ELECROMAGNETIC THEORY FORMULA SHEET

CONTENTS

PART-A (CORE)

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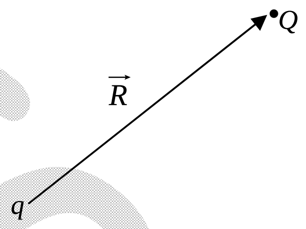
PART 'B' ADVANCED

1. Electrostatics

The electric field at any point due to stationary source charges is called as electrostatic field.

1.1 Coulomb's Law and Superposition Principle

The electric force on a test charge Q due to a single point charge q, which is at rest and a distance R apart is given by Coulomb's law



$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{R^2}\hat{R}.$$

The constant ε_0 is called the *permittivity of free space*.

In mks units, $\varepsilon_0 = 8.85\times 10^{-12}\,\frac{C^2}{N.m^2}$

1.1.1 Electric field

If we have many ***point charges*** $q_1, q_2, ...$ at distances $R_1, R_2, ...$ from test charge Q, then according to the **principle of superposition** the total force on Q is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + ... = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{R_1^2}\hat{R}_1 + \frac{q_2 Q}{R_2^2}\hat{R}_2 + ...\right),$$

$$\Rightarrow \vec{F} = Q\vec{E} \quad \text{where} \quad \vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i^2}\hat{R}_i$$

$\vec{E}$ is called the **electric field** of the source charges. Physically $\vec{E}(P)$ is the force per unit charge that would be exerted on a test charge placed at P.

If ***charge is distributed continuously*** over some region, then

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int_{line}\frac{1}{R^2}\hat{R}\,dq\,.$$

The electric field of a line charge is $(dq = \lambda dl')$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int_{line}\frac{\lambda(r')}{R^2}\hat{R}\,dl' \quad \text{where } \lambda \text{ is charge per unit length.}$$

For surface charge $(dq = \sigma da')$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int_{surface} \frac{\sigma(r')}{R^2} \hat{R} da' \quad \text{where } \sigma \text{ is charge per unit area.}$$

For a volume charge $(dq = \rho d\tau')$

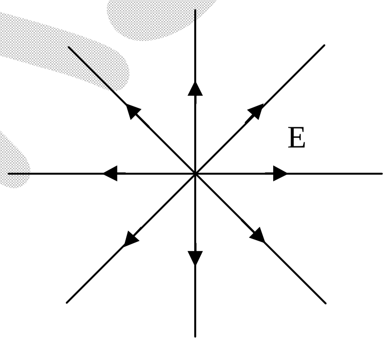
$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int_{volume} \frac{\rho(r')}{R^2} \hat{R} d\tau' \quad \text{where } \rho \text{ is charge per unit volume.}$$

1.2 Gauss's law

1.2.1 Field lines and Electric flux

Consider that a point charge q is situated at the origin: $\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

This field is represented by the **field line** as shown in figure below. The magnitude of the field is indicated by the density of the field lines: it's strong near the center where the field line are close together, and weak farther out, where they are relatively far apart.



The **field strength (*E*)** is proportional to the number of field lines per unit area (area perpendicular to the lines).

The **flux of *E*** through a surface S, $\phi_E = \int_S \vec{E}.d\vec{a}$ is a measure of the "number of field lines" passing through S.

A charge outside the surface would contribute nothing to the total flux, since its field lines go in one side and out the other. It follows, then, that for any closed surface,

$$\oint \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

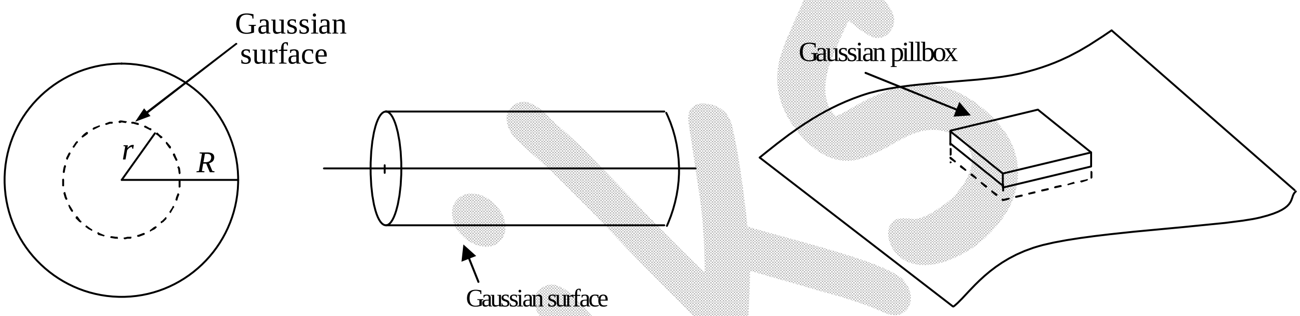
where Q_{enc} is the total charge enclosed within the surface. This is **Gauss's law in integral form.**

Gauss's law in differential form: $\vec{\nabla}.\vec{E} = \frac{1}{\varepsilon_0}\rho$

1.2.2 Applications of Gauss's Law

Gauss's law is always true, but it is not always useful. Gauss's law is useful for only three kinds of symmetry:

1. **Spherical symmetry**. Make your Gaussian surface a concentric sphere.
2. **Cylindrical symmetry**. Make your Gaussian surface a coaxial cylinder.
3. **Plane symmetry**. Make your Gaussian surface a "pillbox," which extends equally above and below the surface.



1.3 Electric Potential

1.3.1 Curl of Electric field

Then integral around a closed path is zero i.e. $\oint \vec{E}.d\vec{l} = 0$

This line integral is independent of path. It depends on two end points.

Applying stokes theorem, we get $\vec{\nabla} \times \vec{E} = 0$. The electric field is not just any vector but only those vector whose curl is zero.

So, we can define a function $V(r) = -\int_{\vartheta}^{P} \vec{E}.d\vec{l}$

where ϑ is some standard reference point. V then depends only on the point r. It is called the **electric potential.**

Evidently, the potential difference between two points a and b is

$$V(b) - V(a) = -\int_a^b \vec{E}.d\vec{l} \quad \Rightarrow \vec{E} = -\vec{\nabla} V$$

Potential obeys the superposition principle.

1.3.2 Potential of localized charges

Potential of a point charge q is $V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$ where R is the distance from the charge.

The potential of a collection of point charge is $V = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i}$.

For continuous volume charge distribution $V(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')}{R}d\tau'$

The potential of line and surface charges are $V(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda(r')}{R}dl'$ and $V(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(r')}{R}da'$.

1.4 Laplace's and Poisson equations

Since $\vec{E} = -\vec{\nabla}V$ and $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$.

This is known as **Poisson's equation**.

In regions where there is no charge, so that $\rho = 0$, Poisson's equation reduces to Laplace's equation, $\nabla^2 V = 0$.

1.5 Electrostatic boundary condition

The boundary between two medium is a thin sheet of surface charge σ .

$$\mathrm{E}_{above}^{\perp} - \mathrm{E}_{below}^{\perp} = \frac{\sigma}{\varepsilon_0} \quad \text{and} \quad E_{above}^{\parallel} = E_{below}^{\parallel} \;\Rightarrow \vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

where $\hat{n}$ is unit vector perpendicular to the surface, pointing upward.

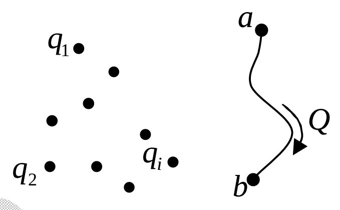
$$V_{above} = V_{below} \quad \Rightarrow \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

where $\frac{\partial V}{\partial n} = \vec{\nabla}V \cdot \hat{n}$ denotes the normal derivative of V (that is the rate of change in the direction perpendicular to the surface.)

1.6 Work and Energy in electrostatics

The work done in moving a test charge Q in an external field $\vec{E}$, from point a to b is

$$W = \int_a^b \vec{F} \cdot d\vec{l} = -Q \int_a^b \vec{E} \cdot d\vec{l} = Q\left[V(b) - V(a)\right]$$



If $a = \infty$ and $b = r$

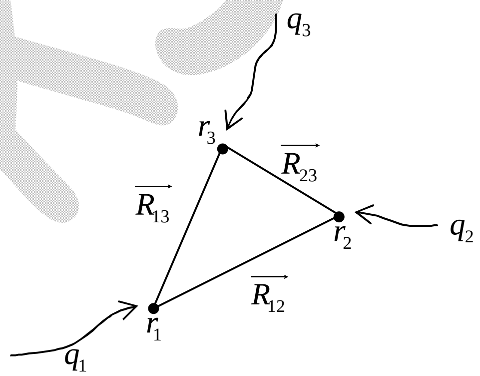
$$\Rightarrow W = Q\left[V(r) - V(\infty)\right] = QV(r) \text{ since } V(\infty) = 0$$

In this sense **potential** is potential energy (the work it takes to create the system) per unit charge (just as the field is the force per unit charge).

1.6.1 The energy of point charge distribution

When the first charge q_1 is placed, no work has been done. When q_2 is placed work done $W_2 = q_2 V_1$ where V_1 is the potential due to q_1 so,

$$W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{R_{12}}\right).$$



Similarly when third charge q_3 is placed

$$W_3 = \frac{1}{4\pi\varepsilon_0} q_3 \left(\frac{q_1}{R_{13}} + \frac{q_2}{R_{23}}\right).$$

The work necessary to assemble the first three charges is $W = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{R_{12}} + \frac{q_1 q_3}{R_{13}} + \frac{q_2 q_3}{R_{23}}\right)$.

In general, $W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{R_{ij}} = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{R_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i V(r_i)$,

where $V(r_i)$ is the potential at point r_i (the position of q_i) due to all other charges.

1.6.2 Energy of continuous charge distribution

For a volume charge density $W = \frac{1}{2}\int \rho V d\tau$ and $W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau$

(i) Energy of a uniformly charged spherical shell of total charge q and radius R is $W = \frac{q^2}{8\pi\varepsilon_0 R}$.

(ii) Energy stored in a uniformly charged solid sphere of radius R and charge q is $W = \frac{1}{4\pi\varepsilon_0}\frac{3q^2}{5R}$.

1.7 Basic properties Conductors

1. $\vec{E} = 0$ inside a conductor.
2. $\rho = 0$ inside a conductor.
3. Any net charge resides on the surface.
4. A conductor is an equipotential.
5. $\vec{E}$ is perpendicular to the surface, just outside a conductor.

Because the field inside a conductor is zero, boundary condition $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ requires that the field immediately outside is $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$.

In terms of potential equation $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$ yields $\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$

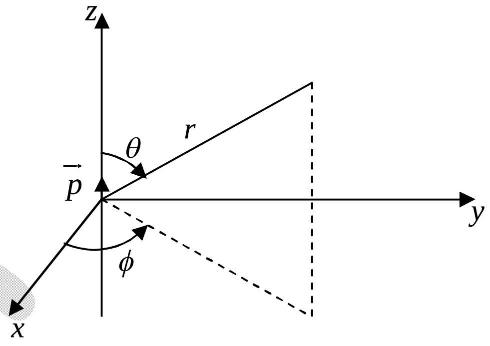
These equations enable us to calculate the surface charge on a conductor, if we can determine $\vec{E}$ or V.

Force per unit area on the conductor is $\vec{f} = \frac{1}{2\varepsilon_0}\sigma^2 \hat{n}$.

This amounts to an outwards *electrostatic pressure* on the surface, tending to draw the conductor into the field, regardless the sign of σ. Expressing the pressure in terms of the field just outside the surface, $P = \frac{\varepsilon_0}{2}E^2$.

1.8 Multiple Expansions

1.8.1 The Electric Potential and Field of a Dipole



If we choose coordinates so that $\vec{p}$ (dipole moment) lies at the origin and points in the z direction, then potential at (r,θ) is:

$$V_{dip}(r,\theta)=\frac{\hat{r}.\vec{p}}{4\pi\varepsilon_o r^2}=\frac{p\cos\theta}{4\pi\varepsilon_o r^2}.$$

$$\Rightarrow \vec{E}_{dip}(r,\theta)=\frac{1}{4\pi\varepsilon_0 r^3}\left[3\left(\vec{p}\cdot\hat{r}\right)\hat{r}-\vec{p}\right]$$

Note:

(a) When a dipole is placed in a uniform electric field ($\vec{E}$), net force on the dipole is zero and it experiences a torque $\vec{\tau}=\vec{p}\times\vec{E}$ where $\vec{p}=q\vec{d}$.

(b) In non-uniform field, dipoles have net force $\vec{F}=\left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}$ and torque $\vec{\tau}=\vec{p}\times\vec{E}$.

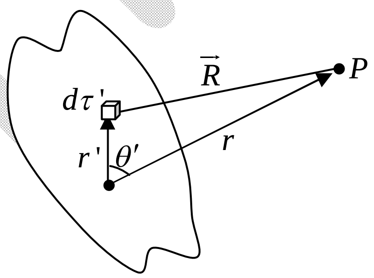
(c) Energy of an ideal dipole $\vec{p}$ in an electric field $\vec{E}$ is $U=-\vec{p}.\vec{E}$.

(d) Interaction energy of two dipoles separated by a distance $\vec{r}$ is

$$U=\frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p_1}\cdot\vec{p_2}-3\left(\vec{p_1}\cdot\hat{r}\right)\left(\vec{p_2}\cdot\hat{r}\right)\right].$$

1.8.2 Approximate potential at large distances

Approximate potential at large distances due to arbitrary localized charge distribution



$$V(r)=\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int\rho\left(r'\right)d\tau'+\frac{1}{r^2}\int r'\cos\theta'\rho\left(r'\right)d\tau'+\frac{1}{r^2}\int\left(r'\right)^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)\rho\left(r'\right)d\tau'+...\right]$$

The first term (n = 0) is the monopole contribution (it goes like $\frac{1}{r}$). The second term (n = 1) is the dipole term (it goes like $\frac{1}{r^2}$). The third term is quadrupole; the fourth octopole and so on.

The lowest nonzero term in the expansion provides the approximate potential at large r and the successive terms tell us how to improve the approximation if greater precision is required.

The monopole and dipole terms

Ordinarily, the multipole expansion is dominated (at large r) by the monopole term:

$$V_{mon}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}.$$

where $Q = \int \rho d\tau$ is the total charge of the configuration.

If the total charge is zero, the dominant term in the potential will be the dipole (unless, of course, it also vanishes):

$$V_{dip}(r) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r' \cos\theta' \rho(r') d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}.\int \vec{r'} \rho(r') d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}.\vec{p}}{r^2},$$

where ***dipole moment*** $\vec{p} = \int \vec{r'} \rho(r') d\tau'$.

The dipole moment is determined by the geometry (size, shape and density) of the charge distribute. The dipole moment of a collection of point charge is

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r_i'}.$$

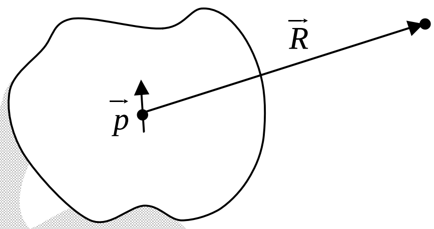
Note: Ordinarily, the dipole moment does change when we shift the origin, but there is an important exception: If the total charge is zero, then the dipole moment is independent of the choice of origin.

1.9 Polarization

$$\text{(Polarization)}\ \vec{P} \equiv \text{ dipole moment per unit volume}$$

1.9.1 The Field of a polarized Object (Bound Charges)

Suppose we have a piece of polarized material with polarization vector $\vec{P}$ containing a lot of microscopic dipoles lined up. Then



$$V\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0} \oiint_S \frac{1}{R} \vec{P}.d\vec{a}' - \frac{1}{4\pi\varepsilon_0} \int_V \left(\frac{1}{R}\right)\left(\vec{\nabla}'.\vec{P}\right) d\tau'.$$

The first term looks like the potential of a ***surface bound charge*** $\sigma_b = \vec{P}.\hat{n}$ (where $\hat{n}$ is the normal unit vector).

The second term looks like the potential of a ***volume bound charge*** $\rho_b = -\vec{\nabla}.\vec{P}$.

Thus $V\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0} \oiint_S \frac{\sigma_b}{R} da' + \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_b}{R} d\tau'$, this means the potential (and hence also the field) of a polarized object is the same as that produced by a volume charge density $\rho_b = -\vec{\nabla}.\vec{P}$ plus a surface charge density $\sigma_b = \vec{P}.\hat{n}$.

1.10 The Electric Displacement

1.10.1 Gauss Law in the Presence of Dielectrics

Within the dielectric, the ***total charge density*** can be written as $\rho = \rho_b + \rho_f$ where ρ_b is volume bound charge ρ_f **free charge** density.

$$\vec{\nabla}.\vec{D} = \rho_f \quad \textit{where} \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$ is known as the **electric displacement.**

Thus Gauss' law reads, $\vec{\nabla}.\vec{D} = \rho_f$

Or, in integral form $\oiint \vec{D}.d\vec{a} = Q_{f_{enc}}$, where $Q_{f_{enc}}$ denotes the total free charge enclosed in the volume.

1.10.2 Linear Dielectrics (Susceptibility, Permittivity, Dielectric Constant)

For any substances, the polarization is proportional to the field provided $\vec{E}$ is not too strong: $\vec{P} \propto \vec{E} \Rightarrow \vec{P} = \varepsilon_0 \chi_e \vec{E}$

(Materials that obey this relation are called linear dielectrics)

The constant of proportionality, χ_e is called the ***electric susceptibility*** of the medium. The value of χ_e depends on the microscopic structure of the substance and also on external conditions such as temperature.

In linear media we have

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 \vec{E}\ (1+\chi_e) = \varepsilon \vec{E}, \quad where \quad \varepsilon = \varepsilon_0 (1+\chi_e)$$

This new constant ε is called the permittivity of the material.

Also $\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0}(1+\chi_e)$ is called ***relative permittivity*** or ***dielectric constant***, of the material.

1.10.3 Boundary Condition on $\vec{D}$

The boundary between two medium is a thin sheet of free surface charge σ_f.

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f \quad \text{and} \quad \vec{D}^{\parallel}_{above} - \vec{D}^{\parallel}_{below} = \vec{P}^{\parallel}_{above} - \vec{P}^{\parallel}_{below}$$

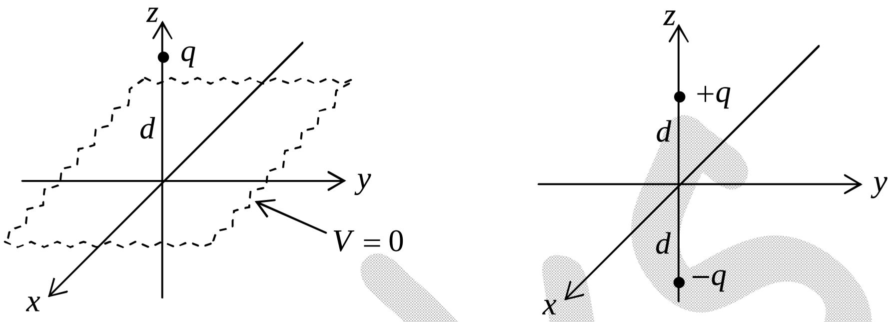
1.10.4 Energy in dielectric system

$$W = \frac{1}{2}\int_{all\,space} \left(\vec{D}\cdot\vec{E}\right) d\tau .$$

1.12 Image problems

1.12.1 The Classic Image problem

Suppose a point charge q is held a distance d above an infinite grounded conducting plane. We can find out what is the potential in the region above the plane.



Forget about the actual problem; we are going to study a *complete different situation*.

The new problem consists of two point charges $+q$ at (0, 0, d) and $-q$ at (0, 0,-d) and no conducting plane.

For this configuration we can easily write down the potential:

$$V(x,y,z)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

(The denominators represent the distances from (x, y, z) to the charges $+q$ and $-q$, respectively.) It follows that

1. $V=0$ when $z=0$ and
2. $V\to 0$ for $x^2+y^2+z^2>>d^2$,

and the only charge in the region $z>0$ is the point charge $+q$ at (0, 0, d). Thus the second configuration produces exactly the same potential as the first configuration, in the upper region $z\geq 0$.

1.12.2 Induced Surface Charge

The surface charge density σ induced on the conductor surface is $\sigma = -\varepsilon_0 \left.\dfrac{\partial V}{\partial z}\right|_{z=0}$

$$\Rightarrow \sigma(x, y) = \frac{-qd}{2\pi\left(x^2 + y^2 + d^2\right)^{3/2}}$$

As expected, the induced charge is negative (assuming q is positive) and greatest at $x=y=0$.

The total induced charge $\quad Q = \int \sigma da = -q$

1.12.3 Force and Energy

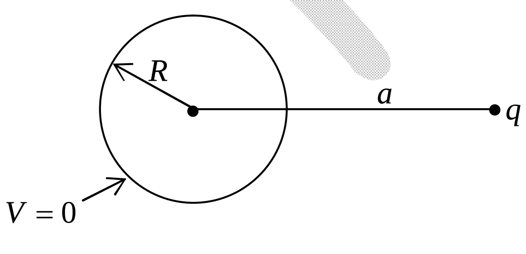
The charge q is attracted towards the plane, because of the negative induced surface charge. The force: $\vec{F} = -\dfrac{1}{4\pi\varepsilon_0}\dfrac{q^2}{(2d)^2}\hat{z}$.

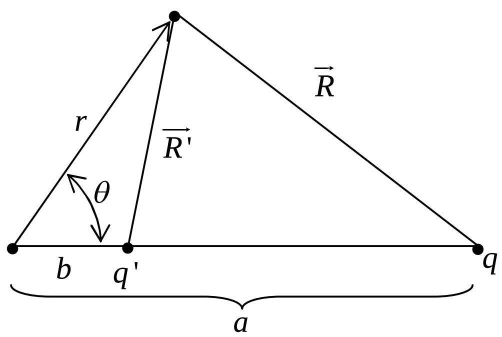
One can determine the energy by calculating the work required to bring q in from infinity.

$$W = \int_{\infty}^{d} \vec{F}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}dz = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{4z}\right)\Bigg|_{\infty}^{d} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$

1.12.4 Other Image Problem

The method just described is not limited to a single point charge; any stationary charge distribution near a grounded conducting plane can be treated in the same way, by introducing its mirror image.





Let us examine the completely different configuration, consisting of the point charge q together with another point charge $q' = -\dfrac{R}{a}q$ placed at a distance $b = \dfrac{R^2}{a}$ to the right of the centre of sphere. No conductor, now-just two point charges.

The potential of this configuration is $V(r,\theta) = \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q}{R} + \dfrac{q'}{R'}\right)$

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{1}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right\}$$

Clearly when $r = R$, $V \to 0$

Force

The force on q, due to the sphere, is the same as the force of the image charge q', thus:

$$F = -\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2 - R^2)^2}$$

Energy

To bring q in from infinity to a, we do work

$$W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{2(a^2 - R^2)}$$

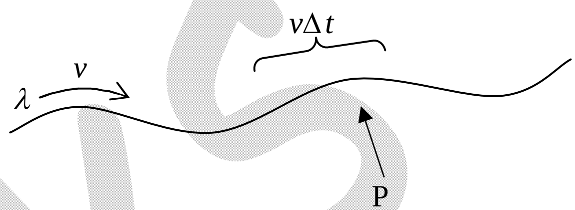
2. MAGNETOSTATICS

The magnetic field at any point due to steady current is called as magnetostatic field.

2.1 Magnetic force on current element

The magnetic force on a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is, $\vec{F}_{mag} = Q\left(\vec{v}\times\vec{B}\right)$. This is known as Lorentz force law. In the presence of both electric and magnetic fields, the net force on Q would be: $\vec{F} = Q\left[\vec{E} + \left(\vec{v}\times\vec{B}\right)\right]$

2.1.1 Current in a wire

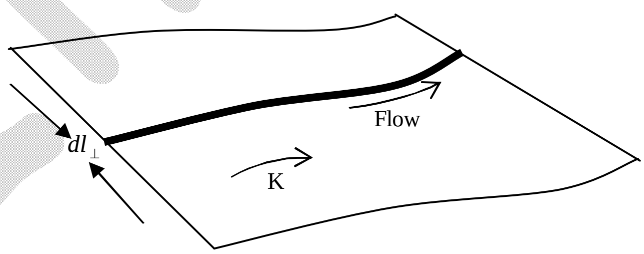


A line charge λ traveling down a wire at a speed $\vec{v}$ constitutes a current $\vec{I} = \lambda\vec{v}$.

Magnetic force on a segment of current-carrying wire is, $\vec{F}_{mag} = I\int\left(d\vec{l}\times\vec{B}\right)$

2.1.2 Surface current density

When charge flows over a surface, we describe it by the surface current $\vec{K}$.

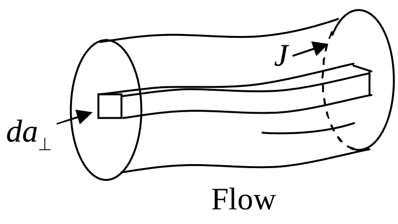


$\vec{K} = \dfrac{d\vec{I}}{dl_\perp}$ is the current per unit width-perpendicular to flow.

Also $\vec{K} = \sigma\vec{v}$ where is σ surface charge density and $\vec{v}$ is its velocity.

Magnetic force on surface current $\vec{F}_{mag} = \int\left(\vec{K}\times\vec{B}\right)da$

2.1.3 Volume current density



$\vec{J} = \dfrac{d\vec{I}}{da_\perp}$ is the current per unit area-perpendicular to flow. Also $\vec{J} = \rho\vec{v}$ where ρ is volume charge density and $\vec{v}$ is its velocity. Magnetic force on volume current $\vec{F}_{mag} = \int\left(\vec{J}\times\vec{B}\right)d\tau$ .Current crossing a surface S is $I = \int_S \vec{J}\cdot d\vec{a}$

2.2 Continuity equation

General continuity equation $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$

This is the precise mathematical statements of **local charge conservation**.

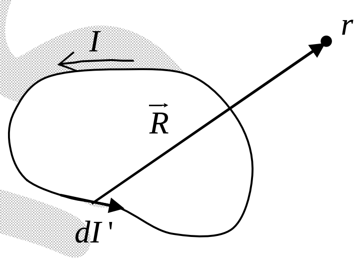
Thus for magnetostatic fields $\frac{\partial\rho}{\partial t} = 0$ and hence the continuity equation becomes:

$$\vec{\nabla}\cdot\vec{J} = 0.$$

2.3 Biot-Savart law (magnetic field of steady line current)

The magnetic field of a steady line current is given by

$$\vec{B}(r) = \frac{\mu_0}{4\pi}\int\frac{\vec{I}\times\hat{R}}{R^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\vec{l}'\times\hat{R}}{R^2}$$



$$where\ \mu_0 = 4\pi\times10^{-7}\frac{N}{A^2}(permeability\ of\ \ free\ space)$$

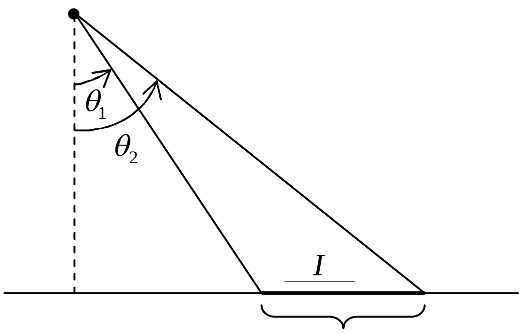
For surface and volume current Biot - Savart law becomes:

$$\mathrm{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{K}(r')\times\hat{R}}{R^2}da'\ and\ \mathrm{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')\times\hat{R}}{R^2}d\tau'.$$

2.3.1 Magnetic field due to wire

Let us find the magnetic field a distance *d* from a long straight wire carrying a steady current *I*.

$$\vec{\mathrm{B}} = \frac{\mu_0 \mathrm{I}}{4\pi \mathrm{d}}(\sin\theta_2 - \sin\theta_1)\hat{\phi}$$



For Infinite wire: $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2} \Rightarrow \vec{\mathrm{B}} = \frac{\mu_0 \mathrm{I}}{2\pi \mathrm{d}}\hat{\phi}$

Note:

1. Force (per unit length) of attraction between two long, parallel wires a distance *d* apart, carrying currents I_1 and I_2 in same direction are: $f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$.

2. If currents are in opposite direction they will repel with same magnitude.

3. The magnetic field a distance d above the center of a circular loop of radius R, which carries a steady current I is $\vec{B}=\frac{\mu_0 I}{2}\frac{R^2}{(R^2+d^2)^{3/2}}\hat{z}$.

At the center of the circle $\vec{B}(0)=\frac{\mu_0 I}{2R}\hat{z}$

2.3.2 Magnetic field due to Solenoid and Toroid

The magnetic field of a very long solenoid, consisting of n closely wound turns per unit length of a cylinder of radius R and carrying a steady current I is:

$$\vec{B}=\begin{cases}\mu_0 nI\,\hat{z} & \text{inside the solenoid}\\ 0 & \text{outside the solenoid.}\end{cases}$$

Magnetic field due to toroid is $\vec{B}=\begin{cases}+\dfrac{\mu_0 NI}{2\pi r}\hat{\phi} & \text{for points inside the coil}\\ 0 & \text{for points outside the coil}\end{cases}$

where N is the total number of turns.

2.4 Ampere's Law

In general we can write $\oint \vec{B}\cdot d\vec{l}=\mu_0 I_{enc}$ where $I_{enc}=\int \vec{J}.d\vec{a}$ is the total current enclosed by the amperian loop.

$$\Rightarrow \vec{\nabla}\times\vec{B}=\mu_0\vec{J}$$

Right hand Rule

If the fingers of your right hand indicate the direction of integration around the boundary, then your thumb defines the direction of a positive current.

2.5 Magnetic Vector Potential $(\vec{A})$: Since $\vec{\nabla}\cdot\vec{B}=0\Rightarrow\vec{B}=\vec{\nabla}\times\vec{A}$

For magnetostatic fields, $\vec{\nabla}\cdot\vec{A}=0$ *and* $\nabla^2\vec{A}=-\mu_0\vec{J}$

If $\vec{J}$ goes to zero at infinity, $\vec{A}(r)=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')}{R}d\tau'$ *for volume current.*

For line and surface currents, $\vec{A}(r)=\frac{\mu_0 I}{4\pi}\int\frac{1}{R}dl'$; $\vec{A}(r)=\frac{\mu_0}{4\pi}\int\frac{\vec{K}}{R}da'$

2.6 Magnetostatic boundary condition (Boundary is sheet of current, $\vec{K}$)

Just as the electric field suffers a discontinuity at a surface charge, so the magnetic field is discontinuous at a surface current. Only this time it is the tangential component that changes.

$$\text{Since } \oint \vec{B}.d\vec{a} = 0 \quad \Rightarrow B^{\perp}_{above} = B^{\perp}_{below}$$

For tangential components

$$\begin{cases} \oint \vec{B}.d\vec{l} = \mu_0 I_{enc} \Rightarrow B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K & \left(\vec{B} \text{ is parallel to surface but } \perp^r \text{ to } \vec{K}\right) \\ \oint \vec{B}.d\vec{l} = \mu_0 I_{enc} \Rightarrow B^{\parallel}_{above} = B^{\parallel}_{below} & \left(\vec{B} \text{ is parallel to surface and along } \vec{K}\right) \end{cases}$$

Thus the component of $\left(\vec{B}\right)$ that is parallel to the surface but perpendicular to the current is discontinuous in the amount $\mu_0 K$. A similar amperian loop running parallel to the current reveals that the parallel component is continuous. The result can be summarized in a single formula:

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \left(\vec{K} \times \hat{n}\right),$$

where $\hat{n}$ is a unit vector perpendicular to the surface, pointing "upward".

Like the scalar potential in electrostatics, the vector potential is continuous across, an boundary: $\vec{A}_{above} = \vec{A}_{below}$

For $\vec{\nabla}.\vec{A} = 0$ guarantees that the normal component is continuous, and $\vec{\nabla} \times \vec{A} = \vec{B}$, in the form $\oint_{line} \vec{A}.d\vec{l} = \int_S \vec{B}.d\vec{a} = \phi$

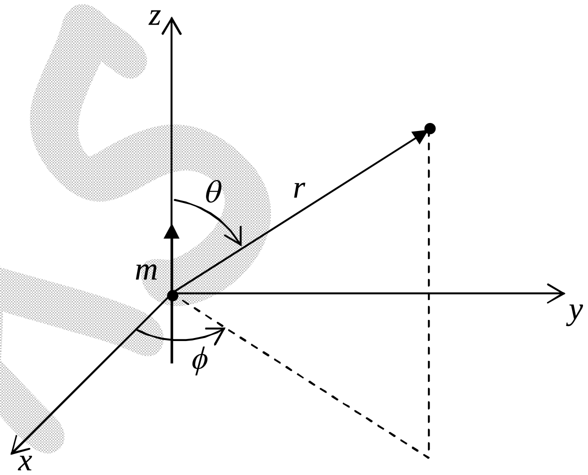
But the derivative of $\left(\vec{A}\right)$ inherits the discontinuity of $\left(\vec{B}\right)$: $\dfrac{\partial \vec{A}_{above}}{\partial n} - \dfrac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$

2.7 Multiple Expansion of Vector Potential

We can always write the potential in the form of a power series in $\frac{1}{r}$, where r is the distance to the point in question. Thus we can always write

$$\vec{A}(r)=\frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}'+\frac{1}{r^2}\oint r'\cos\theta'\, d\vec{l}'+\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)d\vec{l}'+........\right]$$

First term, *monopole* $\Rightarrow \oint d\vec{l}'=0$ (no magnetic monopole)



Second term, *dipole* $\Rightarrow \vec{A}_{dip}(r)=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$

where $\vec{m}$ is the magnetic dipole moment: $\vec{m}=I\int d\vec{a}=I\vec{A}$ where $\vec{A}$ is area vector

Thus $\vec{A}_{dip}(\vec{r})=\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$

Hence

$$\vec{B}_{dip}(\vec{r})=\vec{\nabla}\times\vec{A}=\frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r}+\sin\theta\hat{\theta}\right)=\frac{\mu_0}{4\pi}\cdot\frac{1}{r^3}\left[3\left(\vec{m}\cdot\hat{r}\right)\hat{r}-\vec{m}\right]$$

Note:

(a) When a magnetic dipole is placed in a uniform magnetic field ($\vec{B}$), net force on the dipole is zero and it experiences a torque $\vec{\tau}=\vec{m}\times\vec{B}$.

(b) In non-uniform field, dipoles have net force $\vec{F}=\vec{\nabla}\left(\vec{m}\cdot\vec{B}\right)$ and torque $\vec{\tau}=\vec{m}\times\vec{B}$.

(c) Energy of an ideal dipole $\vec{m}$ in an magnetic field $\vec{B}$ is $U=-\vec{m}.\vec{B}$.

(d) Interaction energy of two dipoles separated by a distance $\vec{r}$ is

$$U=\frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{m}_1\cdot\vec{m}_2-3\left(\vec{m}_1\cdot\hat{r}\right)\left(\vec{m}_2\cdot\hat{r}\right)\right].$$

.

2.8 Magnetisation $\left(\vec{M}\right)$

Magnetization $\left(\vec{M}\right)$ is magnetic dipole moment per unit volume.

2.8.1 The Field of a magnetized Object (Bound Currents)

Consider a piece of magnetized material with magnetization $\left(\vec{M}\right)$.

Then the vector potential is given by

$$\vec{A}(r)=\frac{\mu_0}{4\pi}\int_v \frac{1}{R}\left[\vec{\nabla}'\times\vec{M}(r')\right]d\tau'+\frac{\mu_0}{4\pi}\oint\frac{1}{R}\left[\vec{M}(r')\times d\vec{a}'\right].$$

This means the potential(and hence also the field) of a magnetized object is the same as would be produced by a volume current $\vec{J}_b=\vec{\nabla}\times\vec{M}$ throughout the material, plus a surface current $\vec{K}_b=\vec{M}\times\hat{n}$, on the boundary. We first determine these **bound currents**, and then find the field they produce.

2.9 The Auxiliary field ($\vec{H}$)

2.9.1 Ampere's Law in in presence of magnetic materials

In a magnetized material the total current can be written as $\vec{J}=\vec{J}_b+\vec{J}_f$ where $\vec{J}_b$ is bound current and $\vec{J}_f$ is free current. Thus $\vec{\nabla}\times\vec{H}=\vec{J}_f$ where $\vec{H}=\frac{\vec{B}}{\mu_0}-\vec{M}$

In integral form $\oint\vec{H}\cdot d\vec{l}=I_{f_{enc}}$ where $I_{f_{enc}}$ is the total free current passing through the amperian loop.

$\vec{H}$ plays a role in magnetostatic analogous to $\vec{D}$ in electrostatic: Just as $\vec{D}$ allowed us to write Gauss's law in terms of the free charge alone, $\vec{H}$ permits us to express Ampere's law in terms of the free current alone- and free current is what we control directly.

Note:

When we have to find $\vec{B}$ or $\vec{H}$ in a problem involving magnetic materials, first look for symmetry. If the problem exhibits cylindrical, plane, solenoid, or toroidal symmetry, then we can get $\vec{H}$ directly from the equation $\oint\vec{H}\cdot d\vec{l}=I_{f_{enc}}$.

2.9.2 Magnetic Susceptibility and Permeability

For most substances magnetization is proportional to the field $\vec{H}$, $\Rightarrow \vec{M} = \chi_m \vec{H}$,

where χ_m is magnetic susceptibility of the material.

$\Rightarrow \vec{B} = \mu_0\left(\vec{H} + \vec{M}\right) = \mu_0(1+\chi_m)\vec{H} \Rightarrow \vec{B} = \mu\vec{H}$ where $\mu_0 = \mu_0\mu_r = \mu_0(1+\chi_m)$, is permeability of material.

2.9.3 Boundary Condition ($\vec{H}$)

The boundary between two medium is a thin sheet of free surface current K_f.

The Ampere's law states that

$$\Rightarrow \vec{H}_{above} - \vec{H}_{below} = \vec{K}_f \times \hat{n}.$$

And
$$H^{\perp}_{above} - H^{\perp}_{below} = -\left(M^{\perp}_{above} - M^{\perp}_{below}\right)$$

3. Dynamics of charged particles in static and uniform electromagnetic fields

(The Lorentz Force Law)

The magnetic force on a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is,

$$\vec{F}_{mag} = Q\left(\vec{v} \times \vec{B}\right)$$

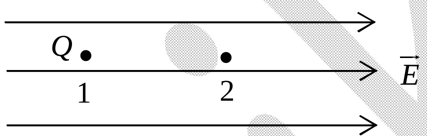
This is known as Lorentz force law.

In the presence of both electric and magnetic fields, the net force on Q would be:

$$\vec{F} = Q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]$$

3.1 Charged particle in static electric field

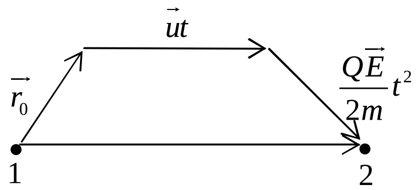
3.1.1 Charged particle enters in the direction of field (Linear motion)



The force on the charge Q in electric field $\vec{E}$ is $\vec{F} = Q\vec{E}$.

Acceleration of the charge particle in the direction of the electric field is $\vec{a} = \frac{\vec{F}}{m} = \frac{Q\vec{E}}{m}$.

If $\vec{r}$ is the position vector at any time t then



Let at $t=0$, $\vec{r} = \vec{r}_0 \Rightarrow C_1 = \vec{r}_0 \Rightarrow \vec{r} = \frac{Q\vec{E}}{2m}t^2 + \vec{u}t + \vec{r}_0$

If initially $\vec{u} = 0, \vec{r}_0 = 0 \Rightarrow \vec{r} = \frac{Q\vec{E}}{2m}t^2$ and $\vec{v} = \frac{Q\vec{E}}{m}t$.

The energy acquired by the charged particle in moving from point 1 to 2 is

$$W = \int_1^2 \vec{F}.d\vec{l} = m\int_1^2 \vec{a}.d\vec{l} = m\int_1^2 \frac{d\vec{v}}{dt}.\vec{v}dt = m\int_1^2 \vec{v}.d\vec{v} \Rightarrow W = \frac{1}{2}m\left(v_2^2 - v_1^2\right)$$

If the potential difference between points 1 to 2 is V then $W = QV = \frac{1}{2}m\left(v_2^2 - v_1^2\right)$

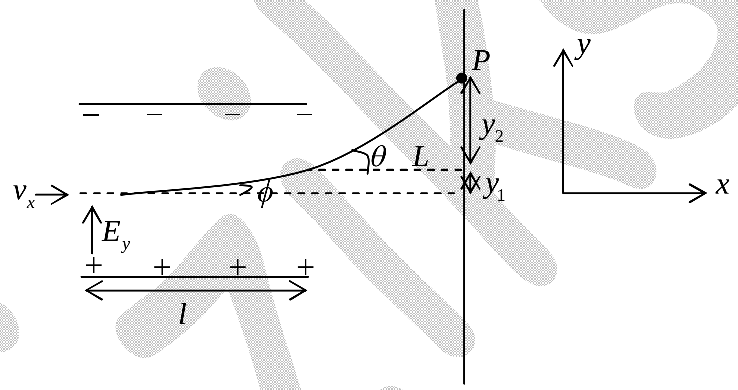
If the particle starts from rest i.e $\left(v_1 = 0\right)$ and final velocity is v then

$$W = QV = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2QV}{m}}$$

Kinetic energy of the particle $K.E. = \frac{1}{2}mv^2 = \frac{1}{2}m.\frac{Q^2E^2}{m^2}t^2 = QE \times \frac{QE}{2m}t^2 = QEr$

3.1.2 Charged particle enters in the direction perpendicular to field (Parabolic motion)

Let us consider a charge particle enters in an electric field region with velocity v_x at t=0. The electric field is in the y-direction and the field region has length l. After traversing a distance l it strikes a point P on a screen which is placed at a distance L from the field region.



Thus $y = \frac{1}{2}a_y t^2 = \frac{QE_y}{2m}\left(\frac{x}{v_x}\right)^2$ and which represents ***parabolic path***.

$y_1 = \frac{QE_y}{2m}\left(\frac{l}{v_x}\right)^2$ and $y_2 == L\tan\theta$

Thus distance of point P from the center of the screen is, $y_1 + y_2 = \frac{QE_y}{2m}\left(\frac{l}{v_x}\right)^2 + L\tan\theta$

Angle of deviation in the field region, $\tan\phi = \frac{dy}{dx} = \frac{QE_y}{mv_x^2}x$

Angle of deviation in the field free region, $\tan\theta = \frac{QE_y}{mv_x^2}l$

3.2 Charged particle in static magnetic field

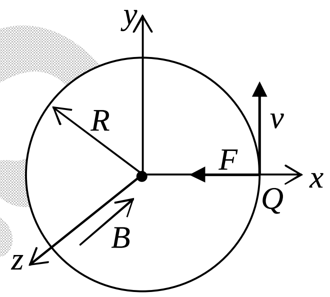
The magnetic force on a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is,

$$\vec{F}_{mag} = Q\left(\vec{v} \times \vec{B}\right)$$

This is known as Lorentz force law.

3.2.1 Charged particle enters in the direction perpendicular to field (Circular motion)

If a charge particle enters in a magnetic field at angle of 90^{o}, then motion will be circular with the magnetic force providing the centripetal acceleration.



For uniform circular motion: $QvB = m\dfrac{v^2}{R} \Rightarrow R = \dfrac{mv}{QB}$

where R is the radius of the circle and m is the mass of the charge particle.

Momentum of the charged particle $p = QBR$

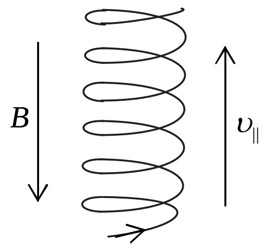
Kinetic energy (KE) $= \dfrac{p^2}{2m} = \dfrac{Q^2B^2R^2}{2m}$

Time period $T = \dfrac{2\pi R}{v} = \dfrac{2\pi m}{QB}$

3.2.2 Charged particle enters in the direction making an angle with the field (Helical motion)

If the charge particle enters in a magnetic field making an angle θ, then motion will be helical.

$v_{\perp} = v\sin\theta$ and $v_{\parallel} = v\cos\theta$,



and the radius of helix is $R = \dfrac{mv_{\perp}}{QB}$.

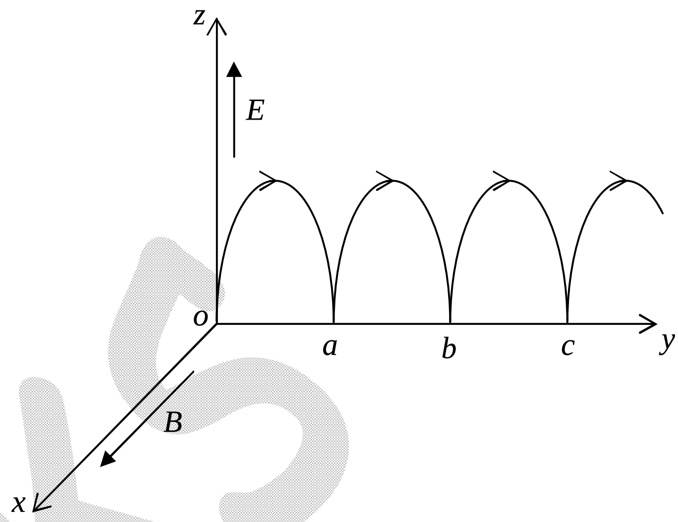
3.3 Charged particle in uniform electric and magnetic field (Cycloid motion)

$$\vec{F} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right) = Q\left(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}\right)$$

$$= m\vec{a} = m\left(\ddot{y}\hat{y} + \ddot{z}\hat{z}\right)$$

$$\Rightarrow \ddot{y} = \omega\dot{z},\ \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

$$where\ \omega = \frac{QB}{m}(cylotron\ \ frequency)$$



Solving above differential equations, we get

$$y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t),\ z(t) = \frac{E}{\omega B}(1 - \cos\omega t)$$

$$\Rightarrow (y - R\omega t)^2 + (z - R)^2 = R^2 \qquad where\ \ R = \frac{E}{\omega B}$$

This is the formula for a circle, of radius R, whose center is $(0, R\omega t, R)$ travels in the y-direction at constant speed, $v = \omega R = \dfrac{E}{B}$

The curve generated in this way is called a cycloid.

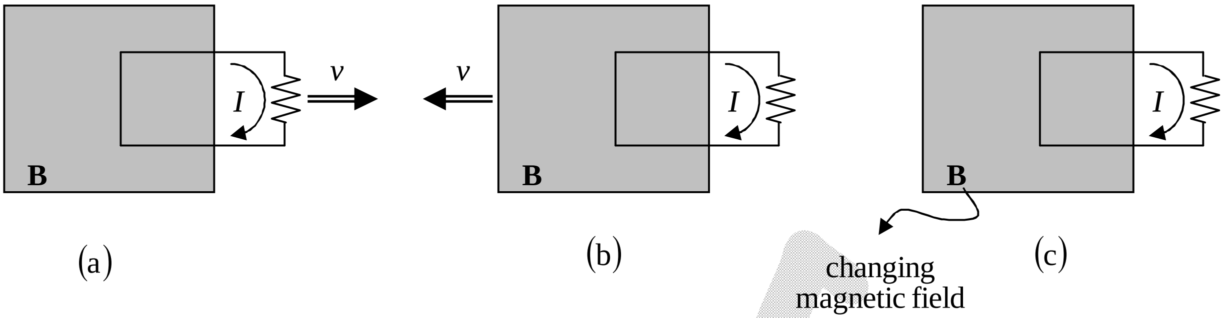
Magnetic forces do not work because $\left(\vec{v} \times \vec{B}\right)$ is perpendicular to $\vec{v}$, so

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = Q\left(\vec{v} \times \vec{B}\right) \cdot \vec{v}dt = 0.$$

Magnetic forces may alter the direction in which a particle moves, but they can not speed up or slow down it.

4. Electromagnetic induction

4.1 Faraday's Law



Experiment 1: He pulled a loop of wire to the right through a magnetic field. A current flowed in the loop (Figure a).

Experiment 2: He moved the magnet to the left, holding the loop still. Again, a current flowed in the loop (Figure b).

Experiment 3: With both the loop and the magnet at rest, he changed the strength of the field (he used an electromagnet, and varied the current in the coil). Once again current flowed in the loop (Figure c).

Thus, universal flux rule is that, whenever (and for whatever reason) the magnetic flux through a loop changes, an e.m.f. (ε) will appear in the loop.

$$\varepsilon=-\frac{d\Phi}{dt} \qquad \text{(Where magnetic flux } \Phi=\int \vec{B}.d\vec{a}\text{)}$$

In experiment 2, *A changing magnetic field induces an electric field.*

It is this "induced" electric field that accounts for the e.m.f. $\varepsilon=\oint \vec{E}\cdot d\vec{l}=-\dfrac{d\Phi}{dt}$

Then $$\oint \vec{E}\cdot d\vec{l}=-\int\frac{\partial \vec{B}}{\partial t}.d\vec{a} \quad \Rightarrow \vec{\nabla}\times\vec{E}=-\frac{\partial \vec{B}}{\partial t}$$

4.1.1 Lenz's Law

In Faraday's law negative sign represents the **Lenz's law.** (The induced current will flow in such a direction that the flux it produces tends to cancel the change).

For example if the magnetic flux is increasing then induced e.m.f will try to reduce and vice versa.

5. Maxwell's equations

5.1 Maxwell's equation in free space

5.1.1 Electrodynamics before Maxwell's

(i) $\vec{\nabla}\cdot\vec{\mathrm{E}}=\dfrac{\rho}{\varepsilon_0}$ (Gauss' Law),

(ii) $\vec{\nabla}\cdot\vec{B}=0$ (No name),

(iii) $\vec{\nabla}\times\vec{\mathrm{E}}=-\dfrac{\partial\vec{B}}{\partial t}$ (Farday's Law),

(iv) $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$ (Ampere's law).

5.1.2 How Maxwell fixed Ampere's Law

From continuity equation and Gauss Law

$$\vec{\nabla}.\vec{J}=\frac{-\partial\rho}{\partial t}=\frac{\partial}{\partial t}(\varepsilon_0\vec{\nabla}.\vec{E})=-\vec{\nabla}.\left(\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\Rightarrow\vec{\nabla}.\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)=0.$$

Thus $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$

A changing electric field induces a magnetic field.

Maxwell called this extra term the **displacement current** $\overrightarrow{J_d}=\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$.

Integral form of Ampere's law $\oint\vec{B}.d\vec{l}=\mu_0 I_{enc}+\mu_0\varepsilon_0\int\left(\dfrac{\partial\vec{E}}{\partial t}\right).d\vec{a}$

5.1.4 Maxwell's equation in free space

(i) $\vec{\nabla}\cdot\vec{\mathrm{E}}=\dfrac{\rho}{\varepsilon_0}$ (Gauss' Law),

(ii) $\vec{\nabla}\cdot\vec{B}=0$ (No name),

(iii) $\vec{\nabla}\times\vec{\mathrm{E}}=-\dfrac{\partial\vec{B}}{\partial t}$ (Farday's Law),

(iv) $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$ (Ampere's law with Maxwell's correction).

5.3 Boundary conditions on the fields at interfaces

$$\left.\begin{array}{l}(a) D_{above}^{\perp}-D_{below}^{\perp}=\sigma_f \\ (b) B_{above}^{\perp}=B_{below}^{\perp}\end{array}\right\} \qquad \left.\begin{array}{l}(c) \vec{E}_{above}^{\|}=\vec{E}_{below}^{\|} \\ (d) \vec{H}_{above}^{\|}-\vec{H}_{below}^{\|}=\vec{K}_f \times \hat{n}\end{array}\right\}$$

In particular, if there is no free charge or free current at the interface between medium1 and medium 2, then

$$\left.\begin{array}{l}(a) \varepsilon_1 E_1^{\perp}-\varepsilon_2 E_2^{\perp}=0 \\ (b) B_1^{\perp}=B_2^{\perp}\end{array}\right\} \text{ and } \left.\begin{array}{l}(c) \vec{E}_1^{\|}=\vec{E}_2^{\|} \\ (d) \dfrac{1}{\mu_1} \vec{B}_1^{\|}-\dfrac{1}{\mu_2} \vec{B}_2^{\|}=0\end{array}\right\}.$$

6. Electromagnetic waves in free space

6.1 Poynting Theorem ("work energy theorem of electrodynamics")

The work necessary to assemble a static charge distribution is

$$W_e=\frac{\varepsilon_0}{2} \int E^2 d\tau \text{, where } E \text{ is the resulting electric field}$$

The work required to get currents going (against the back emf) is

$$W_m=\frac{1}{2\mu_0} \int B^2 d\tau \text{, where } B \text{ is the resulting magnetic field}$$

This suggests that the total energy in the electromagnetic field is

$$U_{em}=\frac{1}{2} \int\left(\varepsilon_0 E^2+\frac{B^2}{\mu_0}\right) d\tau .$$

Suppose we have some charge and current configuration which at time *t*, produces fields $\vec{E}\,\&\,\vec{B}$. In next instant, *dt*, the charges moves around a bit. The work is done by electromagnetic forces acting on these charges in the interval *dt*.

According to Lorentz Force Law, the work done on a charge '*q*' is

$$\vec{F} \cdot d\vec{l}=q(\vec{E}+\vec{v} \times \vec{B}) \cdot \vec{v}\, dt=q \vec{E} \cdot \vec{v}\, dt .$$

Now $q=\rho d\tau$ and $\rho\vec{v}=\vec{J}$, so the rate at which work is done on all the charges in a volume V is

$$\frac{dW}{dt}=-\frac{d}{dt}\int_V \frac{1}{2}\left(\varepsilon_0 E^2+\frac{1}{\mu_0}B^2\right)d\tau-\frac{1}{\mu_0}\oint_S\left(\vec{E}\times\vec{B}\right)\cdot d\vec{a},$$

where S is the surface bounding V.

This is **Poynting's theorem**; it is the "work energy theorem" of electrodynamics.

The ***first integral*** on the right is the total energy stored in the fields, U_{em}.

The ***second term*** evidently, represents the rate at which energy is carries out of V, across its boundary surface, by the electromagnetic fields.

Poynting's theorem says, that, "the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface".

The *energy per unit time, per unit area*, transported by the fields is called the ***Poynting vector***

$$\vec{S}\equiv\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right).$$

$\vec{S}\cdot d\vec{a}$ is the energy per unit time crossing the infinitesimal surface $d\vec{a}$ – the energy or energy flux density.

Momentum density stored in the electromagnetic field is: $\vec{\wp}_{dem}=\mu_0\varepsilon_0\vec{S}$

6.2 Waves in one dimension (Sinusoidal waves)

6.2.1 The wave equation

A wave propagating with speed v in z-direction can be expressed as: $\frac{\partial^2 f}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

The *most general* solution to the wave equation is the sum of a wave to the right (+z direction) and a wave to the left (-z direction):

$$f(z,t)=g(z-vt)+h(z+vt).$$

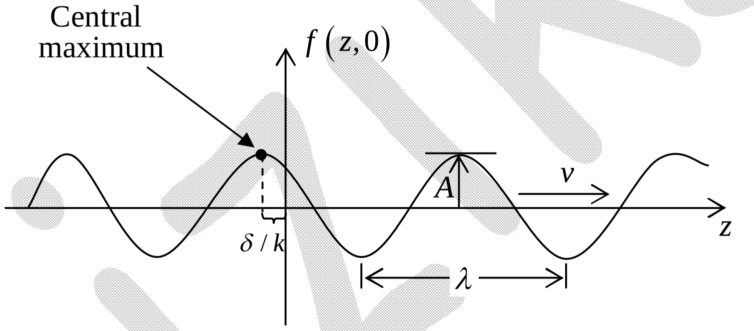
6.2.2 Terminology

Let us consider a function $f(z,t) = A\cos[k(z-vt)+\delta]$

A is the ***amplitude*** of the wave (it is positive, and represents the maximum displacement from equilibrium).

The argument of the cosine is called the ***phase,*** and δ is the ***phase constant*** (normally, we use a value in the range $0 \le \delta < 2\pi$).

Figure given below shows this function at time $t=0$. Notice that at $z = vt - \delta/k$, the phase is zero; let's call this the "central maximum." If $\delta = 0$, central maximum passes the origin at time $t=0$; more generally δ/k is the distance by which the central maximum (and therefore the entire wave) is "***delayed***."



Finally k is the ***wave number***; it is related to the ***wavelength*** λ as $\lambda = \frac{2\pi}{k}$, for when z advances by $\frac{2\pi}{k}$, the cosine executes one complete cycle.

As time passes, the entire wave train ***proceeds to the right***, at speed v. ***Time period*** of one complete cycle is $T = \frac{2\pi}{kv}$.

The **frequency** ν (number of oscillations per unit time) is $\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}$.

The ***angular frequency*** $\omega = 2\pi\nu = kv$

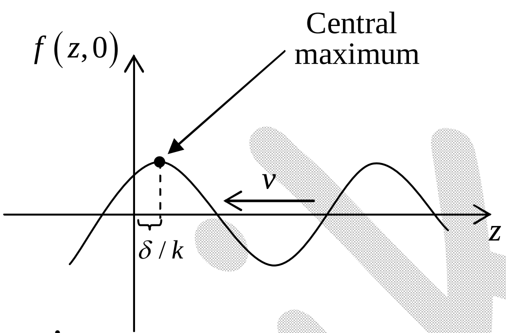
In terms of angular frequency ω, the sinusoidal wave can be represented as

$$f(z,t) = A\cos(kz - \omega t + \delta).$$

A sinusoidal oscillation of wave number k and angular frequency ω ***traveling to the left*** would be written

$$f(z,t) = A\cos(kz + \omega t - \delta).$$

Comparing this with the wave traveling to the right reveals that, in effect, we could *simply switch the sign of* k to produce a wave with the same amplitude, phase constant, frequency, and wavelength, traveling in the opposite direction.



6.2.3 Complex notation

In view of ***Euler's formula***,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

the sinusoidal wave $f(z,t) = A\cos(kz - \omega t + \delta)$ can be written as

$$f(z,t) = \mathrm{Re}\left[Ae^{i(kz - \omega t + \delta)}\right],$$

where $\mathrm{Re}(\eta)$ denotes the real part of the complex number η. This invites us to introduce the ***complex wave function***

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz - \omega t)}$$

with the ***complex amplitude*** $\tilde{A} = Ae^{i\delta}$ absorbing the phase constant.

The actual wave function is the real part of $\tilde{f}$:

$$f(z,t) = \mathrm{Re}\left[\tilde{f}(z,t)\right].$$

The advantage of the complex notation is that exponentials are much easier to manipulate than sines and cosines.

6.2.4 Polarization

In ***longitudinal*** wave, the displacement from the equilibrium is along the direction of propagation. Sound waves, which are nothing but compression waves in air, are longitudinal.

Electromagnetic waves are ***transverse*** in nature. In a transverse wave displacement is perpendicular to the direction of propagation.

There are two dimensions perpendicular to any given line of propagation. Accordingly, transverse waves occur in two independent state of polarization:

"Vertical" polarization $\tilde{f}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{x}$,

"Horizontal" polarization $\tilde{f}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{y}$,

or along any other direction in the *xy* plane $\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n}$

The ***polarization vector*** $\hat{n}$ defines the plane of vibration. Because the waves are transverse, $\hat{n}$ is perpendicular to the direction of propagation:

$$\hat{n}.\hat{z} = 0$$

6.3 Electromagnetic waves in vacuum

6.3.1 The wave equation for $\vec{E}$ *and* $\vec{B}$

Write Maxwell's equations in free space ($\rho = 0$ *and* $\vec{J} = 0$) then,

$$\Rightarrow \nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{and} \qquad \nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

Thus, $\vec{E}$ and $\vec{B}$ satisfy the wave equation $\nabla^2 f = \frac{1}{v}\frac{\partial^2 f}{\partial t^2}$.

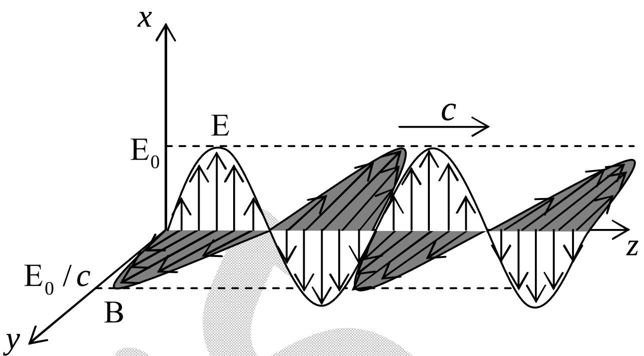
So, EM waves travels with a speed

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8 m/s = c(velocity\ of\ light\ in\ free\ space)$$

$$where \quad \mu_0 = 4\pi\times10^{-7}\ N/A^2,\ \varepsilon_0 = 8.86\times10^{-12}\ C^2/Nm^2$$

6.3.2 Monochromatic plane waves

Suppose waves are traveling in the z-direction and have no x or y dependence; these are called ***plane waves*** because the fields are uniform over every plane perpendicular to the direction of propagation.



The plane waves can be represented as:

$$\tilde{E}(z,t)=\tilde{E}_0 e^{i(kz-\omega t)},\ \tilde{B}(z,t)=\tilde{B}_0 e^{i(kz-\omega t)}$$

where $\tilde{E}_0$ and $\tilde{B}_0$ are the (complex) amplitudes (the physical fields, of course are the real parts of $\tilde{E}$ and $\tilde{B}$).

That is, electromagnetic waves are ***transverse***: the electric and magnetic fields are perpendicular to the direction of propagation.

Also $$\vec{\tilde{B}}_0=\frac{k}{\omega}(\hat{z}\times\vec{\tilde{E}}_0)$$

There is nothing special about the z direction; we can generalize the monochromatic plane waves traveling in an arbitrary direction. The ***propagation vector*** or ***wave vector*** $\vec{k}$ points in the direction of propagation, whose magnitude is the wave number k. The scalar product $\vec{k}.\vec{r}$ is the appropriate generalization of kz, so

$$\vec{\tilde{E}}(\vec{r},t)=\vec{\tilde{E}}_0 e^{i(\vec{k}.\vec{r}-\omega t)}\hat{n},\qquad \vec{\tilde{B}}(\vec{r},t)=\frac{1}{c}\vec{\tilde{E}}_0 e^{i(\vec{k}.\vec{r}-\omega t)}\left(\hat{k}\times\hat{n}\right)=\frac{1}{c}\hat{k}\times\vec{\tilde{E}}$$

where $\hat{n}$ is polarization vector.

The actual (real) electric and magnetic fields in a monochromatic plane wave with propagation vector $\vec{k}$ and polarization $\hat{n}$ are

$$\vec{E}(\vec{r},t)=E_0\cos\left(\vec{k}.\vec{r}-\omega t+\delta\right)\hat{n},\qquad \vec{B}(\vec{r},t)=\frac{1}{c}E_0\cos\left(\vec{k}.\vec{r}-\omega t+\delta\right)\left(\hat{k}\times\hat{n}\right)$$

6.3.3 Energy and Momentum in Electromagnetic Wave

The energy per unit volume stored in electromagnetic field is $u=\frac{1}{2}\left(\varepsilon_0 E^2+\frac{1}{\mu_0}B^2\right)$

In case of monochromatic plane wave $B^2=\frac{E^2}{c^2}=\mu_0\varepsilon_0 E^2$

So the electric and magnetic contributions are equal i.e. $u_E=u_B=\frac{1}{2}\varepsilon_0 E^2=\frac{1}{2\mu_0}B^2$.

$$u=u_E+u_B=\varepsilon_0 E^2 \;=\varepsilon_0 E_0{}^2\cos^2(kz-wt+\delta)\;.$$

As the wave travels, it carries this energy along with it. The energy flux density (energy per unit area, per unit time) transported by the fields is given by the ***Pointing vector***

$$\vec{S}=\frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

For monochromatic plane wave propagating in the z-direction,

$$\vec{S}=c\varepsilon_0 E_0^2\cos^2(kz-wt+\delta)\hat{z}=cu\hat{z}\,.$$

The energy per unit time, per unit area, transported by the wave is therefore uc.

Electromagnetic fields not only carry energy, they also carry ***momentum***. The momentum density stored in the field is $\vec{\wp}=\frac{1}{c^2}\vec{S}$.

For monochromatic plane wave, $\vec{\wp}=\frac{1}{c}\varepsilon_0 E_0{}^2\cos^2(kz-wt+\delta)\hat{z}=\frac{1}{c}u\hat{z}$.

Average energy density $\langle u\rangle=\frac{1}{2}\varepsilon_0 E_0^2$,

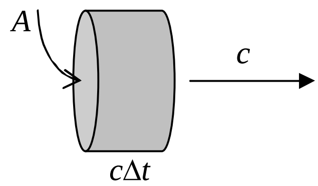
Average of Poynting vector $\langle\vec{S}\rangle=\frac{1}{2}c\varepsilon_0 E_0^2\hat{z}$,

Average momentum density $\langle\vec{\wp}\rangle=\frac{1}{2c}\varepsilon_0 E_0^2\hat{z}$.

The average power per unit area transported by an electromagnetic wave is called the ***intensity*** $I=\langle S\rangle=\frac{1}{2}c\varepsilon_0 E_0^2$.

Note:

(a) When light falls on ***perfect absorber*** it delivers its momentum to the surface. In a time Δt the momentum transfer is $\Delta p = \langle \wp \rangle Ac\Delta t$,



so the ***radiation pressure*** (average force per unit area) is

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\varepsilon_0 E_0^2 = \frac{I}{c}.$$

(b) When light falls on ***perfect reflector***, the ***radiation pressure***

$$P = \frac{2I}{c}$$

because the momentum changes direction, instead of being absorbed.

6.4 Electromagnetic waves in matter

Inside matter, but in regions where there is no free charge or free current. If the medium is linear and homogeneous, $\vec{D} = \varepsilon \vec{E}$ and $\vec{H} = \frac{1}{\mu}\vec{B}$.

Now the wave equation inside matter is $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$ *and* $\nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}$.

Thus EM waves propagate through a linear homogenous medium at a speed

$$v = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{n} \text{ where } n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

thus $n = \sqrt{\varepsilon_r}$ is the **index of refraction** (since $\mu_r = 1$ for non-magnetic material).

The energy density $u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right) \Rightarrow \langle u \rangle = \frac{1}{2}\varepsilon E_0^2$

The Poynting vector $\vec{S} = \frac{1}{\mu}\left(\vec{E} \times \vec{B}\right) \Rightarrow \langle \vec{S} \rangle = \frac{1}{2} v\varepsilon E_0^2 \hat{z}$

Intensity $I = \langle \vec{S} \rangle = \frac{1}{2}\varepsilon v E_0^2$

Thus in a medium $c \to v$, $\varepsilon_0 \to \varepsilon$ and $\mu_0 \to \mu$

6.5 Electromagnetic waves in conductors

Any initial free charge density $\rho_f(0)$ given to conductor dissipate in a characteristic time $\tau \equiv \varepsilon/\sigma$ where σ is conductivity and $\rho_f(t) = e^{-(\sigma/\varepsilon)t}\rho_f(0)$.

This reflects the familiar fact that if we put some free charge on conductor, it will flow out to the edges. The modified wave equation for $\vec{E}$ *and* $\vec{B}$ are,

$$\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t} \qquad and \qquad \nabla^2\vec{B} = \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t} + \mu\sigma\frac{\partial\vec{B}}{\partial t}$$

The admissible plane wave solution is $\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(\tilde{k}z-\omega t)}$, $\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(\tilde{k}z-\omega t)}$

"wave number" $\tilde{k}$ is complex. Let $\tilde{k} = k + i\kappa$ where k and κ are real and imaginary part of $\tilde{k}$.

$$\Rightarrow \qquad k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \quad and \quad \kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-\right]^{1/2}$$

Thus, $\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{-\kappa z}e^{i(kz-\omega t)}$, $\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{-\kappa z}e^{i(kz-\omega t)}$

The distance it takes to reduce the amplitude by a factor of $\frac{1}{e}$ is called the ***skin depth*** **(*d*)**

$$d = \frac{1}{\kappa};$$

it is a measure of how far the wave penetrates into the conductor.

The real part of $\tilde{k}$ determines the wavelength, the propagation speed, and the index of refraction: $\lambda = \frac{2\pi}{k}$, $v = \frac{\omega}{k}$, $n = \frac{ck}{\omega}$

Like any complex number, $\tilde{k}$ can be expressed in terms of its modulus and phase:

$$\tilde{k} = Ke^{i\phi}$$

where $K = |\tilde{k}| = \sqrt{k^2+\kappa^2} = \omega\sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}$ and $\phi \equiv \tan^{-1}\left(k/\kappa\right)$

The complex amplitudes $\tilde{E}_0 = E_0 e^{i\delta_E}$ *and* $\tilde{B}_0 = B_0 e^{i\delta_B}$ are related by

$$\tilde{B}_0 = \frac{\tilde{k}}{\omega}\tilde{E}_0 \Rightarrow B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E}.$$

Evidently the electric and magnetic fields are no longer in phase; in fact $\delta_B - \delta_E = \phi$, the *magnetic field* ***lags behind*** *the electric fields.*

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \left[\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]^{1/2}$$

Thus, $$\vec{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{x}$$
$$\vec{B}(z,t) = B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\hat{y}$$

Note:

(a) In a poor conductor ($\sigma << \omega\varepsilon$)

$$\kappa = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$ i.e independent of frequency.

(b) In a very good conductor ($\sigma >> \omega\varepsilon$)

$$\kappa = \sqrt{\frac{\omega\mu\sigma}{2}}, \quad d = \frac{1}{\kappa} = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f \mu\sigma}}$$

(c) When an electromagnetic wave strikes a perfect conductor ($\sigma \to \infty$) then all waves are reflected back i.e. $\tilde{E}_{0R} = -\tilde{E}_{0I}$ and $\tilde{E}_{0T} = 0$.

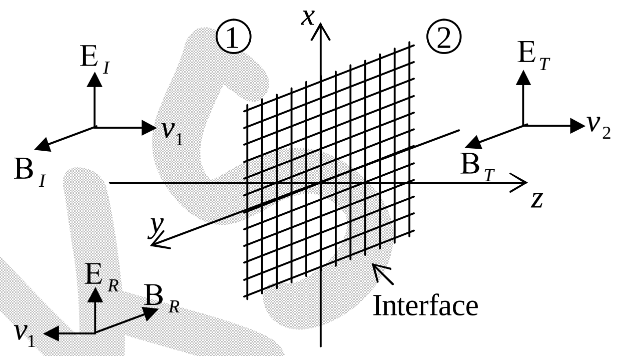
7. Applications of Electromagnetic waves

7.1 Reflection and Refraction

7.1.1 Normal incidence

At $z=0$, the combined field on the left $\tilde{E}_I+\tilde{E}_R$ *and* $\tilde{B}_I+\tilde{B}_R$, must join the fields on the right $\tilde{E}_T$ & $\tilde{B}_T$, in accordance with the **boundary conditions**.

$$\Rightarrow \tilde{E}_{0R}=\left(\frac{v_2-v_1}{v_2+v_1}\right)\tilde{E}_{0I} \quad , \quad \tilde{E}_{0T}=\left(\frac{2v_2}{v_1+v_2}\right)\tilde{E}_{0I}$$



Note:

Reflected wave is ***in phase*** if $v_2>v_1$ *or* $n_2<n_1$ and ***out of phase*** if $v_2<v_1$ *or* $n_2>n_1$.

In terms of indices of refraction the real amplitudes are

$$E_{0R}=\left|\frac{n_1-n_2}{n_1+n_2}\right|E_{0I} \quad , \quad E_{0T}=\left|\frac{2n_1}{n_1+n_2}\right|E_{0I} .$$

Since Intensity $I=\frac{1}{2}\varepsilon v E_0^2$, then the ratio of the reflected intensity to the incident intensity is the *Reflection coefficient* $R=\frac{I_R}{I_I}=\left(\frac{E_{0R}}{E_{0I}}\right)^2=\left(\frac{n_1-n_2}{n_1+n_2}\right)^2$.

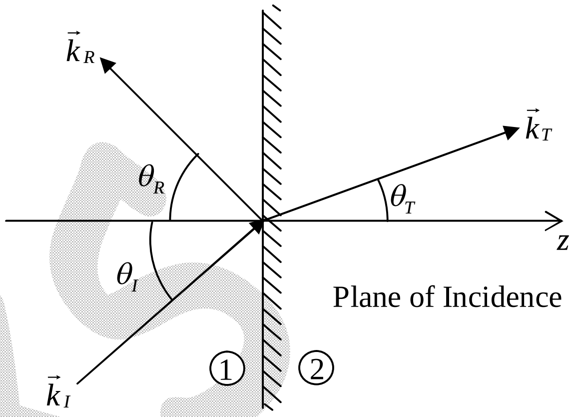
The ratio of the transmitted intensity to the incident intensity is the *Transmission coefficient*

$$T=\frac{I_T}{I_I}=\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2=\frac{4n_1n_2}{(n_1+n_2)^2} .$$

$$\Rightarrow R+T=1$$

7.1.2 Oblique incidence

In oblique incidence an incoming wave meets the boundary at an arbitrary angle θ_I. Of course, normal incidence is really just a special case of oblique incidence with $\theta_I = 0$.



First Law (law of refraction)

The incident, reflected and transmitted wave vectors form a plane (called the *incidence plane*), which also includes normal to the surface.

Second law (law of reflection)

The angle of incidence is equal to the angle of reflection

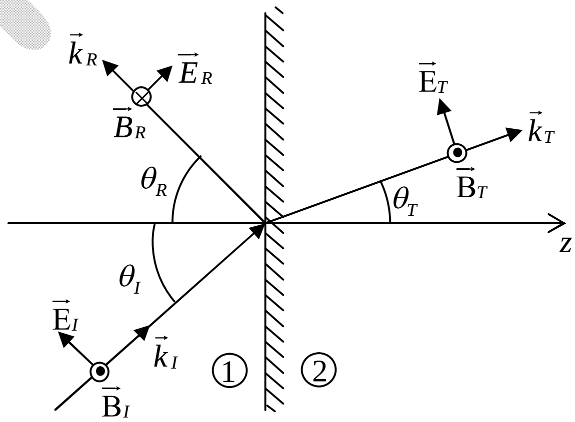
i.e. $\theta_I = \theta_R$

Third Law: (law of refraction, or Snell's law)

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}.$$

7.1.3 Fresnel's relation (Parallel and Perpendicular Polarization)

Case I: (Polarization in the plane of incidence)



Reflected and transmitted amplitudes

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I} \quad \text{and} \quad \tilde{E}_{0T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I}$$

where $\alpha = \dfrac{\cos\theta_T}{\cos\theta_I}$ *and* $\beta = \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\mu_1 n_2}{\mu_2 n_1}$

These are known as **Fresnel's equations**.

Notice that transmitted wave is always in phase with the incident one; the reflected wave is either in phase, if $\alpha > \beta$, or 180^0 out phase if $\alpha < \beta$.

At *Brewster's angle* (θ_B) reflected light is completely extinguished when $\alpha = \beta$, or

$$\tan\theta_B \approx \frac{n_2}{n_1} \quad and \quad \theta_I + \theta_B = 90^o$$

When light enters from denser to rarer medium ($n_1 > n_2$) then after a ***critical angle*** (θ_C) there is total internal reflection.

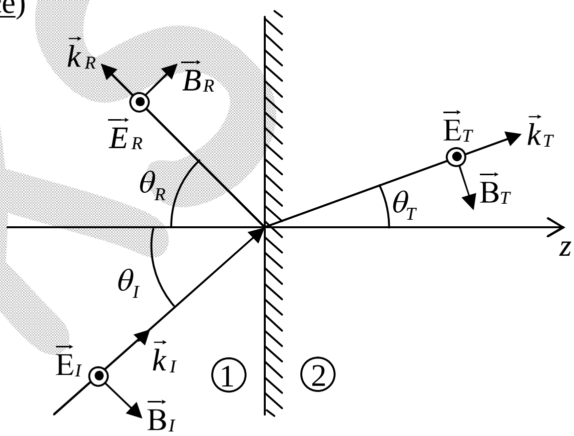
$$\frac{\sin 90^0}{\sin\theta_c} = \frac{n_1}{n_2} \quad \Rightarrow \quad \sin\theta_c = \frac{n_2}{n_1} \quad at \ \ \theta_C, \ \ \theta_T = 90^o .$$

Reflection and Transmission coefficients are $R = \left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)^2$ and $T = \alpha\beta\left(\dfrac{2}{\alpha+\beta}\right)^2$

$$\Rightarrow R + T = 1$$

CaseII: (<u>Polarization perpendicular to plane of incidence</u>)

$$\hat{E}_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\hat{E}_{0I} \quad and \quad \hat{E}_{0T} = \left(\frac{2}{1+\alpha\beta}\right)\hat{E}_{0I}$$



In this case θ_B is not possible.

Thus $R = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)^2$ and $T = \alpha\beta\left(\dfrac{2}{1+\alpha\beta}\right)^2$

$$\Rightarrow \quad R + T = 1$$

8. Potential and Field formulation for Time Varying Fields

8.1 Scalar and vector potentials

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

$$\Rightarrow \boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}}$$

Now from first Maxwell's equation (i)

$$\nabla^2 V + \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = -\frac{\rho}{\varepsilon_0} \text{ ....(1)}$$

From fourth Maxwell's equation

$$\Rightarrow \left(\vec{\nabla}^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J} \text{ ....(2)}$$

Equations (1) and (2) contain all the information of Maxwell's equations. Thus we need to calculate only four components (one for V and three for $\vec{A}$) instead of six components (three for $\vec{E}$ and three for $\vec{B}$).

8.2 Gauge transformation

Suppose we have two sets of potentials, $\left(V, \vec{A}\right)$ and $\left(V', \vec{A'}\right)$, which correspond to the same electric and magnetic fields.

Thus $\quad \vec{A'} = \vec{A} + \vec{\alpha}$ and $V' = V + \beta$.

It follows that
$$\left.\begin{aligned} \vec{A'} &= \vec{A} + \vec{\nabla}\lambda, \\ V' &= V - \frac{\partial \lambda}{\partial t}. \end{aligned}\right\}$$

Conclusion: For any old scalar function λ, we can add $\vec{\nabla}\lambda$ to $\vec{A}$, provided we simultaneously subtract $\frac{\partial \lambda}{\partial t}$ from *V*. None of these will affect the physical quantities $\vec{E}$ and $\vec{B}$. Such changes in V *and* $\vec{A}$ are called **gauge transformations.**

8.3 Coulomb and Lorentz gauge

Coulomb Gauge reads $\vec{\nabla}\cdot\vec{A}=0$.

Lorentz Gauge condition is $\vec{\nabla}\cdot\vec{A}=-\mu_0\varepsilon_0\dfrac{\partial V}{\partial t}$.

Since $\left(\vec{\nabla}^2\vec{A}-\mu_0\varepsilon_0\dfrac{\partial^2\vec{A}}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\dfrac{\partial V}{\partial t}\right)=-\mu_0\vec{J}$ and $\nabla^2 V+\dfrac{\partial}{\partial t}\left(\nabla\cdot\vec{A}\right)=-\dfrac{\rho}{\varepsilon_0}$,

Using Lorentz Gauge condition

$$\Rightarrow \vec{\nabla}^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J} \text{ and } \vec{\nabla}^2 V-\mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2}=-\frac{\rho}{\varepsilon_0}.$$

The virtue of the Lorentz gauge is that it treats V *and* $\vec{A}$ on an equal footing: the same differential operator $\vec{\nabla}^2-\mu_0\varepsilon_0\dfrac{\partial^2}{\partial t^2}\equiv\Box^2$ (called the **d' Alembertian**) occurs in both equations:

$$\text{(i)}\ \Box^2 V=-\frac{\rho}{\varepsilon_0}, \qquad \text{(ii)}\ \Box^2\vec{A}=-\mu_o\vec{J}.$$

9.2 Waveguides

Electromagnetic waves confined to the interior of a hollow pipe or wave guide. The wave guide is a perfect conductor, $\vec{E}=0$ *and* $\vec{B}=0$ inside the material itself, and hence the boundary conditions at the inner wall are:

$$\vec{E}^{\parallel}=0 \text{ and } \vec{B}^{\perp}=0$$

Free charges and currents will be induced on the surface in such a way as to enforce these constraints. Let us assume E.M. Waves that propagate inside the waveguide is represented by:

$$\vec{\tilde{E}}(x,y,z,t)=\vec{\tilde{E}}_0(x,y)e^{i(kz-\omega t)}, \quad \vec{\tilde{B}}(x,y,z,t)=\vec{\tilde{B}}_0(x,y)e^{i(kz-\omega t)}.$$

These electric and magnetic field must satisfies Maxwell's equations in the interior of the waveguide.Since confined waves are not (in general) transverse; in order to fit the boundary conditions we shall have to include longitudinal components $\left(E_z \text{ and } B_z\right)$:

$$\vec{\tilde{E}}_0 = E_x\hat{x} + E_y\hat{y} + E_z\hat{z},\ \vec{\tilde{B}}_0 = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}.$$

Putting this into Maxwells equations(iii) and (iv) and compare R.H.S and L.H.S

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x,\ ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y,\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z$$

$$\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x,\ ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y,\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z$$

Let us rewrite these six equations

(i) $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z$,

(ii) $\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x$,

(iii) $ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y$,

(iv) $\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z$

(v) $\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x$

(vi) $ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y$

Equation (ii), (iii), (v), and (vi) can be solved for E_x, E_y, B_x, and B_y:

(i) $E_x = \frac{i}{\left(\omega/c\right)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$

(ii) $E_y = \frac{i}{\left(\omega/c\right)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$

(iii) $B_x = \frac{i}{\left(\omega/c\right)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$

(iv) $B_y = \frac{i}{\left(\omega/c\right)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$

Finally, we will get coupled differential equation

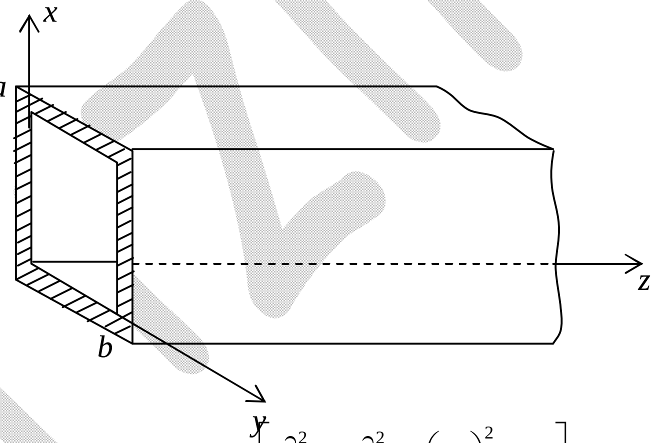
(i) $\left[\frac{\partial^2}{\partial^2 x}+\frac{\partial^2}{\partial^2 y}+\left(\frac{\omega}{c}\right)^2-k^2\right]E_z=0$ (ii) $\left[\frac{\partial^2}{\partial^2 x}+\frac{\partial^2}{\partial^2 y}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z=0$

If $E_z=0$ we call these **TE** ("transverse electric") **waves**; if $B_z=0$ they are called **TM** ("transverse magnetic") **waves**; if both $E_z=0$ and $B_z=0$, we call them **TEM waves**.

Note: It turns out that TEM waves can not occur in a hollow waveguide.

9.2.1 TE Waves in Rectangular Waveguide

Suppose we have a wave guide of rectangular shape with height *a* and with $b\,(a>b)$, and we are interested in the propagation of TE waves.



The problem is to solve $\left[\frac{\partial^2}{\partial^2 x}+\frac{\partial^2}{\partial^2 y}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z=0$ (since $E_z=0$) subject to boundary condition $\vec{B}^{\perp}=0$.

Thus $B_z=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$ where $m=0,1,2.....$ *and* $n=0,1,2......$

This solution is called the TE_{mn} mode. (The first index is conventionally associated with the larger dimension.)

Wave number k is obtaine from equation $-k_x^2 - k_x^2 + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$ by putting k_x and k_y.

So wave number $k = \sqrt{\left(\omega/c\right)^2 - \pi^2\left[\left(m/a\right)^2 + \left(n/b\right)^2\right]}$ where $\omega = 2\pi f$.

If $\omega < c\pi\sqrt{\left[\left(m/a\right)^2 + \left(n/b\right)^2\right]} \equiv \omega_{mn}$ or $f_{mn} = \frac{c}{2}\sqrt{\left[\left(m/a\right)^2 + \left(n/b\right)^2\right]}$,

the wave number is imaginary, and instead of a traveling wave we have exponentially attenuated fields. For this reason ω_{mn} or f_{mn} is called **cutoff frequency** for the mode in question.

The lowest cutoff frequency (fundamental mode) for the waveguide occurs for the mode TE_{10} :

$$\omega_{10} = \frac{c\pi}{a} \quad \text{or} \quad f_{10} = \frac{c}{2a}.$$

Frequencies less than this will not propagate at all.

Wave number can be written more simply in terms of the cutoff frequency

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}.$$

The wave velocity is $v = \frac{\omega}{k} = \frac{c}{\sqrt{1-\left(\omega_{mn}/\omega\right)^2}} = \frac{c}{\sqrt{1-\left(f_{mn}/f\right)^2}} > c$.

Group velocity $v_g = \frac{1}{dk/d\omega} = c\sqrt{1-\left(\omega_{mn}/\omega\right)^2} = c\sqrt{1-\left(f_{mn}/f\right)^2} < c$.

Wavelength inside the waveguide $\lambda_g = \dfrac{\lambda_0}{\sqrt{1-\left(\omega_{mn}/\omega\right)^2}} = \dfrac{\lambda_0}{\sqrt{1-\left(f_{mn}/f\right)^2}}$

Characteristic impedance $\eta_{TE} = \dfrac{377}{\sqrt{1-\left(\dfrac{\omega_{mn}}{\omega}\right)^2}} = \dfrac{377}{\sqrt{1-\left(\dfrac{f_{mn}}{f}\right)^2}}$

9.2.2 TM Waves in Rectangular Waveguide

The problem is to solve $\left[\dfrac{\partial^2}{\partial^2 x}+\dfrac{\partial^2}{\partial^2 y}+\left(\dfrac{\omega}{c}\right)^2-k^2\right]E_z=0$ (since $B_z=0$)

subject to boundary condition $\vec{E}^{\parallel}=0$.

We will get similar expression as we have derived in TE waves. Thus

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad \text{where } m=1,2,3.....\ and\ n=1,2,3......$$

This solution is called the TM_{mn} mode. (The first index is conventionally associated with the larger dimension.)

Formula for cutoff frequency ω_{mn}, wave velocity v, group velocity v_g and λ_g are same as TE waves.

Characteristic impedance $\eta_{TM} = 377\sqrt{1-\left(\dfrac{w_{mn}}{w}\right)^2}$

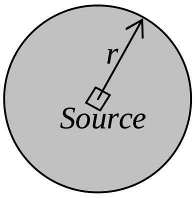
The fundamental mode is TM_{11} and $f_{11}=\dfrac{c}{2}\sqrt{\left(1/a\right)^2+\left(1/b\right)^2}$.

10. Radiation- from moving charges

10.1 Radiation

Accelerating charges and changing current produce electromagnetic waves (radiation).

If the source is localized near the origin, the total power passing out through the spherical shell of radius r is the integral of pointing vector:



$$P(r) = \oiint \vec{S}.d\vec{a} = \frac{1}{\mu_0}\oiint \left(\vec{E}\times\vec{B}\right).d\vec{a}$$

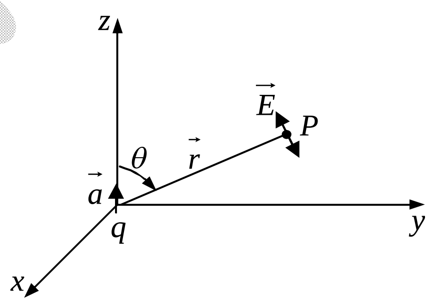
The power radiated is the limit of this quantity as r goes to infinity:

$$P_{rad} \equiv \lim_{r\to\infty} P(r)$$

This is the energy (per unit time) that is transported out to infinity, and never comes back.

10.1.1 Power radiated by a point charge

The electric field $\vec{E}$ and magnetic field $\vec{B}$ of an EM wave due to a point charge q, having acceleration $\vec{a}$ at any point P (position vector $\vec{r}$) is given by



$$\left|\vec{E}\right| \propto \frac{qa\sin\theta}{r}, \left|\vec{B}\right| \propto \frac{qa\sin\theta}{r}$$

(where θ is the angle which $\vec{r}$ makes with z-direction)

Thus $\vec{S} \propto \frac{q^2a^2\sin^2\theta}{r^2} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$

The total power radiated is evidently

$$P = \oint \vec{S}.d\vec{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta\, d\theta\, d\phi = \frac{\mu_0 q^2 a^2}{6\pi c}$$

The direction of $\vec{E}$ can be determined from the following rule:

(i) $\vec{r}$ *and* $\vec{E}$ *is always perpendicular.*

(ii) $\vec{a}$, $\vec{r}$ *and* $\vec{E}$ lies in one plane.