Registration problem in robotics using quaternion representation of $SO(3)$ group

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Abstract

Let $\{\mathbf{a}_i\}_{i=1}^N$ and $\{\mathbf{A}_i\}_{i=1}^N$ be three dimensional position vectors of N points in two coordinate systems S_a and S_A, respectively, that are related to each other by an unknown element g of $SE(3)$, special Euclidean group. The aim is solve the least square problem $\mathbf{A}_i \approx g.\mathbf{r}_i$ for $i = 1, \cdots, N$. In this work the problem is formulated mathematically and the Matlab code is provided.

Data registration problem

Let $\{\mathbf{a}_i\}_{i=1}^N$ and $\{\mathbf{A}_i\}_{i=1}^N$ be two sets of position vectors of N points in $\mathbb{R}^3$ that are measured, respectively, in two coordinate systems S_a and S_A. Let $SE(3)$, $SO(3)$ and $\mathbb{R}^3$ be, respectively, special Euclidean, special orthogonal (rotation) and translation groups in three dimensional Euclidean space. The aim, in data registration algorithm, is to find an element $F = [R, \mathbf{t}] \in SE(3)$, with rotation matrix $R \in SO(3)$ and translation vector $\mathbf{t} \in \mathbb{R}^3$, that solves the least square problem

$$\mathbf{A}_i \approx R\mathbf{a}_i + \mathbf{t}, \qquad i = 1, \cdots, N, \tag{1}$$

or,

$$\min_{[R,\mathbf{t}]\in SE(3)} \sum_{i=1}^{N} \|R\mathbf{a}_i + \mathbf{t} - \mathbf{A}_i\|^2 . \tag{2}$$

To tackle the minimization problem (2), it is useful to firstly define mean values

$$\bar{\mathbf{A}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{A}_i, \qquad \bar{\mathbf{a}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{a}_i, \tag{3}$$

and then the deviation vectors

$$\mathbf{A}'_i = \mathbf{A}_i - \bar{\mathbf{A}}, \qquad \mathbf{a}'_i = \mathbf{a}_i - \bar{\mathbf{a}}. \tag{4}$$

The total error in (2) is rewritten in terms of mean vectors, i.e. $\mathbf{a}_i = \mathbf{a}'_i + \bar{\mathbf{a}}$ and $\mathbf{A}_i = \mathbf{A}'_i + \bar{\mathbf{A}}$, to get,

$$\begin{aligned} E(R,\mathbf{t}) &= \sum_{i=1}^{N} \|R\mathbf{a}_i + \mathbf{t} - \mathbf{A}_i\|^2 = \sum_{i=1}^{N} \left\|R\mathbf{a}'_i + R\bar{\mathbf{a}} + \mathbf{t} - \mathbf{A}'_i - \bar{\mathbf{A}}\right\|^2 \\ &= \sum_{i=1}^{N} \|R\mathbf{a}'_i - \mathbf{A}'_i\|^2 + \sum_{i=1}^{N} \left\|\mathbf{t} - \bar{\mathbf{A}} + R\bar{\mathbf{a}}\right\|^2 + 2\left(\mathbf{t} - \bar{\mathbf{A}} + R\bar{\mathbf{a}}\right) . \sum_{i=1}^{N} (R\mathbf{a}'_i - \mathbf{A}'_i) \\ &= \sum_{i=1}^{N} \|R\mathbf{a}'_i - \mathbf{A}'_i\|^2 + N\left\|\mathbf{t} - \bar{\mathbf{A}} + R\bar{\mathbf{a}}\right\|^2 , \end{aligned} \tag{5}$$

in which the following identity was used, due to the definition of deviation vectors,

$$\sum_{i=1}^{N} (R\mathbf{a}'_i - \mathbf{A}'_i) = R\sum_{i=1}^{N} \mathbf{a}'_i - \sum_{i=1}^{N} \mathbf{A}'_i = \mathbf{0}. \tag{6}$$

It is clear that the minimum error in (5) occurs when $\mathbf{t} = \mathbf{t}^*$ such that

$$\mathbf{t}^* = \bar{\mathbf{A}} - R\,\bar{\mathbf{a}}. \tag{7}$$

Inserting $\mathbf{t} = \mathbf{t}^*$ in (5) leads to, defining $e\,(R) := E\,(R, \mathbf{t} = \mathbf{t}^*)$,

$$e\,(R) = \sum_{i=1}^{N} \|R\mathbf{a}'_i - \mathbf{A}'_i\|^2 = \sum_{i=1}^{N} \|\mathbf{A}'_i\|^2 + \sum_{i=1}^{N} \|\mathbf{a}'_i\|^2 - 2\sum_{i=1}^{N} \mathbf{A}'_i . R\mathbf{a}'_i. \tag{8}$$

The minimum of $e(R)$ in (8) occurs where the term $\sum_{i=1}^{N} \mathbf{A}'_i.R\mathbf{a}'_i$ is maximized. The equivalent problem to be solved is, therefore,

$$\max_{R\in SO(3)} \sum_{i=1}^{N} \mathbf{A}'_i.R\mathbf{a}'_i. \tag{9}$$

Let $q = (q_0, \mathbf{q}) \in \mathbb{H}^4$ be the unit quaternion generating $R \in SO(3)$ and $q^* = (q_0, -\mathbf{q}) \in \mathbb{H}^4$ be its conjugate. For the vectors $\mathbf{a}'_i, \mathbf{A}'_i \in \mathbb{R}^3$, let $a'_i = (0, \mathbf{a}'_i) \in \mathbb{H}^4$ and $A'_i = (0, \mathbf{A}'_i) \in \mathbb{H}^4$ be pure quaternions, i.e. their scalar parts are zero. Also, the multiplication of quanternions $p, q \in \mathbb{H}$ is defined by

$$q.p = (q_0p_0 - \mathbf{q}.\mathbf{p}, q_0\mathbf{p} + p_0\mathbf{q} + \mathbf{q}\times\mathbf{p}).$$

The vector $\mathbf{b}_i := R\mathbf{a}_i$ is the vector part of the quantiernion $b_i := qa'_iq^*$, i.e. $q.a'_i.q^* = \left(0, \mathbf{b}_i^T\right)$. Accordingly, one can extends three-dimenional vectors in (9) as pure quaternions and write corresponding expression

$$\sum_{i=1}^{N} \mathbf{A}'_i.R\mathbf{a}'_i = \sum_{i=1}^{N} (R\mathbf{a}'_i).\mathbf{A}'_i = \sum_{i=1}^{N} (0, R\mathbf{a}'_i).(0, \mathbf{A}'_i) = \sum_{i=1}^{N} (qa'_iq^*).A'_i = \sum_{i=1}^{N} (qa'_i).(A'_iq),$$

where $a'_i = (0, \mathbf{a}'_i) := \left(0, a'_{i,1}, a'_{i,2}, a'_{i,3}\right)$ and $A'_i = (0, \mathbf{A}'_i) := \left(0, A'_{i,1}, A'_{i,2}, A'_{i,3}\right)$.

To transform quaternion multiplication to matrix product, the two following matrices are defined,

$$P_i = \begin{pmatrix} 0 & -a'_{i,1} & -a'_{i,2} & -a'_{i,3} \\ a'_{i,1} & 0 & a'_{i,3} & -a'_{i,2} \\ a'_{i,2} & -a'_{i,3} & 0 & a'_{i,1} \\ a'_{i,3} & a'_{i,2} & -a'_{i,1} & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & -A'_{i,1} & -A'_{i,2} & -A'_{i,3} \\ A'_{i,1} & 0 & -A'_{i,3} & A'_{i,2} \\ A'_{i,2} & A'_{i,3} & 0 & -A'_{i,1} \\ A'_{i,3} & -A'_{i,2} & A'_{i,1} & 0 \end{pmatrix}. \tag{10}$$

The expression (9) then becomes,

$$\begin{aligned} \max_{R\in SO(3)} \sum_{i=1}^{N} \mathbf{A}'_i.R\mathbf{a}'_i &= \max_{q\in\mathbb{H}} \sum_{i=1}^{N} (qa'_i).(A'_iq) = \max_{q\in\mathbb{H}} \sum_{i=1}^{N} (P_iq).(Q_iq) = \max_{q\in\mathbb{H}} \sum_{i=1}^{N} q^TP_i^TQ_iq \\ &= \max_{q\in\mathbb{H}} q^T\left(\sum_{i=1}^{N} P_i^TQ_i\right)q := \max_{q\in\mathbb{H}} q^TMq, \end{aligned} \tag{11}$$

where

$$M = \sum_{i=1}^{N} P_i^TQ_i. \tag{12}$$

It is easy to show, by direct evaluation, that the matrix $P_i^TQ_i$ is symmetric, i.e. $\left(P_i^TQ_i\right)^T = P_i^TQ_i$, hence M is also symmetrc. Therefore, the eigenvalues of M are all real, say $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. Also, let $\{w_1, w_2, w_3, w_4\}$ be set of corresponding unit orthogonal eigenvectors. This set spans $\mathbb{R}^4$, hence evey quaternion in $\mathbb{H} \subset \mathbb{R}^4$ can be expanded in terms of the eigenvectors. Writing $q = c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4$ for unknown coefficients $\{c_i\}_{i=1}^4$, and using eigen-pair equation $M w_i = \lambda_i w_i$, then the equation (11) gives

$$\max_{R \in SO(3)} \sum_{i=1}^{N} \mathbf{A}'_i . R\mathbf{a}'_i = \max_{q \in \mathbb{H}} q^T M q = \max_{q \in \mathbb{H}} \left(\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 + \lambda_4 c_4^2 \right), \tag{13}$$

and the unit quaternion condition is $c_1^2 + c_2^2 + c_3^2 + c_4^2 = 1$, The inequality $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ shows that the solution is $c_1 = 1$ and $c_2 = c_3 = c_4 = 0$. Therefore, $q = w_1$, i.e. the eigenvector of M with largest eigenvalue.

Also, this solution gives the global maximum of $\sum_{i=1}^{N} \mathbf{A}'_i . R\mathbf{a}'_i$ or, equivalenty, the global minimum of total error function (5). To see this, one may take a solution (c_1, c_2, c_3, c_4) such that $c_1 = 1 - c_2^2 - c_3^2 - c_4^2$. Then, the inequality $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ implies that

$$\begin{aligned} F(c_1, c_2, c_3, c_4) &:= \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 + \lambda_4 c_4^2 \\ &= \lambda_1 - (\lambda_1 - \lambda_2)\, c_2^2 - (\lambda_1 - \lambda_3)\, c_3^2 - (\lambda_1 - \lambda_4)\, c_4^2 \\ &\leq \lambda_1 = F(1, 0, 0, 0), \end{aligned}$$

as $\lambda_1 - \lambda_i \geq 0$ or $-(\lambda_1 - \lambda_i)\, c_i^2 \leq 0$ for $i = 2, 3, 4$. This argument shows that $F(c_1, c_2, c_3, c_4) \leq F(1, 0, 0, 0)$ for arbitrary $\{c_i\}_{i=1}^4$ satisfying $c_1^2 + c_2^2 + c_3^2 + c_4^2 = 1$, or, the solution $q = w_1$ is the global maximum of $q^T M q$.

Writing the components $q = w_1 = (q_0, \mathbf{q}) = (q_0, q_1, q_2, q_3)$, the rotation matrix associated to q is

$$R^* = R(q) = \begin{pmatrix} 1 - 2\left(q_2^2 + q_3^2\right) & 2\left(q_1 q_2 - q_3 q_0\right) & 2\left(q_1 q_3 + q_2 q_0\right) \\ 2\left(q_1 q_2 + q_3 q_0\right) & 1 - 2\left(q_1^2 + q_3^2\right) & 2\left(q_2 q_3 - q_1 q_0\right) \\ 2\left(q_1 q_3 - q_2 q_0\right) & 2\left(q_2 q_3 + q_1 q_0\right) & 1 - 2\left(q_1^2 + q_2^2\right) \end{pmatrix}. \tag{14}$$

Inserting the rotation matrix $R = R^*$ in equation (7), i.e. $\mathbf{t} = \mathbf{t}^* = \bar{\mathbf{A}} - R.\bar{\mathbf{a}}$ also gives the translation vector $\mathbf{t}$ that completes the data registration problem.

To estimate the accuracy of the registration problem, the mean Euclidean error is defined by

$$\text{meanError} = \frac{1}{N} \sqrt{\sum_{i=1}^{N} \|\mathbf{e}_i\|^2} = \frac{1}{N} \sqrt{\sum_{i=1}^{N} \|R^* \mathbf{a}_i + \mathbf{t}^* - \mathbf{A}_i\|^2}, \tag{15}$$

that will be used to justify accuracy of the registration problem.