INTRO TO MATRICES

Reference: “3D Math Primer for Graphics and Game Development”, chapter 4

MATRICES VS. VECTORS

- A vector is a group of ***scalars***.
  - $\vec{v} = \begin{bmatrix} 7 & 2 & -4 \end{bmatrix}$
- A matrix is a group of ***vectors***.
  - $M = \begin{bmatrix} 4 & 3 & -1 \\ 2 & 0 & 9.5 \end{bmatrix}$

Two ways of looking at this:

$$\begin{bmatrix} 4 & 3 & -1 \\ 2 & 0 & 9.5 \end{bmatrix}$$

Three Vector2's – Column-Major

Used in Right-handed systems

$$\begin{bmatrix} 4 & 3 & -1 \\ 2 & 0 & 9.5 \end{bmatrix}$$

Two Vector3's – Row-Major

Used in Left-handed systems (and in the book)

DIMENSIONS OF A MATRIX (4.1)

- Take this matrix: $M = \begin{bmatrix} 0 & 4 & 9 \\ -1 & 2 & 5 \end{bmatrix}$
- The dimension of M is:

  2x3 (2 rows, 3 columns – always put the row number first – even in left-handed systems)
- The elements are referred to like this:
  - 0 is **m00** 4 is **m01** 9 is **m02**
  - -1 is **m10** 2 is **m11** 5 is **m12**
  - Note: The book uses m11 for 0 (the "index numbers" are 1-based; I use 0-based "index numbers")
  - $M_{row0} = \begin{bmatrix} 0 \\ 4 \\ 9 \end{bmatrix}$ $M_{row1} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ $M_{col0} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $M_{col1} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $M_{col2} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$

SQUARE + IDENTITY MATRICES (4.1.2)

- A **square matrix** has dimensions n x n
  - In other words, the #rows = #columns
  - We'll mainly use 4x4 and 3x3 matrices
    - There are some "Shortcuts" you can take for *some* square matrices.
- A special square matrix is the **identity matrix**
  - For 3x3…

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - For 4x4…

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

VECTORS & MATRICES (4.1.3)

- A Vector is really just a special case of matrix:

$$\vec{v} = \begin{bmatrix} 0 & 1 & -3 \end{bmatrix}$$

But…it's also a 1x3 matrix

- Note: In a right-handed system we would write $\vec{v}$ as …

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

Which is also a 3x1 matrix

TRANSPOSITION (4.1.4)

- “Flips” the matrix along the diagonal.
- Transpose is indicated by using a “T” super-script
  - This doesn’t mean raise the matrix to the “T” power.

TRANSPOSITION, CONT.

- Example:

$$M = \begin{bmatrix} 4 & 5 \\ -3 & 0 \\ 1 & -9 \end{bmatrix} \qquad M^T = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 0 & -9 \end{bmatrix}$$

The first column of M becomes the first row of M^T.
The second column of M becomes the second row of M^T.

Said another way:

$$M_{CR}{}^T = M_{RC}$$

Another observation: if the dimension of M is n x m, the dimension of M^T is m x n.

MATRIX-MATRIX MULTIPLICATION (4.1.6)

- First, you must decide if you can even do the multiplication:
  - The first matrix must have #cols = the second matrix's #rows.
  - Said mathematically,

$$C = A_{mxn} * B_{nxp}$$

Must be the same.

Dimension of result.

- The result (C) will be a matrix of dimension **m x p**.
- NOTE: Matrix multiplication is NOT commutative.

MATRIX-MATRIX MULTIPLICATION, CONT.

Example: $C_{mxp} = A_{mxn} * B_{nxp}$

$$A = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 5 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 1.5 & -8 & 0 \\ 0.5 & 9 & -3 & 1 \\ -2 & -5 & 10 & 7 \end{bmatrix} \qquad C = A * B$$

Q: Can we do A * B?

A: “2x3” times “3x4”. Yes A’s #cols=B’s #rows

Q: What is the dimensions of the result?

A: “2x4”

The of value of the element in the R’th row and the C’th column is:

$$\boxed{C_{RC} = A_{rowR} \bullet B_{colC}}$$

MATRIX-MATRIX MULTIPLICATION, CONT.

$$\boxed{C_{RC} = A_{rowR} \bullet B_{colC}}$$

$$A = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 5 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 1.5 & -8 & 0 \\ 0.5 & 9 & -3 & 1 \\ -2 & -5 & 10 & 7 \end{bmatrix}$$

$$\begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} \\ c_{10} & c_{11} & c_{12} & c_{13} \end{bmatrix} = C$$

$c_{00} = A_{row0} \bullet B_{col0} = 17.5$ $\quad c_{01} = A_{row0} \bullet B_{col1} = -18$ $\quad c_{02} = A_{row0} \bullet B_{col2} = 1$ $\quad c_{03} = A_{row0} \bullet B_{col3} = 20$

$c_{10} = A_{row1} \bullet B_{col0} = 14.5$ $\quad c_{11} = A_{row1} \bullet B_{col1} = 48$ $\quad c_{12} = A_{row1} \bullet B_{col2} = -31$ $\quad c_{13} = A_{row1} \bullet B_{col3} = 5$

$$\boxed{C = \begin{bmatrix} 17.5 & -18 & 1 & 20 \\ 14.5 & 48 & -31 & 5 \end{bmatrix}}$$

MATRIX-MATRIX MULTIPLICATION, CONT.

Example, with a vector this time:

$$A = \begin{bmatrix} 5 & 0 & -2 \\ 1 & 4 & 3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -1 \\ 9 \\ 10 \end{bmatrix} \qquad C = A * \vec{v}$$

Q: Can we do A * v?

A: “2x3” times “**3x1**”. Yes A’s #cols=B’s #rows

Q: What is the dimensions of the result?

A: “2x1”

MATRIX-MATRIX MULTIPLICATION, CONT.

$$\boxed{C_{IJ} = A_{rowI} \bullet B_{colJ}}$$

$$A = \begin{bmatrix} 5 & 0 & -2 \\ 1 & 4 & 3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -1 \\ 9 \\ 10 \end{bmatrix}$$

$$C = \vec{C} = \begin{bmatrix} c_{00} \\ c_{10} \end{bmatrix}$$

$$c_{00} = A_{row0} \bullet \vec{v}_{col0} = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 9 \\ 10 \end{bmatrix} = -25$$

$$c_{10} = A_{row1} \bullet \vec{v}_{col0} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 9 \\ 10 \end{bmatrix} = 65$$

$$\vec{C} = \begin{bmatrix} -25 \\ 65 \end{bmatrix}$$

Note: I'll soon use a term for this operation:

"v was TRANSFORMED into c by A"

"A is a TRANSFORMATION matrix"

IDENTITY MATRIX MULTIPLICATION

- Multiplying a matrix A by an Identity matrix is equivalent to A (try it!).

$$A = A * I = I * A$$

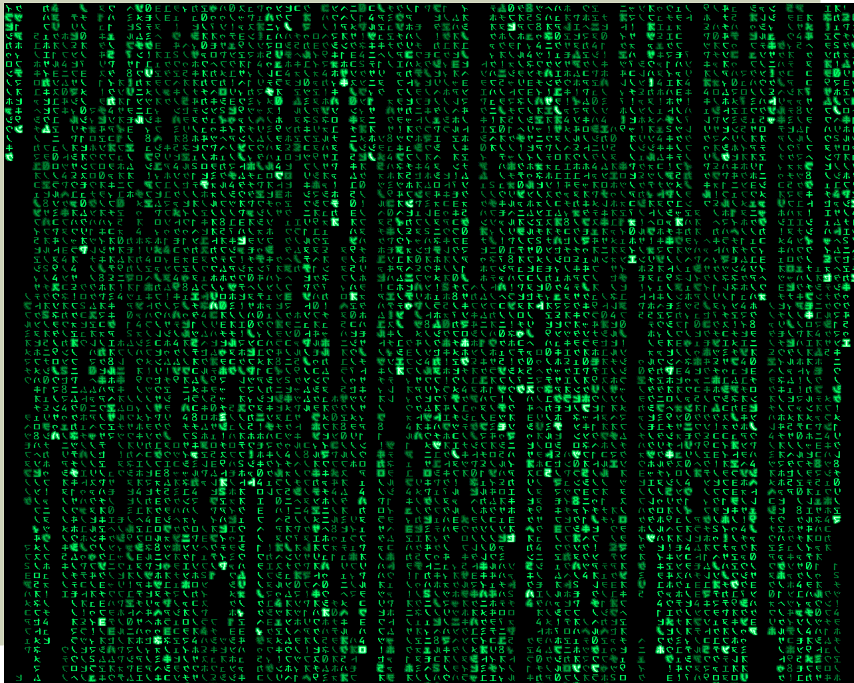
- Multiplying a vector v by an Identity matrix is equivalent to v.

$$\vec{v} = I * \vec{v}$$

- Note: I can be whatever (square) dimension is appropriate for the given vector/matrix.
- Compare this to the *scalar* multiplicative identity (1.0):
  - 5 * **1.0** = 5
  - **1.0** * -4.7 = -4.7

OK, BUT WHAT *IS* THE MATRIX?

- A collection of linear equations.
  - E.g. I went to the store twice.
    - The first time, I bought 5 apples and 2 pears for \$4
    - The next time, I bought 1 apple and 3 pears for \$2.75
  - Q: What is the unit price of each?



<u>As a set of linear equations</u>

5a + 2p = 4

a + 3p = 2.75

<u>As a matrix equation</u>

$$\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ p \end{bmatrix} = \begin{bmatrix} 4 \\ 2.75 \end{bmatrix}$$

MATRIX INVERSE

- Another matrix, which, if multiplied by the original, gives you the identity matrix.
  - Only square matrices have inverses
  - Only matrices whose rows aren't multiples of another row have matrices.
  - [Lots of other "tests" for an invertible matrix]
- Take our original matrix: $\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ p \end{bmatrix} = \begin{bmatrix} 4 \\ 2.75 \end{bmatrix}$

  Or $M\vec{x} = \vec{b}$

  If can calculate the inverse of M (denoted M^{-1}), then

$$M^{-1}M\vec{x} = M^{-1}\vec{b}$$
$$\vec{x} = M^{-1}\vec{b}$$

MATRIX INVERSE

- Take our original matrix: $\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ p \end{bmatrix} = \begin{bmatrix} 4 \\ 2.75 \end{bmatrix}$

  Or $M\vec{x} = \vec{b}$

- We'll see that the inverse of M (denoted M^{-1}) is about $\begin{bmatrix} 0.231 & -0.154 \\ -0.077 & 0.385 \end{bmatrix}$
- We can use it to solve for the price of apples (a) and price of pears (p)

$$M^{-1}M\vec{x} = M^{-1}\vec{b}$$
$$I\vec{x} = M^{-1}\vec{b}$$
$$\vec{x} = M^{-1}\vec{b}$$
$$\vec{x} = \begin{bmatrix} 0.5 \\ 0.75 \end{bmatrix}$$

- So…apples are $0.50 and pears are $0.75.

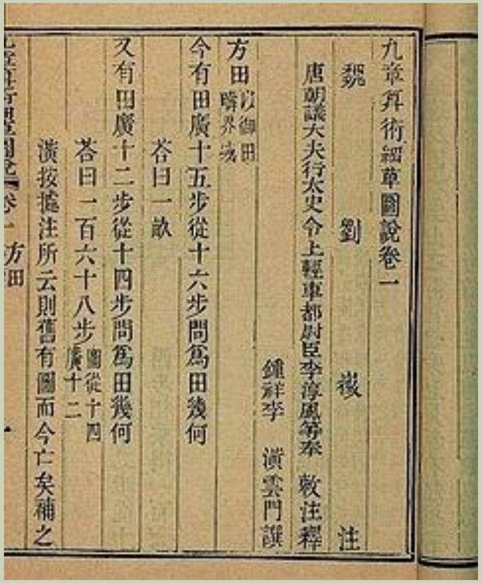
MATRIX INVERSE, CONT.

- Some things to note when inverting:
  - Multiplying a row (and the right-side) by a scalar doesn't change the meaning.
    - E.g. "I bought 5 apples and 2 pears for $4" gives us the same info as "I bought 10 apples and 4 pears for $8"
  - Interchanging rows doesn't change anything.
    - A matrix is just a list of equations.

INVERTING A MATRIX (GAUSSIAN ELIMINATION)

Carl Friedrich Gauss
(1777 – 1855AD)

- Repeat for each value of i from 0 to n-1 (where the matrix is n x n):
  a. Find the row (looking at rows i to n-1) with the biggest value in column i. The row number is the **pivot** and the biggest value is the **pivot value.**
     i. If the pivot value is 0, there is no inverse. STOP.
  b. Multiply row# pivot by 1.0 / pivot_value.
  c. For every row except that of the pivot index, add a multiple of the pivot row such that we get 0 in the pivot column.
  d. If necessary, swap the pivot row so that it is in row i.
- The inverse of the matrix is the final n columns of the augmented matrix.



九章算術細草圖說卷一
魏 劉 徽 注
唐朝議大夫行太史令上輕車都尉臣李淳風等奉 敕注釋
鍾祥李 潢雲門譔
方田 以御田疇界域
今有田廣十五步從十六步問為田幾何
荅曰一畝
又有田廣十二步從十四步問為田幾何
荅曰一百六十八步 圖從十四廣十二
潢按據注所云則舊有圖而今亡矣補之

"Nine Chapters on the Mathematical Arts" (179BC)

- $M = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ $n = 3$
- Augment M
- $A = \begin{bmatrix} 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & 3 & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 4 & -3 & 8 & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$
- Start first iteration. i = 0
- [Step a]. Pivot = 2, PivotVal = 4
- [Step b]. Multiply row 2 by ¼
- $A = \begin{bmatrix} 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & 3 & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & -\frac{3}{4} & 2 & \mathbf{0} & \mathbf{0} & \mathbf{\frac{1}{4}} \end{bmatrix}$
- [Step c]. "Cancelling" row0: nothing to do!
- [Step c]. "Cancelling" row1:
- $A = \begin{bmatrix} 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & \frac{3}{4} & 1 & \mathbf{0} & \mathbf{1} & -\mathbf{\frac{1}{4}} \\ 1 & -\frac{3}{4} & 2 & \mathbf{0} & \mathbf{0} & \mathbf{\frac{1}{4}} \end{bmatrix}$

- [from previous slide]
- $A = \begin{bmatrix} 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & \frac{3}{4} & 1 & \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{4}} \\ 1 & -\frac{3}{4} & 2 & \mathbf{0} & \mathbf{0} & \frac{\mathbf{1}}{\mathbf{4}} \end{bmatrix}$
- [Step d] Swap row2 and row0
- $A = \begin{bmatrix} 1 & -\frac{3}{4} & 2 & \mathbf{0} & \mathbf{0} & \frac{\mathbf{1}}{\mathbf{4}} \\ 0 & \frac{3}{4} & 1 & \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{4}} \\ 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$
- Start second iteration. i = 1
- [Step a] Pivot = 2. PivotVal = 1
- [Step b] Multiply row 2 by 1/1 (no effect)
- [Step c] Add 3/4 times row 2 to row 0
- $A = \begin{bmatrix} 1 & 0 & \frac{7}{2} & \frac{\mathbf{3}}{\mathbf{4}} & \mathbf{0} & \frac{\mathbf{1}}{\mathbf{4}} \\ 0 & \frac{3}{4} & 1 & \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{4}} \\ 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

- [from previous slide]
- $A = \begin{bmatrix} 1 & 0 & \frac{7}{2} & \mathbf{\frac{3}{4}} & \mathbf{0} & \mathbf{\frac{1}{4}} \\ 0 & \frac{3}{4} & 1 & \mathbf{0} & \mathbf{1} & \mathbf{-\frac{1}{4}} \\ 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$
- [Step c] Add -3/4 times row2 to row1
- $A = \begin{bmatrix} 1 & 0 & \frac{7}{2} & \mathbf{\frac{3}{4}} & \mathbf{0} & \mathbf{\frac{1}{4}} \\ 0 & 0 & -\frac{1}{2} & \mathbf{-\frac{3}{4}} & \mathbf{1} & \mathbf{-\frac{1}{4}} \\ 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$
- [Step d] Swap row2 and row1
- $A = \begin{bmatrix} 1 & 0 & \frac{7}{2} & \mathbf{\frac{3}{4}} & \mathbf{0} & \mathbf{\frac{1}{4}} \\ 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & -\frac{1}{2} & \mathbf{-\frac{3}{4}} & \mathbf{1} & \mathbf{-\frac{1}{4}} \end{bmatrix}$

- [from previous slide]
- $A = \begin{bmatrix} 1 & 0 & \frac{7}{2} & \mathbf{\frac{3}{4}} & \mathbf{0} & \mathbf{\frac{1}{4}} \\ 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & -\frac{1}{2} & -\mathbf{\frac{3}{4}} & \mathbf{1} & -\mathbf{\frac{1}{4}} \end{bmatrix}$
- Start last iteration. i = 2
- [Step a] Pivot = 2. PivotVal = -1/2
- [Step b] Multiply row 2 by -2.0
- $A = \begin{bmatrix} 1 & 0 & \frac{7}{2} & \mathbf{\frac{3}{4}} & \mathbf{0} & \mathbf{\frac{1}{4}} \\ 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{\frac{3}{2}} & -\mathbf{2} & \mathbf{\frac{1}{2}} \end{bmatrix}$
- [Step c] Add -7/2*row2 to row0
- $A = \begin{bmatrix} 1 & 0 & 0 & -\mathbf{\frac{18}{4}} & \mathbf{7} & -\mathbf{\frac{3}{2}} \\ 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{\frac{3}{2}} & -\mathbf{2} & \mathbf{\frac{1}{2}} \end{bmatrix}$

- [from previous slide]
- $A = \begin{bmatrix} 1 & 0 & 0 & -\mathbf{\frac{18}{4}} & \mathbf{7} & -\mathbf{\frac{3}{2}} \\ 0 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{\frac{3}{2}} & -\mathbf{2} & \mathbf{\frac{1}{2}} \end{bmatrix}$
- [Step c] Add -2 times row2 to row1
- $A = \begin{bmatrix} 1 & 0 & 0 & -\mathbf{\frac{18}{4}} & \mathbf{7} & -\mathbf{\frac{3}{2}} \\ 0 & 1 & 0 & -\mathbf{2} & \mathbf{4} & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{\frac{3}{2}} & -\mathbf{2} & \mathbf{\frac{1}{2}} \end{bmatrix}$
- The inverse is in the last 3 columns.

- $M^{-1} = \begin{bmatrix} -\frac{18}{4} & 7 & -\frac{3}{2} \\ -2 & 4 & 1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$

- **Double-check: M * M^{-1} should be the Identity matrix.**

- $\begin{bmatrix} -\frac{18}{4} & 7 & -\frac{3}{2} \\ -2 & 4 & 1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} * \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

PRACTICE

- Invert this matrix:

$$\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$$

AN OPTIMIZATION

- For **orthonormal** (which are common in graphics)…
  - ortho = orthogonal (all columns are perpendicular)
  - normal = all columns are unit-length
- …the inverse is just the transpose!