Finding the Mounting Position of a Sensor by Solving a Homogeneous Transform Equation of the Form AX=XB

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ABSTRACT

In order to use a wrist-mounted sensor (such as a camera, a range sensor, or a tactile sensor) for a robot task, the position of the sensor with respect to T_6 (wrist of robot) must be known. We can find the mounting position of the sensor by moving the robot and observing the resulting motion of the sensor. This yields a homogeneous transform equation of the form AX=XB, where A is the change in T_6 due to the arm movement, B is the resulting sensor displacement, and X is the sensor position relative to T_6. A and B are known, since A can be computed from the encoder values and B can be found by the sensor system. The solution to an equation of this form has one degree of rotational freedom and one degree of translational freedom. In order to solve for X (the sensor position) uniquely, it is necessary to make two arm movements and form a system of two equations of the form: $A_1X=XB_1$ and $A_2X=XB_2$. A closed-form solution to this system of equations is presented. The necessary condition for a unique solution is that the axes of rotation of A_1 and A_2 are neither parallel or antiparallel to one another. The theory is supported by simulation results.

1. Introduction

The investigation into the solution of the homogeneous transform equation of the form A X = X B, where A are B are known and X is unknown, is motivated by a need to solve for the position between a wrist-mounted sensor and the manipulator wrist center (T_6).

Much research has been done on using a sensor to locate an object. The three-dimensional position and orientation of an object can be found by monocular vision, stereo vision, dense/sparse range sensing, or tactile sensing. Monocular vision locates an object using a single view, and the object dimensions are assumed to be known apriori [2,6,8,10,13,22,29,31,32]. Stereo vision uses two views instead of one so that the range information of feature points can be found [1,6,12,14,20,24,32]. A dense range sensor scans a region of the world and there are as many sensed points as its resolution allows [3,7,17,25]. A sparse range sensor scans only a few points, and if the sensed points are not sufficient to locate the object, additional points will be sensed [5,15,16]. Tactile sensing is similar to sparse range sensing in that it obtains the same information: range and surface normal of the sensed points [4,15,16].

A sensing system refers to object positions with respect to a coordinate frame attached to the sensor, but robot motions are specified by the wrist positions. In order to use the sensor information for a robot task, the relative position between the sensor and the wrist must be known.

Direct measurements are difficult because there may be obstacles to obstruct the measurement path, the points of interests may be inside a solid and unreachable, and the coordinate frames may differ in their orientations. The measurement path can be obstructed by the geometry of the sensor or the robot, the sensor mount, wires, etc. The coordinate frames that are unreachable includes T_6 and the camera frame: T_6 is unreachable because it is the intersection of three link axes, the camera frame is unreachable because its origin is at the focal point, inside the camera. Instead of direct measurement, we can compute the camera position by displacing the robot and observing the changes in the sensor frame using the sensor system. This method works for any sensors capable of finding the three-dimensional position and orientation of an object. Figures 1.1 and 1.2 show the cases of a monocular vision system and a robot hand with tactile sensors. In both examples, a homogeneous transform equation of the form A X=X B is formed, where A is the movement of the robot, B is the resulting displacement of the sensor, and X is the unknown transform between the sensor and the robot wrist. Let T_{6_1} and T_{6_2} be the robot wrist positions before and after the motion, respectively. Since T_{6_1} and T_{6_2} can be calculated by the robot controller from the joint encoder values, A can be calculated by $A=T_{6_1}^{-1}T_{6_2}$. Similarly, if OBJ_1 and OBJ_2 are the object positions with respect ot the sensor before and after the motion, the displacement of the sensor can be calculated by $B=OBJ_1OBJ_2^{-1}$.

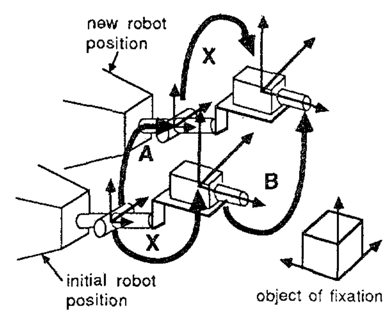

Figure 1.1. Finding the mounting position of a camera by solving a homogeneous transform equation of the form AX=XB, where A is the robot motion, B is the resulting camera motion, and X is the camera mounting position.

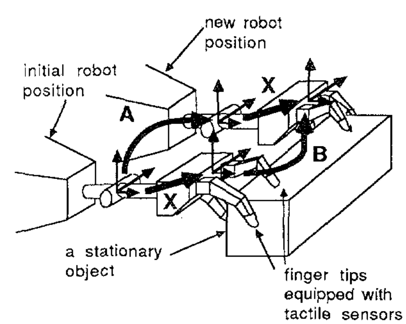

Figure 1.2. Finding the mounting position of a robot hand equipped with tactile sensors, by solving a homogeneous transform equation of the form AX=XB. A is the robot motion, B is the resulting motion of the hand coordinate frame, and X is the mounting position of the hand.

Matrix equations of the form $A X = X B$ have been discussed in linear algebra [11]; however, no known work has been done for homogeneous transform equations, in which case the matrices are much more restricted and a geometric interpretation is desired. Using Gantmacher's method, the solution to the 3×3 rotational part of X (R_X) is any linear combination of n linearly independent matrices: $R_X = k_1 M_1 + \cdots + k_n M_n$, where n is determined by properties of eigenvalues of R_A and R_B (rotational parts of A and B), $k_1, \cdots, k_n$ are arbitrary constants, and $M_1, \cdots M_n$ are linearly independent matrices. Gantmacher's solution is for general matrices; the given solution may not be a homogeneous transform. To restrict the solution to homogeneous transforms, we must impose the conditions that the 3×3 rotational part of the solution be orthonormal and that the right-handed screw rule is satisfied. These restrictions will result in non-linear equations in terms of $k_1, \cdots k_n$. There are many disadvantages using this method: (1) Only iterative solutions are possible, since non-linear equations are involved. (2) Iterative solutions are not useful when there are infinite number of solutions. Different initial estimates will converge to different solutions, and it is not possible to generalize the solutions. (3) The solution cannot be expressed symbolically. (4) Geometric interpretation of the solution is not possible and the conditions for uniqueness cannot be specified in geometrical terms.

2. Homogeneous Transforms and Rotation about an Arbitrary Axis

Homogeneous transforms [28] can be viewed as the relative position and orientation of a coordinate frame with respect to another coordinate frame. The elements of a homogeneous transform T is usually denoted as follows:

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.1}$$

We also denote $[n_x,n_y,n_z]^T$ as $\mathbf{a}$, $[o_x,o_y,o_z]^T$ as $\mathbf{o}$, and $[a_x,a_y,a_z]^T$ as $\mathbf{a}$. $\mathbf{n}$, $\mathbf{o}$, and $\mathbf{a}$ can be interpreted as unit vectors which indicate the $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ directions of coordinate frame T; p can be viewed as the origin of T. The vectors $\mathbf{n}$, $\mathbf{o}$, $\mathbf{a}$ and $\mathbf{p}$ are referenced with respect to a frame represented by a transform to which T is post-multiplied. If there is no frame to the left of T, then $\mathbf{n}$, $\mathbf{o}$, $\mathbf{a}$, and $\mathbf{p}$ will be vectors relative to the world or absolute frame.

We will refer to the upper-left 3×3 submatrix of T as the rotational submatrix since it contains information about the orientation of the coordinate frame. A rotational submatrix can be expressed as a rotation around an arbitrary axis. From [28], the matrix representing a right-hand-rule rotation of θ around an axis $[k_x,k_y,k_z]^T$ is :

$$\mathrm{Rot}(k,\theta)=\begin{bmatrix} k_xk_x\mathrm{vers}\theta+\cos\theta & k_yk_x\mathrm{vers}\theta-k_z\sin\theta & k_zk_x\mathrm{vers}\theta+k_y\sin\theta \\ k_xk_y\mathrm{vers}\theta+k_z\sin\theta & k_yk_y\mathrm{vers}\theta+\cos\theta & k_zk_y\mathrm{vers}\theta-k_x\sin\theta \\ k_xk_z\mathrm{vers}\theta-k_y\sin\theta & k_yk_z\mathrm{vers}\theta+k_z\sin\theta & k_zk_z\mathrm{vers}\theta+\cos\theta \end{bmatrix}$$

$$\text{where } \mathrm{vers}\theta = (1-\cos\theta). \tag{2.2}$$

In this paper, we will follow the convention that $0 \le \theta \le \pi$. From Paul's text [28], we have the following two equations:

$$\cos\theta = \frac{1}{2}(n_x+o_y+a_z-1) \tag{2.3}$$

and

$$\sin\theta = \pm\frac{1}{2}\sqrt{(o_z-a_y)^2+(a_x-n_z)^2+(n_y-o_x)^2}. \tag{2.4}$$

Since $0 \le \theta \le \pi$, we only take the positive sign of Equation 2.4. Thus, we have only one solution for θ:

$$\theta = \mathrm{atan2}(\sqrt{(o_z-a_y)^2+(a_x-n_z)^2+(n_y-o_x)^2}, n_x+o_y+a_z-1). \tag{2.5}$$

We can now find $\mathbf{k}$ using θ computed by Equation 2.5. The set of equations used depends on whether n_x, o_y, or a_z is most positive. From Paul's text, if n_x is most positive,

$$k_x=\mathrm{sgn}(o_z-a_y)\sqrt{\frac{n_x-\cos\theta}{\mathrm{vers}\theta}}, \tag{2.6a}$$

$$k_y=\frac{n_y+o_x}{2k_x\mathrm{vers}\theta},\quad k_z=\frac{a_x+n_z}{2k_x\mathrm{vers}\theta}, \tag{2.6b,c}$$

where $\mathrm{sgn}(e)=+1$ if $e \ge 0$ and $\mathrm{sgn}(e)=-1$ if $e<0$. (Note that our definition of sgn(e) is different from that in Paul's text. We will discuss this further on.) If o_y is the most positive,

$$k_y=\mathrm{sgn}(a_x-n_z)\sqrt{\frac{o_y-\cos\theta}{\mathrm{vers}\theta}}, \tag{2.7a}$$

$$k_x=\frac{n_y+o_x}{2k_y\mathrm{vers}\theta},\quad k_z=\frac{o_z+a_y}{2k_y\mathrm{vers}\theta}. \tag{2.7b,c}$$

Finally, if a_z is the most positive,

$$k_z=\mathrm{sgn}(n_y-o_x)\sqrt{\frac{a_z-\cos\theta}{\mathrm{vers}\theta}}, \tag{2.8a}$$

$$k_x=\frac{a_x+n_z}{2k_z\mathrm{vers}\theta},\quad k_y=\frac{o_z+a_y}{2k_z\mathrm{vers}\theta}. \tag{2.8b,c}$$

From a geometric point of view, when $\theta=\pi$, there are two solutions to $\mathbf{k}$, one opposite to the other. Also, when $\theta=\pi$, we can see from Equation 2.2 that $o_z-a_y=0$, $a_x-n_z=0$, and $n_y-o_x=0$. In this case, we can use either $\mathrm{sgn}(0)=+1$ or $\mathrm{sgn}(0)=-1$ for Equation 2.6a, Equation 2.7a, and Equation 2.8a; we have two solutions for $\mathbf{k}$. However, it is desirable to use some convention so that we can solve for $\mathbf{k}$ uniquely even when $\theta=\pi$. To do this, we can modify Paul's definition for the sgn function. In Paul's text, sgn(0) can be either +1 or -1. We define $\mathrm{sgn}(0)=+1$, so that we have unique θ and $\mathbf{k}$ for each rotational matrix.

A general rotational matrix can be represented as the exponent of a skew-symmetric matrix [26,23]:

$$\mathrm{Rot}(\mathbf{k},\theta) = e^{K\theta},\quad \text{where } K = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}. \tag{2.9}$$

Lemma 1: The eigenvalues of a general rotation matrix not equal to identity are 1, $e^{j\theta}$, and $e^{-j\theta}$. Let $e^{j\theta}$ and $e^{-j\theta}$ be denoted by λ and $\bar{\lambda}$. Then θ can be calculated by:

$$\theta = \mathrm{atan2}(\left|\mathrm{Re}(\lambda-\bar{\lambda})\right|, \lambda+\bar{\lambda}). \tag{2.10}$$

Proof: This is a well known property of a rotational matrix. See [33], or both [21] and [28].
□

Lemma 2: For a general rotation matrix not equal to identity, the eigenvector corresponding to the eigenvalue 1 can be expressed as a vector with real components and is either parallel or antiparallel to the axis of rotation. The other two eigenvectors cannot be expressed as real vectors.

Proof: Follows from Lemma 1 and Fisher's thesis [9]. More details can be found in a technical report [33]. □

Lemma 3: Non-identity rotational matrices are commutative if and only if their axes of rotation are parallel or antiparallel to one another.

Proof: Follows from Lemmas 1 and 2. See [33] for more details. □

3. Solution to the Equation $A X = X B$

Dividing a homogeneous transform into its rotational and translational components, $A X = X B$ becomes

$$\begin{bmatrix} \mathbf{R}_A & \mathbf{P}_A \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R}_X & \mathbf{P}_X \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_X & \mathbf{P}_X \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R}_B & \mathbf{P}_B \\ \mathbf{0} & 1 \end{bmatrix}, \tag{3.1}$$

where $\mathbf{R}$ is a 3×3 rotational matrix, $\mathbf{P}$ is a 3×1 translation vector,
and $\mathbf{0}$ is a row of 3 zeros. Multiplying out and equating the first row of Equation 3.1, we have

$$\mathbf{R}_A\mathbf{R}_X = \mathbf{R}_X\mathbf{R}_B \text{ and } \mathbf{R}_A\mathbf{P}_X + \mathbf{P}_A = \mathbf{R}_X\mathbf{P}_B + \mathbf{P}_X. \qquad (3.2\text{-}3)$$

Also, if R_X is fixed, P_X has one degree of freedom.

Lemma 4: If R_A and R_B are rotation matrices such that $R_A\, R = R\, R_B$ for any rotation matrix R, then R_A and R_B must have the same angle of rotation. □

Proof: From Lemma 1, the product of the eigenvalues of a rotational matrix is 1. Thus a rotational matrix has a determinant of 1 and is always invertible. R_A and R_B are similar, since $R_A = R\, R_B\, R^{-1}$. R_A and R_B must have the same eigenvalues since similar matrices have the same eigenvalues [26]. From Lemma 1, R_A and R_B must have the same angle of rotation. □

Before we formally state and prove the solution to $R_AR_X=R_XR_B$ in Theorem 1, we first examine the geometry of the problem. Let us rewrite R_A and R_B as $\mathrm{Rot}(\mathbf{k}_A,\theta)$ and $\mathrm{Rot}(\mathbf{k}_B,\theta)$ respectively. Notice that R_A and R_B have the same angle of rotation from Lemma 4. We can now rewrite (3.2) as

$$\mathrm{Rot}(\mathbf{k}_A,\theta)R_X = R_X\mathrm{Rot}(\mathbf{k}_B,\theta). \qquad (3.4)$$

Using the geometrical interpretation of post-multiplication of homogeneous transforms [28], Equation 3.4 can be interpreted as follows: R_X is a coordinate frame such that rotating R_X about a vector ${}^{base}\mathbf{k}_A$ by β is equivalent to rotating R_X about ${}^{R_X}\mathbf{k}_B$ by the same amount, where ${}^{base}\mathbf{k}_A$ is referenced with respect to the base frame (the world frame), and ${}^{R_X}\mathbf{k}_B$ is referenced with respect to R_X. This is shown in Figure 3.1. In order that rotating β about ${}^{base}\mathbf{k}_A$ being the same as rotating about ${}^{R_X}\mathbf{k}_B$, ${}^{base}\mathbf{k}_A$ and ${}^{R_X}\mathbf{k}_B$ must be the same physical vector in the base frame. Rotating R_X in the figure about the axis of rotation by any angle will still satisfy the constraints posted by Equation 3.4. Thus the solution to Equation 3.2 has one degree of freedom. The general solution can be either written as $R_{XP}\mathrm{Rot}(\mathbf{k}_B,\beta)$ or $\mathrm{Rot}(\mathbf{k}_A,\beta)R_{XP}$, where R_{XP} is a particular solution. We will use the later form for the rest of the paper. It is not difficult to see that $\mathrm{Rot}(\mathbf{k}_A,\beta)R_{XP}$ is a solution for any β, if R_{XP} is a particular solution. Since R_{XP} is a particular solution, $\mathrm{Rot}(\mathbf{k}_A,\theta)R_{XP}=R_{XP}\mathrm{Rot}(\mathbf{k}_B,\theta)$. Also, since $\mathrm{Rot}(\mathbf{k}_A,-\beta)\mathrm{Rot}(\mathbf{k}_A,\beta)=I$,
$\mathrm{Rot}(\mathbf{k}_A,\theta)\mathrm{Rot}(\mathbf{k}_A,-\beta)\mathrm{Rot}(\mathbf{k}_A,\beta)R_{XP}=R_{XP}\mathrm{Rot}(\mathbf{k}_B,\theta)$. Using the commutative properties of rotational matrices with a common axis of rotation and that $\mathrm{Rot}(\mathbf{k}_A,-\beta)^{-1}=\mathrm{Rot}(\mathbf{k}_A,\beta)$, we have $\mathrm{Rot}(\mathbf{k}_A,\theta)\mathrm{Rot}(\mathbf{k}_A,\beta)R_{XP}=\mathrm{Rot}(\mathbf{k}_A,\beta)R_{XP}\mathrm{Rot}(\mathbf{k}_B,\theta)$, from which we can see that $\mathrm{Rot}(\mathbf{k}_A,\beta)R_{XP}$ is a solution. The geometric interpretation of the general solution is shown in Figure 3.2.

Definition: A homogeneous transform equation of the form AX=XB is *solvable* if there exist a homogeneous transform U such that $B=U^{-1}AU$.

Theorem 1: The general solution to the rotational part of a solvable homogeneous transform equation of the form

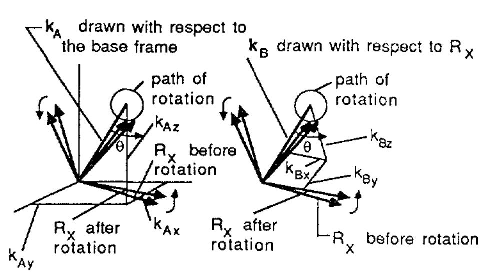

Figure 3.1. Rotating R_X about ${}^{base}\mathbf{k}_A$ by θ is equivalent to rotating R_X about ${}^{RX}\mathbf{k}_B$ by the same amount. $\mathbf{k}_A$ is the axis of rotation of A and $\mathbf{k}_B$ is the axis of rotation of B in the homogeneous transform equation AX=XB.

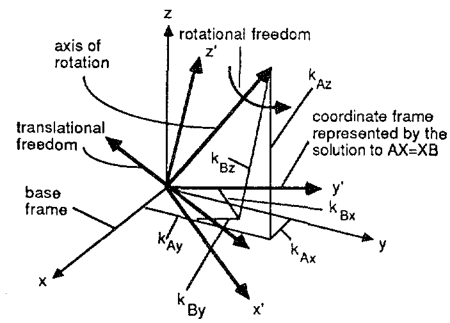

Figure 3.2. Geometric interpretation of the general solution to the equation AX=XB and the two degrees of freedom of the solution. $[k_{Ax}, k_{Ay}, k_{Az}]$ is the axis of rotation of A, and $[k_{Bx}, k_{By}, k_{Bz}]$ is the axis of rotation of B.

$R_AR_X=R_XR_B$, where $R_A\neq I$ and $R_B\neq I$, is

$$R_X=\mathrm{Rot}(\mathbf{k}_A,\beta)R_{XP}, \qquad (3.5)$$

where $\mathbf{k}_A$ is the axis of rotation of R_A, R_{XP} is a particular solution to the equation, and β is any arbitrary angle.

Proof: Assume $\mathrm{Rot}(\mathbf{k}_A)R_{XP}$ is not a general solution. Then, there must exist some rotation matrix R' such that

$$R_AR'=R'R_B, \qquad (3.6)$$

and $R'\neq\mathrm{Rot}(\mathbf{k}_A,\beta)R_{XP}$ for any β. Since R_{XP} is a particular solution to Equation 3.2, $R_AR_{XP} = R_{XP}R_B$, or $R_B=R_{XP}{}^{-1}R_AR_{XP}$. Substituting this into Equation 3.6, we have

$$R'^{-1}R_AR' = R_{XP}{}^{-1}R_AR_{XP}. \qquad (3.7)$$

Substituting $\mathrm{Rot}(\mathbf{k}_A,\beta)$ into R_A and rearranging, we have

$$\mathrm{Rot}(\mathbf{k}_A,\beta)R'R_{XP}{}^{-1} = R'R_{XP}{}^{-1}\mathrm{Rot}(\mathbf{k}_A,\beta). \qquad (3.8)$$

Thus, $\mathrm{Rot}(\mathbf{k}_A,\beta)$ and $R'R_{XP}{}^{-1}$ are commutative. If $R'R_{XP}{}^{-1}\neq I$, from Lemma 3, the axis of rotation of $R'R_{XP}{}^{-1}$ must be parallel or antiparallel to $\mathbf{k}_A$. Thus there must exist a γ such that $R'R_{XP}{}^{-1}=\mathrm{Rot}(\mathbf{k}_A,\gamma)$. We have $R'=\mathrm{Rot}(\mathbf{k}_A,\gamma)R_{XP}$, which is a contradiction. If $R'R_{XP}{}^{-1}=I$, $R'=R_{XP}\mathrm{Rot}(\mathbf{k}_A,0)$, which is also a contradiction. □

Next we will look at the translational part of the equation AX=XB. It has one degree of freedom, as shown in Figure 3.2. From Equation 3.3, we have

$$(R_A-I)P_X = R_XP_B-P_A. \qquad (3.9)$$

Theorem 2: The translational part (P_X) of the solution to a solvable homogeneous transform equation AX=XB, where $R_A\neq I$ and $R_B\neq I$, has one degree of freedom.

Proof: We can see that R_A-I is similar to a matrix of rank two if $R_A\neq I$:

$$R_A-I = E\Lambda_AE^{-1}-EIE^{-1} = E\begin{bmatrix}0 & 0 & 0\\ 0 & \lambda-1 & 0\\ 0 & 0 & \bar{\lambda}-1\end{bmatrix}E^{-1}. \qquad (3.10)$$

Thus R_A-I must have a rank of two. Thus, from Equation 3.9, there may be no solution or there are infinite number of solutions to P_X. The first case is ruled out since the physical system guarantees the existence of a solution. The solution must exist and consist of all the vectors in the null space of R_A-I translated by a particular solution to Equation 3.9 [30]. The null space of R_A-I has a dimension of $3-\mathrm{rank}(R_A-I)$ thus the solution to Equation 3.9 has one degree of freedom. □

Finally, we need to find a particular solution to AX=XB. From the geometric interpretation of the general solution, we will show that any transformation that transform $\mathbf{k}_B$ into $\mathbf{k}_A$ is a solution.

Lemma 5:

$$\mathrm{Rot}(\mathrm{R}\mathbf{k},\theta)=\mathrm{R}\ \mathrm{Rot}(\mathbf{k},\theta)\ \mathrm{R}^{-1} \tag{3.11}$$

for any axis of rotation $\mathbf{k}$, any $\theta \subset [0,\pi]$, and any 3×3 rotation matrix R.

Proof: For the purpose of this proof, we will represent a rotation matrix in a form used by [23]. Let $[\mathbf{n}\ \mathbf{o}\ \mathbf{a}]$ be a homogeneous transform and $[\mathbf{n}'\ \mathbf{o}'\ \mathbf{a}']$ be the former transform rotated by $\mathrm{Rot}(\mathbf{k},\theta)$. Thus

$$\mathrm{Rot}(\mathbf{k},\theta)=[\mathbf{n}\ \mathbf{o}\ \mathbf{a}][\mathbf{n}'\ \mathbf{o}'\ \mathbf{a}']^{-1}. \tag{3.12}$$

If we premultiply $\mathbf{n}$, $\mathbf{o}$, $\mathbf{a}$, $\mathbf{n}'$, $\mathbf{o}'$, $\mathbf{a}'$, and $\mathbf{k}$ by R, the angular relationship between $\mathrm{R}\mathbf{n}$, $\mathrm{R}\mathbf{o}$, $\mathrm{R}\mathbf{a}$, $\mathrm{R}\mathbf{n}'$, $\mathrm{R}\mathbf{o}'$, $\mathrm{R}\mathbf{a}'$, and $\mathrm{R}\mathbf{k}$ will be the same as before the premultiplication, because of the angular preservation property of R as a rotational matrix. Since $\mathbf{n}'=\mathrm{Rot}(\mathbf{k},\theta)\mathbf{n}$ before the premultiplication, $\mathrm{R}\mathbf{n}'=\mathrm{Rot}(\mathrm{R}\mathbf{k},\theta)\mathrm{R}\mathbf{n}$. Similar relationships hold for other vectors as well; therefore, $[\mathrm{R}\mathbf{n}'\ \mathrm{R}\mathbf{o}'\ \mathrm{R}\mathbf{a}']=\mathrm{Rot}(\mathrm{R}\mathbf{k},\theta)[\mathrm{R}\mathbf{n}\ \mathrm{R}\mathbf{o}\ \mathrm{R}\mathbf{a}]$ and

$$\mathrm{Rot}(\mathrm{R}\mathbf{k},\theta)=[\mathrm{R}\mathbf{n}'\ \mathrm{R}\mathbf{o}'\ \mathrm{R}\mathbf{a}'][\mathrm{R}\mathbf{n}\ \mathrm{R}\mathbf{o}\ \mathrm{R}\mathbf{a}]^{-1}. \tag{3.13}$$

From Equation 3.13, $\mathrm{Rot}(\mathrm{R}\mathbf{k},\theta) = \mathrm{R}\ [\mathbf{n}'\ \mathbf{o}'\ \mathbf{a}']\ [\mathbf{n}\ \mathbf{o}\ \mathbf{a}]^{-1}\mathrm{R}^{-1} = \mathrm{R}\ \mathrm{Rot}(\mathbf{k},\theta)\ \mathrm{R}^{-1}$. □

Theorem 3: Any rotation matrix R that satisfies

$$\mathbf{k}_A = \mathrm{R}\ \mathbf{k}_B \tag{3.14}$$

is a solution to

$$\mathrm{R}_A\mathrm{R}_X=\mathrm{R}_X\mathrm{R}_B, \tag{3.15}$$

where $\mathbf{k}_A$ is the axes of rotation of R_A and $\mathbf{k}_B$ is the axes of rotation of R_B.

Proof: Let us rewrite Equation 3.15 as

$$\mathrm{Rot}(\mathbf{k}_A,\theta)\mathrm{R}_X=\mathrm{R}_X\mathrm{Rot}(\mathbf{k}_B,\theta). \tag{3.16}$$

Substituting R into R_X and $\mathrm{R}\mathbf{k}_B$ into $\mathbf{k}_A$, the left hand side becomes $\mathrm{Rot}(\mathrm{R}\mathbf{k}_B,\theta)\mathrm{R}$. By Lemma 5, this becomes $\mathrm{R}\mathrm{Rot}(\mathbf{k}_B,\theta)\mathrm{R}^{-1}\mathrm{R} = \mathrm{R}\mathrm{Rot}(\mathbf{k}_B,\theta)$, which is the same as the right hand side when R_X is replaced by R. □

Since any rotational matrix R such that $\mathbf{k}_A=\mathrm{R}\mathbf{k}_B$ is a particular solution, one particular solution is a rotation about an axis perpendicular to both $\mathbf{k}_B$ and $\mathbf{k}_A$. Thus,

$$\mathrm{R}_{XP}=\mathrm{Rot}(\mathbf{v},\omega),\ \text{where } \mathbf{v}=\mathbf{k}_B\times\mathbf{k}_A, \tag{3.17}$$

$$\text{and } \omega=\mathrm{atan2}(\left|\mathbf{k}_B\times\mathbf{k}_A\right|,\mathbf{k}_B\cdot\mathbf{k}_A).$$

4. Solving for a Unique Solution Using Two Simultaneous Equations

We have seen that the solution to a homogeneous transform equation of the form AX=XB has two degrees of freedom. We can find a unique solution to this equation if we have two equations of the form

$$\mathrm{A}_1\mathrm{X}=\mathrm{X}\mathrm{B}_1 \text{ and } \mathrm{A}_2\mathrm{X}=\mathrm{X}\mathrm{B}_2. \tag{4.1,2}$$

In order to obtain two such equations, we need to move the robot twice and use the vision system to find the corresponding changes in the camera frame.

A unique solution to R_X (the rotational part of X) can be found by associating the general solutions of the two equations $\mathrm{R}_{A_1}\mathrm{R}_X=\mathrm{R}_X\mathrm{R}_{B_1}$ and $\mathrm{R}_{A_2}\mathrm{R}_X=\mathrm{R}_X\mathrm{R}_{B_2}$. Let $\mathrm{R}_{XP_1}\mathrm{Rot}(\mathbf{k}_{A_1},\beta_1)$ and $\mathrm{R}_{XP_2}\mathrm{Rot}(\mathbf{k}_{A_2},\beta_2)$ be the general solutions to the above two equations, we then have

$$\mathrm{Rot}(\mathbf{k}_{A_1},\beta_1)\mathrm{R}_{XP_1}=\mathrm{Rot}(\mathbf{k}_{A_2},\beta_2)\mathrm{R}_{XP_2}. \tag{4.3}$$

Let the particular solutions be written as follows:

$$\mathrm{R}_{XP_i} = \begin{bmatrix} n_{x_i} & o_{x_i} & a_{x_i} & p_{x_i} \\ n_{y_i} & o_{y_i} & a_{y_i} & p_{y_i} \\ n_{z_i} & o_{z_i} & a_{z_i} & p_{z_i} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad i=1,2. \tag{4.4}$$

Rearranging and writing it in more condensed form, we have

$$\begin{bmatrix}
-n_{x_1}+k_{x_1}\mathbf{n}_1\cdot\mathbf{k}_{A_1} & (\mathbf{n}_1\times\mathbf{k}_{A_1})_x & n_{x_2}-k_{x_2}\mathbf{n}_2\cdot\mathbf{k}_{A_2} & (-\mathbf{n}_2\times\mathbf{k}_{A_2})_x \\
-o_{x_1}+k_{x_1}\mathbf{o}_1\cdot\mathbf{k}_{A_1} & (\mathbf{o}_1\times\mathbf{k}_{A_1})_x & o_{x_2}-k_{x_2}\mathbf{o}_2\cdot\mathbf{k}_{A_2} & (-\mathbf{o}_2\times\mathbf{k}_{A_2})_x \\
-a_{x_1}+k_{x_1}\mathbf{a}_1\cdot\mathbf{k}_{A_1} & (\mathbf{a}_1\times\mathbf{k}_{A_1})_x & a_{x_2}-k_{x_2}\mathbf{a}_2\cdot\mathbf{k}_{A_2} & (-\mathbf{a}_2\times\mathbf{k}_{A_2})_x \\
-n_{y_1}+k_{y_1}\mathbf{n}_1\cdot\mathbf{k}_{A_1} & (\mathbf{n}_1\times\mathbf{k}_{A_1})_y & n_{y_2}-k_{y_2}\mathbf{n}_2\cdot\mathbf{k}_{A_2} & (-\mathbf{n}_2\times\mathbf{k}_{A_2})_y \\
-o_{y_1}+k_{y_1}\mathbf{o}_1\cdot\mathbf{k}_{A_1} & (\mathbf{o}_1\times\mathbf{k}_{A_1})_y & o_{y_2}-k_{y_2}\mathbf{o}_2\cdot\mathbf{k}_{A_2} & (-\mathbf{o}_2\times\mathbf{k}_{A_2})_y \\
-a_{y_1}+k_{y_1}\mathbf{a}_1\cdot\mathbf{k}_{A_1} & (\mathbf{a}_1\times\mathbf{k}_{A_1})_y & a_{y_2}-k_{y_2}\mathbf{a}_2\cdot\mathbf{k}_{A_2} & (-\mathbf{a}_2\times\mathbf{k}_{A_2})_y \\
-n_{z_1}+k_{z_1}\mathbf{n}_1\cdot\mathbf{k}_{A_1} & (\mathbf{n}_1\times\mathbf{k}_{A_1})_z & n_{z_2}-k_{z_2}\mathbf{n}_2\cdot\mathbf{k}_{A_2} & (-\mathbf{n}_2\times\mathbf{k}_{A_2})_z \\
-o_{z_1}+k_{z_1}\mathbf{o}_1\cdot\mathbf{k}_{A_1} & (\mathbf{o}_1\times\mathbf{k}_{A_1})_z & o_{z_2}-k_{z_2}\mathbf{o}_2\cdot\mathbf{k}_{A_2} & (-\mathbf{o}_2\times\mathbf{k}_{A_2})_z \\
-a_{z_1}+k_{z_1}\mathbf{a}_1\cdot\mathbf{k}_{A_1} & (\mathbf{a}_1\times\mathbf{k}_{A_1})_z & a_{z_2}-k_{z_2}\mathbf{a}_2\cdot\mathbf{k}_{A_2} & (-\mathbf{a}_2\times\mathbf{k}_{A_2})_z
\end{bmatrix}
\begin{bmatrix} \cos\beta_1 \\ \sin\beta_1 \\ \cos\beta_2 \\ \sin\beta_2 \end{bmatrix}$$

$$= \begin{bmatrix}
-k_{x_2}\mathbf{n}_2\cdot\mathbf{k}_{A_2}+k_{x_1}\mathbf{n}_1\cdot\mathbf{k}_{A_1} \\
-k_{x_2}\mathbf{o}_2\cdot\mathbf{k}_{A_2}+k_{x_1}\mathbf{o}_1\cdot\mathbf{k}_{A_1} \\
-k_{x_2}\mathbf{a}_2\cdot\mathbf{k}_{A_2}+k_{x_1}\mathbf{a}_1\cdot\mathbf{k}_{A_1} \\
-k_{y_2}\mathbf{n}_2\cdot\mathbf{k}_{A_2}+k_{y_1}\mathbf{n}_1\cdot\mathbf{k}_{A_1} \\
-k_{y_2}\mathbf{o}_2\cdot\mathbf{k}_{A_2}+k_{y_1}\mathbf{o}_1\cdot\mathbf{k}_{A_1} \\
-k_{y_2}\mathbf{a}_2\cdot\mathbf{k}_{A_2}+k_{y_1}\mathbf{a}_1\cdot\mathbf{k}_{A_1} \\
-k_{z_2}\mathbf{n}_2\cdot\mathbf{k}_{A_2}+k_{z_1}\mathbf{n}_1\cdot\mathbf{k}_{A_1} \\
-k_{z_2}\mathbf{o}_2\cdot\mathbf{k}_{A_2}+k_{z_1}\mathbf{o}_1\cdot\mathbf{k}_{A_1} \\
-k_{z_2}\mathbf{a}_2\cdot\mathbf{k}_{A_2}+k_{z_1}\mathbf{a}_1\cdot\mathbf{k}_{A_1}
\end{bmatrix}, \tag{4.5}$$

where the notation $(\mathbf{u}\times\mathbf{v})_w$ denotes the w component of the cross product $\mathbf{u}\times\mathbf{v}$. Equation 4.5 is a system of linear equations involving $\cos\beta_1$, $\sin\beta_1$, $\cos\beta_2$ and $\sin\beta_2$. Once these values are solved for, we can find β_1 and β_2 by $\beta_1=\mathrm{atan2}(\sin\beta_1,\cos\beta_1)$ and $\beta_2=\mathrm{atan2}(\sin\beta_2,\cos\beta_2)$. Since we have more equations than unknown, from the point of view of linear algebra, we can have a system of inconsistent equations. However, in an ideal environment where there is no noise, the equations must be consistent because they originated from physical situations. Since the linear equations are physically constrained to be consistent, there are either a unique solution or an infinite number or solutions. Let us abbreviate Equation 4.5 to CY=D, if rank(C)=4, we can find four linearly independent rows of C to solve for Y uniquely. However, in real applications where noise is present, we can find a least-square-fit solution $\hat{\mathrm{Y}}$ by

$$\hat{\mathrm{Y}} = (\mathrm{C}^T\mathrm{C})^{-1}\mathrm{C}^T\mathrm{D}. \tag{4.6}$$

The translational part of X is constrained by Equation 3.3; thus, we have $\mathrm{R}_{A_1}\mathrm{P}_X+\mathrm{P}_{A_1}=\mathrm{R}_X\mathrm{P}_{B_1}+\mathrm{P}_X$ and $\mathrm{R}_{A_2}\mathrm{P}_X+\mathrm{P}_{A_2}=\mathrm{R}_X\mathrm{P}_{B_2}+\mathrm{P}_X$. Combining these two equations, we can solve for P_X by

$$\begin{bmatrix} \mathrm{R}_{A_1}-\mathrm{I} \\ \mathrm{R}_{A_2}-\mathrm{I} \end{bmatrix}\mathrm{P}_X = \begin{bmatrix} \mathrm{R}_X\mathrm{P}_{B_1}-\mathrm{P}_{A_1} \\ \mathrm{R}_X\mathrm{P}_{B_2}-\mathrm{P}_{A_2} \end{bmatrix}. \tag{4.7}$$

Rewriting Equation 4.7 as $\mathrm{E}\mathrm{P}_X=\mathrm{F}$, a least-square fit solution can be calculated by

$$\hat{\mathrm{P}}_X=(\mathrm{E}^T\mathrm{E})^{-1}\mathrm{E}^T\mathrm{F}. \tag{4.8}$$

Before we go into the necessary conditions for uniqueness, we need to prove two more lemmas.

Lemma 6: If R is a 3×3 rotational part of a homogeneous transform and $\mathrm{R}\neq\mathrm{I}$, any row of $(\mathrm{R}-\mathrm{I})$ is a linear combination of the transposes of the two eigenvectors corresponding to the two non-unity eigenvalues of R.

Proof: From Equation 3.10, we have

$$\mathrm{R-I}=\begin{bmatrix}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & \lambda-1 & 0\\ 0 & 0 & \bar{\lambda}-1\end{bmatrix}\begin{bmatrix}\mathbf{e}_1^{\mathrm{T}}\\ \mathbf{e}_2^{\mathrm{T}}\\ \mathbf{e}_3^{\mathrm{T}}\end{bmatrix}, \tag{4.9}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the eigenvectors of R corresponding to the eigenvalues 1, λ and $\bar{\lambda}$. Writing $\mathbf{e}_i^{\mathrm{T}}$ as $(e_{i_x}, e_{i_y}, e_{i_z})$ and rearranging Equation 4.9, we have

$$\mathrm{R-I}=(\lambda-1)\begin{bmatrix}e_{2_x}\mathbf{e}_2^{\mathrm{T}}\\ e_{2_y}\mathbf{e}_2^{\mathrm{T}}\\ e_{2_z}\mathbf{e}_2^{\mathrm{T}}\end{bmatrix}+(\bar{\lambda}-1)\begin{bmatrix}e_{3_x}\mathbf{e}_3^{\mathrm{T}}\\ e_{3_y}\mathbf{e}_3^{\mathrm{T}}\\ e_{3_z}\mathbf{e}_3^{\mathrm{T}}\end{bmatrix}. \ \square \tag{4.10}$$

Lemma 7: For two rotational matrices R_1 and R_2 whose axes of rotation are neither parallel nor antiparallel to one another, it is impossible that the sets of vectors $\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_2\}$ and $\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_3\}$ are both linearly dependent, where $\mathbf{e}_2$ and $\mathbf{e}_3$ are the eigenvectors of R_1 corresponding to the non-unity eigenvalues of R_1, and $\mathbf{f}_2$ and $\mathbf{f}_3$ are the eigenvectors of R_2 corresponding to the non-unity eigenvalues of R_2.

Proof: For any rotational matrix R and its hermitian R^H, $RR^H=R^HR=I$; hence R is a normal matrix [27]. From Key Theorem 9.2 in Noble's text, the eigenvectors of a normal matrix is hermitian. Let $\mathbf{e}_1$ be the eigenvector of R_1 corresponding to the unity eigenvalue. Note that $\mathbf{e}_1.\mathbf{f}_2$ and $\mathbf{e}_1.\mathbf{f}_3$ cannot be zero simultaneously. If they are simultaneously zero, we will have a system of two linearly independent homogeneous equations which will constraint $\mathbf{e}_1$ except for a scaling factor. Since the eigenvectors of R_2 are hermitian, $\mathbf{f}_1.\mathbf{f}_2$ and $\mathbf{f}_1.\mathbf{f}_3$ are zero. Similarly, this will constraint $\mathbf{f}_1$ up to a scaling factor. Thus $\mathbf{f}_1$ and $\mathbf{e}_1$ must be scalar product of one another. However, this contradicts the assumption that the axes of rotation ($\mathbf{e}_1$ and $\mathbf{f}_1$) are neither parallel nor antiparallel to one another. Therefore, the two dot products cannot be zero simultaneously. To prove that $\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{f}_2\}$ is linearly independent, We need to prove that $k_1=k_2=k_3=0$ if

$$k_1\mathbf{e}_2+k_2\mathbf{e}_3+k_3\mathbf{f}_2=0. \tag{4.11}$$

Taking the dot product of both sides of Equation 4.11 and using the fact that eigenvalues of a rotation matrix are hermitian to each another, we will have $k_3\mathbf{e}_1.\mathbf{f}_2=0$. If $\mathbf{e}_1.\mathbf{f}_2\neq 0$, then $k_3=0$. Equation 4.11 simplifies to

$$k_1\mathbf{e}_2+k_2\mathbf{e}_3=0. \tag{4.12}$$

Since $\mathbf{e}_2$ are $\mathbf{e}_3$ are hermitian, we have $k_1=k_2=0$. Therefore, $\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{f}_2\}$ are linearly independent if $\mathbf{e}_1.\mathbf{f}_2\neq 0$. When $\mathbf{e}_1.\mathbf{f}_2=0$, $\mathbf{e}_1.\mathbf{f}_3$ must be non-zero , from a previous argument in this proof. In this case, we can use a similar method to prove that $\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{f}_3\}$ is linearly independent. $\square$

Theorem 4: A consistent system of two solvable homogeneous transform equations of the form $A_1X=XB_1$ and $A_2X=XB_2$ has a unique solution if the axes of rotation for A_1 and A_2, $\mathbf{k}_{A_1}$ and $\mathbf{k}_{A_2}$, are neither parallel nor antiparallel to one another.

Proof for the rotational part: We have already seen that the general solution to AX=XB has one degree of rotational freedom; any solution revolving about $\mathbf{k}_A$ is still a solution. The solution to the system of homogeneous transform equations $A_1X=XB_1$ and $A_2X=XB_2$ consists of the overlap of the solutions to the individual equations , $\mathrm{Rot}(\mathbf{k}_{A_1},\beta_1)$ and $\mathrm{Rot}(\mathbf{k}_{A_2},\beta_2)$. Since this overlap is independent of the choices of the particular solutions, we can simplify Equation 4.5 by choosing a particular solution in the overlap region to be the particular solutions to both equations; i.e., $R_{XP_0}=R_{XP_1}=R_{XP_2}$. After replacing R_{XP_1} and R_{XP_2} in Equation 4.5 by R_{XP_0}, R_{XP_0} cancels out and we have

$$\begin{bmatrix}1-kx_1^2 & 0 & kx_2^2-1 & 0\\ -kx_1ky_1 & -kz_1 & kx_2ky_2 & kz_2\\ -kx_1kz_1 & ky_1 & kx_2kz_2 & -ky_2\\ -kx_1ky_1 & kz_1 & kx_2ky_2 & -kz_2\\ 1-ky_1^2 & 0 & ky_2^2-1 & 0\\ -ky_1kz_1 & -kx_1 & ky_2kz_2 & kx_2\\ -kx_1kz_1 & -ky_1 & kx_2kz_2 & ky_2\\ -ky_1kz_1 & kx_1 & ky_2kz_2 & -kx_2\\ 1-kz_1^2 & 0 & kz_2^2-1 & 0\end{bmatrix}\begin{bmatrix}\cos\beta_1\\ \sin\beta_1\\ \cos\beta_2\\ \sin\beta_2\end{bmatrix}=\begin{bmatrix}kx_2^2-kx_1^2\\ kx_2ky_2-kx_1ky_1\\ kx_2kz_2-kx_1kz_1\\ kx_2ky_2-kx_1ky_1\\ ky_2-ky_1\\ ky_2kz_2-ky_1kz_1\\ kx_2kz_2-kx_1kz_1\\ ky_2kz_2-ky_1kz_1\\ kz_2^2-kz_1^2\end{bmatrix}. \tag{4.13}$$

Let us abbreviate Equation 4.13 as $C'Y'=D'$. With the assumption of consistency, a unique solution exist if and only if the rank of Y' is 4, in which case we can pick 4 linearly rows to form 4 equations to solve for the same number of unknowns. Since the rank of C' is the same as the rank of $C'^{T}C'$ and that the later is a 4 by 4 matrix, C' has a rank of 4 if and only if $C'^{T}C'$ has full rank . Thus, we will have a unique solution iff the determinant of $C'^{T}C'$ is not equal to zero. We have used the SMP program [19] to express the determinant of $C'^{T}C'$ in symbolic form and have simplified it by making the following substitutions:

(1) $k_{x_i}^2+k_{y_i}^2+k_{z_i}^2=1$, i=1,2.

(2) $k_{x_1}k_{x_2}+k_{y_1}k_{y_2}+k_{z_1}k_{z_2}=\mathbf{k}_{A_1}.\mathbf{k}_{A_2}$.

(3) $1-k_{x_1}^2k_{x_2}^2-k_{y_1}^2k_{y_2}^2-k_{z_1}^2k_{z_2}^2-2k_{x_1}k_{x_2}k_{y_1}k_{y_2}-2k_{x_1}k_{x_2}k_{z_1}k_{z_2}-2k_{y_1}k_{y_2}k_{z_1}k_{z_2}=\sin^2\theta_{12}$.

The third substitution comes from the fact that $\left|\mathbf{k}_{A_1}\times\mathbf{k}_{A_2}\right|$ equals $\left|\mathbf{k}_{A_1}\right|\left|\mathbf{k}_{A_2}\right|\sin\theta_{12}$. The determinant is finally simplified to

$$\det(C'^{T}C')=4\sin^2\theta_{12}(\sin^2\theta_{12}-4)(\mathbf{k}_{A_1}.\mathbf{k}_{A_2}+1)(\mathbf{k}_{A_1}.\mathbf{k}_{A_2}-1). \tag{4.14}$$

The determinant is zero when $\sin\theta_{12}=\pm 2$, which is impossible, when $\sin\theta_{12}=0$, and when $\mathbf{k}_{A_1}.\mathbf{k}_{A_2}=\pm 1$. Thus we will have a non-unique solution only when $\mathbf{k}_{A_1}$ and $\mathbf{k}_{A_2}$ are parallel or antiparallel to one another. $\square$

Proof for the translational part: Since E is a 6 by 3 matrix, we have 6 equations and 3 unknowns. We know that these equations cannot be inconsistent since they originated from physical conditions. Therefore, we have a unique solution for P_X if and only if matrix E has a rank of 3, in which case we can pick 3 linearly independent rows for E to solve for P_X. From Lemma 6, any row of $(R_{A_1}-I)$ is a linear combination of the transposes of the eigenvectors $\mathbf{e}_2^{\mathrm{T}}$ and $\mathbf{e}_3^{\mathrm{T}}$ corresponding to the non-unity eigenvalues, and any row of $(R_{A_2}-I)$ is a linear combination of the transposes of the eigenvectors $\mathbf{f}_2^{\mathrm{T}}$ and $\mathbf{f}_3^{\mathrm{T}}$ corresponding to the non-unity eigenvalues. Since the rank of R_{A_1} is two (from the proof of Theorem 2), we can pick two linear independent rows from it, both are linear combinations of $\mathbf{e}_2$ and $\mathbf{e}_3$. We can also pick a row from R_{A_2}, which is a linear combination of $\mathbf{f}_2$ and $\mathbf{f}_3$, and combine it with the two rows from R_{A_1}. Since we know that if k_1 is not aligned with k_2, from Lemma 7, at least one of $\mathbf{f}_2^{\mathrm{T}}$ and $\mathbf{f}_3^{\mathrm{T}}$ must be linearly independent from $\mathbf{e}_2^{\mathrm{T}}$ and $\mathbf{e}_3^{\mathrm{T}}$. Say a row from R_{X_2} is $a\mathbf{f}_2^{\mathrm{T}}+b\mathbf{f}_3^{\mathrm{T}}$. We can always pick a row where $a\neq 0$ or a row where $b\neq 0$ since $\mathrm{rank}(R_{A_2})=2$. Thus, we can always find a row from R_{A_2} and combine it with two rows from R_{A_1} to form three linearly independent rows. We can use the corresponding three equations from Equation 4.7 to solve for a unique P_X. $\square$

5. Results

We have written a program calling IMSL routines [18] to test our method. A double-precision version is used on a VAX

780 machine. We have solved for the sensor position relative to the robot wrist by moving the robot twice and observing the changes in the sensor positions. From several test cases, we found that the solution was correct when the two robot movements have distinct axes of rotation. These results agreed with the theory developed in this paper. Due to page limits, a numerical example will not be presented here. One example can be found in a technical report [33].

6. Conclusions

We have described a method to find the position of a wrist-mounted sensor relative to a robot wrist, without using direct measurements. This will be useful for calibrating vision systems, range sensing systems and tactile sensing systems. The process can be automated and does not require any measuring equipment other than the sensor system itself.

Our method requires the solution to a homogeneous transform equation of the form AX=XB. We found that the solution is not unique; it has one degree of rotational freedom and one degree of translational freedom. We propose that we use two simultaneous equations of the form $A_1X=XB_1$ and $A_2X=XB_2$. Physically, this means that we move the robot twice and observe the changes in the sensor frame twice. The necessary condition for a unique solution is that the axes of rotation of A_1 and A_2 are neither parallel nor antiparallel to one another. A computer program was written for the proposed method. We have generated several test cases in which the condition for uniqueness was satisfied; all the computed solutions were found to be correct.

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