VECTOR DOT, CROSS PRODUCT, AND RAYS

- References
  - Chapter 2.11 - 2.13, 9.2.1

DOT PRODUCT (2.11)

- Symbolically: $\vec{v} \bullet \vec{w}$
- Numeric Interpretation #1: $\boxed{\vec{v} \bullet \vec{w} = \overrightarrow{v_1} * \overrightarrow{w_1} + \overrightarrow{v_2} * \overrightarrow{w_2} + \cdots + \overrightarrow{v_n} * \overrightarrow{w_n}}$
- Numeric Example:

$$\begin{bmatrix} 4 & 3 & -7 \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 5 \end{bmatrix}$$
$$= 4 * 0 + 3 * 2 + (-7) * 5$$
$$= 0 + 6 - 35$$
$$= 29$$

- Observations
  - Result is a scalar
  - Commutative
  - Obeys the distributive rule:

$$\vec{v} = \begin{bmatrix} 4 & 3 & -7 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 & 2 & 5 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 & -8 & 6 \end{bmatrix}$$
$$\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$$
$$\begin{bmatrix} 1 & -8 & 6 \end{bmatrix} \bullet \begin{bmatrix} 4 & 5 & -2 \end{bmatrix} = (4 - 24 - 42) + (0 - 16 + 30)$$
$$(4 - 40 - 12) = -62 + 14 = -48$$

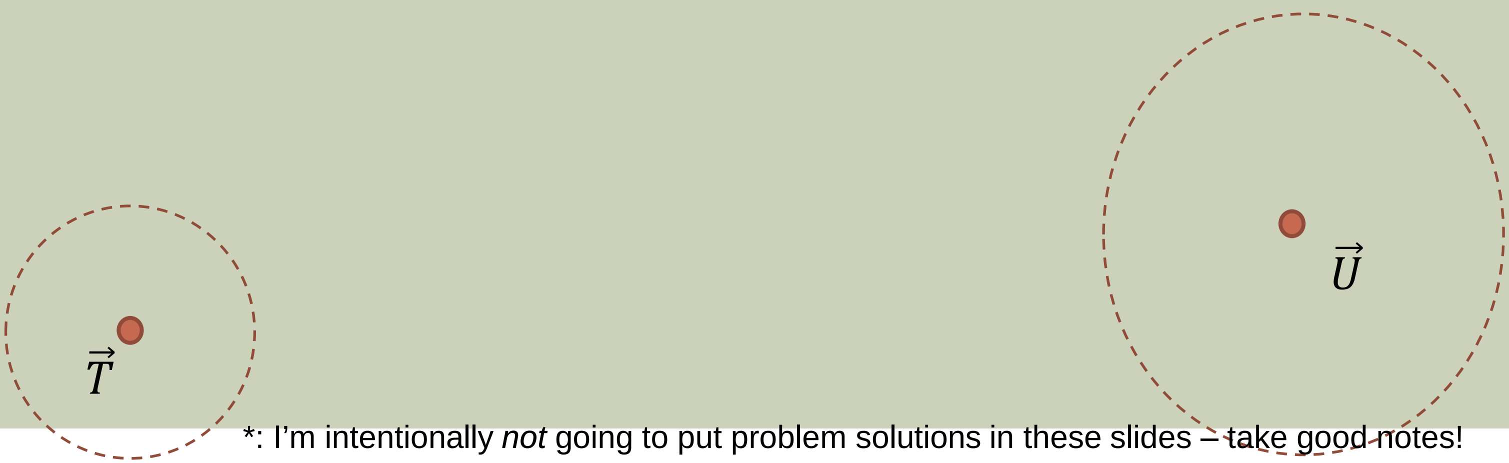
D.P. APPLICATION #1

- What is $\vec{v} \bullet \vec{v}$?
  - $v_1 * v_1 + \ldots + v_n * v_n$
  - Simply the magnitude of the vector squared!

$$\boxed{\vec{v} \bullet \vec{v} = \|\vec{v}\|^2}$$

PROBLEM*

- Given:
  - $\vec{T}$ and $\vec{U}$: two spheres (or circle, or hyper-sphere) centers.
  - t and u: the radii of the spheres
- Determine: does T intersect U, but…
  - We need to improve our frame-rate and square roots are a <u>big</u> bottleneck
  - so no square-roots!
  - squares are OK, but minimize them.



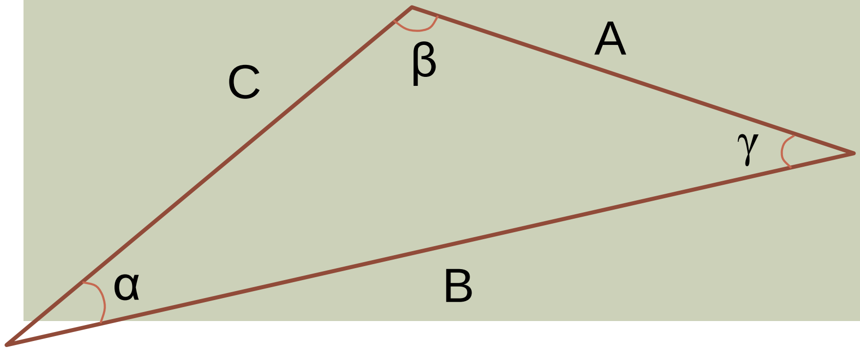
*: I'm intentionally *not* going to put problem solutions in these slides – take good notes!

DOT PRODUCT, CONT.

- **Numeric Interpretation #2:**

$$\boxed{\vec{v} \bullet \vec{w} = \|\vec{v}\| * \|\vec{w}\| * \cos(\theta)}$$

- Where θ is the *angle* between v and w (if their tails are together)
- The <u>SAME</u> number as above, just a different way of interpreting / calculating it.
- Why???
- It's due to the **Law of Cosines** (Geometry)…

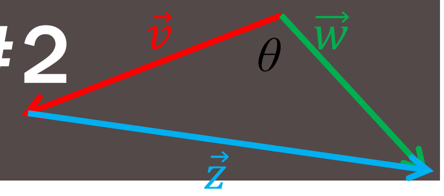


Law of Cosines is like the Pythagorean Theorem for **any** type of triangle (not just right triangles)

$$\boxed{C^2 = A^2 + B^2 - 2AB\cos(\gamma)}$$

Note: for right triangles, the last term is 0…

DERIVATION OF INTERP. #2



1. Let $\vec{z} = \vec{w} - \vec{v}$
2. $\|\vec{z}\,\|^2 = \|\vec{w}\,\|^2 + \|\vec{v}\,\|^2$ - 2$\|\vec{v}\,\|\|\vec{w}\,\|cos(\theta)$ *Law of Cosines*
3. (Lemma) $\|\vec{a}\|^2 = \vec{a}\bullet\vec{a}$
4. $\vec{z} \bullet \vec{z}$ = $\vec{w} \bullet \vec{w}$ + $\vec{v} \bullet \vec{v}$ - 2$\|\vec{v}\,\|\|\vec{w}\,\|$ cos(θ)

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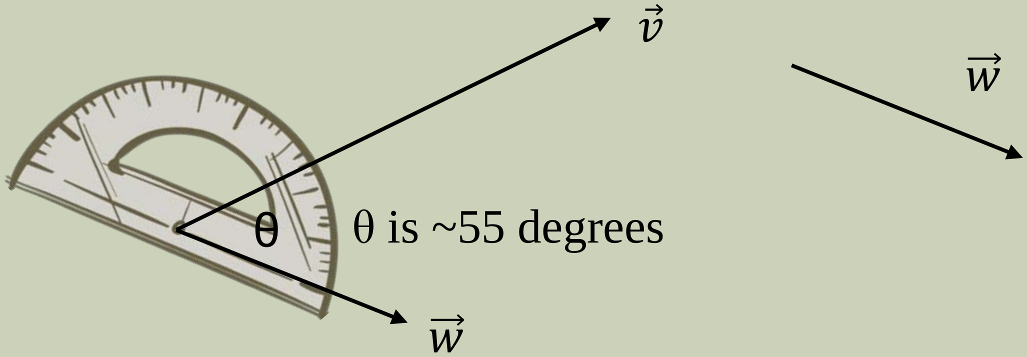
5. $\vec{z} \bullet \vec{z}$ = $(\vec{w} - \vec{v})\bullet(\vec{w} - \vec{v})$ *"squaring" step1, using step3*
6. $\vec{z} \bullet \vec{z}$ = $\vec{w} \bullet \vec{w}$ + $\vec{v} \bullet \vec{v}$ - 2$(\vec{w} \bullet \vec{v}\,)$ *D.P. follows distributive rule & step 5 (F.O.I.L)*
7. $\vec{w} \bullet \vec{w}$ + $\vec{v} \bullet \vec{v}$ - 2$(\vec{w} \bullet \vec{v}\,)$ = $\vec{w} \bullet \vec{w}$ + $\vec{v} \bullet \vec{v}$ - 2$\|\vec{v}\,\|\|\vec{w}\,\|$ cos(θ)
   a. Step4 and Step6 are equal (but different) definitions of z-dot-z.
   b. Set them equal to each other.
8. - 2$(\vec{w} \bullet \vec{v}\,)$ = - 2$\|\vec{v}\,\|\|\vec{w}\,\|$ cos(θ)
9. $\vec{w} \bullet \vec{v} = \|\vec{v}\,\|\|\vec{w}\,\|$ cos(θ)
10. QED

"Quod Erat Demonstrandum", or "which had to be demonstrated", or
this to a mathematician…

NOT CONVINCED?

- Let $\vec{v} = [10 \quad 5 \quad 0]$ and $\vec{w} = [5 \quad -3 \quad 0]$
- Let's draw a picture



EXAMPLE, CONTINUED

$\vec{v} = [10 \quad 5 \quad 0]$ and $\vec{w} = [5 \quad -3 \quad 0]$

Theta is ~55 degrees.

Interpretation#1: $\vec{v} \bullet \vec{w} = 10 * 5 + 5 * (-3) + 0 * 0 = 50 - 15 = 35$

Interpretation#2: $\vec{v} \bullet \vec{w} = \|\vec{v}\| * \|\vec{w}\| * \cos(\theta)$

$$\|\vec{v}\| = \sqrt{10^2 + 5^2 + 0^2} = \sqrt{100 + 25} = \sqrt{125} \approx 11.18$$

$$\|\vec{w}\| = \sqrt{5^2 + (-3)^2 + 0^2} = \sqrt{25 + 9} = \sqrt{31} \approx 5.57$$

$$\vec{v} \bullet \vec{w} = \|\vec{v}\| * \|\vec{w}\| * \cos\theta = 11.18 * 5.57 * \cos(55)$$

$$= 35.72$$

(we estimated the angle (it’s more like 55.8 degrees) and rounded off the lengths, otherwise they'd be identical)

APPLICATION OF D.P #2 (CALCULATION OF Θ)

- We can come up with an <u>exact</u> value for θ, given any two vectors using a little algebra and our two definitions of dot product.

$$\vec{v} \bullet \vec{w} = \|\vec{v}\| * \|\vec{w}\| * \cos(\theta)$$

$$\frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\| * \|\vec{w}\|} = \frac{\cancel{\|\vec{v}\| * \|\vec{w}\|} * \cos(\theta)}{\cancel{\|\vec{v}\| * \|\vec{w}\|}}$$

$$\frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\| * \|\vec{w}\|} = \cos(\theta)$$

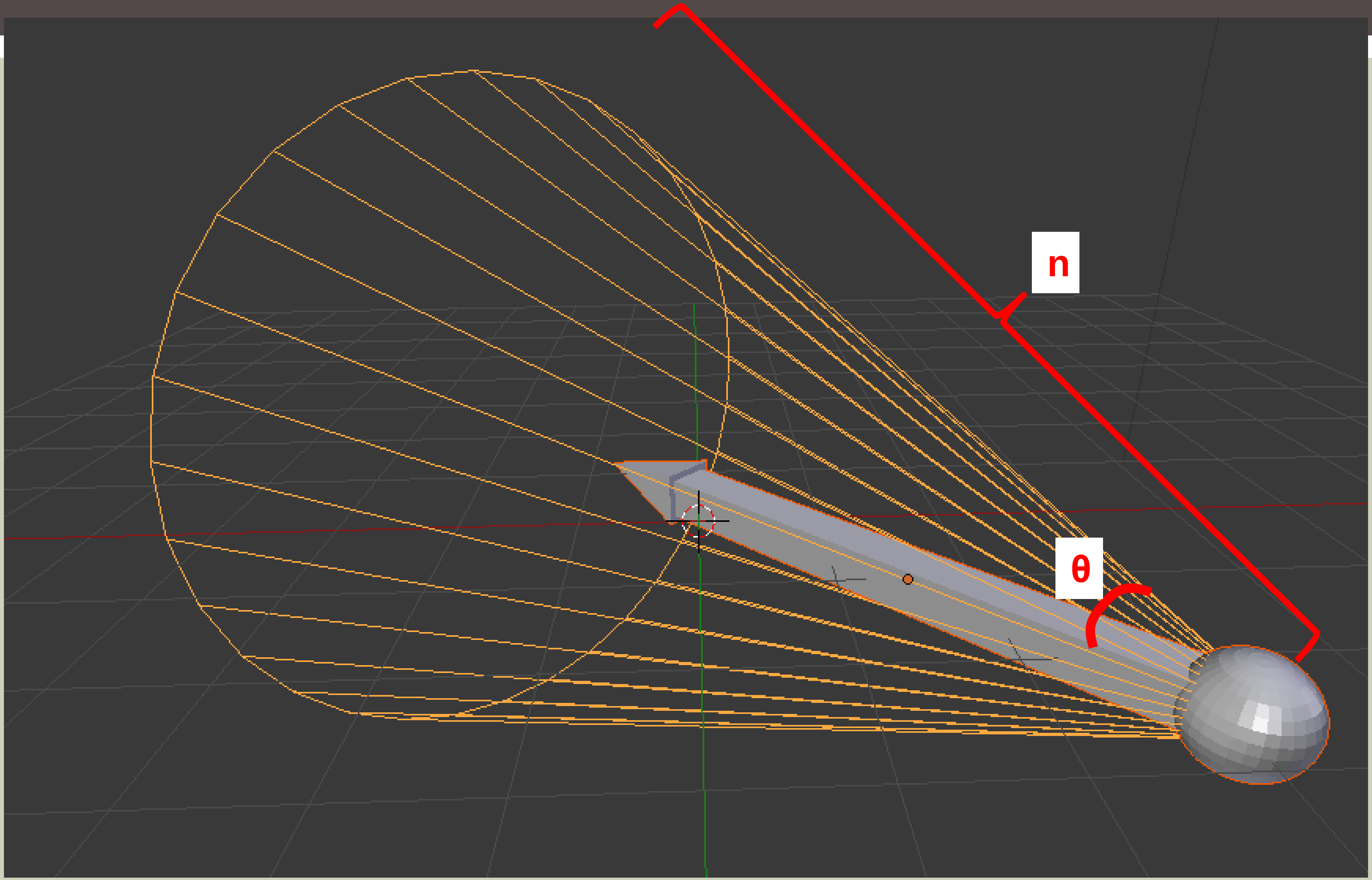
$$\cos^{-1}\left(\frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\| * \|\vec{w}\|}\right) = \cos^{-1}(\cos(\theta))$$

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\| * \|\vec{w}\|}\right)}$$

PROBLEM

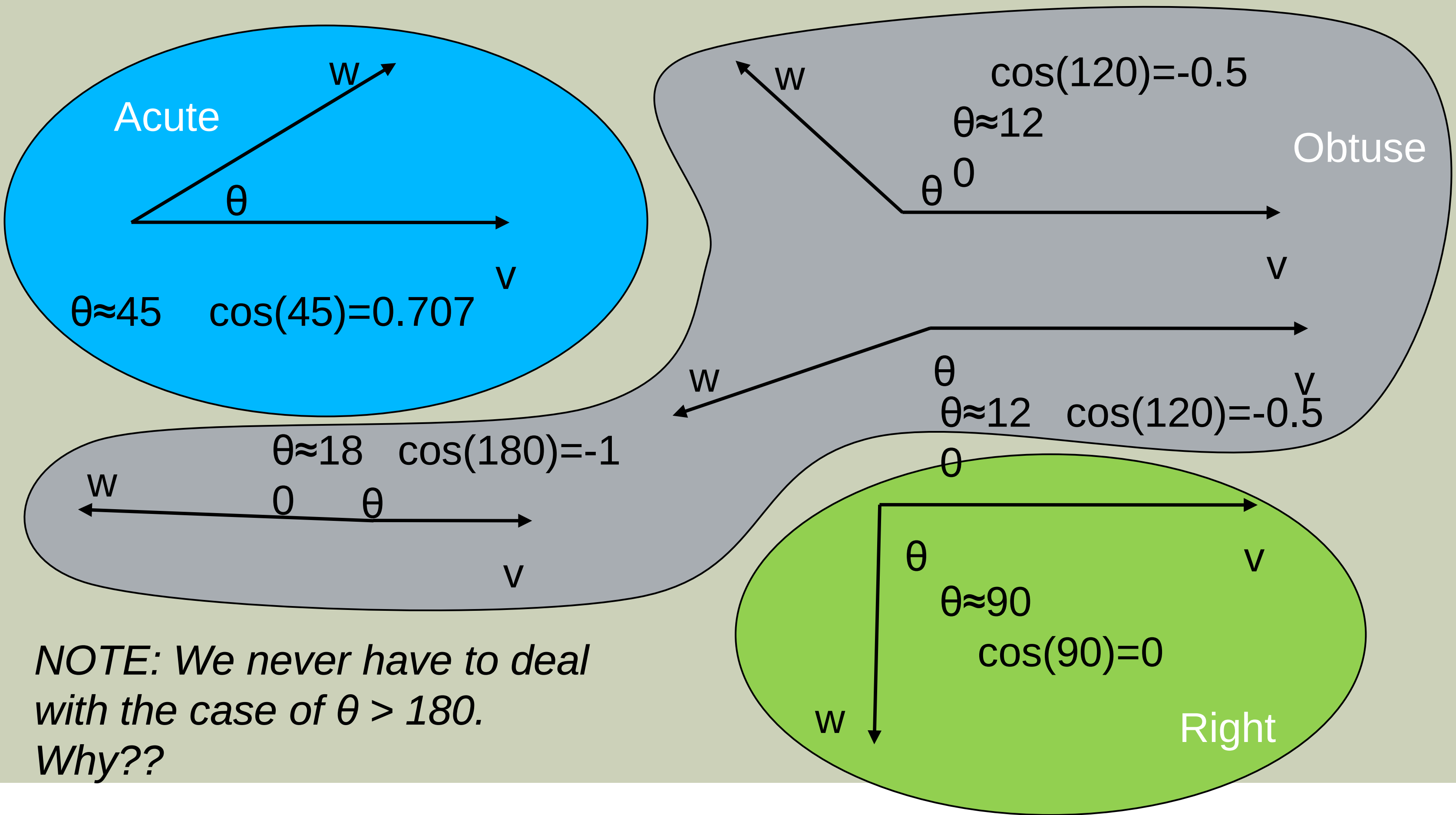
- You are given:
  - $\vec{C}$: the position of your character
  - $\hat{D}$: the direction your character is facing
  - $\vec{E}$: the position of an enemy
  - **Θ**: the half-spread of a shotgun blast (around D). $< 45^\circ$ (the full blast-spread is 2 Θ)
  - **n**: the distance at which the shotgun does no damage
  - **m**: the damage the shotgun blast does at point-blank range (the damage falls off linearly the farther away we go)
- Problem: Symbolically find the actual amount of damage the shotgun does to the enemy. Bonus: minimize the use of trig functions and don't use any inverse trig (both are <u>very</u> slow).

PROBLEM (PICTURE)



APPLICATION OF DOT PRODUCT #3

- θ is the angle between v and w. In each of these cases, think of what cos(θ) would be…



NOTE: We never have to deal with the case of θ > 180. Why??

D.P. APPLICATION #3

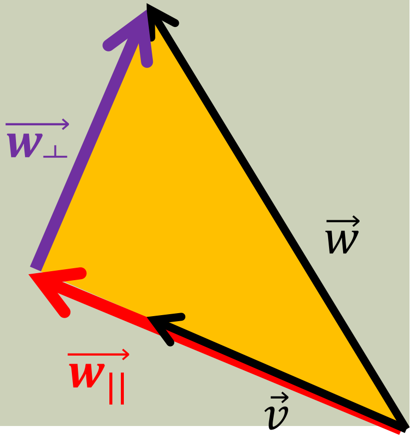
- We can classify the relationship between vectors cheaply:
  - If $\vec{v} \bullet \vec{w} < 0$, the vectors make an obtuse angle
  - If $\vec{v} \bullet \vec{w} > 0$, the vectors make an acute angle
  - If $\vec{v} \bullet \vec{w} = 0$, the vectors make a right angle
- If v and w happen to be unit-length, we can make more observations:
  - $-1 \leq \hat{v} \bullet \hat{w} \leq 1$
  - $\hat{v} \bullet \hat{w}$ = 1 if the vectors are equal
  - $\hat{v} \bullet \hat{w}$ = -1 if the vectors are negations of one another.

PROBLEM

- Given:
  - $\vec{G}$: a guard's position
  - θ: the guard's orientation (if in 2d, this is the angle he is facing; if in 3d, this is a rotation about the y axis, where y is up and the x/z plane is the ground).
    - Note the guard can see to her left / right (and up/down if in 3d) forming a line (or plane in 3d) dividing the non-visible from visible areas).
  - $\vec{P}$: the player's position
- Problem: Symbolically determine if the guard can see the player.
  - Restriction: no inverse trig calls!

APPLICATION #4 (PROJECTION)

- When you project one vector onto another, you produce two new vectors: a perpendicular and a parallel.
- Say we're projecting $\vec{w}$ onto $\vec{v}$
- We'll get:
  - $\overrightarrow{\boldsymbol{w}_{||}}$: the parallel projection of w onto v.
  - $\overrightarrow{\boldsymbol{w}_{\perp}}$: the perpendicular projection of …
  - such that $\vec{w} = \overrightarrow{\boldsymbol{w}_{||}} + \overrightarrow{\boldsymbol{w}_{\perp}}$

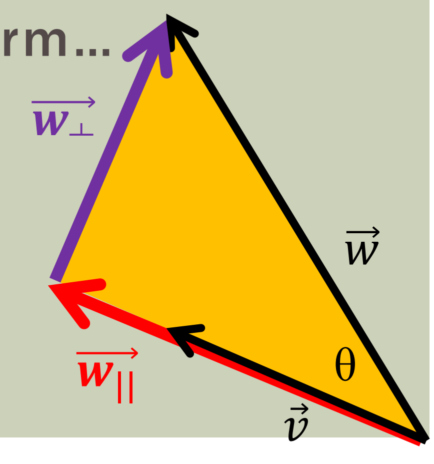


APPLICATION #4 (PROJECTION)

- Recall our trig identities:

$$\sin(\theta) = \frac{O}{H} \quad \cos(\theta) = \frac{A}{H} \quad \tan(\theta) = \frac{O}{A}$$

- So…in the picture below:
  - $\|\vec{w}\|$ is the length of the hypotenuse (H)
  - $\|\overrightarrow{w_{\perp}}\|$ is the length of the opposite side (O)
  - $\left\|\overrightarrow{w_{||}}\right\|$is the length of the adjacent side (A)
- Also recall that $\vec{v} \bullet \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$
- If we substitute $\cos(\theta) = \frac{A}{H} = \frac{\left\|\overrightarrow{w_{||}}\right\|}{\|\vec{w}\|}$ into the d.p. form…
  - $\vec{v} \bullet \vec{w} = \|\vec{v}\|\|\vec{w}\| * \frac{\left\|\overrightarrow{w_{||}}\right\|}{\|\vec{w}\|}$
  - [cont. on next slide]



APPLICATION #4 (PROJECTION)

- [from last slide…]
- $\vec{v} \bullet \vec{w} = \|\vec{v}\| \|\vec{w}\| * \frac{\|\overrightarrow{w_{||}}\|}{\|\vec{w}\|}$
  - **Now, some algebra produces:**

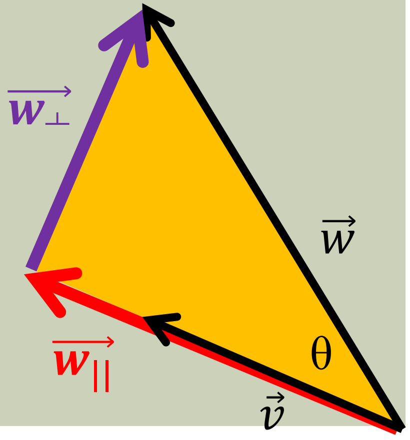
$$\boxed{\|\overrightarrow{w_{||}}\| = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|}}$$

If we actually want the vector $\overrightarrow{w_{||}}$…

$$\boxed{\overrightarrow{w_{||}} = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|} * \hat{v}} = \boxed{\frac{\vec{v} \bullet \vec{w}}{\vec{v} \bullet \vec{v}} \vec{v}}$$

- **Recall (from vector addition):**
  - $\vec{w} = \overrightarrow{w_{||}} + \overrightarrow{w_{\perp}}$
  - **Which makes it easy to calculate $\overrightarrow{w_{\perp}}$:**

$$\boxed{\overrightarrow{w_{\perp}} = \vec{w} - \overrightarrow{w_{||}}}$$

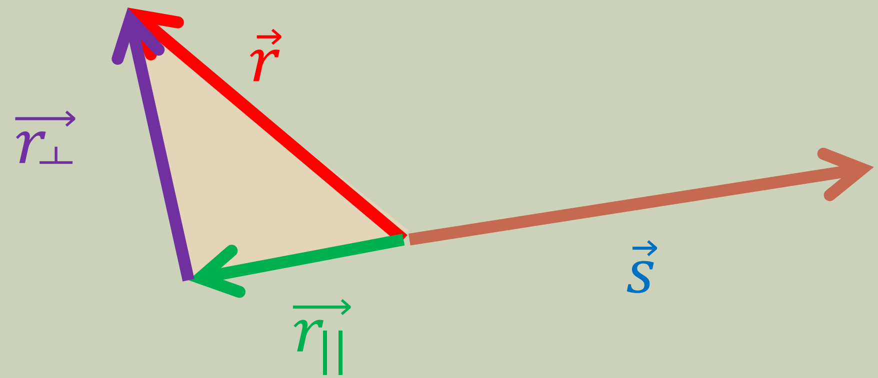


PROBLEM

- **Given**:
  - o $\vec{B}$: a "beamos"'s position
  - o α: beamos's rotation
  - o $\vec{L}$: link's (feet) position
  - o n: link's bounding radius
- o **Problem**: Symbolically determine if the beamos's beam hits link (regardless of distance). You can assume this is a 2d problem (although if in 3d, think of α as the y-axis rotation of the beamos)
- o **Bonus**: Determine the closest intersection point, $\vec{P}$, where the laser hits link's bounding circle / sphere.

APPLICATION #4, CONT.

- It works even if they make an obtuse angle



CROSS PRODUCT (2.12)

- **Symbolically:** $\vec{v}\ x\ \vec{w}$
  - Again, NOT, NOT, NOT vector multiplication!
  - The result is a **vector.**
  - Note: cross product is **not** commutative.
  - Only makes sense in 3d (at least for us)
- **Numerically:**

$$[\overrightarrow{v_x} \quad \overrightarrow{v_y} \quad \overrightarrow{v_z}]\ x\ [\overrightarrow{w_x} \quad \overrightarrow{w_y} \quad \overrightarrow{w_z}] = \begin{bmatrix} \overrightarrow{v_y} * \overrightarrow{w_z} - \overrightarrow{v_z} * \overrightarrow{w_y} \\ \overrightarrow{v_z} * \overrightarrow{w_x} - \overrightarrow{v_x} * \overrightarrow{w_z} \\ \overrightarrow{v_x} * \overrightarrow{w_y} - \overrightarrow{v_y} * \overrightarrow{w_x} \end{bmatrix}^T$$

(the little T means "transpose" – flip it sideways [it's easier to read this way])

  - Tips to memorizing this:
    - determinant of a matrix (next slide)
    - xyzzy (slide after that)
    - … something else.

C.P. MNEMONIC #1

- This is actually where the formula comes from.
  - The **determinant** of the following **matrix**
- Suppose we're calculating $\vec{r}$, where $\vec{r} = \vec{v}\ x\ \vec{w}$

$$\vec{\boldsymbol{r}} = \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix}$$

r_x	v_x	w_x
r_y	v_y	w_y
r_z	v_z	w_z
r_x	v_x	w_x
r_y	v_y	w_y
r_z	v_z	w_z

subtract these…

add these…

C.P. MNEMONIC #2

$$result_x = v_y * w_z - v_z * w_y$$

Memorize Me

Increase the subscripts by 1, "wrapping" around from z=>x

$$result_y = v_z * w_x - v_x * w_z$$

Increase the subscripts by 1, "wrapping" around from z=>x

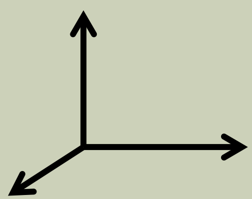
$$result_z = v_x * w_y - v_y * w_x$$

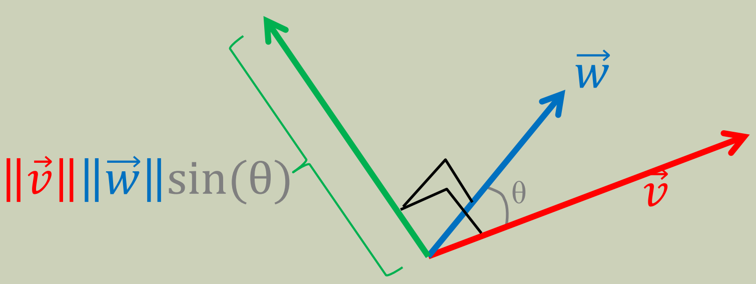
NOTE ABOUT PARALLEL VECTORS

- Note: $\vec{v}\ x\ \vec{v}$ is the zero vector
- Actually any two vectors that are in the same or opposite directions will give a zero vector if crossed)

CROSS PRODUCT, CONT.

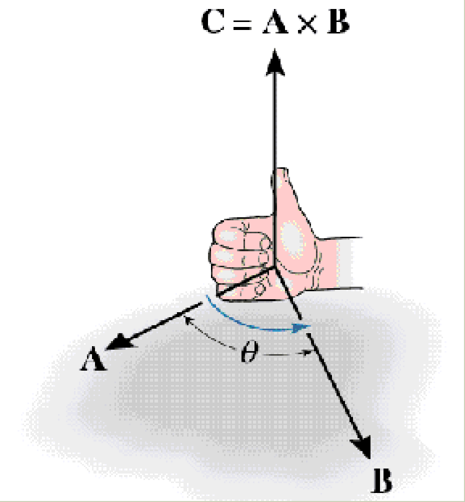
- **Graphically:**
  - Note that $\vec{v}$ and $\vec{w}$ (3d, non-parallel, non-zero) define a plane.
    - If v and w or zero or (anti-) parallel, you'll get a zero vector as the result.
  - The cross product is a vector ***perpendicular to that plane***.
  - The *length* of that vector is: $\|\vec{v}\|\|\vec{w}\|\sin(\theta)$





DIRECTION

- Cross product is *anti-commutative*
  - Meaning: $\vec{v}\ x\ \vec{w} = -(\vec{w}\ x\ \vec{v})$
- Determining direction of result:
  - If in a right-handed system, use your right-hand (if in left, use left hand).
  - Line your palm up with the first vector.
  - Curl all your fingers but your thumb in the shortest arc you can towards the second vector.
  - Your thumb points in the direction of the result.

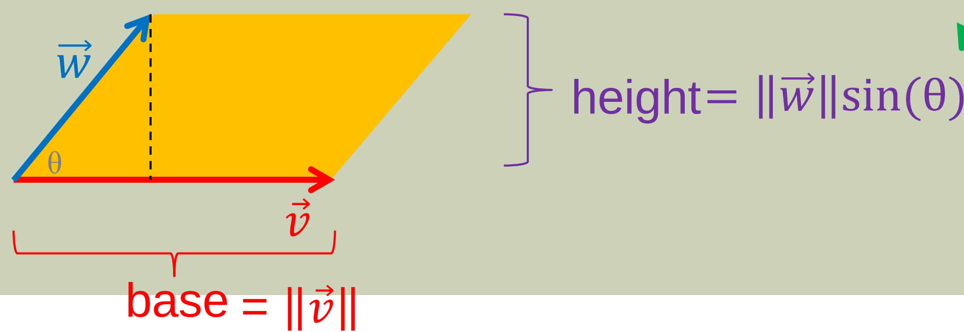


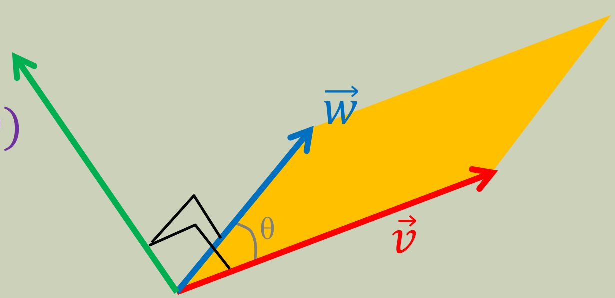
from http://web.aeromech.usyd.edu.au/statics/doc/math3.htm

ADDITIONAL PROPERTIES

- Imagine a parallelogram with sides $\vec{v}$ and $\vec{w}$
- Recall: area of a parallelogram is base * height
- So…the area is $\|\vec{v}\| * (\|\vec{w}\| \sin(\theta))$
- Which is just $\|\boldsymbol{\vec{v}}\ \boldsymbol{x}\ \boldsymbol{\vec{w}}\|$

the parallelogram viewed along the green arrow

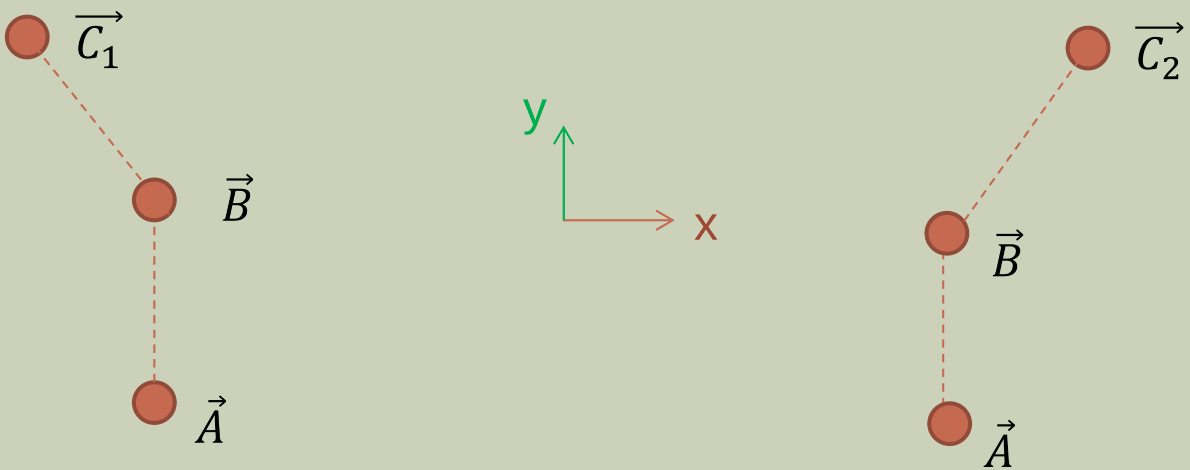




PRACTICE PROBLEM

- Given: $\vec{A}$, $\vec{B}$, and $\vec{C}$, which are three ordered points along a character's path (all of them are on the xy plane).
- Problem: Did the character take a left / right / straight turn at point B?

<u>Turned left</u> <= Which is it? => <u>Turned right</u>



RAYS (9.2.1)

- Description
  - [Compare to (directed) line (segments)]
  - straightforward definition:
    - $\vec{O}$: An origin (a point)
    - $\widehat{D}$: A (unit-length) direction (a vector)
- **Parametric form**
  - Vector notation:

$$\overrightarrow{P(t)} = \vec{O} + t\widehat{D}$$

  - "Component" notation

$$\overrightarrow{P(t)_x} = \overrightarrow{O_x} + t\widehat{D_x}$$
$$\overrightarrow{P(t)_y} = \overrightarrow{O_y} + t\widehat{D_y}$$
$$\ldots$$