sum_{i=1}^{N} e^{-i\nu_j k} \psi_T(\nu_j)\eta \quad (3)$$

Demonstration:

$$C_T(k) = \lim_{a \to \infty} \frac{exp(-\alpha k)}{\pi} \int_0^a e^{-i\nu k} \psi_T(\nu) d\nu$$

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Trapezoid rule

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Demonstration:

$$C_T(k) = \lim_{a \to \infty} \frac{exp(-\alpha k)}{\pi} \int_0^a e^{-i\nu k} \psi_T(\nu) d\nu$$

Thanks to the Trapezoidal rule, we can write:

$$C_T(k) = \lim_{a \to \infty} \frac{exp(-\alpha k)}{\pi} \frac{a}{N} \sum_{j=2}^{N} e^{-ik(j-1)\frac{a}{N}} \psi_T((j-1)\frac{a}{N})$$

$$+ \frac{a}{2N}(\psi_T(0) + e^{-ika}\psi_T(a))$$

Trapezoid rule

Final result for $C_T(k)$

$$C_T(k) \approx \frac{exp(-\alpha k)}{\pi} \sum_{i=1}^{N} e^{-i\nu_j k} \psi_T(\nu_j)\eta \tag{3}$$

Demonstration:

And because $\frac{a}{N} = \eta$ and $\lim_{a \to \infty} \psi_T(a) = 0$, we have:

$$C_T(k) = \frac{exp(-\alpha k)}{\pi} \frac{a}{N} \sum_{i=1}^{N} e^{-i\nu_j k} \psi_T(\nu_j)\eta + \epsilon$$

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Simpson's rule

Final result for $C_T(k)$

$$C_T(k) \approx \frac{exp(-\alpha k)}{\pi} \sum_{i=1}^{N} e^{-i\nu_j k} \psi_T(\nu_j) \frac{\eta}{3} (3 + (-1)^j - \delta_{j-1}) \qquad (4)$$

Demonstration:

Thanks to the Simpson's rule, we get

$$C_T(k) = \lim_{a \to \infty} \frac{exp(-\alpha k)}{\pi} \int_0^a e^{-i\nu k} \psi_T(\nu) d\nu$$

Simpson's rule

Final result for $C_T(k)$

$$C_T(k) \approx \frac{exp(-\alpha k)}{\pi} \sum_{i=1}^{N} e^{-i\nu_j k} \psi_T(\nu_j) \frac{\eta}{3}(3 + (-1)^j - \delta_{j-1}) \quad (4)$$

Demonstration:

$$\int_0^a e^{-i\nu k} \psi_T(\nu) d\nu = \frac{2\eta}{6} \sum_{j=0}^{N-1} e^{-ik2j\frac{\eta}{2}} \psi_T(2j\frac{\eta}{2}) + \frac{4\eta}{6} \sum_{j=0}^{N-1} e^{-ik(2j+1)\frac{\eta}{2}} \psi_T(2j+1)\frac{\eta}{2} + \frac{\eta}{6}(\psi_T(0) + e^{-ika}\psi_T(a))$$

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Simpson's rule

Final result for $C_T(k)$

$$C_T(k) \approx \frac{exp(-\alpha k)}{\pi} \sum_{i=1}^{N} e^{-i\nu_j k} \psi_T(\nu_j) \frac{\eta}{3}(3 + (-1)^j - \delta_{j-1}) \quad (4)$$

Demonstration:
Then we replace $\frac{\eta}{2}$ with η' and therefore N by $\frac{N'}{2}$, and we get the result by defining $\epsilon = \frac{\eta}{2}\psi_T(0)$

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Future goals

- We can main compute the call value using the FFT.
- We can compare the fact that Simpson's rule is more powerful the Trapezoidal rule.
- We will use this two different rules to evaluate the call value with different characteristic functions, for instance the Black-Scholes model or the VG model, and we will compare them to each other and to the result we have when we calculate the Fourier transform of the different models.

Bibliography I

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