SOLUTIONS MANUAL FOR

SIGNALS AND SYSTEMS

A MATLAB® Integrated Approach

by

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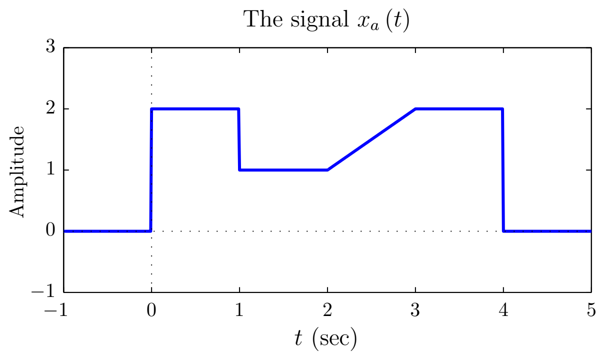
Chapter 1

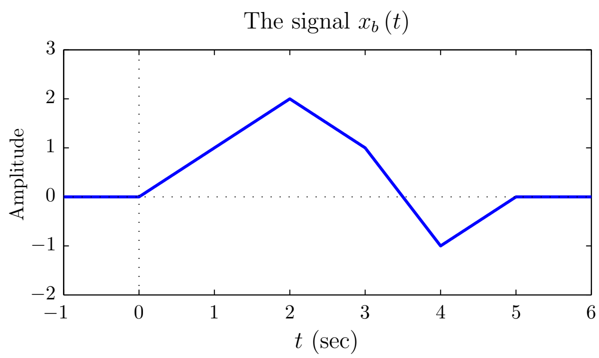
Signal Representation and Modeling

1.1.

a.



b.



1.2.

a.

$$x_a(t) = \begin{cases} 0\ , & t < -1 \text{ or } t > 3 \\ 2t+2\ , & -1 < t < 0 \\ -t+2\ , & 0 < t < 1 \\ 1\ , & 1 < t < 2 \\ -t+3\ , & 2 < t < 3 \end{cases}$$

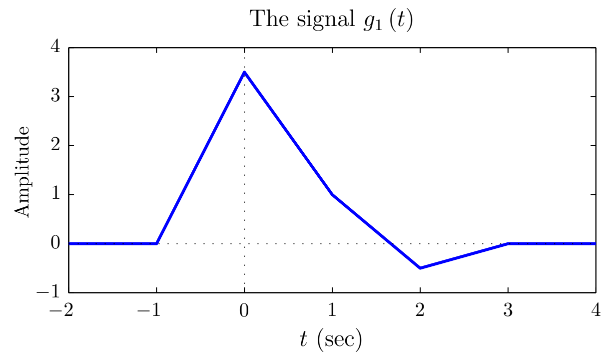
b.

$$x_a(t) = \begin{cases} 0\ , & t < -1 \text{ or } t > 3 \\ 1.5t+1.5\ , & -1 < t < 0 \\ -1.5t+1.5\ , & 0 < t < 2 \\ 1.5t-4.5\ , & 2 < t < 3 \end{cases}$$

1.3.

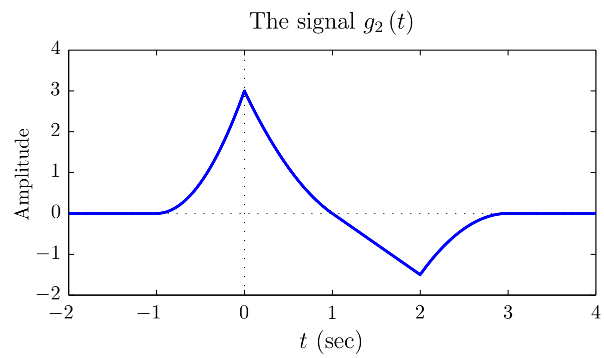
a.

$$g_1(t) = \begin{cases} 0\ , & t < -1 \text{ or } t > 3 \\ 3.5t + 3.5\ , & -1 < t < 0 \\ -2.5t + 3.5\ , & 0 < t < 1 \\ -1.5t + 2.5\ , & 1 < t < 2 \\ 0.5t - 1.5\ , & 2 < t < 3 \end{cases}$$



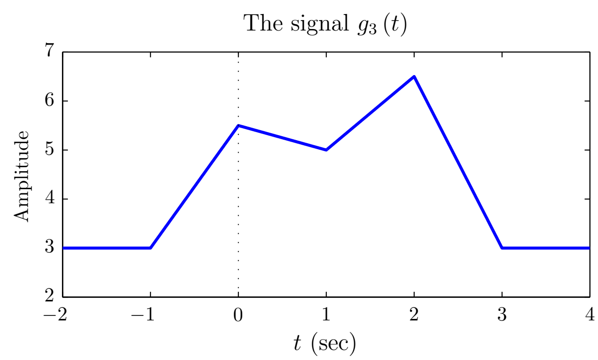
b.

$$g_2(t) = \begin{cases} 0\ , & t < -1 \text{ or } t > 3 \\ 3t^2 + 6t + 3\ , & -1 < t < 0 \\ 1.5t^2 - 4.5t + 3\ , & 0 < t < 1 \\ -1.5t + 1.5\ , & 1 < t < 2 \\ -1.5t^2 + 9t - 13.5\ , & 2 < t < 3 \end{cases}$$



c.

$$g_3(t) = \begin{cases} 3\ , & t < -1 \text{ or } t > 3 \\ 2.5t + 5.5\ , & -1 < t < 0 \\ -0.5t + 5.5\ , & 0 < t < 1 \\ 1.5t + 3.5\ , & 1 < t < 2 \\ -3.5t + 13.5\ , & 2 < t < 3 \end{cases}$$

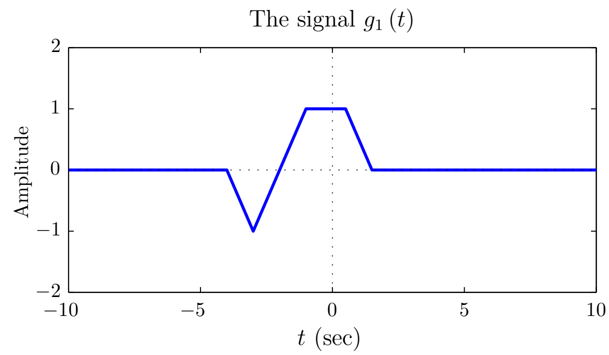


1.4.

a.

Time reversal

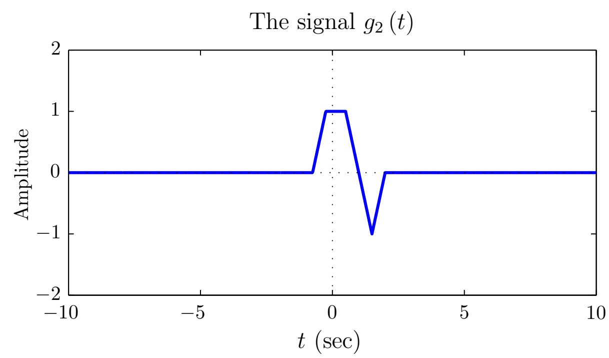
$$g_1(t) = x(-t)$$



b.

Time scaling

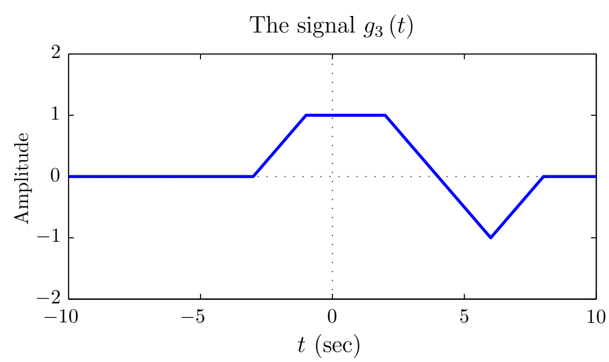
$$g_2(t) = x(2t)$$



c.

Time scaling

$$g_3(t) = x\left(\frac{t}{2}\right)$$



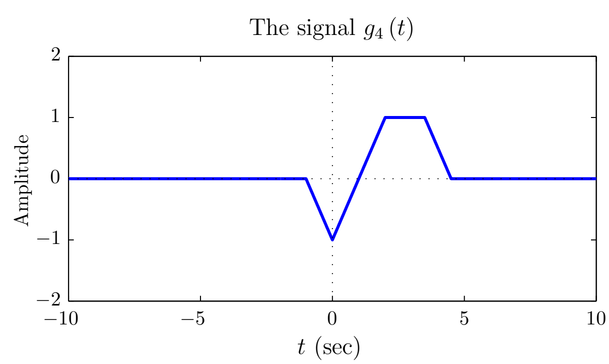
d.

Step 1: Time reversal

$$g_{4a}(t) = x(-t)$$

Step 2: Time shifting

$$g_4(t) = g_{4a}(t-3) = x(-t+3)$$



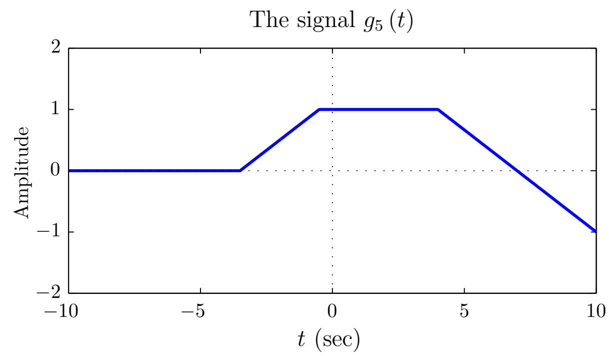
e.

Step 1: Time scaling

$$g_{5a}(t) = x\left(\frac{t}{3}\right)$$

Step 2: Time shifting

$$g_5(t) = g_{5a}(t-1) = x\left(\frac{(t-1)}{3}\right)$$



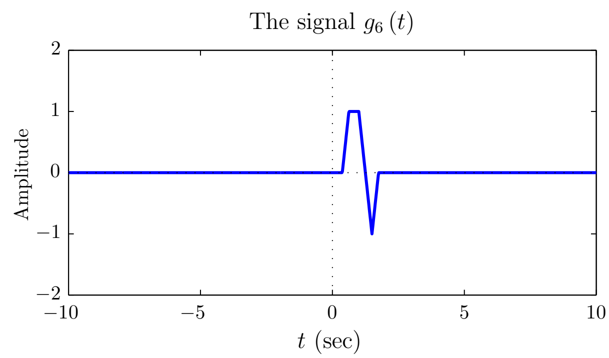
f.

Step 1: Time scaling

$$g_{6a}(t) = x(4t)$$

Step 2: Time shifting

$$g_6(t) = g_{6a}(t-3/4) = x(4t-3)$$



g.

Step 1: Time scaling

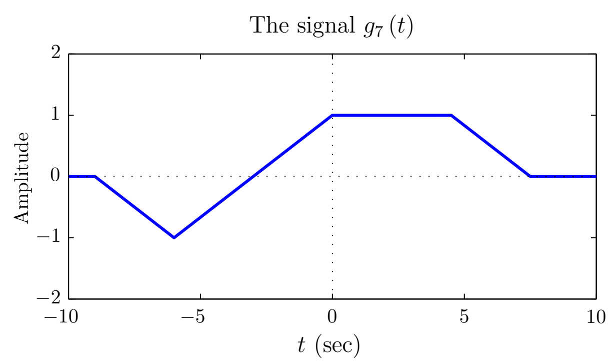
$$g_{7a}(t) = x\left(\frac{t}{3}\right)$$

Step 2: Time reversal

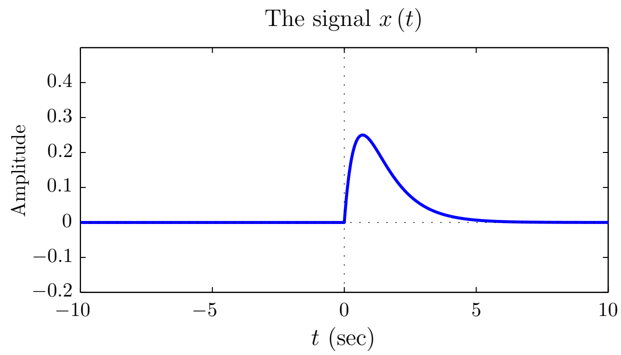
$$g_{7b}(t) = g_{7a}(-t) = x\left(= \frac{t}{3}\right)$$

Step 3: Time shifting

$$g_y(t) = g_{7b}(t-3) = x\left(1 - \frac{t}{3}\right)$$



1.5.



a.

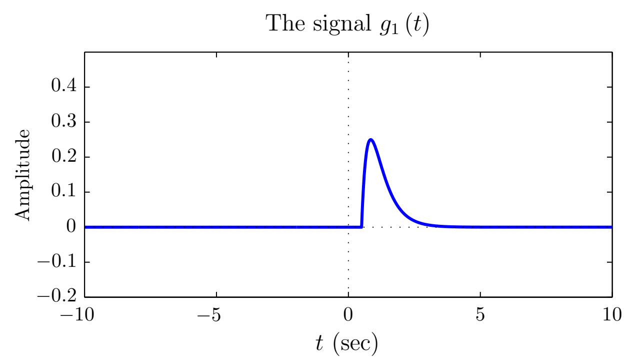
$$g_1(t) = x(2t-1)$$

Step 1: Time scaling

$$g_{1a}(t) = x(2t)$$

Step 2: Time shifting

$$g_1(t) = g_{1a}(t-0.5) = x(2t-1)$$



b.

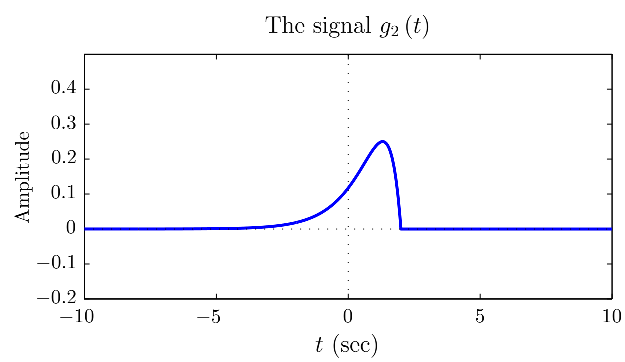
$$g_2(t) = x(-t+2)$$

Step 1: Time reversal

$$g_{2a}(t) = x(-t)$$

Step 2: Time shifting

$$g_2(t) = g_{2a}(t-2) = x(-t+2)$$



c.

$$g_3(t) = x(-3t+5)$$

Step 1: Time scaling

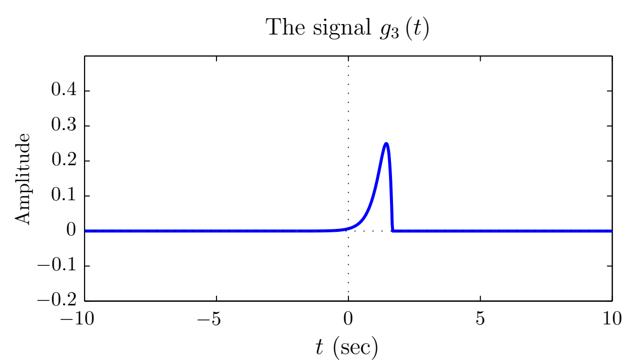
$$g_{3a}(t) = x(3t)$$

Step 2: Time reversal

$$g_{3b}(t) = g_{3a}(-t) = x(-3t)$$

Step 3: Time shifting

$$g_3(t) = g_{3b}(t-5/3) = x(-3t+5)$$



d.

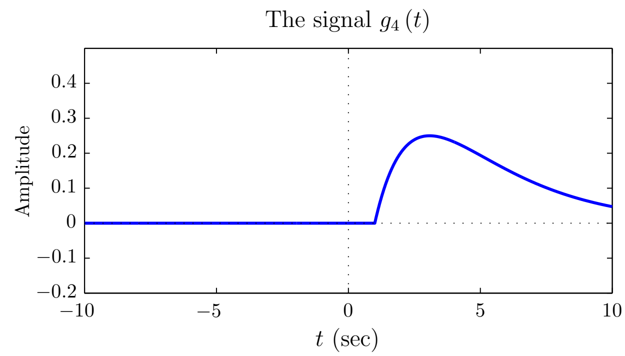
$$g_4(t) = x\left(\frac{t-1}{3}\right)$$

Step 1: Time scaling

$$g_{4a}(t) = x\left(\frac{t}{3}\right)$$

Step 2: Time shifting

$$g_4(t) = g_{4a}(t-1) = x\left(\frac{t-1}{3}\right)$$



1.6.

Let $q(t)$ be a rectangular pulse with height $1/a$ and width a.

$$q(t) = \frac{1}{a}\Pi\left(\frac{t}{a}\right)$$

and the unit-impulse function can be obtained through

$$\delta(t) = \lim_{a\to\infty}\left[q(t)\right] = \lim_{a\to\infty}\left[\frac{1}{a}\Pi\left(\frac{t}{a}\right)\right]$$

It follows that

$$\delta(bt) = \lim_{a\to\infty}\left[q(bt)\right] = \lim_{a\to\infty}\left[\frac{1}{a}\Pi\left(\frac{bt}{a}\right)\right]$$

Let $\tilde{a} = a/b$ so that

$$\frac{1}{a}\Pi\left(\frac{bt}{a}\right) = \frac{1}{\tilde{a}b}\Pi\left(\frac{t}{\tilde{a}}\right)$$

Therefore

$$\begin{aligned}\delta(bt) &= \lim_{a\to\infty}\left[\frac{1}{\tilde{a}b}\Pi\left(\frac{t}{\tilde{a}}\right)\right]\\ &= \frac{1}{b}\lim_{a\to\infty}\left[\frac{1}{\tilde{a}}\Pi\left(\frac{t}{\tilde{a}}\right)\right]\\ &= \frac{1}{b}\delta(t)\end{aligned}$$

1.7.

Given that

$$q(t) = \frac{1}{a}\Pi\left(\frac{t}{a}\right) = \begin{cases}\dfrac{1}{a}\,, & -\dfrac{a}{2} < t < \dfrac{a}{2}\\ 0\,, & \text{otherwise}\end{cases}$$

the time shifted pulse $q(t-t_1)$ is

$$q(t-t_1) = \frac{1}{a}\Pi\left(\frac{t-t_1}{a}\right) = \begin{cases}\dfrac{1}{a}\,, & t_1-\dfrac{a}{2} < t < t_1+\dfrac{a}{2}\\ 0\,, & \text{otherwise}\end{cases}$$

and the integral can be written as

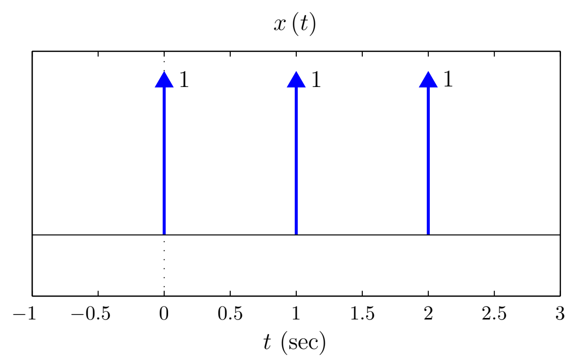
$$\int_{-\infty}^{\infty} f(t)\, q(t-t_1)\, dt = \frac{1}{a} \int_{t_1-a/2}^{t_1+a/2} f(t)\, dt$$

If $f(t)$ is continuous in the vicinity of $t = t_1$ then, for small values of a, the value of the integral above is approximately equal to the area of a rectangle with height equal to $f(t_1)$ and width equal to a, that is,

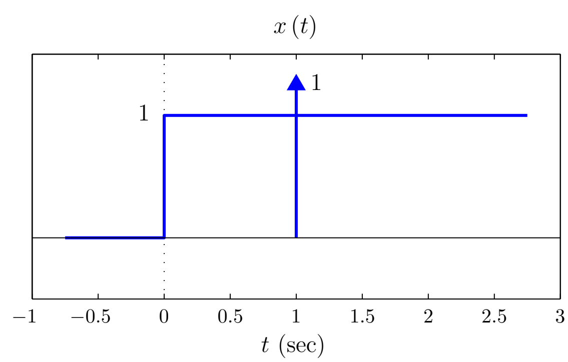
$$\frac{1}{a} \int_{t_1-a/2}^{t_1+a/2} f(t)\, dt \approx \frac{1}{a} \left[f(t_1)\, a \right] = f(t_1)$$

1.8.

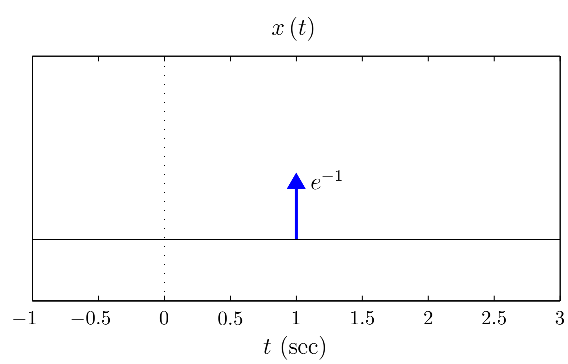
a.



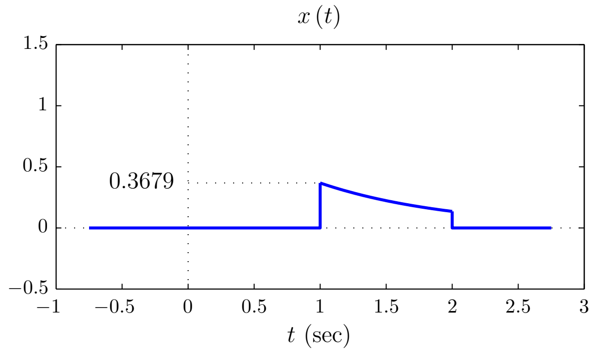
b.



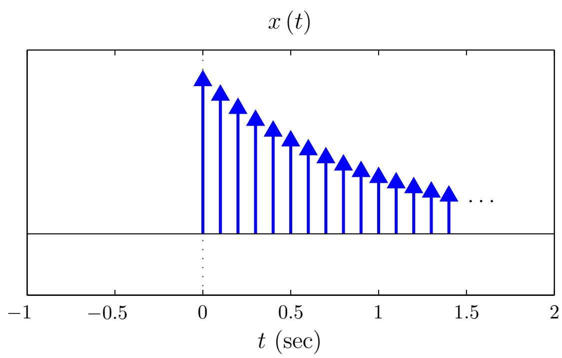
c.



d.

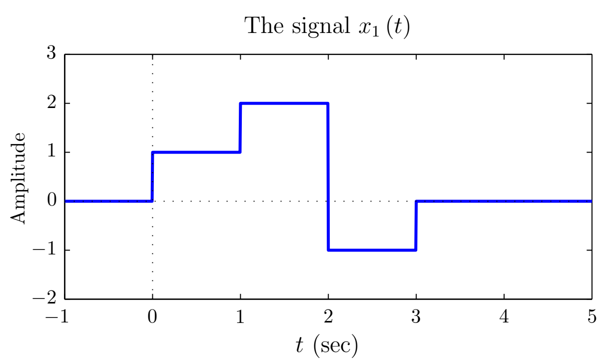


e.

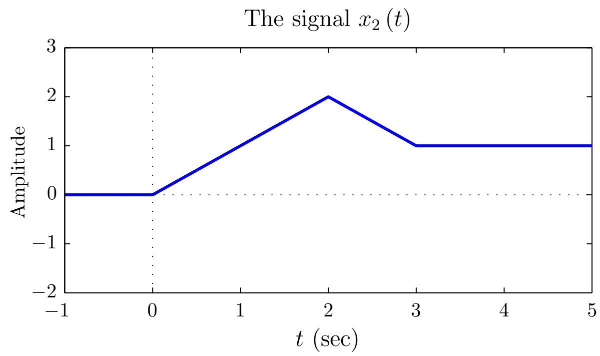


1.9.

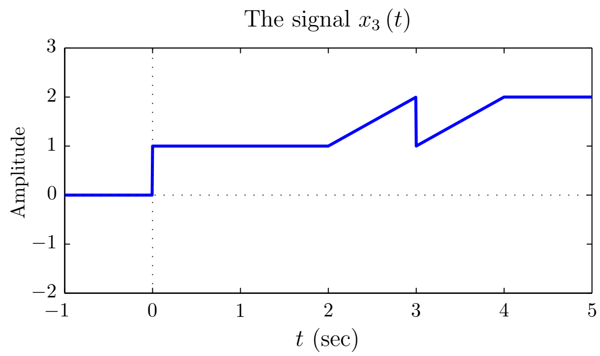
a.



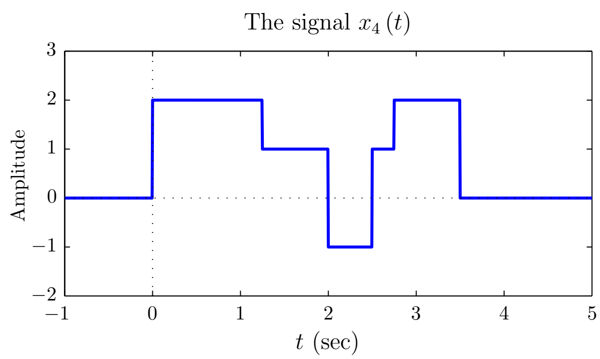
b.


The signal $x_2(t)$
Amplitude
t (sec)
3
2
1
0
−1
−2
−1
0
1
2
3
4
5

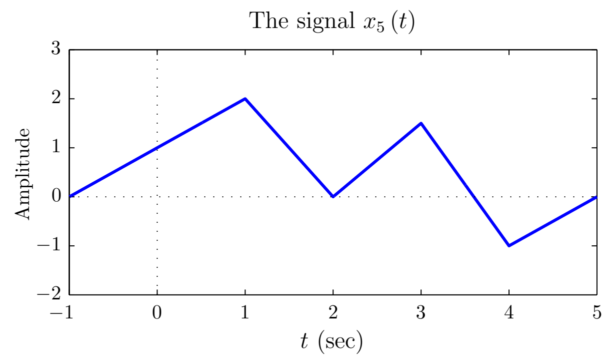
c.


The signal $x_3(t)$
Amplitude
t (sec)
3
2
1
0
−1
−2
−1
0
1
2
3
4
5

d.


The signal $x_4(t)$
Amplitude
t (sec)
3
2
1
0
−1
−2
−1
0
1
2
3
4
5

e.



1.10.

a. $x_a(t) = u(t+2) + 0.5\,u(t+1) - 2\,u(t-1.5) + 0.5\,u(t)$

b. $x_b(t) = 1.5\,u(t+1) - 2.5\,u(t) + u(t-1)$

1.11.

a. $x_a(t) = \Pi(t+1.5) + 1.5\,\Pi\left(\frac{t-0.25}{2.5}\right) - 0.5\,\Pi\left(\frac{t-2.25}{1.5}\right)$

b. $x_b(t) = 1.5\,\Pi(t+0.5) - \Pi(t-0.5)$

1.12.

a. $x(t) = -r(t+1) + 3.5\,r(t) - 3\,r(t-1) - 0.5\,r(t-2) + r(t-3)$

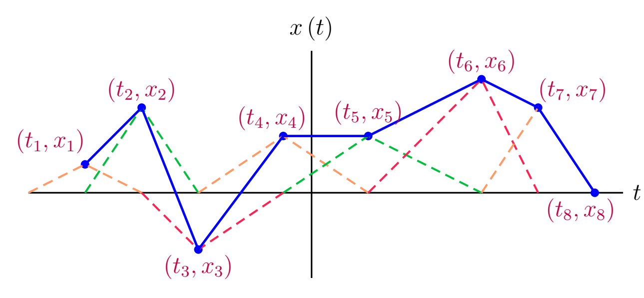
b. $x(t) = -\Lambda(t) + 1.5\,\Lambda(t-1) + \Lambda(t-2)$

1.13.

For each data point (t_i, x_i) use a triangle $x_i\,\Lambda_s(t, a_i, b_i)$ with parameters $a_i = t_i - t_{i-1}$ and $b_i = t_{i+1} - t_i$. The signal $x(t)$ can be expressed as

$$x(t) = \sum_i x_i\,\Lambda_s(t, a_i, b_i)$$

$$= \sum_i x_i\,\Lambda_s(t, t_i - t_{i-1}, t_{i+1} - t_i)$$



1.14.

Integration by parts:

$$\int_a^b u(t)\, dv(t) = u(t)\, v(t)\Big|_a^b - \int_a^b v(t)\, du(t)$$

Let $u(t) = f(t)$ and $dv(t) = \delta'(t)\, dt$:

$$\int_{-\infty}^{\infty} f(t)\, \delta'(t)\, dt = f(t)\, \delta'(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t)\, f'(t)\, dt$$

Using the sifting property of the unit impulse function yields

$$\int_{-\infty}^{\infty} f(t)\, \delta'(t)\, dt = -f'(0)$$

1.15.

Using

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \qquad \text{and} \qquad u(-t) = \begin{cases} 1, & t < 1 \\ 0, & t > 0 \end{cases}$$

the signum function can be written as

$$\operatorname{sgn}(t) = -u(-t) + u(t)$$

or as

$$\operatorname{sgn}(t) = -1 + 2\, u(t)$$

1.16.

a.

$$e^{ja} = \cos(a) + j\, \sin(a)$$

$$e^{-ja} = \cos(a) - j\, \sin(a)$$

Therefore

$$\frac{1}{2}\, e^{ja} + \frac{1}{2}\, e^{-ja} = \cos(a)$$

b.

$$e^{ja} = \cos(a) + j\, \sin(a)$$

$$e^{-ja} = \cos(a) - j\, \sin(a)$$

Therefore

$$\frac{1}{2}\, e^{ja} - \frac{1}{2}\, e^{-ja} = j\, \sin(a) \qquad \Rightarrow \qquad \sin(a) = \frac{1}{2j}\, e^{ja} - \frac{1}{2j}\, e^{-ja}$$

c.

$$\frac{d}{da}\left[\frac{1}{2}\,e^{ja}+\frac{1}{2}\,e^{-ja}\right]=\frac{j}{2}\,e^{ja}-\frac{j}{2}\,e^{-ja}=-\frac{1}{2j}\,e^{ja}+\frac{j}{2j}\,e^{-ja}=-\sin\left(a\right)$$

d.

$$\begin{aligned}\cos\left(a+b\right)=&\frac{1}{2}\,e^{j(a+b)}+\frac{1}{2}\,e^{-j(a+b)}\\=&\frac{1}{2}\left[e^{ja}\,e^{jb}+e^{-ja}\,e^{-jb}\right]\\=&\frac{1}{2}\left[\left[\cos\left(a\right)+j\,\sin\left(a\right)\right]\left[\cos\left(b\right)+j\,\sin\left(b\right)\right]+\left[\cos\left(a\right)-j\,\sin\left(a\right)\right]\left[\cos\left(b\right)-j\,\sin\left(b\right)\right]\right]\\=&\cos\left(a\right)\,\cos\left(b\right)-\sin\left(a\right)\,\sin\left(b\right)\end{aligned}$$

e.

$$\frac{d}{da}\left[\frac{1}{2}\,e^{ja}+\frac{1}{2}\,e^{-ja}\right]=\frac{j}{2}\,e^{ja}-\frac{j}{2}\,e^{-ja}=-\frac{1}{2j}\,e^{ja}+\frac{j}{2j}\,e^{-ja}=-\sin\left(a\right)$$

d.

$$\begin{aligned}\sin\left(a+b\right)=&\frac{1}{2j}\,e^{j(a+b)}-\frac{1}{2j}\,e^{-j(a+b)}\\=&\frac{1}{2j}\left[e^{ja}\,e^{jb}-e^{-ja}\,e^{-jb}\right]\\=&\frac{1}{2j}\left[\left[\cos\left(a\right)+j\,\sin\left(a\right)\right]\left[\cos\left(b\right)+j\,\sin\left(b\right)\right]-\left[\cos\left(a\right)-j\,\sin\left(a\right)\right]\left[\cos\left(b\right)-j\,\sin\left(b\right)\right]\right]\\=&\sin\left(a\right)\,\cos\left(b\right)+\cos\left(a\right)\,\sin\left(b\right)\end{aligned}$$

f.

$$\begin{aligned}\cos^2\left(a\right)=&\left[\frac{1}{2}\,e^{ja}+\frac{1}{2}\,e^{-ja}\right]^2\\=&\frac{1}{4}\,e^{j2a}+\frac{1}{2}+\frac{1}{4}\,e^{-j2a}\\=&\frac{1}{2}+\frac{1}{2}\,\cos\left(2a\right)\end{aligned}$$

1.17.

a. Periodic.

$$2\pi f_0=2\qquad\Rightarrow\qquad f_0=\frac{1}{\pi}\ \text{Hz}\ ,\qquad T_0=\frac{1}{f_0}=\pi\ \text{sec}$$

b. Periodic.

$$2\pi f_0=\sqrt{20}\qquad\Rightarrow\qquad f_0=\frac{\sqrt{20}}{2\pi}=\frac{\sqrt{5}}{\pi}\ \text{Hz}\ ,\qquad T_0=\frac{1}{f_0}=\frac{\pi}{\sqrt{5}}\ \text{sec}$$

c. Not periodic due to the factor $u(t)$.

d. Periodic.

$$2\pi f_0 = 3 \quad \Rightarrow \quad f_0 = \frac{3}{2\pi} \text{ Hz}, \quad T_0 = \frac{1}{f_0} = \frac{2\pi}{3} \text{ sec}$$

e. Not periodic due to the factor $e^{-|t|}$.

f. Not periodic.

g. Periodic.

$$x(t) = cos(2t + \pi/10) + j\,sin(2t + \pi/10)$$

$$2\pi f_0 = 2 \quad \Rightarrow \quad f_0 = \frac{1}{\pi} \text{ Hz}, \quad T_0 = \frac{1}{f_0} = \pi \text{ sec}$$

h. Not periodic.

1.18.

a.

$$f_1 = \frac{5}{2\pi} \text{ Hz}, \quad f_2 = \frac{5}{2\pi} \text{ Hz} \quad \Rightarrow \quad f_0 = \frac{5}{2\pi} \text{ Hz}, \quad T_0 = \frac{2\pi}{5} \text{ sec}$$

b.

$$f_1 = 5 \text{ Hz}, \quad f_2 = 15 \text{ Hz} \quad \Rightarrow \quad f_0 = 5 \text{ Hz}, \quad T_0 = \frac{1}{5} = 0.2 \text{ sec}$$

c.

$$f_1 = \frac{\sqrt{2}}{2\pi} \text{ Hz}, \quad f_2 = \frac{2}{2\pi} \text{ Hz}$$

For periodicity we require two integers m_1 and m_2 to be found such that

$$\frac{m_1}{f_1} = \frac{m_2}{f_2} \quad \Rightarrow \quad \frac{m_1}{\sqrt{2}} = \frac{m_2}{2}$$

No two integers can be found; therefore the signal is not periodic.

d.

$$f_1 = 22.5 \text{ Hz}, \quad f_2 = 27.5 \text{ Hz} \quad \Rightarrow \quad f_0 = 2.5 \text{ Hz}, \quad T_0 = \frac{1}{2.5} = 0.4 \text{ sec}$$

1.19.

a. The energy of the signal $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$$

The energy of $g(t)$ is found as

$$E_g = \int_{-\infty}^{\infty} = \left|g(t)\right|^2 dt = \int_{-\infty}^{\infty} |A\,x(t)|^2 \, dt = |A|^2 \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Thus we have

$$E_g = |A|^2\, E_x$$

If $x(t)$ is an energy signal, $x(t)$ is also an energy signal.

b. The power in the signal $x(t)$ is

$$P_x = \left\langle \left|x(t)\right|^2 \right\rangle$$

The power in $g(t)$ is found as

$$P_g = \left\langle \left|g(t)\right|^2 \right\rangle = \left\langle \left|A\,x(t)\right|^2 \right\rangle = |A|^2 \left\langle \left|x(t)\right|^2 \right\rangle$$

Thus we have

$$P_g = |A|^2\, P_x$$

If $x(t)$ is a power signal, $x(t)$ is also a power signal.

1.20.

a. We will assume that parameters A and B are real-valued. Given that $E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt$, attempting to compute the normalized energy of the signal $g(t)$ results in

$$E_g = \int_{-\infty}^{\infty} \left|A\,x(t) + B\right|^2\, dt$$

which does not converge for $B \neq 0$. Therefore $g(t)$ is a power signal.

b. The normalized average power in $g(t)$ is

$$\begin{aligned} P_g &= \left\langle \left|g(t)\right|^2 \right\rangle \\ &= \left\langle\, \big(A\,x(t) + B\big)\big(A\,x(t) + B\big)^* \right\rangle \\ &= A^2 \left\langle\, |x(t)|^2 \right\rangle + B^2 + 2AB \left\langle\, \mathrm{Re}\left\{x(t)\right\} \right\rangle \end{aligned}$$

Since $x(t)$ is an energy signal, we have

$$P_g = B^2$$

1.21.

a. The signal $x(t)$ can be written as

$$x(t) = e^{-2|t|} = \begin{cases} e^{2t}, & t < 0 \\ e^{-2t}, & t > 0 \end{cases}$$

The energy of the signal is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{0} e^{4t}\, dt + \int_{0}^{\infty} e^{-4t}\, dt = \frac{1}{2}$$

b.

$$E_x = \int_{-\infty}^{\infty} \left|e^{-2t}\right|^2 \, u(t) \, dt = \int_0^{\infty} e^{-4t} \, dt = \frac{1}{4}$$

c.

$$\begin{aligned} \left|x(t)\right|^2 &= \left|e^{-2t}\right|^2 \, \left|\cos(5t)\right|^2 \, u(t) \\ &= e^{-4t} \cos^2(5t) \, u(t) \end{aligned}$$

Remembering that

$$\cos^2(5t) = \frac{1}{2} + \frac{1}{2} \cos(10t)$$

we have

$$\begin{aligned} \left|x(t)\right|^2 &= \frac{1}{2} e^{-4t} \, u(t) + \frac{1}{2} e^{-4t} \cos(10t) \, u(t) \\ &= \frac{1}{2} e^{-4t} \, u(t) + \frac{1}{4} e^{-4t} \, e^{j10t} \, u(t) + \frac{1}{4} e^{-4t} \, e^{-j10t} \, u(t) \\ &= \frac{1}{2} e^{-4t} \, u(t) + \frac{1}{4} e^{(-4+j10)\,t} \, u(t) + \frac{1}{4} e^{(-4-j10)\,t} \, u(t) \end{aligned}$$

The normalized energy is

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} \left|x(t)\right|^2 \, dt \\ &= \frac{1}{2} \int_0^{\infty} e^{-4t} \, dt + \frac{1}{4} \int_0^{\infty} e^{(-4+j10)\,t} \, dt + \frac{1}{4} \int_0^{\infty} e^{(-4-j10)\,t} \, dt \\ &= \frac{1}{8} + \frac{1}{58} = 0.1422 \end{aligned}$$

1.22.

a.

$$x_a(t) = \begin{cases} 2t+2\,, & -1 < t < 0 \\ -t+2\,, & 0 < t < 1 \\ 1\,, & 1 < t < 2 \\ -t+3\,, & 2 < t < 3 \\ 0;\,, & \text{otherwise} \end{cases}$$

$$E_x = \int_{-1}^{0} (2t+2)^2 \, dt + \int_0^1 (-t+2)^2 \, dt + \int_1^2 (1)^2 \, dt + \int_2^3 (-t+3)^2 \, dt = 5$$

b.

$$x_b(t) = \begin{cases} 1.5t+1.5\,, & -1 < t < 0 \\ -1.5t+1.5\,, & 0 < t < 2 \\ 1.5t-4.5\,, & 2 < t < 3 \\ 0\,, & \text{otherwise} \end{cases}$$

$$E_x = \int_{-1}^{0} (1.5t + 1.5)^2 \, dt + \int_{0}^{2} (-1.5t + 1.5)^2 \, dt + \int_{2}^{3} (1.5t - 4.5)^2 \, dt = 3$$

1.23.

a.

$$P_x = \left\langle \left| x(t) \right|^2 \right\rangle = \int_0^{0.5} (1)^2 \, dt = \frac{1}{3}$$

b.

$$P_x = \left\langle \left| x(t) \right|^2 \right\rangle = \int_0^{1} t^2 \, dt = \left. \frac{t^3}{3} \right|_0^1 = 0.5$$

c.

$$P_x = \int_0^1 \sin^2(\pi t) \, dt = \int_0^1 \left(\frac{1}{2} - \frac{1}{2} \cos(2\pi t) \right) dt = \frac{1}{2}$$

1.24.

The two terms with the same frequency need to be combined. Consider that

$$A \cos\left(2\pi f_1 t + \theta\right) = A \cos(\theta) \cos\left(2\pi f_1 t\right) - A \sin(\theta) \sin\left(2\pi f_1 t\right)$$

Let $A \cos(\theta) = 2$ and $-A \sin(\theta) = 3$. Solving the two equations we get

$$A = 3.6056 \qquad \text{and} \qquad \theta = -0.9828 \ \text{radians}$$

The signal is

$$x(t) = 3.6056 \cos\left(2\pi f_1 t - 0.9858\right) + 6 \cos\left(2\pi f_2 t\right)$$

and its RMS value is

$$x_{RMS} = \sqrt{\frac{3.6056^2}{2} + \frac{6^2}{2}} = 4.9497$$

1.25.

a. Even

b. Odd

c. Neither even nor odd

d. Even

e. Odd

f. Neither even nor odd

1.26.

a.

$$\int_{-\lambda}^{\lambda} x(t)\,dt = \int_{-\lambda}^{0} x(t)\,dt \int_{0}^{\lambda} x(t)\,dt$$

For the first integral, apply the variable change $t = -\alpha$ to obtain

$$\begin{aligned}\int_{-\lambda}^{\lambda} x(t)\,dt &= \int_{\lambda}^{0} x(-\alpha)\,(-d\alpha) + \int_{0}^{\lambda} x(t)\,dt \\ &= -\int_{\lambda}^{0} x(-\alpha)\,d\alpha + \int_{0}^{\lambda} x(t)\,dt \\ &= \int_{0}^{\lambda} x(-\alpha)\,d\alpha + \int_{0}^{\lambda} x(t)\,dt\end{aligned}$$

Since $x(t)$ is even, $x(-\alpha) = x(\alpha)$ and

$$\begin{aligned}\int_{-\lambda}^{\lambda} x(t)\,dt &= \int_{0}^{\lambda} x(\alpha)\,d\alpha + \int_{0}^{\lambda} x(t)\,dt \\ &= 2\int_{0}^{\lambda} x(t)\,dt\end{aligned}$$

b. From part (a) we have

$$\int_{-\lambda}^{\lambda} x(t)\,dt = \int_{0}^{\lambda} x(-\alpha)\,d\alpha + \int_{0}^{\lambda} x(t)\,dt$$

Since $x(t)$ is odd, $x(-\alpha) = -x(\alpha)$ and

$$\begin{aligned}\int_{-\lambda}^{\lambda} x(t)\,dt &= -\int_{0}^{\lambda} x(\alpha)\,d\alpha + \int_{0}^{\lambda} x(t)\,dt \\ &= 0\end{aligned}$$

1.27.

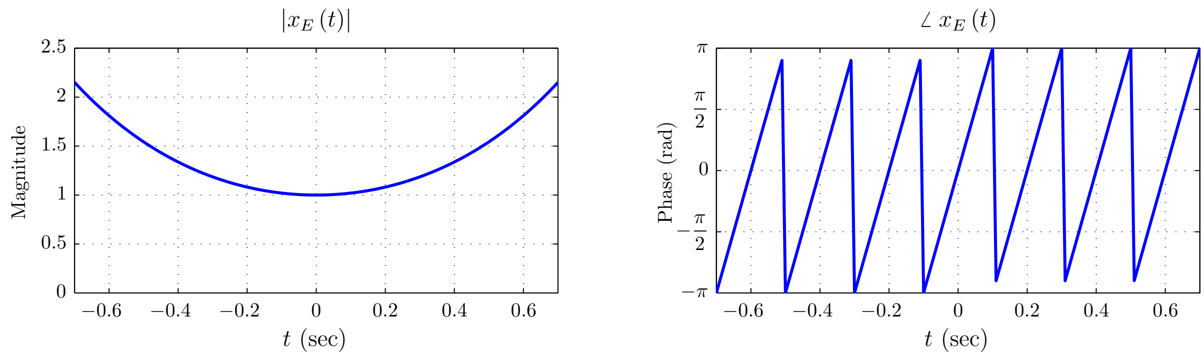
a. $x(-t) = x_1(-t)\,x_2(-t) = x_1(t)\,x_2(t) = x(t)$

b. $x(-t) = x_1(-t)\,x_2(-t) = [-x_1(t)]\,[-x_2(t)] = x_1(t)\,x_2(t) = x(t)$

c. $x(-t) = x_1(-t)\,x_2(-t) = x_1(t)\,[-x_2(t)] = -x_1(t)\,x_2(t) = -x(t)$

1.28.

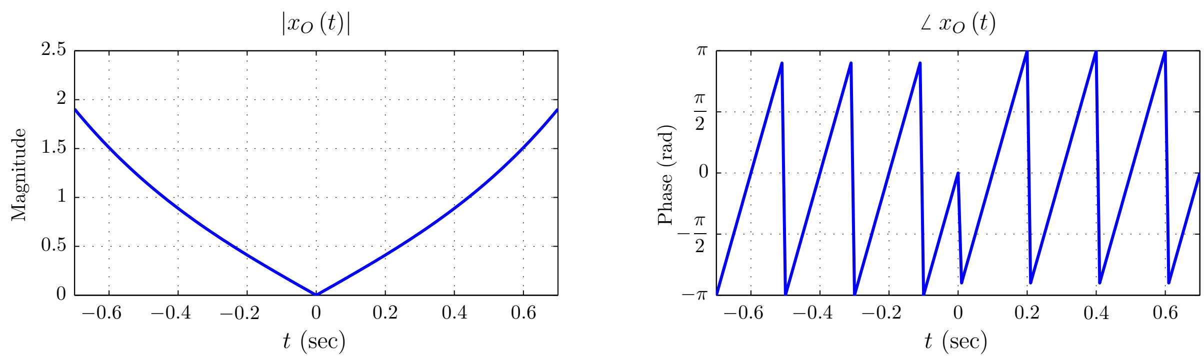
$$\begin{aligned}x_E(t) &= \frac{1}{2}\,e^{(-2+j10\pi)t} + \frac{1}{2}\,e^{(-2-j10\pi)(-t)} \\ &= \frac{1}{2}\left(e^{-2t} + e^{2t}\right)e^{j10\pi t}\end{aligned}$$



(a) **(b)**

$$x_O(t) = \frac{1}{2}\, e^{(-2+j10\pi)t} - \frac{1}{2}\, e^{(-2-j10\pi)(-t)}$$

$$= \frac{1}{2}\left(e^{-2t} - e^{2t}\right) e^{j10\pi t}$$



(c) **(d)**

1.29.

a.

$$x_e(t) = \frac{1}{2}\, e^{-5t} \sin(t)\; u(t) + \frac{1}{2}\, e^{5t} \sin(-t)\; u(-t)$$

$$= \frac{1}{2} \sin(t) \left[e^{-5t}\, u(t) - e^{5t}\, u(-t)\right]$$

$$x_o(t) = \frac{1}{2}\, e^{-5t} \sin(t)\; u(t) - \frac{1}{2}\, e^{5t} \sin(-t)\; u(-t)$$

$$= \frac{1}{2} \sin(t) \left[e^{-5t}\, u(t) + e^{5t}\, u(-t)\right]$$

b.

$$x_e(t) = \frac{1}{2}\,e^{-3|t|}\cos(t) + \frac{1}{2}\,e^{-3|-t|}\cos(-t)$$
$$= e^{-3|t|}\cos(t)$$

$$x_o(t) = \frac{1}{2}\,e^{-3|t|}\cos(t) - \frac{1}{2}\,e^{-3|-t|}\cos(-t)$$
$$= 0$$

c.

$$x_e(t) = \frac{1}{2}\,e^{-3|t|}\sin(t) + \frac{1}{2}\,e^{-3|-t|}\sin(-t)$$
$$= 0$$

$$x_o(t) = \frac{1}{2}\,e^{-3|t|}\sin(t) - \frac{1}{2}\,e^{-3|-t|}\sin(-t)$$
$$= e^{-3|t|}\sin(t)$$

d.

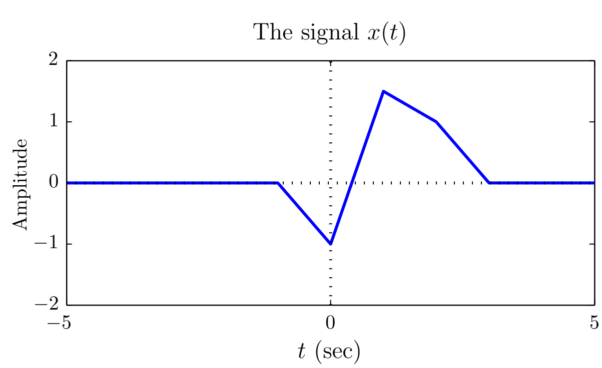
$$x_e(t) = \frac{1}{2}\left(t\,e^{-3t}+2\right)u(t) + \frac{1}{2}\left(-t\,e^{3t}+2\right)u(-t)$$
$$= \frac{1}{2}te^{-3t}\,u(t) - \frac{1}{2}te^{3t}\,u(-t) + 1$$
$$x_o(t) = \frac{1}{2}\left(t\,e^{-3t}+2\right)u(t) - \frac{1}{2}\left(-t\,e^{3t}+2\right)u(-t)$$

e.

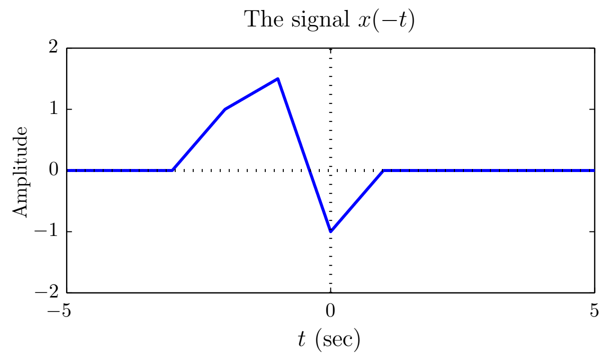
$$x_e(t) = \frac{1}{2}\,e^{-2|t-1|} + \frac{1}{2}\,e^{-2|-t-1|}$$
$$x_o(t) = \frac{1}{2}\,e^{-2|t-1|} - \frac{1}{2}\,e^{-2|-t-1|}$$

1.30.

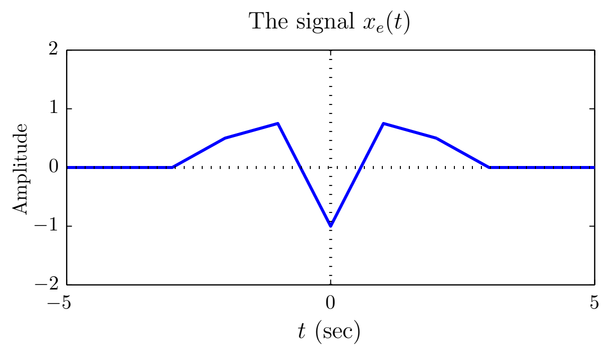
$$x(t) = \begin{cases} -t-1\,, & -1<t<0 \\ 2.5\,t-1\,, & 0<t<1 \\ -0.5\,t+2\,, & 1<t<2 \\ -t+3\,, & 2<t<3 \\ 0\,, & \text{otherwise} \end{cases}$$



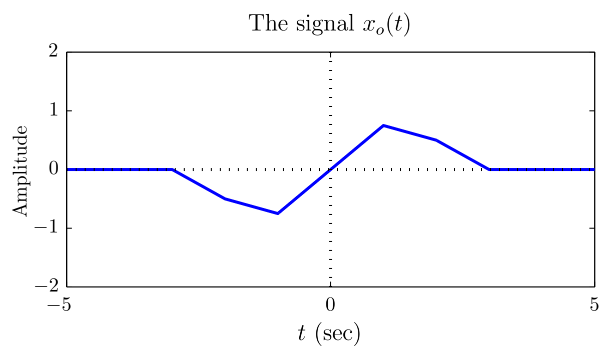
$$x(-t) = \begin{cases} t+3\,, & -3 < t < -2 \\ 0.5\,t+2\,, & -2 < t < -1 \\ -2.5\,t-1\,, & -1 < t < 0 \\ t-1\,, & 0 < t < 1 \\ 0\,, & \text{otherwise} \end{cases}$$



$$x_e(t) = \frac{x(t)+x(-t)}{2}$$



$$x_o(t) = \frac{x(t)-x(-t)}{2}$$



1.31.

a. $\boldsymbol{X} = 3\,e^{j\,0^\circ}\,, \quad \omega_0 = 200\pi \ \text{rad/s}$

b. $x(t) = 7\cos(100\pi t - \pi/2) \quad \Rightarrow \quad \boldsymbol{X} = 7\,e^{-j\pi/2}\,, \quad \omega_0 = 100\pi \ \text{rad/s}$

c.

$$x(t) = 2\cos(10\pi t - \pi/2) + 5\cos(10\pi t + \pi)$$

$$\begin{aligned} \boldsymbol{X} =& 2\,e^{-j\pi/2} + 5\,e^{j\pi} \\ =& 5.3852\,e^{-j\,2.7611}\,, \quad \omega_0 = 10\pi \ \text{rad/s} \end{aligned}$$

1.32.

$$f_0 = 10 \text{ Hz} \qquad \Rightarrow \qquad \omega_0 = 20\pi \text{ rad/s}$$

a.

$$x(t) = \text{Re}\left\{\boldsymbol{X}\, e^{j\omega_0 t}\right\} = 5\cos\left(20\pi t + 14^\circ\right)$$

b.

$$\begin{aligned} x(t) =& \text{Re}\left\{\boldsymbol{X}\, e^{j\omega_0 t}\right\} = \text{Re}\left\{2\, e^{j28^\circ}\, e^{j20\pi t}\right\} + \text{Re}\left\{3\, e^{j18^\circ}\, e^{j20\pi t}\right\} \\ =& 2\cos\left(20\pi t + 28^\circ\right) + 3\cos\left(20\pi t + 18^\circ\right) \end{aligned}$$

Alternatively

$$\boldsymbol{X_1} = 2\, e^{j28^\circ}\ , \quad \boldsymbol{X_2} = 3\, e^{j18^\circ} \qquad \text{and} \qquad \boldsymbol{X} = \boldsymbol{X_1} + \boldsymbol{X_2} = 4.9817\, e^{j22^\circ}$$

$$x(t) = \text{Re}\left\{\boldsymbol{X}\, e^{j\omega_0 t}\right\} = \text{Re}\left\{4.9817\, e^{j22^\circ}\, e^{j20\pi t}\right\} = 4.9817\cos\left(20\pi t + 22^\circ\right)$$

c.

$$\begin{aligned} x(t) =& \text{Re}\left\{\boldsymbol{X}\, e^{j\omega_0 t}\right\} = \text{Re}\left\{2\, e^{j28^\circ}\, e^{j20\pi t}\right\} - \text{Re}\left\{3\, e^{j18^\circ}\, e^{j20\pi t}\right\} \\ =& 2\cos\left(20\pi t + 28^\circ\right) - 3\cos\left(20\pi t + 18^\circ\right) \end{aligned}$$

Alternatively

$$\boldsymbol{X_1} = 2\, e^{j28^\circ}\ , \quad \boldsymbol{X_2} = -3\, e^{j18^\circ} \qquad \text{and} \qquad \boldsymbol{X} = \boldsymbol{X_1} + \boldsymbol{X_2} = 1.0873\, e^{j179.4^\circ}$$

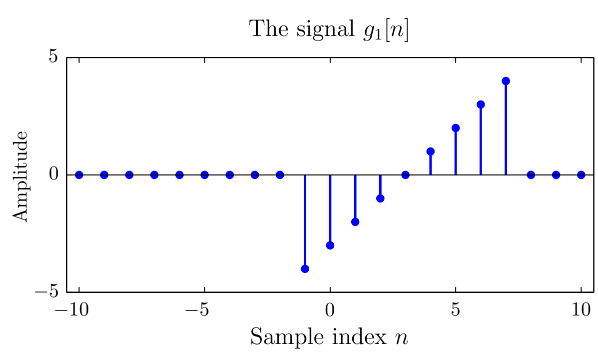
$$x(t) = \text{Re}\left\{\boldsymbol{X}\, e^{j\omega_0 t}\right\} = \text{Re}\left\{1.0873\, e^{j179.4^\circ}\, e^{j20\pi t}\right\} = 1.0873\cos\left(20\pi t + 179.4^\circ\right)$$

1.33.

a.

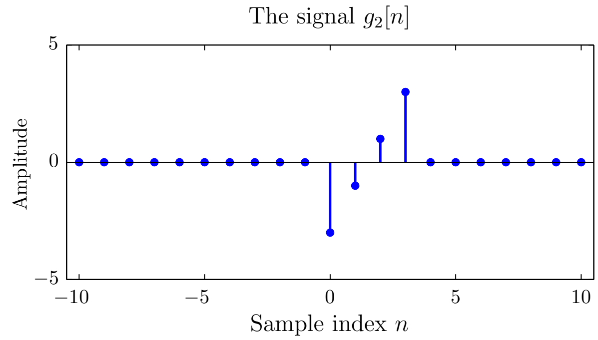
Time shifting

$$g_1[n] = x[n-3]$$



b.

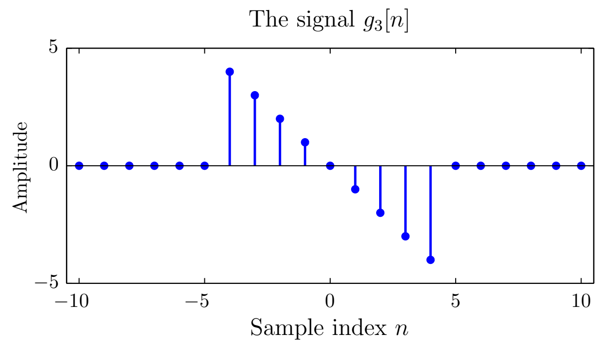
$$g_2[n] = x[2n-3]$$



c.

Time reversal

$$g_3[n] = x[-n]$$



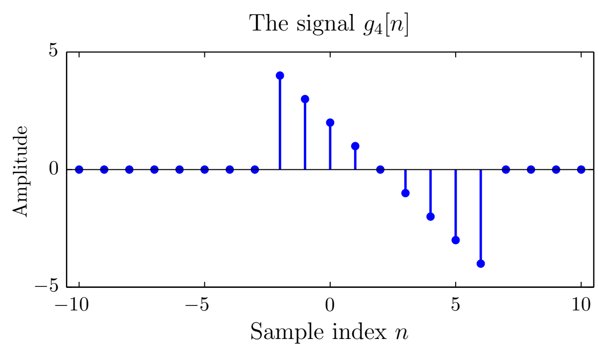
d.

Step 1: Time reversal

$$g_{4a}[n] = x[-n]$$

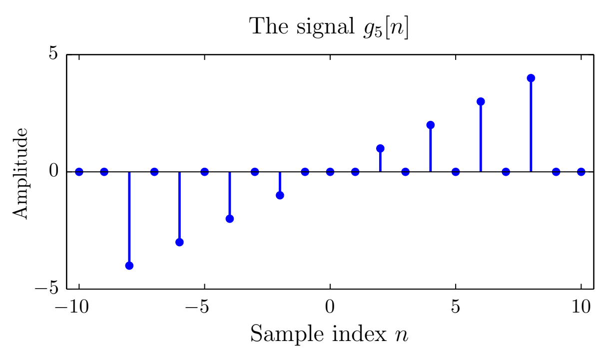
Step 2: Time shifting

$$g_4[n] = g_{4a}[n-2] = x[2-n]$$



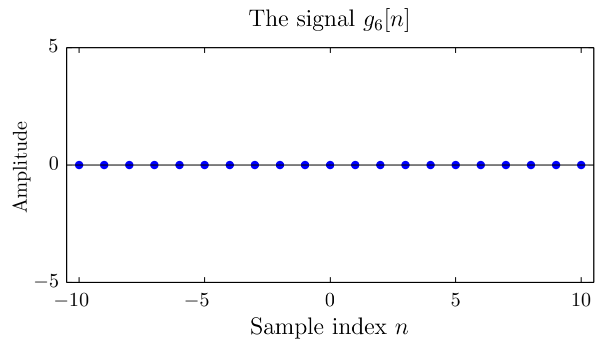
e.

$$g_5[n] = \begin{cases} x[n/2]\,, & \text{if } n/2 \text{ is integer} \\ 0\,, & \text{otherwise} \end{cases}$$



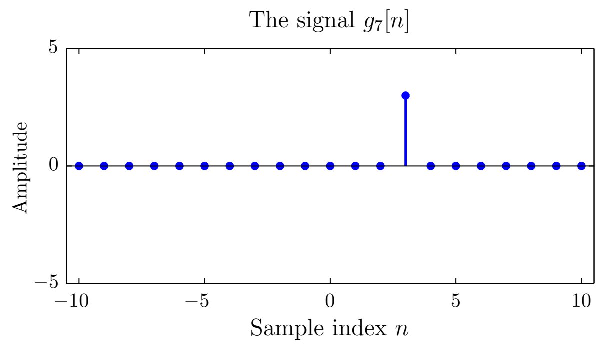
f.

$$g_7(t) = x[n]\,\delta[n-3]$$
$$= \begin{cases} x[n]\,, & n = 0 \\ 0\,, & \text{otherwise} \end{cases}$$



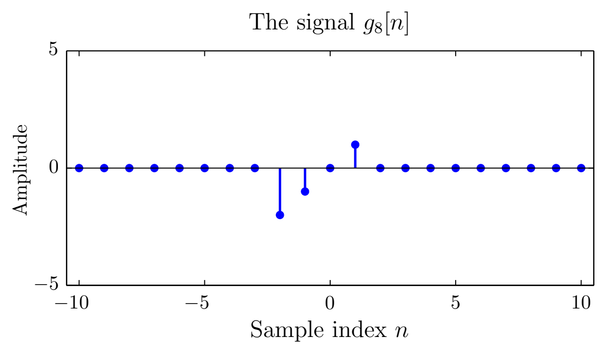
g.

$$g_7(t) = x[n]\,\delta[n]$$
$$= \begin{cases} x[n]\,, & n = 3 \\ 0\,, & \text{otherwise} \end{cases}$$



h.

$$g_8[n] = x[n]\,\{u[n+2] - u[n-2]\}$$
$$= \begin{cases} x[n]\,, & n = -2, \ldots, 1 \\ 0\,, & \text{otherwise} \end{cases}$$



1.34. We have $2\pi F_0 = 2\pi/23$, therefore the normalized frequency is

$$F_0 = \frac{3}{46}$$

The period of the signal is

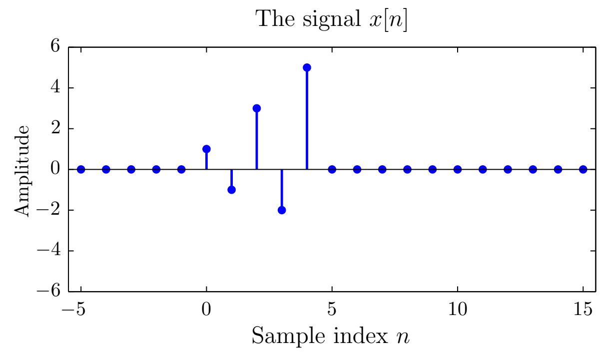
$$N = \frac{k}{F_0} = \frac{46k}{3}$$

if it can be made integer. For $k = 3$ we get $N = 46$. The signal is periodic with a period of 46 samples.

1.35.

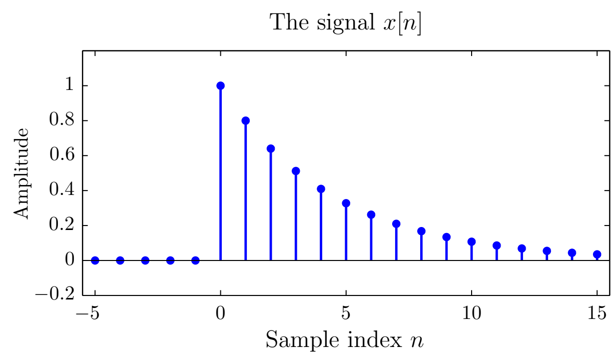
a.

$$x[n] = \{\underset{\uparrow}{1}, -1, 3, -2, 5\}$$



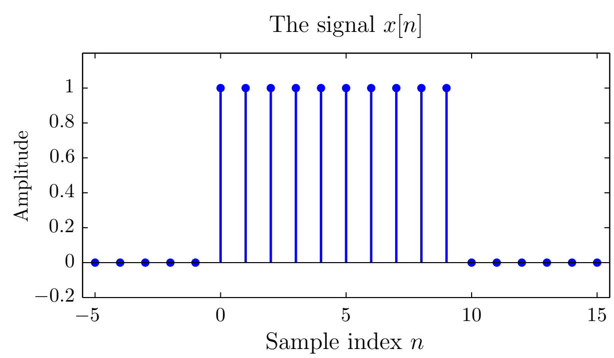
b.

$$x[n] = (0.8)^n\, u[n]$$



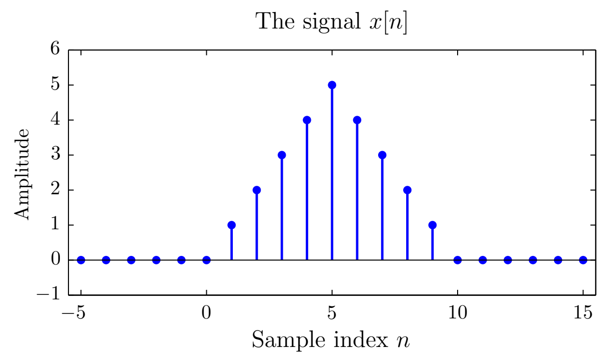
c.

$$\begin{aligned} x[n] &= u[n] - u[n-10] \\ &= \begin{cases} 1\,, & n = 0, \ldots, 9 \\ 0\,, & \text{otherwise} \end{cases} \end{aligned}$$



d.

$$
\begin{aligned}
x[n] &= r[n] - 2r[n-5] + r[n-10] \\
&= \begin{cases} n\,, & n = 0,\ldots,5 \\ 10-n\,, & n = 6,\ldots,10 \\ 0\,, & \text{otherwise} \end{cases}
\end{aligned}
$$



1.36.

a.

$$E_x = (1)^2 + (-1)^2 + (3)^2 + (-2)^2 + (5)^2 = 40$$

b.

$$E_x = \sum_{n=0}^{\infty} (0.8)^{2n} \sum_{n=0}^{\infty} (64)^n = \frac{1}{1-0.64} = 2.7708$$

c.

$$E_x = \sum_{n=0}^{9} (1)^2 = 10$$

d.

$$E_x = \sum_{n=0}^{5} n^2 + \sum_{n=6}^{10} (10-n)^2$$

Using the variable change $m = 10 - n$ in the second summation we obtain

$$
\begin{aligned}
E_x &= \sum_{n=0}^{5} n^2 + \sum_{m=4}^{0} m^2 = \sum_{n=0}^{5} n^2 + \sum_{n=0}^{4} n^2 \\
&= x[5] + 2\sum_{n=0}^{4} n^2 = 2\,(5)^2 + \left[\,(0)^2 + (1)^2 + (2)^2 + (3)^2 + (4)^2\right] = 85
\end{aligned}
$$

1.37.

a.

$$\sum_{n=-M}^{M} x[n] = \sum_{n=-M}^{-1} x[n] + x[0] + \sum_{n=1}^{M} x[n]$$

In the first summation, apply the variable change $n = -k$:

$$\sum_{n=-M}^{-1} x[n] = \sum_{k=M}^{1} x[-k]$$

Since $x[n]$ is even, $x[-k] = x[k]$, and therefore

$$\sum_{n=-M}^{-1} x[n] = \sum_{k=1}^{M} x[k]$$

It follows that

$$\sum_{n=-M}^{M} x[n] = \sum_{n=1}^{M} x[n] + x[0] + \sum_{n=1}^{M} x[n]$$

$$= x[0] + 2\sum_{n=1}^{M} x[n]$$

b.

$$\sum_{n=-M}^{M} x[n] = \sum_{n=-M}^{-1} x[n] + x[0] + \sum_{n=1}^{M} x[n]$$

If $x[n]$ is odd, then $x[-0] = -x[0]$, and consequently we must have $x[0] = 0$. Thus

$$\sum_{n=-M}^{M} x[n] = \sum_{n=-M}^{-1} x[n] + \sum_{n=1}^{M} x[n]$$

In the first summation, apply the variable change $n = -k$:

$$\sum_{n=-M}^{-1} x[n] = \sum_{k=M}^{1} x[-k]$$

Since $x[n]$ is even, $x[-k] = -x[k]$, and therefore

$$\sum_{n=-M}^{-1} x[n] = -\sum_{k=1}^{M} x[k]$$

It follows that

$$\sum_{n=-M}^{M} x[n] = -\sum_{n=1}^{M} x[n] + \sum_{n=1}^{M} x[n] = 0$$

1.38.

a. Since $x_1[n]$ and $x_2[n]$ are both even, we have

$$x_1[-n] = x_1[n] \qquad \text{and} \qquad x_2[-n] = x_2[n]$$

and

$$x[-n] = x_1[-n]\,x_2[-n] = x_1[n]\,x_2[n] = x[n]$$

Therefore, $x[n]$ is even.

b. Since $x_1[n]$ and $x_2[n]$ are both odd, we have

$$x_1[-n] = -x_1[n] \qquad \text{and} \qquad x_2[-n] = -x_2[n]$$

and

$$x[-n] = x_1[-n]\, x_2[-n] = \big(-x_1[n]\big)\big(-x_2[n]\big) = x[n]$$

Therefore, $x[n]$ is even.

c. Since $x_1[n]$ is even and $x_2[n]$ is odd, we have

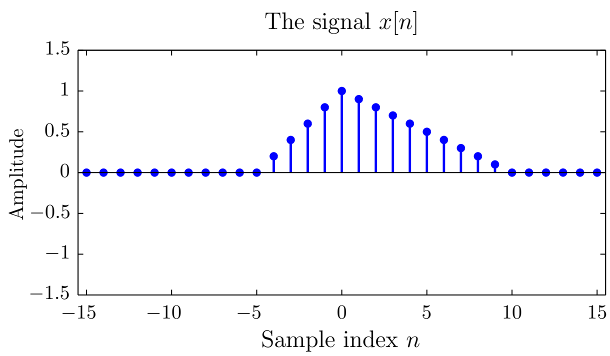
$$x_1[-n] = x_1[n] \qquad \text{and} \qquad x_2[-n] = -x_2[n]$$

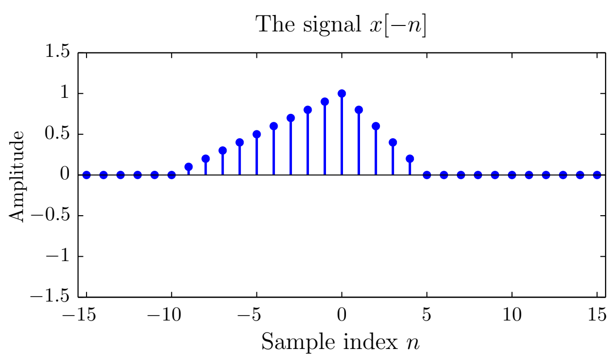
and

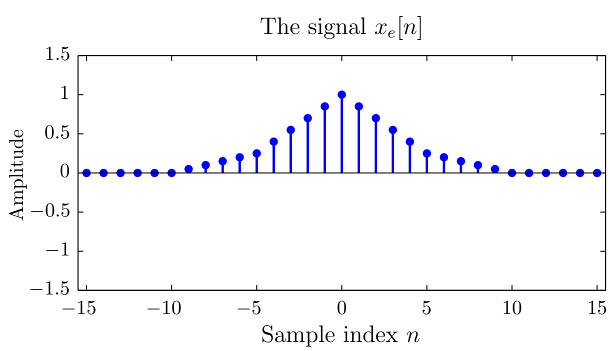
$$x[-n] = x_1[-n]\, x_2[-n] = x_1[n]\big(-x_2[n]\big) = -x[n]$$

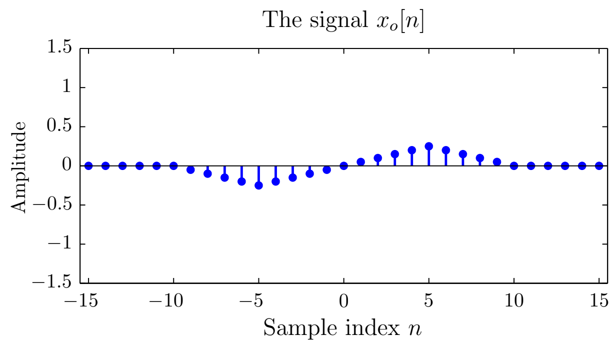
Therefore, $x[n]$ is odd.

1.39.









1.40.

a.

```
xa = @(t)...
    (2).*((t>=0)&(t<1))+...
    (1).*((t>=1)&(t<2))+...
    (t-1).*((t>=2)&(t<3))+...
    (2).*((t>=3)&(t<4));
t = [-1:0.01:6];
plot(t,xa(t));
axis([-1,6,-1,3]);
title('The signal x_{a}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

b.

```
% Define the signal
xb = @(t)...
    (t).*((t>=0)&(t<2))+...
    (-t+4).*((t>=2)&(t<3))+...
    (-2*t+7).*((t>=3)&(t<4))+...
    (t-5).*((t>=4)&(t<5));
t = [-1:0.01:6];
plot(t,xb(t));
axis([-1,6,-2,3]);
title('The signal x_{b}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

1.41.

a. Compute and graph $x_a(t)$:

```
% Define the signal xa(t) using data points in vectors tp and xp
tp = [-2,-1,0,1,2,3,4];
xp = [0,0,2,1,1,0,0];
% Obtain signal xa(t) through interpolation
t = [-2:0.01:4];
xa = interp1(tp,xp,t,'linear');
plot(t,xa,'b-',tp,xp,'ro');
axis([-2,4,-1,3]);
title('The signal x_{a}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $x_b(t)$:

```
% Define the signal xb(t) using data points in vectors tp and xp
tp = [-2,-1,0,2,3,4];
xp = [0,0,1.5,-1.5,0,0];
% Obtain signal xb(t) through interpolation
t = [-2:0.01:4];
xb = interp1(tp,xp,t,'linear');
plot(t,xb,'b-',tp,xp,'ro');
axis([-2,4,-2,2]);
title('The signal x_{b}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

b. Compute and graph $g_1(t)$:

```
g1 = xa+xb;
plot(t,g1);
axis([-2,4,-1,4]);
title('The signal g_{1}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_2(t)$:

```
g2 = xa.*xb;
plot(t,g2);
axis([-2,4,-2,4]);
title('The signal g_{2}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_3(t)$:

```
g3 = 2*xa-xb+3;
plot(t,g3);
axis([-2,4,2,7]);
title('The signal g_{3}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

1.42.

```
t = [-1:0.01:4];
g1 = (0.5*t+2).*((t>=0)&(t<1))+...
     (0.5*t+1).*((t>=1)&(t<2))+...
     (-2*t+4).*((t>=2)&(t<3))+...
     (t-4).*((t>=3)&(t<4));
subplot(2,1,1);
plot(t,g1);
title('g_{1}(t)');
xlabel('t');
grid;
subplot(2,1,2);
g2 = (t).*((t>=0)&(t<1))+...
     (0.5*t).*((t>=1)&(t<2))+...
     (2*t-5).*((t>=2)&(t<3));
plot(t,g2);
title('g_{2}(t)');
xlabel('t');
grid;
```

1.43.

a. Compute and graph $x_a(t)$:

```
% Define the signal xa(t) using data points in vectors tp and xp
tp = [-100,-1.5,-0.5,1,3,4,100];
xp = [0,0,1,1,-1,0,0];
x = @(t) interp1(tp,xp,t,'linear');
% Obtain signal xa(t) through interpolation
t = [-10:0.02:10];
plot(t,x(t),'b-',tp,xp,'ro');
axis([-10,10,-2,2]);
title('The signal x(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

b. Compute and graph $g_1(t)$:

```
g1 = x(-t);
plot(t,g1);
axis([-10,10,-2,2]);
title('The signal g_{1}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_2(t)$:

```
g2 = x(2*t);
plot(t,g2);
axis([-10,10,-2,2]);
title('The signal g_{2}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_3(t)$:

```
g3 = x(t/2);
plot(t,g3);
axis([-10,10,-2,2]);
title('The signal g_{3}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_4(t)$:

```
g4 = x(-t+3);
plot(t,g4);
axis([-10,10,-2,2]);
title('The signal g_{4}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_5(t)$:

```
g5 = x((t-1)/3);
plot(t,g5);
axis([-10,10,-2,2]);
title('The signal g_{5}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_6(t)$:

```
g6 = x(4*t-3);
plot(t,g6);
axis([-10,10,-2,2]);
title('The signal g_{6}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_7(t)$:

```
g7 = x(1-t/3);
plot(t,g7);
axis([-10,10,-2,2]);
title('The signal g_{7}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

1.44.

a. Compute and graph $x(t)$:

```
x = @(t) (exp(-t)-exp(-2*t)).*(t>=0);
t = [-10:0.02:10];
plot(t,x(t));
axis([-10,10,-0.2,0.5]);
title('The signal x(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

b.

Compute and graph $g_1(t)$:

```
g1 = @(t) x(2*t-1);
plot(t,g1(t));
axis([-10,10,-0.2,0.5]);
title('The signal g_{1}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_2(t)$:

```
g2 = @(t) x(-t+2);
plot(t,g2(t));
axis([-10,10,-0.2,0.5]);
title('The signal g_{2}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_3(t)$:

```
g3 = @(t) x(-3*t+5);
plot(t,g3(t));
axis([-10,10,-0.2,0.5]);
title('The signal g_{3}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $g_4(t)$:

```
g4 = @(t) x((t-1)/3);
plot(t,g4(t));
axis([-10,10,-0.2,0.5]);
title('The signal g_{4}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

1.45.

a.

```
t = [-2:0.01:8];
x = ss_step(t)+ss_step(t-1)-3*ss_step(t-2)+ss_step(t-3);
plot(t,x);
axis([-2,8,-5,5]);
title('The signal x(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

b.

```
x = ss_ramp(t)-2*ss_ramp(t-2)+ss_ramp(t-3);
plot(t,x);
axis([-2,8,-5,5]);
title('The signal x(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

c.

```
x = ss_step(t)+ss_ramp(t-2)-ss_step(t-3)-ss_ramp(t-4);
plot(t,x);
axis([-2,8,-5,5]);
title('The signal x(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

d.

```
x = ss_pulse((t-1)/2)-ss_pulse((t-2)/1.5)+2*ss_pulse(t-3);
plot(t,x);
axis([-2,8,-5,5]);
title('The signal x(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

e.

```
x = ss_tri(t)+2*ss_tri(t-1)+1.5*ss_tri(t-3)-ss_tri(t-4);
plot(t,x);
axis([-2,8,-5,5]);
title('The signal x(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

1.46.

a. Compute and graph $x(t)$ using unit-ramp functions.

```
x = @(t) -ss_ramp(t+1)+3.5*ss_ramp(t)...
        -3*ss_ramp(t-1)-0.5*ss_ramp(t-2)+ss_ramp(t-3);
t = [-5:0.01:5];
plot(t,x(t));
axis([-5,5,-2,2]);
title('The signal x(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

b. Compute and graph $x(t)$ using unit-triangle functions.

```
x = @(t) -ss_tri(t)+1.5*ss_tri(t-1)+ss_tri(t-2);
t = [-5:0.01:5];
```

```
plot(t,x(t));
axis([-5,5,-2,2]);
title('The signal x(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

c. Compute and graph $x_e(t)$, the even component of $x(t)$.

```
xe = 0.5*x(t)+0.5*x(-t);
plot(t,xe);
axis([-5,5,-2,2]);
title('The signal x_{e}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $x_o(t)$, the even component of $x(t)$.

```
xo = 0.5*x(t)-0.5*x(-t);
plot(t,xo);
axis([-5,5,-2,2]);
title('The signal x_{o}(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

1.47.

a.

```
x = @(n) n.*((n>=-4)&(n<=4));
n = [-10:10];
stem(n,x(n));
axis([-10.5,10.5,-5,5]);
```

b.

```
g1 = x(n-3);
stem(n,g1);
axis([-10.5,10.5,-5,5]);
```

```
g2 = x(2*n-3);
stem(n,g2);
axis([-10.5,10.5,-5,5]);
```

```
g3 = x(-n);
stem(n,g3);
axis([-10.5,10.5,-5,5]);
```

```
g4 = x(2-n);
stem(n,g4);
axis([-10.5,10.5,-5,5]);
```

```
g5 = zeros(size(n));
for k = 1:length(g5),
  nn = k-11;
  if (mod(nn,2)==0),
    g5(k) = x(nn/2);
  end;
end;
stem(n,g5);
axis([-10.5,10.5,-5,5]);
```

```
g6 = x(n).*(n==0);
stem(n,g6);
axis([-10.5,10.5,-5,5]);
```

```
g7 = x(n).*(n==3);
stem(n,g7);
axis([-10.5,10.5,-5,5]);
```

```
g8 = x(n).*(ss_step(n+2)-ss_step(n-2));
stem(n,g8);
axis([-10.5,10.5,-5,5]);
```

1.48.

a.

```
x = @(n) 0.2*ss_ramp(n+5)-0.3*ss_ramp(n)+0.1*ss_ramp(n-10);
n = [-10:10];
stem(n,x(n));
axis([-10.5,10.5,-1.5,1.5]);
```

b.

```
xe = 0.5*x(n)+0.5*x(-n);
xo = 0.5*x(n)-0.5*x(-n);
subplot(2,1,1);
stem(n,xe);
axis([-10.5,10.5,-1.5,1.5]);
subplot(2,1,2);
stem(n,xo);
axis([-10.5,10.5,-1.5,1.5]);
```

Chapter 2

Analyzing Continuous-Time Systems in the Time Domain

2.1.

a.

$$y_1(t) = \text{Sys}\{x_1(t)\} = |x_1(t)| + x_1(t)$$

$$y_2(t) = \text{Sys}\{x_2(t)\} = |x_2(t)| + x_2(t)$$

Using $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$ as input we obtain

$$\begin{aligned} y(t) &= \text{Sys}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} \\ &= |\alpha_1 x_1(t) + \alpha_2 x_2(t)| + \alpha_1 x_1(t) + \alpha_2 x_2(t) \\ &\neq \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

The system is not linear.

$$\text{Sys}\{x_1(t-\tau)\} = |x_1(t-\tau)| + x_1(t-\tau) = y_1(t-\tau)$$

The system is time-invariant.

b.

$$y_1(t) = \text{Sys}\{x_1(t)\} = t\, x_1(t)$$

$$y_2(t) = \text{Sys}\{x_2(t)\} = t\, x_2(t)$$

Using $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$ as input we obtain

$$\begin{aligned} y(t) &= \text{Sys}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} \\ &= t\left[\alpha_1 x_1(t) + \alpha_2 x_2(t)\right] \\ &= \alpha_1 t\, x_1(t) + \alpha_2 t\, x_2(t) \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

The system is linear.

$$\text{Sys}\{x_1(t-\tau)\} = t\, x_1(t-\tau) \neq y_1(t-\tau)$$

The system is not time-invariant.

c.

$$y_1(t) = \text{Sys}\{x_1(t)\} = e^{-t} x_1(t)$$

$$y_2(t) = \text{Sys}\{x_2(t)\} = e^{-t}\, x_2(t)$$

Using $x(t) = \alpha_1\, x_1(t) + \alpha_2\, x_2(t)$ as input we obtain

$$\begin{aligned} y(t) =& \text{Sys}\{\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\} \\ =& e^{-t}\left[\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\right] \\ =& \alpha_1\, e^{-t}\, x_1(t) + \alpha_2\, e^{-t}\, x_2(t) \\ =& \alpha_1\, y_1(t) + \alpha_2\, y_2(t) \end{aligned}$$

The system is linear.

$$\text{Sys}\{x_1(t-\tau)\} = e^{-t}\, x_1(t-\tau) \neq y_1(t-\tau)$$

The system is not time-invariant.

d.

$$y_1(t) = \text{Sys}\{x_1(t)\} = \int_{-\infty}^{t} x_1(\lambda)\, d\lambda$$

$$y_2(t) = \text{Sys}\{x_2(t)\} = \int_{-\infty}^{t} x_2(\lambda)\, d\lambda$$

Using $x(t) = \alpha_1\, x_1(t) + \alpha_2\, x_2(t)$ as input we obtain

$$\begin{aligned} y(t) =& \text{Sys}\{\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\} \\ =& \int_{-\infty}^{t} \left[\alpha_1\, x_1(\lambda) + \alpha_2\, x_2(\lambda)\right] d\lambda \\ =& \alpha_1 \int_{-\infty}^{t} x_1(\lambda)\, d\lambda + \alpha_2 \int_{-\infty}^{t} x_2(\lambda)\, d\lambda \\ =& \alpha_1\, y_1(t) + \alpha_2\, y_2(t) \end{aligned}$$

The system is linear.

$$\text{Sys}\{x_1(t-\tau)\} = \int_{-\infty}^{t} x_1(\lambda-\tau)\, d\lambda$$

Let $\gamma = \lambda - \tau$. It follows that $d\gamma = d\lambda$. Substituting these into the integral and adjusting the limits yields

$$\text{Sys}\{x_1(t-\tau)\} = \int_{-\infty}^{t-\tau} x_1(\gamma)\, d\gamma = y_1(t-\tau)$$

The system is time-invariant.

e.

$$y_1(t) = \text{Sys}\{x_1(t)\} = \int_{t-1}^{t} x_1(\lambda)\, d\lambda$$

$$y_2(t) = \text{Sys}\{x_2(t)\} = \int_{t-1}^{t} x_2(\lambda)\, d\lambda$$

Using $x(t) = \alpha_1\, x_1(t) + \alpha_2\, x_2(t)$ as input we obtain

$$\begin{aligned}
y(t) &= \text{Sys}\left\{\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\right\} \\
&= \int_{t-1}^{t} \left[\alpha_1\, x_1(\lambda) + \alpha_2\, x_2(\lambda)\right] d\lambda \\
&= \alpha_1 \int_{t-1}^{t} x_1(\lambda)\, d\lambda + \alpha_2 \int_{t-1}^{t} x_2(\lambda)\, d\lambda \\
&= \alpha_1\, y_1(t) + \alpha_2\, y_2(t)
\end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1(t-\tau)\right\} = \int_{t-1}^{t} x_1(\lambda-\tau)\, d\lambda$$

Let $\gamma = \lambda - \tau$. It follows that $d\gamma = d\lambda$. Substituting these into the integral and adjusting the limits yields

$$\text{Sys}\left\{x_1(t-\tau)\right\} = \int_{t-\tau-1}^{t-\tau} x_1(\gamma)\, d\gamma = y_1(t-\tau)$$

The system is time-invariant.

f.

$$y_1(t) = \text{Sys}\left\{x_1(t)\right\} = (t+1) \int_{-\infty}^{t} x_1(\lambda)\, d\lambda$$

$$y_2(t) = \text{Sys}\left\{x_2(t)\right\} = (t+1) \int_{-\infty}^{t} x_2(\lambda)\, d\lambda$$

Using $x(t) = \alpha_1\, x_1(t) + \alpha_2\, x_2(t)$ as input we obtain

$$\begin{aligned}
y(t) &= \text{Sys}\left\{\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\right\} \\
&= (t+1) \int_{-\infty}^{t} \left[\alpha_1\, x_1(\lambda) + \alpha_2\, x_2(\lambda)\right] d\lambda \\
&= \alpha_1\, (t+1) \int_{-\infty}^{t} x_1(\lambda)\, d\lambda + \alpha_2\, (t+1) \int_{-\infty}^{t} x_2(\lambda)\, d\lambda \\
&= \alpha_1\, y_1(t) + \alpha_2\, y_2(t)
\end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1(t-\tau)\right\} = (t+1) \int_{-\infty}^{t} x_1(\lambda-\tau)\, d\lambda$$

Let $\gamma = \lambda - \tau$. It follows that $d\gamma = d\lambda$. Substituting these into the integral and adjusting the limits yields

$$\text{Sys}\left\{x_1(t-\tau)\right\} = (t+1) \int_{-\infty}^{t-\tau} x_1(\gamma)\, d\gamma \neq y_1(t-\tau)$$

The system is not time-invariant.

2.2.

a.

$$\begin{aligned} w(t) &= 3\,x(t) \\ y(t) &= w(t-2) = 3\,x(t-2) \end{aligned}$$

b.

$$\begin{aligned} \bar{w}(t) &= x(t-2) \\ \bar{y}(t) &= 3\,\bar{w}(t) = 3\,x(t-2) \end{aligned}$$

Input-output relationship of the system does not change when the order of the two subsystems is changed.

2.3.

a. Using the first configuration:

$$\begin{aligned} w(t) &= 3\,x(t) \\ y(t) &= t\,w(t) = 3t\,x(t) \end{aligned}$$

Using the second configuration:

$$\begin{aligned} \bar{w}(t) &= t\,x(t) \\ \bar{y}(t) &= 3\,\bar{w}(t) = 3t\,x(t) \end{aligned}$$

Input-output relationship of the system does not change when the order of the two subsystems is changed.

b. Using the first configuration:

$$\begin{aligned} w(t) &= 3\,x(t) \\ y(t) &= w(t) + 5 = 3\,x(t) + 5 \end{aligned}$$

Using the second configuration:

$$\begin{aligned} \bar{w}(t) &= x(t) + 5 \\ \bar{y}(t) &= 3\,\bar{w}(t) = 3\left[x(t)+5\right] = 3\,x(t) + 15 \end{aligned}$$

Input-output relationship of the system changes when the order of the two subsystems is changed.

2.4.

Writing the KVL around the loop on the left yields

$$\begin{aligned} x(t) &= R\left[i_L(t) + i_C(t)\right] + y(t) \\ &= R\,i_L(t) + R\,i_C(t) + y(t) \end{aligned}$$

Recognizing that

$$i_C(t) = C\,\frac{dv_c(t)}{dt} = C\,\frac{dy(t)}{dt}$$

we have

$$x(t) = R\,i_L(t) + RC\,\frac{dy(t)}{dt} + y(t)$$

Differentiating both sides of this result and recognizing that

$$y(t) = v_L(t) = L\,\frac{di_L(t)}{dt}$$

we get

$$\begin{aligned}\frac{dx(t)}{dt} &= R\,\frac{di_L(t)}{dt} + RC\,\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt}\\ &= \frac{R}{L}\,y(t) + RC\,\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt}\end{aligned}$$

Thus the differential equation for the circuit is

$$\frac{d^2y(t)}{dt^2} + \frac{1}{RC}\,\frac{dy(t)}{dt} + \frac{1}{LC}\,y(t) = \frac{1}{RC}\,\frac{dx(t)}{dt}$$

Initial conditions are found through

$$y(0) = v_C(0) = 2$$

and

$$R\,i_L(0) + RC\left.\frac{dy(t)}{dt}\right|_{t=0} + y(0) = x(0) \quad\Rightarrow\quad \begin{aligned}\left.\frac{dy(t)}{dt}\right|_{t=0} &= -\frac{1}{RC}\,y(0) - \frac{1}{C}\,i_L(0) - \frac{1}{RC}\,x(0)\\ &= -\frac{2}{RC} - \frac{1}{C} - \frac{1}{RC}\,x(0)\end{aligned}$$

2.5. Let the currents of the two capacitors be $i_1(t)$ and $i_2(t)$. Begin by writing the nodal equations for the circuit:

$$\frac{v_1(t) - x(t)}{R_1} + \frac{v_1(t) - v_2(t)}{R_2} + i_1(t) = 0$$

$$\frac{v_2(t) - v_1(t)}{R_2} + i_2(t) = 0$$

Using the relationships

$$v_2(t) = y(t)\,, \quad i_1(t) = C_1\,\frac{dv_1(t)}{dt}\,,\text{and} \quad i_2(t) = C_2\,\frac{dv_2(t)}{dt} = C_2\,\frac{dy(t)}{dt}$$

nodal equations become

$$\frac{v_1(t) - x(t)}{R_1} + \frac{v_1(t) - y(t)}{R_2} + C_1\,\frac{dv_1(t)}{dt} = 0 \qquad \text{(P.2.5.1)}$$

$$\frac{y(t) - v_1(t)}{R_2} + C_2\,\frac{dy(t)}{dt} = 0 \qquad \text{(P.2.5.2)}$$

Next, let us solve for $v_1(t)$ from Eqn. (P2.5.2)

$$v_1(t) = y(t) + R_2 C_2 \frac{dy(t)}{dt}$$

and differentiate both sides to obtain

$$\frac{dv_1(t)}{dt} = \frac{dy(t)}{dt} + R_2 C_2 \frac{d^2 y(t)}{dt^2}$$

Substituting the last two results into Eqn. ((P.2.5.1) and simplifying the differential equation obtained yields

$$R_1 R_2 C_1 C_2 \frac{d^2 y(t)}{dt^2} + \left[R_1 (C_1 + C_2) + R_2 C_2\right] \frac{dy(t)}{dt} + y(t) = x(t)$$

The initial conditions are

$$y() = v_2(0) = 2 \text{ V} \qquad \text{and} \qquad \left.\frac{dy(t)}{dt}\right|_{t=0} = \frac{1}{R_2 C_2}\left[v_1(0) - v_2(0)\right]$$

2.6.

Let $w(t)$ be the number of encounters between prey and predators at time t:

$$w(t) = K\, x(t)\, y(t)$$

The growth rate of prey is

$$\begin{aligned} \frac{dx(t)}{dt} =& A\, x(t) - C\, w(t) \\ =& A\, x(t) - \bar{C}\, x(t)\, y(t) \end{aligned}$$

The growth rate of predators is

$$\begin{aligned} \frac{dy(t)}{dt} =& -B\, y(t) + D\, w(t) \\ =& -B\, y(t) + \bar{D}\, x(t)\, y(t) \end{aligned}$$

The differential equations derived form a nonlinear system.

2.7.

Using Eqn. (2.57) with $t_0 = 0$ yields the solution

$$\begin{aligned} y(t) =& e^{-4t}\, y(0) + \int_0^t e^{-4(t-\tau)}\, r(\tau)\, dt \\ =& e^{-4t}\, y(0) + 4 \int_0^t e^{-4(t-\tau)}\, u(\tau)\, dt \\ =& e^{-4t}\, y(0) + 4\, e^{-4t} \int_0^t e^{4\tau}\, dt \\ =& e^{-4t}\, y(0) + 1 - e^{-4t} \end{aligned}$$

a. $y(t) = 1 - e^{-4t}\ , \qquad t \geq 0$

b. $y(t) = 1 + 4\,e^{-4t}\ , \qquad t \geq 0$

c. $y(t) = 1\ , \qquad t \geq 0$

d. $y(t) = 1 - 2\,e^{-4t}\ , \qquad t \geq 0$

e. $y(t) = 1 - 4\,e^{-4t}\ , \qquad t \geq 0$

2.8.

a.

$$\begin{aligned} y(t) &= e^{-4t}\,(-1) + e^{-4t}\int_0^t e^{4\tau}\,u(\tau)\,d\tau \\ &= e^{-4t}\,(-1) + e^{-4t}\int_0^t e^{4\tau}\,d\tau \\ &= \frac{1}{4} - \frac{5}{4}\,e^{-4t}\ , \qquad t \geq 0 \end{aligned}$$

b.

$$y(t) = e^{-2t}\,(2) + e^{-2t}\int_0^t e^{2\tau}\,(2)\,[u(\tau) - u(\tau-5)]\,d\tau$$

If $0 < t < 5$ then

$$\begin{aligned} y(t) &= 2\,e^{-2t} + 2\,e^{-2t}\int_0^t e^{2\tau}\,d\tau \\ &= 1 + e^{-2t} \end{aligned}$$

If $t > 5$, then

$$\begin{aligned} y(t) &= 2\,e^{-2t} + 2\,e^{-2t}\int_0^5 e^{2\tau}\,d\tau \\ &= \left[e^{10} + 1\right]\,e^{-2t} \end{aligned}$$

Therefore, the complete solution is

$$y(t) = \begin{cases} 1 + e^{-2t}\,, & 0 < t < 5 \\ \left[e^{10}+1\right]\,e^{-2t}\,, & t > 5 \end{cases}$$

c.

$$\begin{aligned} y(t) &= e^{-5t}\,(0.5) + e^{-5t}\int_0^t 3\,e^{5\tau}\,\delta(\tau)\,d\tau \\ &= 0.5\,e^{-5t} + 3\,e^{-5t} = 3.5\,e^{-5t}\ , \qquad t > 0 \end{aligned}$$

d.

$$\begin{aligned} y(t) =& e^{-5t}(-4) + e^{-5t}\int_0^t e^{5\tau}\, 3\tau\, u(\tau)\, d\tau \\ =& -4\, e^{-5t} + 3\, e^{-5t}\int_0^t \tau\, e^{5\tau}\, d\tau \end{aligned}$$

Using Eqn. (B.16) from Appendix B.2 we get

$$\int_0^t \tau\, e^{5\tau}\, d\tau = \frac{1}{25}\left[5\, t\, e^{5t} - e^{5t} + 1\right]$$

and

$$y(t) = \frac{3}{5}\, t - \frac{3}{25} - \frac{97}{25}\, e^{-5t}\;, \qquad t \ge 0$$

e.

$$\begin{aligned} y(t) =& e^{-t}(-1) + e^{-t}\int_0^t e^{\tau}\, 2\, e^{-2\tau}\, u(\tau)\, d\tau \\ =& -e^{-t} + 2\, e^{-t}\int_0^t e^{-\tau}\, d\tau \\ =& e^{-t} - 2\, e^{-2t}\;, \quad t \ge 0 \end{aligned}$$

2.9.

a. Characteristic equation is

$$s^2 + 3s + 2 = 0 \qquad \Rightarrow \qquad (s+1)\,(s+2) = 0$$

The solutions of the characteristic equation are $s_1 = -1$ and $s_2 = -2$. The homogeneous solution is in the form

$$y(t) = c_1\, e^{-t} + c_2\, e^{-2t}\;, \qquad t \ge 0$$

In order to satisfy the initial conditions we need

$$y(0) = c_1 + c_2 = 3$$

and

$$\left.\frac{d\,y(t)}{dt}\right|_{t=0} = -c_1 - 2\, c_2 = 0$$

which can be solved to yield $c_1 = 6$ and $c_2 = -3$. The homogeneous solution is

$$y(t) = 6\, e^{-t} - 3\, e^{-2t}\;, \qquad t \ge 0$$

b. Characteristic equation is

$$s^2 + 4s + 3 = 0 \qquad \Rightarrow \qquad (s+1)\,(s+3) = 0 \qquad \Rightarrow \qquad s_{1,2} = -1\,,\ -3$$

The solutions of the characteristic equation are $s_1 = -1$ and $s_2 = -3$. The homogeneous solution is in the form

$$y(t) = c_1\, e^{-t} + c_2\, e^{-3t}\ , \qquad t \ge 0$$

In order to satisfy the initial conditions we need

$$y(0) = c_1 + c_2 = -2$$

and

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = -c_1 - 3\,c_2 = 1$$

which can be solved to yield $c_1 = -5/2$ and $c_2 = 1/2$. The homogeneous solution is

$$y(t) = -\frac{5}{2}\, e^{-t} + \frac{1}{2}\, e^{-3t}\ , \qquad t \ge 0$$

c. Characteristic equation is

$$s^2 - 1 = 0 \qquad \Rightarrow \qquad (s+1)\,(s+2) = 0$$

The solutions of the characteristic equation are $s_1 = 1$ and $s_2 = -1$. The homogeneous solution is in the form

$$y(t) = c_1\, e^{t} + c_2\, e^{-t}\ , \qquad t \ge 0$$

In order to satisfy the initial conditions we need

$$y(0) = c_1 + c_2 = 1$$

and

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = c_1 - c_2 = -2$$

which can be solved to yield $c_1 = -1/2$ and $c_2 = 3/2$. The homogeneous solution is

$$y(t) = -\frac{1}{2}\, e^{t} + \frac{3}{2}\, e^{-t}\ , \qquad t \ge 0$$

d. Characteristic equation is

$$s^3 + 6s^2 + 6s + 2 = 0 \qquad \Rightarrow \qquad (s+1)\,(s+2)\,(s+3) = 0$$

The solutions of the characteristic equation are $s_1 = -1$, $s_2 = -2$ and $s_3 = -3$. The homogeneous solution is in the form

$$y(t) = c_1\, e^{-t} + c_2\, e^{-2t} + c_3\, e^{-3t}\ , \qquad t \ge 0$$

In order to satisfy the initial conditions we need

$$y(0) = c_1 + c_2 + c_3 = 2$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = -c_1 - 2\,c_2 - 3\,c_3 = -1$$

and

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=0} = c_1 + 4\,c_2 + 9\,c_3 = 1$$

which can be solved to yield $c_1 = 4$, $c_2 = -3$ and $c_3 = 1$. The homogeneous solution is

$$y(t) = 4\,e^{-t} - 3\,e^{-2t} + e^{-3t}\ , \qquad t \ge 0$$

2.10.

a. The characteristic equation is

$$s^2 + 3 = 0 \qquad \Rightarrow \qquad \left(s + j\sqrt{3}\right)\left(s - j\sqrt{3}\right) = 0$$

Therefore the homogeneous solution is in the form

$$y(t) = d_1\,\cos\left(\sqrt{3}\,t\right) + d_2\,\sin\left(\sqrt{3}\,t\right)\ , \qquad t \ge 0$$

Coefficients d_1 and d_2 are determined through the initial conditions.

$$y(0) = d_1 = 2$$

$$\frac{dy(t)}{dt} = -\sqrt{3}\,d_1\,\sin\left(\sqrt{3}\,t\right) + \sqrt{3}\,d_2\,\cos\left(\sqrt{3}\,t\right)$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = \sqrt{3}\,d_2 = 0 \qquad \Rightarrow \qquad d_2 = 0$$

Therefore

$$y(t) = 2\,\cos\left(\sqrt{3}\,t\right)\ , \qquad t \ge 0$$

b. The characteristic equation is

$$s^2 + 2s + 2 = 0 \qquad \Rightarrow \qquad (s+1)^2 + 1 = 0 \qquad \Rightarrow \qquad (s + 1 + j)\,(s + 1 - j) = 0$$

Therefore the homogeneous solution is in the form

$$y(t) = d_1\,e^{-t}\cos(t) + d_2\,e^{-t}\,\sin(t)\ , \qquad t \ge 0$$

Coefficients d_1 and d_2 are determined through the initial conditions.

$$y(0) = d_1 = -2$$

$$\frac{dy(t)}{dt} = e^{-t}\,(d_2 - d_1)\,\cos(t) + e^{-t}\,(-d_1 - d_2)\,\sin(t)$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = d_2 - d_1 = -1 \qquad \Rightarrow \qquad d_2 - 1 + d_1 = -3$$

Therefore

$$y(t) = -2\,e^{-t}\,\cos(t) - 3\,e^{-t}\,\sin(t)\ , \qquad t \ge 0$$

c. The characteristic equation is

$$s^2+4s+13=0 \qquad \Rightarrow \qquad (s+2)^2+9=0 \qquad \Rightarrow \qquad \left(s+2+j3\right)\left(s+2-j3\right)=0$$

Therefore the homogeneous solution is in the form

$$y(t)=d_1\,e^{-2t}\cos(3t)+d_2\,e^{-2t}\sin(3t)\ , \qquad t\ge 0$$

Coefficients d_1 and d_2 are determined through the initial conditions.

$$y(0)=d_1=5$$

$$\frac{dy(t)}{dt}=e^{-2t}\,(-2\,d_1+3\,d_2)\,\cos(3t)+e^{-2t}\,(-3\,d_1-3\,d_2)\,\sin(t)$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0}=-2\,d_1+3\,d_2=0 \qquad \Rightarrow \qquad d_2=\frac{2}{3}\,d_1=\frac{10}{3}$$

Therefore

$$y(t)=5\,e^{-2t}\cos(3t)+\frac{10}{3}\,e^{-2t}\sin(3t)\ , \qquad t\ge 0$$

d. The characteristic equation is

$$s^3+3\,s^2+4s+2=0 \qquad \Rightarrow \qquad (s+1)\left(s+1+j\right)\left(s+1-j\right)=0$$

Therefore the homogeneous solution is in the form

$$y(t)=c_1\,e^{-t}+d_2\,e^{-t}\cos(t)+d_3\,e^{-t}\sin(t)\ , \qquad t\ge 0$$

The derivatives are

$$\frac{dy(t)}{dt}=-c_1\,e^{-t}+(d_3-d_2)\,e^{-t}\cos(t)+(-d_3-d_2)\,e^{-t}\sin(t)$$

and

$$\frac{d^2y(t)}{dt^2}=c_1\,e^{-t}-2d_3\,e^{-t}\cos(t)+2d_2\,e^{-t}\sin(t)$$

Imposing the initial conditions yields

$$y(0)=c_1+d_2=1$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0}=-c_1-d_2+d_3=0$$

and

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=0}=c_1-2d_3=-2$$

Coefficient values are $c_1=0$, $d_2=1$ and $d_3=1$. The homogeneous solution is

$$y(t)=e^{-t}\cos(t)+e^{-t}\sin(t)\ , \qquad t\ge 0$$

2.11.

a. The characteristic equation is

$$s^2+2s+1=0 \qquad \Rightarrow \qquad (s+1)^2=0$$

Therefore the homogeneous solution is in the form

$$y(t)=c_1\,e^{-t}+c_2\,t\,e^{-t}\;, \qquad t\ge 0$$

Coefficients c_1 and c_2 are determined through the initial conditions.

$$y(0)=c_1=1$$

$$\frac{dy(t)}{dt}=-c_1\,e^{-t}+c_2\,e^{-t}-c_2\,t\,e^{-t}$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0}=-c_1+c_2=0 \qquad \Rightarrow \qquad c_2=1$$

Therefore

$$y(t)=e^{-t}+t\,e^{-t}\;, \qquad t\ge 0$$

b. The characteristic equation is

$$s^3+7s^2+16s+12=0 \qquad \Rightarrow \qquad (s+2)^2\,(s+3)=0$$

Therefore the homogeneous solution is in the form

$$y(t)=c_1\,e^{-2t}+c_2\,t\,e^{-2t}+c_3\,e^{-3t}\;, \qquad t\ge 0$$

Coefficients c_1, c_2 and c_3 are determined through the initial conditions.

$$y(0)=c_1+c_3=1 \tag{P.2.11.1}$$

$$\frac{dy(t)}{dt}=-2\,c_1\,e^{-2t}+c_2\,e^{-2t}-2\,c_2\,t\,e^{-2t}-3\,c_3\,e^{-3t}$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0}=-2\,c_1+c_2-3\,c_3=-2 \tag{P.2.11.2}$$

$$\frac{d^2y(t)}{dt^2}=4\,c_1\,e^{-2t}-2\,c_2\,e^{-2t}-2\,c_2\,e^{-2t}+4\,c_2\,t\,e^{-2t}+9\,c_3\,e^{-3t}$$

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=0}=4\,c_1-4\,c_2+9\,c_3=1 \tag{P.2.11.3}$$

Solving Eqns. (P.2.11.1), (P.2.11.2) and (P.2.11.3) for the coefficients leads to

$$c_1=4\;, \qquad c_2=-3\;, \qquad c_3=-3\;,$$

Therefore

$$y(t)=4\,e^{-2t}-3\,t\,e^{-2t}-3\,e^{-3t}\;, \qquad t\ge 0$$

c. The characteristic equation is

$$s^3+6s^2+12s+8=0 \qquad \Rightarrow \qquad (s+2)^3=0$$

Therefore the homogeneous solution is in the form

$$y(t)=c_1\,e^{-2t}+c_2\,t\,e^{-2t}+c_3\,t^2\,e^{-2t}\ , \qquad t\ge 0$$

Coefficients c_1, c_2 and c_3 are determined through the initial conditions.

$$y(0)=c_1=-1 \tag{P.2.11.4}$$

$$\frac{d\,y(t)}{d\,t}=-2\,c_1\,e^{-2t}+c_2\,e^{-2t}-2\,c_2\,t\,e^{-2t}+2\,c_3\,t\,e^{-2t}-2\,c_3\,t^2\,e^{-2t}$$

$$\left.\frac{d\,y(t)}{d\,t}\right|_{t=0}=-2\,c_1+c_2=0 \qquad \Rightarrow \qquad c_2=2\,c_1=-2 \tag{P.2.11.5}$$

$$\frac{d^2\,y(t)}{d\,t^2}=4\,c_1\,e^{-2t}-2\,c_2\,e^{-2t}-2\,c_2\,e^{-2t}+4\,c_2\,t\,e^{-2t}+2\,c_3\,e^{-2t}-4\,c_3\,t\,e^{-2t}-4\,c_3\,t\,e^{-2t}+4\,c_3\,t^2\,e^{-2t}$$

$$\left.\frac{d^2\,y(t)}{d\,t^2}\right|_{t=0}=4\,c_1-4\,c_2+2\,c_3=1 \tag{P.2.11.6}$$

Solving Eqns. (P.2.11.1), (P.2.11.2) and (P.2.11.3) for the coefficients leads to

$$c_1=-1\ , \qquad c_2=-2\ , \qquad c_3=-1.5;\ ,$$

Therefore

$$y(t)=-e^{-2t}+-2\,t\,e^{-2t}-1.5\,e^{-3t}\ , \qquad t\ge 0$$

2.12.

The particular solution is in the form

$$y_p=k_1\,t+k_2$$

Since it must satisfy the differential equation, we have

$$k_1+4\left[k_1\,t+k_2\right]=4t$$

which leads to coefficient values $k_1=1$ and $k_2=-1/4$. The characteristic equation is

$$s+4=0$$

Therefore the homogeneous solution is in the form

$$y_h(t)=c_1\,e^{-4t}\ , \qquad t\ge 0$$

and the total solution is in the form

$$y(t)=c_1\,e^{-4t}+t-\frac{1}{4}\ , \qquad t\ge 0$$

The total solution must satisfy the initial conditions.

$$y(0) = c_1 - \frac{1}{4} = 0 \qquad \Rightarrow \qquad c_1 = \frac{1}{4}$$

Therefore

$$\begin{aligned} y(t) =& \frac{1}{4}\, e^{-4t} + t - \frac{1}{4} \\ =& t - \frac{1}{4}\left[1 - e^{-4t}\right] \;, \qquad t \ge 0 \end{aligned}$$

2.13.

a. The particular solution is in the form

$$y_p = k_1$$

Since it must satisfy the differential equation, we have

$$4\,k_1 = 1 \qquad \Rightarrow \qquad k_1 = -\frac{1}{4}$$

The characteristic equation is

$$s + 4 = 0$$

Therefore the homogeneous solution is in the form

$$y_h(t) = c_1\, e^{-4t} \;, \qquad t \ge 0$$

and the total solution is in the form

$$y(t) = c_1\, e^{-4t} + \frac{1}{4} \;, \qquad t \ge 0$$

The total solution must satisfy the initial condition.

$$y(0) = c_1 + \frac{1}{4} = -1 \qquad \Rightarrow \qquad c_1 = -\frac{5}{4}$$

Therefore

$$y(t) = -\frac{5}{4}\, e^{-4t} \;, \qquad t \ge 0$$

b. The particular solution is in the form

$$y_p = k_1 \sin(2t) + k_2 \cos(2t) + k_3 \cos(t) + k_4 \sin(t) \tag{P.2.13.1}$$

The particular solution must satisfy the differential equation.

$$\frac{d\,y_p(t)}{dt} = 2\,k_1 \cos(2t) - 2\,k_2 \sin(2t) - k_3 \sin(t) + k_4 \cos(t) \tag{P.2.13.2}$$

Using Eqns. (P.2.13.1) and (P.2.13.2) in the differential equation we have

$$\begin{aligned} \frac{d\,y_p(t)}{dt} + 2\,y_p(t) =& \left[2\,k_1 + 2\,k_2\right] \cos(2t) + \left[2\,k_1 - 2\,k_2\right] \sin(2t) + \left[2\,k_3 + k_4\right] \cos(t) + \left[-k_3 + 2\,k_4\right] \sin(t) \\ =& 2 \sin(2t) + 4 \cos(t) \end{aligned}$$

which leads to the set of equations

$$2k_1 + 2k_2 = 0$$

$$2k_1 - 2k_2 = 2$$

$$2k_3 + k_4 = 4$$

$$-k_3 + 2k_4 = 0$$

and can be solved to yield

$$k_1 = \frac{1}{2}\ , \qquad k_2 = -\frac{1}{2}\ , \qquad k_3 = \frac{8}{5}\ , \qquad k_4 = \frac{4}{5}$$

The characteristic equation is

$$s + 2 = 0$$

Therefore the homogeneous solution is in the form

$$y_h(t) = c_1\, e^{-2t}\ , \qquad t \geq 0$$

and the total solution is in the form

$$y(t) = c_1\, e^{-2t} + \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t) + \frac{8}{5}\cos(t) + \frac{4}{5}\sin(t)\ , \qquad t \geq 0$$

The total solution must satisfy the initial condition.

$$y(0) = c_1 - \frac{1}{2} + \frac{8}{5} = 2 \qquad \Rightarrow \qquad c_1 = \frac{9}{10}$$

Therefore

$$y(t) = \frac{9}{10}\, e^{-2t} + \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t) + \frac{8}{5}\cos(t) + \frac{4}{5}\sin(t)\ , \qquad t \geq 0$$

c. The particular solution is in the form

$$y_p = k_1\, t + k_2$$

The particular solution must satisfy the differential equation.

$$k_1 + 5\left[k_1\, t + k_2\right] = 3\, t$$

We obtain the set of equations

$$5\, k_1 = 3$$

$$k_1 + 5\, k_2 = 0$$

The coefficients of the particular solution are $k_1 = 3/5$ and $k_2 = -3/25$. The characteristic equation is

$$s + 5 = 0$$

Therefore the homogeneous solution is in the form

$$y_h(t) = c_1\, e^{-5t}\ , \qquad t \geq 0$$

and the total solution is in the form

$$y(t) = c_1\, e^{-5t} + \frac{3}{5}\, t - \frac{3}{25}\ , \qquad t \ge 0$$

The total solution must satisfy the initial condition.

$$y(0) = c_1 - \frac{3}{25} = -4 \qquad \Rightarrow \qquad c_1 = -\frac{97}{25}$$

Therefore

$$y(t) = -\frac{97}{25}\, e^{-5t} + \frac{3}{5}\, t - \frac{3}{25}\ , \qquad t \ge 0$$

d. The particular solution is in the form

$$y_p = k_1\, e^{-2t} \tag{P.2.13.1}$$

The particular solution must satisfy the differential equation.

$$-2\, k_1\, e^{-2t} + k_1\, e^{-2t} - 2\, e^{-2t} \tag{P.2.13.1}$$

which leads to $k_1 = -2$. The characteristic equation is

$$s + 1 = 0$$

Therefore the homogeneous solution is in the form

$$y_h(t) = c_1\, e^{-t}\ , \qquad t \ge 0$$

and the total solution is in the form

$$y(t) = c_1\, e^{-t} - 2\, e^{-2t}\ , \qquad t \ge 0$$

The total solution must satisfy the initial condition.

$$y(0) = c_1 - 2 = -1 \qquad \Rightarrow \qquad c_1 = 1$$

Therefore

$$y(t) = e^{-t} - 2\, e^{-2t}\ , \qquad t \ge 0$$

2.14.

a. The particular solution is in the form

$$y_p = k_1$$

Since it must satisfy the differential equation, we have $k_1 = 1/2$. The characteristic equation is

$$s^2 + 3s + 2 = 0 \qquad \Rightarrow \qquad (s+1)\,(s+2) = 0$$

Therefore the homogeneous solution is in the form

$$y_h(t) = c_1\, e^{-t} + c_2\, e^{-2t}\ , \qquad t \ge 0$$

and the total solution is in the form

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2} , \qquad t \geq 0$$

The total solution must satisfy the initial conditions.

$$y(0) = c_1 + c_2 + \frac{1}{2} = 3 \qquad \Rightarrow \qquad c_1 + c_2 = \frac{5}{2}$$

$$\frac{dy(t)}{dt} = -c_1 e^{-t} - 2 c_2 e^{-2t}$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = -c_1 - 2 c_2 = 0$$

The coefficients are found as $c_1 = 5$ and $c_2 = -5/2$. Therefore

$$y(t) = 5 e^{-t} - \frac{5}{2} e^{-2t} + \frac{1}{2} , \qquad t \geq 0$$

b. The particular solution is in the form

$$y_p = k_1 t + k_2$$

The particular solution must satisfy the differential equation.

$$\frac{dy_p(t)}{dt} = k_1$$

$$4 k_1 + 3 (k_1 t + k_2) = t + 1 \qquad \Rightarrow \qquad k_1 = \frac{1}{3} , \quad k_2 = -\frac{1}{9}$$

The characteristic equation is

$$s^2 + 4s + 3 = 0 \qquad \Rightarrow \qquad (s+1)(s+3) = 0$$

Therefore the homogeneous solution is in the form

$$y_h(t) = c_1 e^{-t} + c_2 e^{-3t} , \qquad t \geq 0$$

and the total solution is in the form

$$y(t) = c_1 e^{-t} + c_2 e^{-3t} + \frac{1}{3} t - \frac{1}{9} , \qquad t \geq 0$$

The total solution must satisfy the initial conditions.

$$y(0) = c_1 + c_2 - \frac{1}{9} = 2 \qquad \Rightarrow \qquad c_1 + c_2 = \frac{19}{9}$$

$$\frac{dy(t)}{dt} = -c_1 e^{-t} - 3 c_2 e^{-3t} + \frac{1}{3}$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = -c_1 - 3 c_2 + \frac{1}{3} = 0 \qquad \Rightarrow \qquad -c_1 - 3 c_2 = -\frac{1}{3}$$

The coefficients are found as $c_1 = 3$ and $c_2 = -8/9$. Therefore

$$y(t) = 3\,e^{-t} - \frac{8}{9}\,e^{-3t} + \frac{1}{3}\,t - \frac{1}{9}\;, \qquad t \ge 0$$

c. The particular solution is in the form

$$y_p = k_1$$

Since it must satisfy the differential equation, we have $k_1 = 1/3$. The characteristic equation is

$$s^2 + 3 = 0 \qquad \Rightarrow \qquad \left(s + j\sqrt{3}\right)\left(s - j\sqrt{3}\right) = 0$$

Therefore the homogeneous solution is in the form

$$y_h(t) = d_1\,\cos\left(\sqrt{3}\,t\right) + d_2\,\sin\left(\sqrt{3}\,t\right)\;, \qquad t \ge 0$$

and the total solution is in the form

$$y(t) = d_1\,\cos\left(\sqrt{3}\,t\right) + d_2\,\sin\left(\sqrt{3}\,t\right) + \frac{1}{3}\;, \qquad t \ge 0$$

The total solution must satisfy the initial conditions.

$$y(0) = d_1 + \frac{1}{3} = 1 \qquad \Rightarrow \qquad d_1 = \frac{2}{3}$$

$$\frac{dy(t)}{dt} = -\sqrt{3}\,d_1\,\sin\left(\sqrt{3}\,t\right) + \sqrt{3}\,d_2\,\cos\left(\sqrt{3}\,t\right)$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = \sqrt{3}\,d_2 = 0 \qquad \Rightarrow \qquad d_2 = 0$$

Therefore

$$y(t) = \frac{2}{3}\,\cos\left(\sqrt{3}\,t\right) + \frac{1}{3}\;, \qquad t \ge 0$$

d. The particular solution is in the form

$$y_p = k_1\,e^{-2t}$$

Since it must satisfy the differential equation, we have

$$4\,k_1\,e^{-2t} - 4\,k_1\,e^{-2t} + k_1\,e^{-2t} = e^{-2t}$$

leading to $k_1 = 1$. The characteristic equation is

$$s^2 + 2s + 1 = 0 \qquad \Rightarrow \qquad (s+1)^2 = 0$$

Therefore the homogeneous solution is in the form

$$y_h(t) = c_1\,e^{-t} + c_2\,t\,e^{-t}\;, \qquad t \ge 0$$

and the total solution is in the form

$$y(t) = c_1\,e^{-t} + c_2\,t\,e^{-t} + e^{-2t}\;, \qquad t \ge 0$$

The total solution must satisfy the initial conditions.

$$y(0) = c_1 + 1 = 1 \qquad \Rightarrow \qquad c_1 = 0$$

$$\frac{dy(t)}{dt} = -c_1\,e^{-t} + c_2\,e^{-t} - c_2\,t\,e^{-t} - 2\,e^{-2t}$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = -c_1 + c_2 - 2 = 0 \qquad \Rightarrow \qquad c_2 = 2$$

Therefore

$$y(t) = 2t\,e^{-t} + e^{-2t}\,, \qquad t \ge 0$$

2.15. Using the intermediate variable $w(t)$ we have

$$\frac{d^2w(t)}{dt^2} + 4\,\frac{dw(t)}{dt} + 3\,w(t) = x(t)$$

and the output signal $y(t)$ is computed as

$$y(t) = \frac{dw(t)}{dt} - 2\,w(t)$$

Using the output equation, the initial conditions can be expressed as

$$y(0) = \left.\frac{dw(t)}{dt}\right|_{t=0} - 2\,w(0) = -2 \tag{P.2.15.1}$$

and

$$\left.\frac{dy(0)}{dt}\right|_{t=0} = \left.\frac{d^2w(t)}{dt^2}\right|_{t=0} - 2\,\left.\frac{dw(t)}{dt}\right|_{t=0} = 1 \tag{P.2.15.2}$$

The second derivative in Eqn. (P.2.15.2) can be resolved as

$$\left.\frac{d^2w(t)}{dt^2}\right|_{t=0} = -4\,\left.\frac{dw(t)}{dt}\right|_{t=0} - 3\,w(0) + x(0)$$

which can be used in Eqn. (P.2.15.2) to yield

$$-6\,\left.\frac{dw(t)}{dt}\right|_{t=0} - 3\,w(0) = 1 \tag{P.2.15.3}$$

where we have assumed that $x(0) = 0$. To simplify the notation, let

$$a = \left.\frac{dw(t)}{dt}\right|_{t=0} \qquad \text{and} \qquad b = w(0)$$

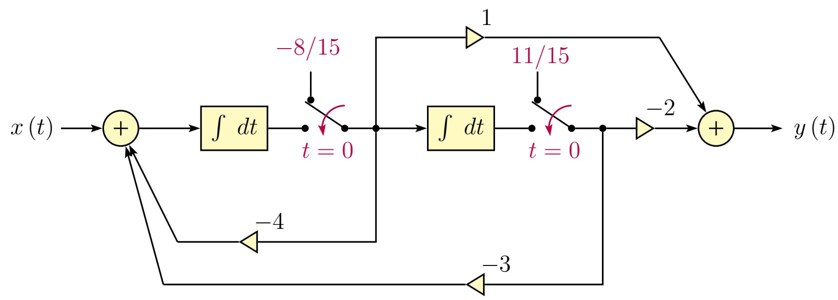
so that the Eqns. (P.2.15.1) and (P.2.15.3) become

$$a - 2b = -2$$

$$-6a - 3b = 1$$

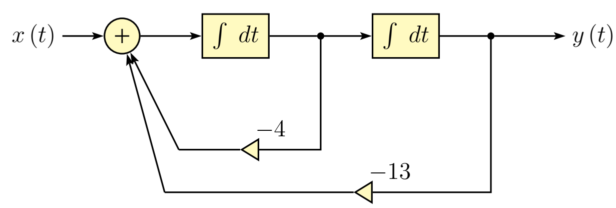
with the solutions

$$a = \left.\frac{dw(t)}{dt}\right|_{t=0} = -8/15 \qquad \text{and} \qquad b = w(0) = 11/15$$



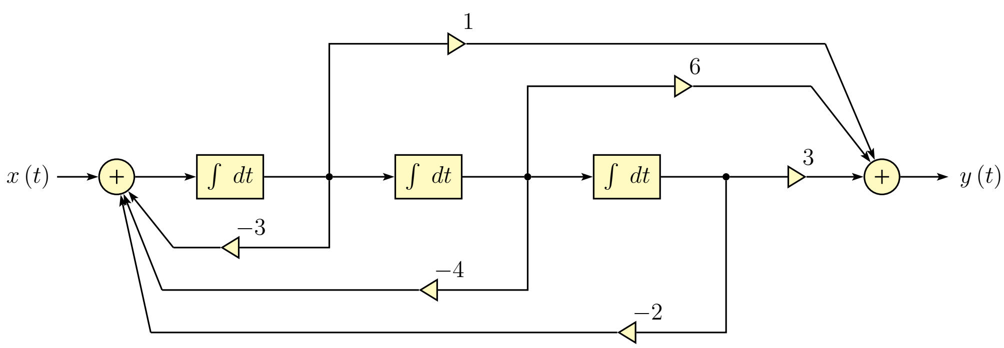
$x(t)$
$\int dt$
$-8/15$
$t=0$
$11/15$
1
-2
-4
-3
$y(t)$

2.16. a.



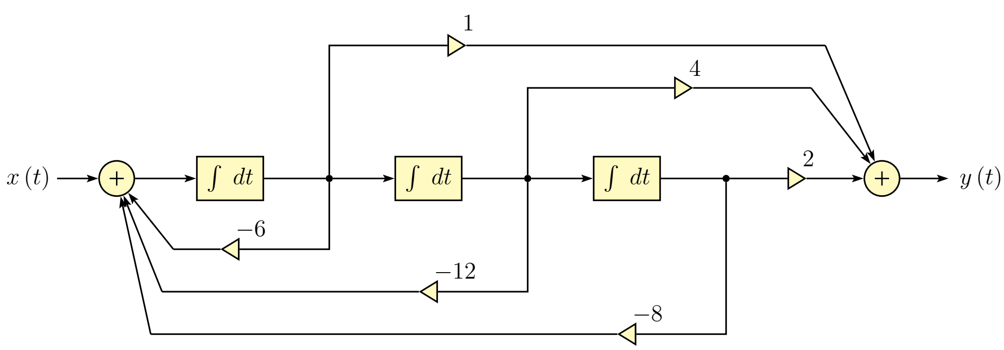
$x(t)$
$\int dt$
-4
-13
$y(t)$

b.



$x(t)$
$\int dt$
1
6
3
-3
-4
-2
$y(t)$

c.



$x(t)$
$\int dt$
1
4
2
-6
-12
-8
$y(t)$

2.17.

a.

$$\begin{aligned} w(t) &= h_1(t) * x(t) \\ y(t) &= h_2(t) * w(t) \\ &= h_2(t) * \left[h_1(t) * x(t)\right] \\ &= \left[h_2(t) * h_1(t)\right] * x(t) \end{aligned}$$

Therefore

$$h_{eq}(t) = h_2(t) * h_1(t) = h_1(t) * h_2(t)$$

b.

$$h_{eq}(t) = \int_{-\infty}^{\infty} \Pi(\tau - 0.5)\, \Pi(t - \tau - 0.5)\, d\tau$$

Since

$$\Pi(\tau - 0.5) = \begin{cases} 1\,, & 0 < \tau < 1 \\ 0\,, & \text{otherwise} \end{cases}$$

and

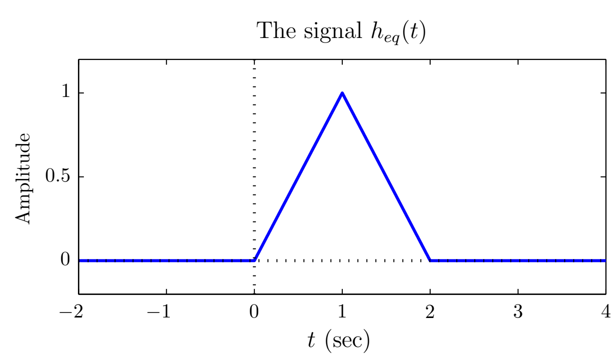
$$\Pi(t - \tau - 0.5) = \begin{cases} 1\,, & t-1 < \tau < t \\ 0\,, & \text{otherwise} \end{cases}$$

the convolution integral can be written as follows:

$$\begin{aligned} t < 0: &\quad h_{eq}(t) = 0 \\ 0 < t < 1: &\quad h_{eq}(t) = \int_0^t (1)\,(1)\, d\tau = t \\ 1 < t < 2: &\quad h_{eq}(t) = \int_{t-1}^1 (1)\,(1)\, d\tau = 2 - t \\ t > 2: &\quad h_{eq}(t) = 0 \end{aligned}$$

The equivalent impulse response is

$$h_{eq}(t) = \begin{cases} t\,, & 0 < t < 1 \\ 2 - t\,, & 1 < t < 2 \\ 0\,, & \text{otherwise} \end{cases}$$



c.

$$w(t) = \int_{-\infty}^{\infty} h_1(\tau)\, u(t - \tau)\, d\tau$$

Since

$$u(t-\tau) = \begin{cases} 1\,, & \tau < t \\ 0\,, & \tau > t \end{cases}$$

the convolution integral can be written as

$$w(t) = \int_{-\infty}^{t} h_1(\tau)\, d\tau$$

and can be evaluated as

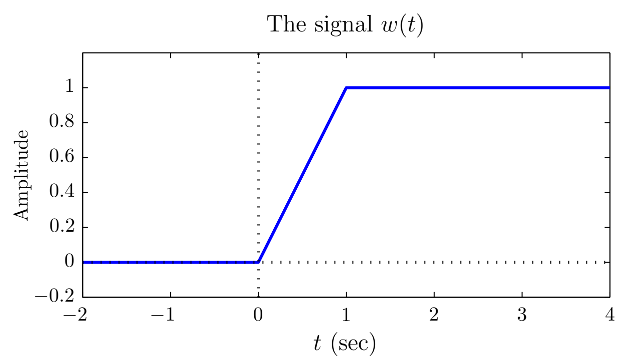
$$t<0: \qquad w(t) = 0$$

$$0<t<1: \qquad w(t) = \int_0^t (1)\, d\tau = t$$

$$t>1: \qquad w(t) = \int_0^1 (1)\, d\tau = 1$$

The signal $w(t)$ is

$$w(t) = \begin{cases} 0\,, & t<0 \\ t\,, & 0<t<1 \\ 1\,, & t>1 \end{cases}$$



Similarly

$$y(t) = \int_{-\infty}^{\infty} h_{eq}(\tau)\, u(t-\tau)\, d\tau = \int_{-\infty}^{t} h_{eq}(\tau)\, d\tau$$

which can be evaluated as

$$t<0: \qquad y(t) = 0$$

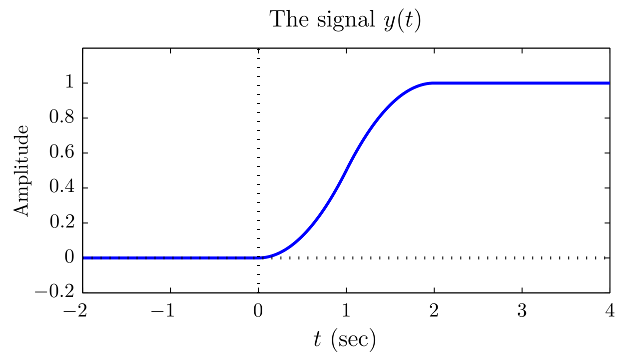
$$0<t<1: \qquad y(t) = \int_0^t \tau\, d\tau = \frac{t^2}{2}$$

$$1<t<2: \qquad y(t) = \int_0^1 \tau\, d\tau + \int_1^t (2-\tau)\, d\tau = -\frac{t^2}{2} + 2t - 1$$

$$t>2: \qquad y(t) = \int_0^1 \tau\, d\tau + \int_1^2 (2-\tau)\, d\tau = 1$$

The response of the system is

$$y(t) = \begin{cases} 0\,, & t<0 \\ \dfrac{t^2}{2}\,, & 0<t<1 \\ -\dfrac{t^2}{2}+2t-1\,, & 1<t<2 \\ 1\,, & t>2 \end{cases}$$



2.18.

a.

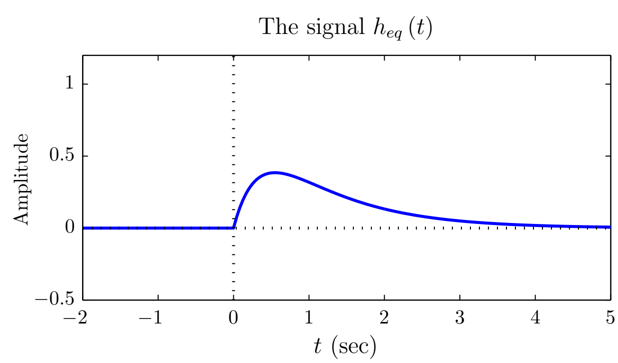
$$\begin{aligned} y_1(t) &= h_1(t) * x(t) \\ y_2(t) &= h_2(t) * x(t) \\ y(t) &= y_1(t) + y_2(t) \\ &= h_1(t) * x(t) + h_2(t) * x(t) \\ &= \left[h_1(t) + h_2(t)\right] * x(t) \end{aligned}$$

Therefore

$$h_{eq}(t) = h_1(t) + h_2(t)$$

b.

$$h_{eq}(t) = \left(e^{-t} - e^{-3t}\right) u(t)$$



c.

$$y_1(t) = \int_{-\infty}^{\infty} h_1(\tau)\, u(t-\tau)\, d\tau$$

Since

$$u(t-\tau) = \begin{cases} 1\,, & \tau<t \\ 0\,, & \tau>t \end{cases}$$

the convolution integral can be written as

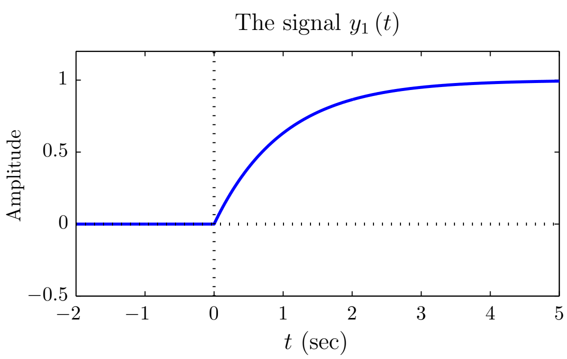
$$y_1(t) = \int_{-\infty}^{t} h_1(\tau)\, d\tau = \int_{-\infty}^{t} e^{-\tau}\, d\tau = 1 - e^{-t}\,, \qquad t \ge 0$$

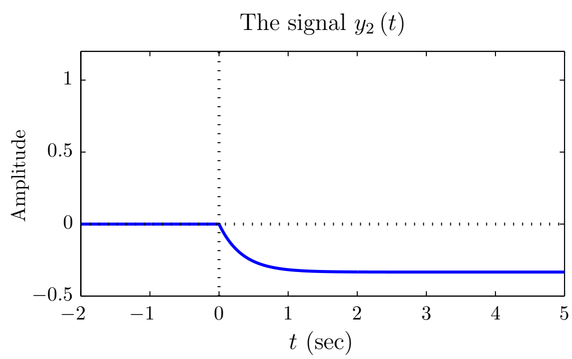
Similarly for $y_2(t)$ we obtain

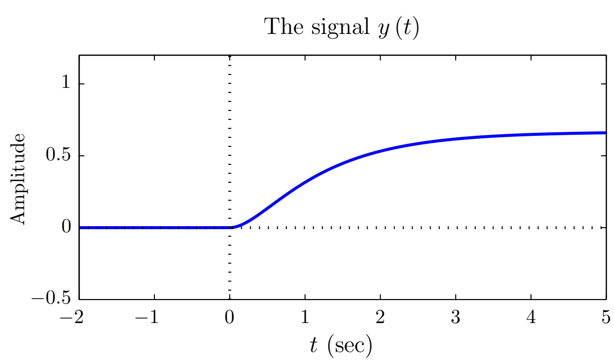
$$y_2(t) = \int_{-\infty}^{t} h_2(\tau)\, d\tau = \int_{-\infty}^{t} -e^{-3\tau}\, d\tau = -\frac{1}{3}\left[1 - e^{-3t}\right]\,, \qquad t \ge 0$$

and the output signal is

$$y(t) = y_1(t) + y_2(t) = \left[\frac{2}{3} - e^{-t} + \frac{1}{3}e^{-3t}\right] u(t)$$







2.19.

a.

$$y_1(t) = h_1(t) * x(t)$$
$$w(t) = h_2(t) * x(t)$$
$$y_3(t) = h_3(t) * w(t) = h_2(t) * h_3(t) * x(t)$$

The output signal is

$$\begin{aligned} y(t) &= y_1(t) + y_2(t) \\ &= \left[h_1(t) + h_2(t) * h_3(t)\right] * x(t) \end{aligned}$$

and the equivalent impulse response is

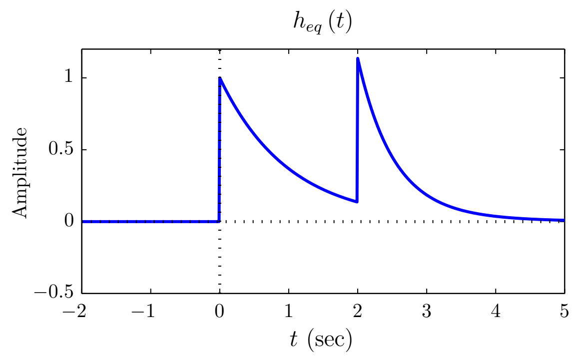
$$h_{eq}(t) = h_1(t) + h_2(t) * h_3(t)$$

b. Carrying out convolution operation we obtain

$$h_2(t) * h_3(t) = h_3(t-2) = e^{-2(t-2)}\, u(t-2)$$

and the equivalent impulse response is

$$h_{eq}(t) = e^{-t}\, u(t) + e^{-2(t-2)}\, u(t-2)$$



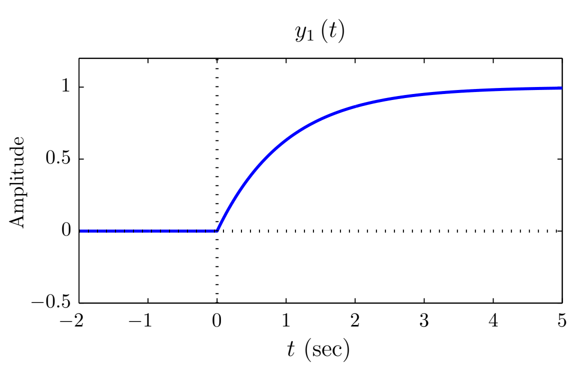
c.

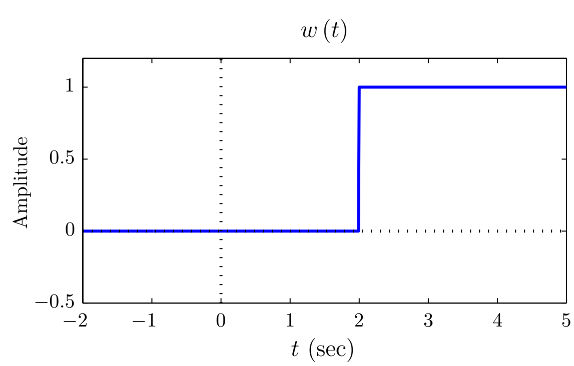
$$y_1(t) = u(t) * h_1(t) = \left[1 - e^{-t}\right] u(t)$$

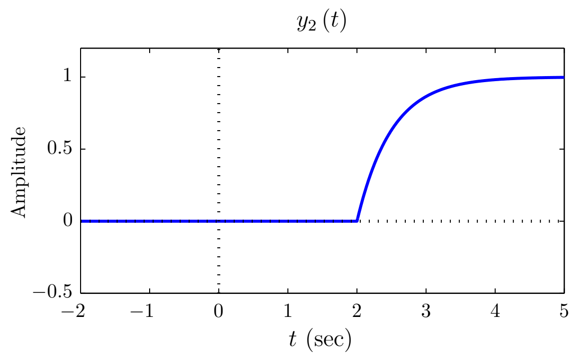
$$w(t) = u(t) * h_2(t) = u(t-2)$$

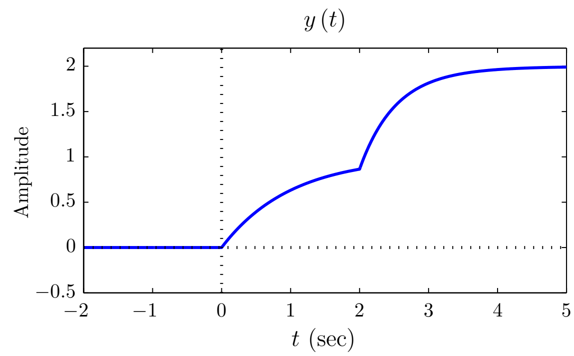
$$y_2(t) = w(t) * h_3(t) = \left[1 - e^{-2(t-2)}\right] u(t-2)$$

$$y(t) = \left[1 - e^{-t}\right] u(t) + \left[1 - e^{-2(t-2)}\right] u(t-2)$$









2.20.

a.

$$
\begin{aligned}
y_1(t) &= h_1(t) * x(t) \\
w(t) &= h_2(t) * x(t) \\
y_3(t) &= h_3(t) * w(t) = h_2(t) * h_3(t) * x(t) \\
y_4(t) &= h_4(t) * w(t) = h_2(t) * h_4(t) * x(t)
\end{aligned}
$$

The output signal is

$$
\begin{aligned}
y(t) &= y_1(t) + y_3(t) + y_4(t) \\
&= \left[h_1(t) + h_2(t) * h_3(t) + h_2(t) * h_4(t)\right] * x(t)
\end{aligned}
$$

and the equivalent impulse response is

$$h_{eq}(t) = h_1(t) + h_2(t) * h_3(t) + h_2(t) * h_4(t)$$

b.

Carrying out convolution operations we obtain

$$h_2(t) * h_3(t) = \begin{cases} t\,, & 0 \le t < 1 \\ 2-t\,, & 1 \le t < 2 \\ 0\,, & \text{otherwise} \end{cases} \qquad \Rightarrow \qquad h_2(t) * h_3(t) = \Lambda(t-1)$$

and

$$h_2(t) * h_4(t) = u(t-1) - u(t-2) \qquad \Rightarrow \qquad h_2(t) * h_4(t) = \Pi(t-1.5)$$

The equivalent impulse response is

$$h_{eq}(t) = e^{-t}\,u(t) + \Lambda(t-1) + \Pi(t-1.5)$$

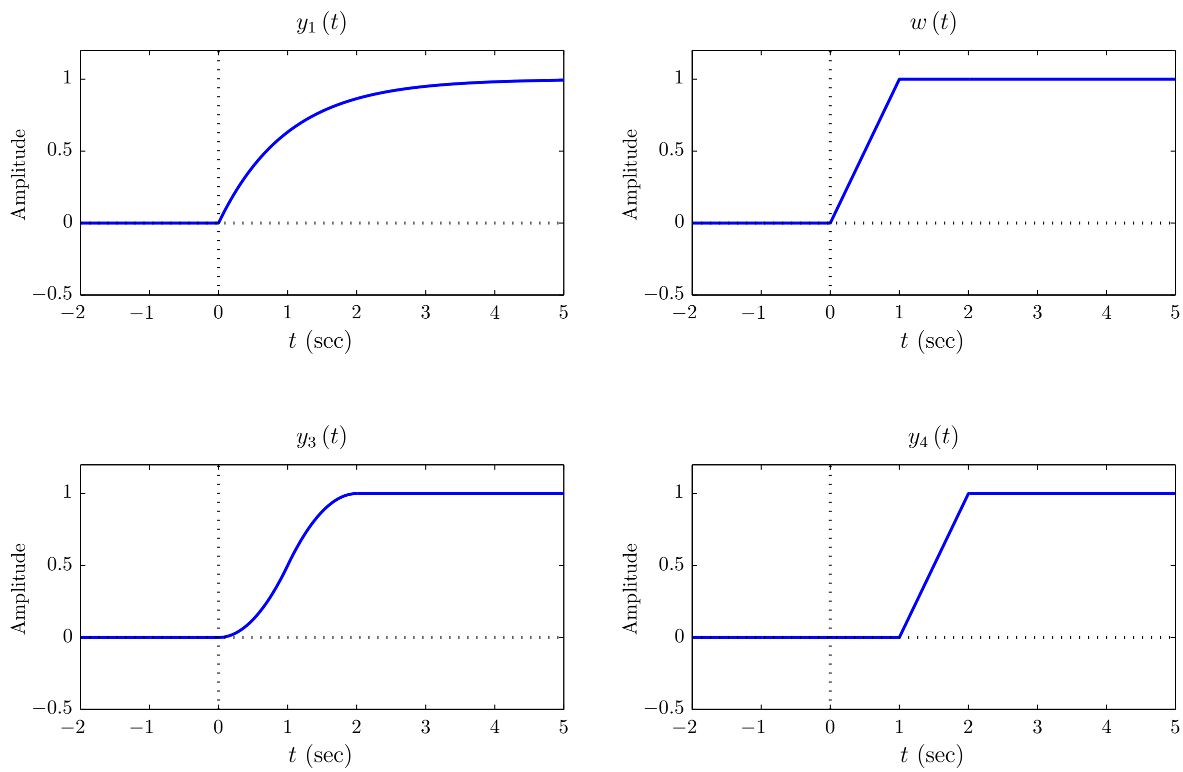
c.

$$y_1(t) = u(t) * h_1(t) = \left(1-e^{-t}\right) u(t)$$

$$w(t) = u(t) * h_2(t) = \begin{cases} t\,, & 0 \le t < 1 \\ 1\,, & t \ge 1 \\ 0\,, & \text{otherwise} \end{cases}$$

$$y_3(t) = w(t) * h_3(t) = \begin{cases} t^2/2\,, & 0 \le t < 1 \\ -t^2/2 + 2t - 1\,, & 1 \le t < 2 \\ 1\,, & t \ge 2 \\ 0\,, & \text{otherwise} \end{cases}$$

$$y_4(t) = h_4(t) * w(t) = w(t-1)$$



2.21.

$$y(t) = \text{Sys}\left\{x(t)\right\} = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Let $w(t) = \dfrac{dx(t)}{dt}$.

$$\text{Sys}\left\{\frac{dx(t)}{dt}\right\} = \text{Sys}\left\{w(t)\right\} = \int_{-\infty}^{\infty} h(\tau)\, w(t-\tau)\, d\tau$$

$$\begin{aligned}\frac{dy(t)}{dt} &= \frac{d}{dt}\left[\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau\right]\\
&= \int_{-\infty}^{\infty} \frac{d}{dt}\left[h(\tau)\, x(t-\tau)\right] d\tau\\
&= \int_{-\infty}^{\infty} h(\tau)\, \frac{d}{dt}\left[x(t-\tau)\right] d\tau\\
&= \int_{-\infty}^{\infty} h(\tau)\, w(t-\tau)\, d\tau = \mathrm{Sys}\left\{\frac{dx(t)}{dt}\right\}\end{aligned}$$

2.22.

a. $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau = x(t)$

b. $x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-t_0-\tau)\, d\tau = x(t-t_0)$

c.

$$x(t) * u(t-2) = \int_{-\infty}^{\infty} x(\tau)\, u(t-2-\tau)\, d\tau$$

Since

$$u(t-2-\tau) = \begin{cases} 1\,, & \tau < t-2 \\ 0\,, & \tau > t-2 \end{cases}$$

the convolution integral can be written as

$$x(t) * u(t-2) = \int_{-\infty}^{t-2} x(\tau)\, d\tau$$

d.

$$x(t) * u(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, u(t-t_0-\tau)\, d\tau$$

Since

$$u(t-t_0-\tau) = \begin{cases} 1\,, & \tau < t-t_0 \\ 0\,, & \tau > t-t_0 \end{cases}$$

the convolution integral can be written as

$$x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$$

e.

$$x(t) * \Pi\left(\frac{t-t_0}{T}\right) = \int_{-\infty}^{\infty} x(\tau)\, \Pi\left(\frac{t-t_0-\tau}{T}\right) d\tau$$

Since

$$\Pi(t-t_0-\tau) = \begin{cases} 1\,, & t-t_0-T/2 < \tau < t-t_0+T/2 \\ 0\,, & \text{otherwise} \end{cases}$$

the convolution integral can be written as

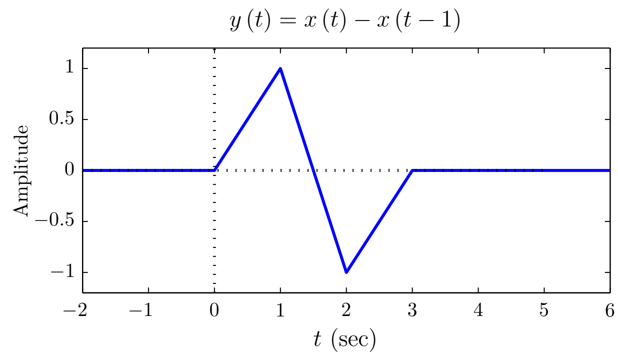
$$x(t) * \Pi\left(\frac{t-t_0}{T}\right) = \int_{t-t_0-T/2}^{t-t_0+T/2} x(\tau)\, d\tau$$

2.23.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \left[\delta(\tau) - \delta(\tau-1)\right] x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \delta(\tau)\, x(t-\tau)\, d\tau - \int_{-\infty}^{\infty} \delta(\tau-1)\, x(t-\tau)\, d\tau \end{aligned}$$

Using the sifting property of the unit-impulse function, we have

$$y(t) = x(t) - x(t-1)$$

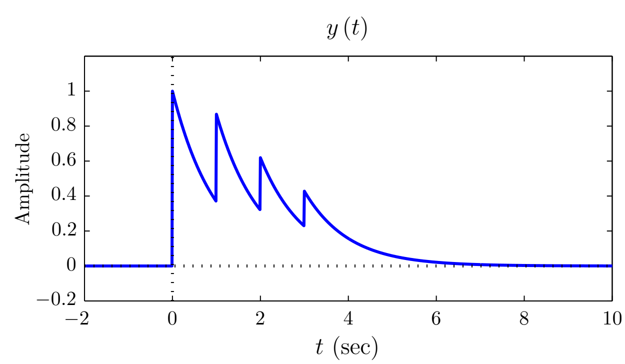


2.24.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \left[\delta(\tau) + 0.5\,\delta(\tau-1) + 0.3\,\delta(\tau-2) + 0.2\,\delta(\tau-3)\right] x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \delta(\tau)\, x(t-\tau)\, d\tau + 0.5\int_{-\infty}^{\infty} \delta(\tau-1)\, x(t-\tau)\, d\tau \\ &\quad + 0.3\int_{-\infty}^{\infty} \delta(\tau-2)\, x(t-\tau)\, d\tau + 0.2\int_{-\infty}^{\infty} \delta(\tau-3)\, x(t-\tau)\, d\tau \end{aligned}$$

Using the sifting property of the unit-impulse function, we have

$$y(t) = x(t) + 0.5\,x(t-1) + 0.3\,x(t-2) + 0.2\,x(t-3)$$



2.25.

$$\begin{aligned}\tilde{x}(t) &= \left[\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\right] * x(t) \\ &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} \delta(\tau-nT_s)\right] x(t-\tau)\, d\tau\end{aligned}$$

Changing the order of integration and summation yields

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\tau-nT_s)\, x(t-\tau)\, d\tau$$

Using the sifting property of the unit-impulse function on each integral leads to the result

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(t-nT_s)\, d\tau$$

which is clearly a periodic extension of the signal $x(t)$.

2.26.

a.

$$y(t) = \int_{-\infty}^{\infty} u(\lambda)\, e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda = \int_{0}^{\infty} e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda$$

Case 1: $t<0$

$$y(t)=0$$

Case 2: $t\ge 0$

$$\int_0^t e^{-2(t-\lambda)}\, d\lambda = \frac{1}{2}\left(1-e^{-2t}\right)$$

b.

$$y(t) = \int_{-\infty}^{\infty} u(\lambda) \left[e^{-(t-\lambda)} - e^{-2(t-\lambda)}\right] u(t-\lambda)\, d\lambda = \int_{0}^{\infty} \left[e^{-(t-\lambda)} - e^{-2(t-\lambda)}\right] u(t-\lambda)\, d\lambda$$

Case 1: $t<0$

$$y(t)=0$$

Case 2: $t\ge 0$

$$y(t) = \int_0^t \left[e^{-(t-\lambda)} - e^{-2(t-\lambda)}\right] d\lambda = 1-e^{-t} - \frac{1}{2}\left(1-e^{-2t}\right)$$

c.

$$y(t) = \int_{-\infty}^{\infty} u(\lambda-2)\, e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda = \int_{2}^{\infty} e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda$$

Case 1: $t<2$

$$y(t)=0$$

Case 2: $t\ge 2$

$$\int_2^t e^{-2(t-\lambda)}\, d\lambda = \frac{1}{2}\left(1-e^{-2(t-2)}\right)$$

d.

$$y(t) = \int_{-\infty}^{\infty} \left[u(\lambda) - u(\lambda-2)\right] e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda = \int_{0}^{2} e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda$$

<u>Case 1: $t<0$</u>

$$y(t) = 0$$

<u>Case 2: $0 \le t < 2$</u>

$$\int_0^t e^{-2(t-\lambda)}\, d\lambda = \frac{1}{2}\left(1-e^{-2t}\right)$$

<u>Case 3: $t \ge 2$</u>

$$\int_0^2 e^{-2(t-\lambda)}\, d\lambda = \frac{1}{2}\left(e^4-1\right) e^{-2t}$$

e.

$$y(t) = \int_{-\infty}^{\infty} e^{-\lambda}\, u(\lambda)\, e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda = \int_0^{\infty} e^{-2t+\lambda}\, u(t-\lambda)\, d\lambda$$

<u>Case 1: $t<0$</u>

$$y(t) = 0$$

<u>Case 2: $t \ge 0$</u>

$$y(t) = \int_0^t e^{-2t+\lambda}\, d\lambda = e^{-t} - e^{-2t}$$

2.27.

a.

$$y(t) = \int_{-\infty}^{\infty} \Pi\left(\frac{\lambda-2}{4}\right) u(t-\lambda)\, d\lambda = \int_0^4 u(t-\lambda)\, d\lambda$$

<u>Case 1: $t<0$</u>

$$y(t) = 0$$

<u>Case 2: $0 \le t < 4$</u>

$$y(t) = \int_0^t (1)\, d\lambda = t$$

<u>Case 3: $t > 4$</u>

$$y(t) = \int_0^4 (1)\, d\lambda = 4$$

b.

$$y(t) = \int_{-\infty}^{\infty} 3\,\Pi\left(\frac{\lambda-2}{4}\right) e^{-(t-\lambda)}\, u(t-\lambda)\, d\lambda = \int_0^4 3\, e^{-(t-\lambda)}\, u(t-\lambda)\, d\lambda$$

<u>Case 1: $t<0$</u>

$$y(t) = 0$$

<u>Case 2: $0 \le t < 4$</u>

$$y(t) = \int_0^t 3\, e^{-(t-\lambda)}\, d\lambda = 3\left(1-e^{-t}\right)$$

Case 3: $t > 4$

$$y(t) = \int_0^4 3\, e^{-(t-\lambda)}\, d\lambda = 3\, e^{-t}\left(e^4 - 1\right)$$

c.

$$y(t) = \int_{-\infty}^{\infty} \Pi\left(\frac{\lambda-2}{4}\right) \Pi\left(\frac{t-\lambda-2}{4}\right) d\lambda = \int_0^4 \Pi\left(\frac{t-\lambda-2}{4}\right) d\lambda$$

Case 1: $t < 0$

$$y(t) = 0$$

Case 2: $0 \le t < 4$

$$y(t) = \int_0^t (1)\, d\lambda = t$$

Case 3: $4 \le t < 8$

$$y(t) = \int_{t-4}^4 (1)\, d\lambda = 8 - t$$

Case 4: $t > 8$

$$y(t) = 0$$

d.

$$y(t) = \int_{-\infty}^{\infty} \Pi\left(\frac{\lambda-2}{4}\right) \Pi\left(\frac{t-\lambda-3}{6}\right) d\lambda = \int_0^4 \Pi\left(\frac{t-\lambda-3}{6}\right) d\lambda$$

Case 1: $t < 0$

$$y(t) = 0$$

Case 2: $0 \le t < 4$

$$y(t) = \int_0^t (1)\, d\lambda = t$$

Case 3: $4 \le t < 6$

$$y(t) = \int_0^4 (1)\, d\lambda = 4$$

Case 4: $6 \le t < 10$

$$y(t) = \int_{t-6}^4 (1)\, d\lambda = 10 - t$$

Case 5: $t > 10$

$$y(t) = 0$$

2.28.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Since $x(t) = 0$ outside the interval $t_1 < t < t_2$, the convolution integral can be written as

$$y(t) = x(t) * h(t) = \int_{t_1}^{t_2} x(\tau)\, h(t-\tau)\, d\tau$$

The nonzero range of the signal $h(t)$ is specified to be $t_3 < t < t_4$.

$$h(t) = 0 \quad \text{except for } \; t_3 < t < t_4$$

and

$$h(t-\tau) = 0 \quad \text{except for } \; t_3 < t-\tau < t_4$$

Equivalently

$$h(t-\tau) = 0 \quad \text{except for } \; t-t_4 < \tau < t-t_3$$

For the integrand to be nonzero, we need

$$t - t_3 > t1 \qquad \text{and} \qquad t - t_4 < t_2$$

which can also be expressed as

$$t_1 + t_3 < t < t_2 + t_4$$

Therefore

$$t_5 = t_1 + t_3 \qquad \text{and} \qquad t_6 = t_2 + t_4$$

2.29.

Using the convolution integral, the output signal is

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

First, let us assume that $|h(t)| < \infty$ for all t, and we can select the input signal to be $x(t) = h^*(-t)$ so that

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, h^*(\lambda - t)\, d\lambda$$

At $t = 0$ the output signal is

$$y(0) = \int_{-\infty}^{\infty} h(\lambda)\, h^*(\lambda)\, d\lambda = \int_{-\infty}^{\infty} \left|h(\lambda)\right|^2 d\lambda$$

If the integral

$$\int_{-\infty}^{\infty} \left|h(\lambda)\right| d\lambda$$

does not converge, then neither does the integral

$$y(0) = \int_{-\infty}^{\infty} \left|h(\lambda)\right|^2 d\lambda$$

If the assumption $|h(t)| < \infty$ is not valid, then there is at least one value of t for which $h(t)$ is infinitely large. In that case choosing the input signal to be $x(t) = \delta(t)$ leads to the output signal

$$y(t) = h(t)$$

which is also infinitely large for at least one value of t.

2.30.

a. Let the input signal to the system be $x_1(t)$.

$$y_1(t) = \text{Sys}\left\{x_1(t)\right\} = x_1(t) + \alpha_1\, x_1(t-\tau_1) + \alpha_2\, x_1(t-\tau_2)$$

Similarly, if the input signal is $x_2(t)$

$$y_2(t) = \text{Sys}\left\{x_2(t)\right\} = x_2(t) + \alpha_1\, x_2(t-\tau_1) + \alpha_2\, x_2(t-\tau_2)$$

The response of the system to the input signal $x(t) = \beta_1\, x_1(t) + \beta_2\, x_2(t)$ is

$$\begin{aligned}\text{Sys}\left\{\beta_1\, x_1(t) + \beta_2\, x_2(t)\right\} &= \beta_1\left[x_1(t) + \alpha_1\, x_1(t-\tau_1) + \alpha_2\, x_1(t-\tau_2)\right] + \beta_2\left[x_2(t) + \alpha_1\, x_2(t-\tau_1) + \alpha_2\, x_2(t-\tau_2)\right]\\ &= \beta_1\, y_1(t) + \beta_2\, y_2(t)\end{aligned}$$

The system is linear.

b. The response to $x_1(t-a)$ is

$$\text{Sys}\left\{x_1(t-a)\right\} = x_1(t-a) + \alpha_1\, x_1(t-\tau_1-a) + \alpha_2\, x_1(t-\tau_2-a) = y_1(t-a)$$

The system is time-invariant.

c. The system is causal provided that $\tau_1 > 0$ and $\tau_2 > 0$.

d. The system is stable provided that $\alpha_1,\ \alpha_2 < \infty$.

2.31.

a. Let $x(t) = \delta(t)$.

$$h(t) = \text{Sys}\left\{\delta(t)\right\} = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda = \begin{cases} 1\,, & t > 0\\ 0\,, & \text{otherwise}\end{cases}$$

Therefore

$$h(t) = u(t)$$

Since $h(t) = 0$ for $t < 0$, the system is causal. However, since $h(t)$ is not absolute summable, the system is not stable.

b. Let $x(t) = \delta(t)$.

$$h(t) = \text{Sys}\left\{\delta(t)\right\} = \int_{t-T}^{t} \delta(\lambda)\, d\lambda = \begin{cases} 1\,, & 0 < t < T\\ 0\,, & \text{otherwise}\end{cases}$$

Therefore

$$h(t) = \Pi\left(\frac{t-T/2}{T}\right)$$

Since $h(t) = 0$ for $t < 0$, the system is causal. Also, since $h(t)$ is absolute summable, the system is stable.

c. Let $x(t) = \delta(t)$.

$$h(t) = \text{Sys}\left\{\delta(t)\right\} = \int_{t-T}^{t+T} \delta(\lambda)\, d\lambda = \begin{cases} 1\,, & -T < t < T\\ 0\,, & \text{otherwise}\end{cases}$$

Therefore

$$h(t) = \Pi\left(\frac{t}{2T}\right)$$

Since $h(t)$ has nonzero values for some $t < 0$, the system is not causal. It is stable, however, since $h(t)$ is absolute summable.

2.32.

a.

```
x = @(t) exp(-t).*cos(2*t).*(t>=0);
t = [-1:0.01:5];
```

b. Compute and graph $w(t)$:

```
w = @(t) 3*x(t);
plot(t,w(t));
axis([-1,5,-1,4]);
title('w(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $y(t)$:

```
y = @(t) w(t-2);
plot(t,y(t));
axis([-1,5,-1,4]);
title('y(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

c. Compute and graph $\bar{w}(t)$:

```
wbar = @(t) x(t-2);
plot(t,wbar(t));
axis([-1,5,-1,4]);
title('wbar(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $\bar{y}(t)$:

```
ybar = @(t) 3*wbar(t);
plot(t,ybar(t));
axis([-1,5,-1,4]);
```

```
title('ybar(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

2.33.

Create an anonymous function to compute the signal $x(t)$:

```
x = @(t) exp(-t).*cos(2*t).*(t>=0);
t = [-1:0.01:5];
```

a. Compute and graph $w(t)$:

```
w = @(t) 3*x(t);
plot(t,w(t));
axis([-1,5,-2,4]);
title('w(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $y(t)$:

```
y = @(t) t.*w(t);
plot(t,y(t));
axis([-1,5,-2,4]);
title('y(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $\bar{w}(t)$:

```
wbar = @(t) t.*x(t);
plot(t,wbar(t));
axis([-1,5,-2,4]);
title('wbar(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $\bar{y}(t)$:

```
ybar = @(t) 3*wbar(t);
plot(t,ybar(t));
axis([-1,5,-2,4]);
title('ybar(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

b. Compute and graph $w(t)$:

```
w = @(t) 3*x(t);
plot(t,w(t));
axis([-1,5,-1,20]);
title('w(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $y(t)$:

```
y = @(t) w(t)+5;
plot(t,y(t));
axis([-1,5,-1,20]);
title('y(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $\bar{w}(t)$:

```
wbar = @(t) x(t)+5;
plot(t,wbar(t));
axis([-1,5,-1,20]);
title('wbar(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Compute and graph $\bar{y}(t)$:

```
ybar = @(t) 3*wbar(t);
plot(t,ybar(t));
axis([-1,5,-1,20]);
title('ybar(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

2.34.

a.

```
t = [0:0.001:2];
% Compute the exact solution
y = 0.25-1.25*exp(-4*t);
% Compute the approximate solution using Euler method
Ts = 1/40;
```

```
t2 = [0:Ts:2];
yhat = zeros(size(t2));
yhat(1) = -1;  % Initial value
for k=1:length(yhat)-1,
  g = -4*yhat(k)+1;
  yhat(k+1) = yhat(k)+Ts*g;  % Eqn. (2.185)
end;
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','SouthEast');
```

b.

```
t = [0:0.001:10];
% Compute the exact solution
y = (1+exp(-2*t)).*((t>=0)&(t<5))...
    +(exp(10)+1)*exp(-2*t).*(t>=5);
% Compute the approximate solution using Euler method
Ts = 1/20;
t2 = [0:Ts:10];
yhat = zeros(size(t2));
yhat(1) = 2;  % Initial value
for k=1:length(yhat)-1,
  if ((k-1)*Ts<5),
    g = -2*yhat(k)+2;
  else
    g = -2*yhat(k);
  end;
  yhat(k+1) = yhat(k)+Ts*g;  % Eqn. (2.185)
end;
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','NorthEast');
```

c.

```
t = [0:0.001:2];
% Compute the exact solution
y = 3.5*exp(-5*t);
% Compute the approximate solution using Euler method
Ts = 1/50;
t2 = [0:Ts:2];
```

```
yhat = zeros(size(t2));
yhat(1) = 0.5;  % Initial value
for k=1:length(yhat)-1,
  if (k==1),
    g = -5*yhat(k)+3/Ts;  % Approximate unit impulse with rectangle
                          % that has a width of Ts and area of 1.
  else
    g = -5*yhat(k);
  end;
  yhat(k+1) = yhat(k)+Ts*g;  % Eqn. (2.185)
end;
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','NorthEast');
```

d.

```
t = [0:0.001:2];
% Compute the exact solution
y = 0.6*t-0.12-3.88*exp(-5*t);
% Compute the approximate solution using Euler method
Ts = 1/50;
t2 = [0:Ts:2];
yhat = zeros(size(t2));
yhat(1) = -4;  % Initial value
for k=1:length(yhat)-1,
  x = (k-1)*Ts;
  g = -5*yhat(k)+3*x;
  yhat(k+1) = yhat(k)+Ts*g;  % Eqn. (2.185)
end;
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','SouthEast');
```

e.

```
t = [0:0.001:2];
% Compute the exact solution
y = exp(-t)-2*exp(-2*t);
% Compute the approximate solution using Euler method
Ts = 1/10;
t2 = [0:Ts:2];
```

```
yhat = zeros(size(t2));
yhat(1) = -1;  % Initial value
for k=1:length(yhat)-1,
  x = exp(-2*(k-1)*Ts);
  g = -yhat(k)+2*x;
  yhat(k+1) = yhat(k)+Ts*g;  % Eqn. (2.185)
end;
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','SouthEast');
```

2.35.

a.

```
t = [0:0.001:2];
% Compute the exact solution
y = 0.25-1.25*exp(-4*t);
% Compute the approximate solution using Euler method
Ts = 1/40;
t2 = [0:Ts:2];
ga = @(t,yhat) -4*yhat+1;
[t2,yhat] = ode45(ga,t2,-1);
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','SouthEast');
```

b.

```
t = [0:0.001:10];
% Compute the exact solution
y = (1+exp(-2*t)).*((t>=0)&(t<5))...
    +(exp(10)+1)*exp(-2*t).*(t>=5);
% Compute the approximate solution using Euler method
Ts = 1/20;
t2 = [0:Ts:10];
gb = @(t,yhat) -2*yhat+2*((t>=0)&(t<=5));
[t2,yhat] = ode45(gb,t2,2);
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
```

```
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','NorthEast');
```

c.

```
t = [0:0.001:2];
% Compute the exact solution
y = 3.5*exp(-5*t);
% Compute the approximate solution using Euler method
Ts = 1/50;
t2 = [0:Ts:2];
gc = @(t,yhat) -5*yhat+3/Ts*(t==0);
[t2,yhat] = ode45(gc,t2,3.5);
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','NorthEast');
```

d.

```
t = [0:0.001:2];
% Compute the exact solution
y = 0.6*t-0.12-3.88*exp(-5*t);
% Compute the approximate solution using Euler method
Ts = 1/50;
t2 = [0:Ts:2];
gd = @(t,yhat) -5*yhat+3*t.*(t>=0);
[t2,yhat] = ode45(gd,t2,-4);
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','SouthEast');
```

e.

```
t = [0:0.001:2];
% Compute the exact solution
y = exp(-t)-2*exp(-2*t);
% Compute the approximate solution using Euler method
Ts = 1/10;
```

```
t2 = [0:Ts:2];
ge = @(t,yhat) -yhat+2*exp(-2*t).*(t>=0);
[t2,yhat] = ode45(ge,t2,-1);
% Graph exact and approximate solutions
clf;
plot(t,y,'-',t2,yhat,'r.'); grid;
title('Exact and approximate solutions');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution',...
  'Location','SouthEast');
```

2.36.

a. The unit-ramp function can be expressed with an anonymous function as

```
xr = @(t) t.*(t>=0);
```

b. The unit-ramp response of the circuit is

$$y_r(t) = \left[t - \frac{1}{4} + \frac{1}{4}e^{-4t}\right] u(t)$$

and can be expressed with an anonymous function as

```
yr = @(t) (t-0.25+0.25*exp(-4*t)).*(t>=0);
```

c. The input signal can be expressed using unit-ramp functions as

$$x(t) = r(t-1) - r(t-1) - r(t-2) + r(t-3)$$

which can be produced with MATLAB statements

```
t = [-1:0.001:5];
inp = xr(t)-xr(t-1)-xr(t-2)+xr(t-3);
```

d. The response of the circuit to the signal $x(t)$ is

$$y(t) = y_r(t-1) - y_r(t-1) - y_r(t-2) + y_r(t-3)$$

and can be computed in MATLAB using

```
t = [-1:0.001:5];
out = yr(t)-yr(t-1)-yr(t-2)+yr(t-3);
```

e. The input and the output signals can be graphed with the following statements:

```
plot(t,inp,'b',t,out,'r');
title('x(t) and y(t)');
xlabel('t (sec)');
ylabel('Amplitude');
legend('x(t)','y(t)');
axis([-1,5,-0.2,1.2]);
grid;
```

2.37.

a. The input signal can be expressed as

$$x(t) = r(t-1) - 2\,u(t-1) + r(t-2) - r(t-3)$$

b. The unit-step response of the circuit is

$$y_u(t) = \left[1 - e^{-4t}\right] u(t)$$

The unit-ramp response of the circuit is

$$y_r(t) = \left[t - \frac{1}{4} + \frac{1}{4}\,e^{-4t}\right] u(t)$$

The output signal $y(t)$ can be computed through the following statements:

```
xu = @(t) 1*(t>=0);
xr = @(t) t.*(t>=0);
yu = @(t) (1-exp(-4*t)).*(t>=0);
yr = @(t) (0.25*exp(-4*t)+t-0.25).*(t>=0);
t = [-1:0.001:5];
inp = xr(t)-xr(t-1)-2*xu(t-1)+xr(t-2)-xr(t-3);
out = yr(t)-yr(t-1)-2*yu(t-1)+yr(t-2)-yr(t-3);
```

c. Use the following statements to graph the input and the output signals:

```
plot(t,inp,'b',t,out,'r');
axis([-1,5,-1.2,1.2]);
title('x(t) and y(t)');
xlabel('t (sec)');
ylabel('Amplitude');
legend('x(t)','y(t)');
grid;
```

2.38.

```
x = @(t) ss_tri(t-1);
t = [-1:0.01:5];
y = x(t)-x(t-1);
plot(t,y);
axis([-1,5,-1.2,1.2]);
title('y(t)=x(t)-x(t-1)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

2.39.

```
x = @(t) exp(-t).*(t>=0);
t = [-1:0.01:7];
y = x(t)+0.5*x(t-1)+0.3*x(t-2)+0.2*x(t-3);
plot(t,y);
axis([-1,7,-0.2,1.2]);
title('y(t)');
xlabel('t (sec)');
ylabel('Amplitude');
grid;
```

Chapter 3

Analyzing Discrete-Time Systems in the Time-Domain

3.1.

a.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = x_1[n]\,u[n]$$
$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = x_2[n]\,u[n]$$

Using $x[n] = \alpha_1\,x_1[n] + \alpha_2\,x_2[n]$ as input we obtain

$$\begin{aligned} y[n] =& \text{Sys}\left\{\alpha_1\,x_1[n] + \alpha_2\,x_2[n]\right\} \\ =& \left(\alpha_1\,x_1[n] + \alpha_2\,x_2[n]\right)\,u[n] \\ =& \alpha_1\,y_1[n] + \alpha_2\,y_2[n] \end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = x_1[n-m]\,u[n] \neq y_1[n-m]$$

The system is not time-invariant.

b.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = 3\,x_1[n] + 5$$
$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = 3\,x_2[n] + 5$$

Using $x[n] = \alpha_1\,x_1[n] + \alpha_2\,x_2[n]$ as input we obtain

$$\begin{aligned} y[n] =& \text{Sys}\left\{\alpha_1\,x_1[n] + \alpha_2\,x_2[n]\right\} \\ =& 3\,\left(\alpha_1\,x_1[n] + \alpha_2\,x_2[n]\right) + 5 \\ \neq& \alpha_1\,y_1[n] + \alpha_2\,y_2[n] \end{aligned}$$

The system is not linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = 3\,x_1[n-m] + 5 = y_1[n-m]$$

The system is time-invariant.

c.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = 3\,x_1[n] + 5\,u[n]$$
$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = 3\,x_2[n] + 5\,u[n]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] =& \text{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\ =& 3\left(\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right) + 5\, u[n] \\ \neq& \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is not linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = 3\, x_1[n-m] + 5\, u[n] \neq y_1[n-m]$$

The system is not time-invariant.

d.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = n\, x_1[n]$$

$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = n\, x_2[n]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] =& \text{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\ =& n\left(\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right) \\ =& \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = n\, x_1[n-m] \neq y_1[n-m]$$

The system is not time-invariant.

e.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = \cos(0.2\pi n)\; x_1[n]$$

$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = \cos(0.2\pi n)\; x_2[n]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] =& \text{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\ =& \cos(0.2\pi n)\left(\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right) \\ =& \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = \cos(0.2\pi n)\; x_1[n-m] \neq y_1[n-m]$$

The system is not time-invariant.

f.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = x_1[n] + 3\, x_1[n-1]$$

$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = x_2[n] + 3\, x_2[n-1]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] =& \text{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\ &= \left(\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right) + 3\, \left(\alpha_1\, x_1[n-1] + \alpha_2\, x_2[n-1]\right) \\ &= \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = x_1[n-m] + 3\, x_1[n-m-1] = y_1[n-m]$$

The system is not time-invariant.

g.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = x_1[n] + 3\, x_1[n-1]\, x_1[n-2]$$

$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = x_2[n] + 3\, x_2[n-1]\, x_2[n-2]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] =& \text{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\ &= \left(\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right) + 3\, \left(\alpha_1\, x_1[n-1] + \alpha_2\, x_2[n-1]\right)\, \left(\alpha_1\, x_1[n-2] + \alpha_2\, x_2[n-2]\right) \\ &\neq \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is not linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = x_1[n-m] + 3\, x_1[n-m-1]\, x_1[n-m-2] = y_1[n-m]$$

The system is time-invariant.

3.2.

a.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = \sum_{k=-\infty}^{n} x_1[k]$$

$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = \sum_{k=-\infty}^{n} x_2[k]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] =& \text{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\ &= \sum_{k=-\infty}^{n} \left(\alpha_1\, x_1[k] + \alpha_2\, x_2[k]\right) \\ &= \alpha_1 \sum_{k=-\infty}^{n} x_1[k] + \alpha_2 \sum_{k=-\infty}^{n} x_2[k] \\ &= \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is linear.

$$\mathrm{Sys}\left\{x_1[n-m]\right\} = \sum_{k=-\infty}^{n} x_1[k-m]$$

Let $k-m=r$. The last relationship can be written as

$$\mathrm{Sys}\left\{x_1[n-m]\right\} = \sum_{r=-\infty}^{n-m} x_1[r] = y_1[n-m]$$

The system is time-invariant.

b.

$$y_1[n] = \mathrm{Sys}\left\{x_1[n]\right\} = \sum_{k=0}^{n} x_1[k]$$

$$y_2[n] = \mathrm{Sys}\left\{x_2[n]\right\} = \sum_{k=0}^{n} x_2[k]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] =& \mathrm{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\ =& \sum_{k=0}^{n} \left(\alpha_1\, x_1[k] + \alpha_2\, x_2[k]\right) \\ =& \alpha_1 \sum_{k=0}^{n} x_1[k] + \alpha_2 \sum_{k=0}^{n} x_2[k] \\ =& \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is linear. For testing time invariance, let us assume that $n>0$.

$$\mathrm{Sys}\left\{x_1[n-m]\right\} = \sum_{k=0}^{n} x_1[k-m]$$

Let $k-m=r$. The last relationship can be written as

$$\mathrm{Sys}\left\{x_1[n-m]\right\} = \sum_{r=-m}^{n-m} x_1[r] \neq y_1[n-m]$$

The system is not time-invariant.

c.

$$y_1[n] = \mathrm{Sys}\left\{x_1[n]\right\} = \sum_{k=n-2}^{n} x_1[k]$$

$$y_2[n] = \mathrm{Sys}\left\{x_2[n]\right\} = \sum_{k=n-2}^{n} x_2[k]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned}
y[n] &= \text{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\
&= \sum_{k=n-2}^{n} \left(\alpha_1\, x_1[k] + \alpha_2\, x_2[k]\right) \\
&= \alpha_1 \sum_{k=n-2}^{n} x_1[k] + \alpha_2 \sum_{k=n-2}^{n} x_2[k] \\
&= \alpha_1\, y_1[n] + \alpha_2\, y_2[n]
\end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = \sum_{k=n-2}^{n} x_1[k-m]$$

Let $k - m = r$. The last relationship can be written as

$$\text{Sys}\left\{x_1[n-m]\right\} = \sum_{r=n-2-m}^{n-m} x_1[r] = y_1[n-m]$$

The system is time-invariant.

d.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = \sum_{k=n-2}^{n+2} x_1[k]$$

$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = \sum_{k=n-2}^{n+2} x_2[k]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned}
y[n] &= \text{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\
&= \sum_{k=n-2}^{n+2} \left(\alpha_1\, x_1[k] + \alpha_2\, x_2[k]\right) \\
&= \alpha_1 \sum_{k=n-2}^{n+2} x_1[k] + \alpha_2 \sum_{k=n-2}^{n+2} x_2[k] \\
&= \alpha_1\, y_1[n] + \alpha_2\, y_2[n]
\end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = \sum_{k=n-2}^{n+2} x_1[k-m]$$

Let $k - m = r$. The last relationship can be written as

$$\text{Sys}\left\{x_1[n-m]\right\} = \sum_{r=n-2-m}^{n+2-m} x_1[r] = y_1[n-m]$$

The system is time-invariant.

3.3.

a.

$$w[n] = 3\,x[n]$$
$$y[n] = w[n-2] = 3\,x[n-2]$$

b.

$$\bar{w}[n] = x[n-2]$$
$$\bar{y}[n] = 3\,\bar{w}[n] = 3\,x[n-2]$$

Input-output relationship of the system does not change when the order of the two subsystems is changed.

3.4.

a. Using the first configuration:

$$w[n] = 3\,x[n]$$
$$y[n] = n\,w[n] = 3n\,x[n]$$

Using the second configuration:

$$\bar{w}[n] = n\,x[n]$$
$$\bar{y}[n] = 3\,\bar{w}[n] = 3n\,x[n]$$

Input-output relationship of the system does not change when the order of the two subsystems is changed.

b. Using the first configuration:

$$w[n] = 3\,x[n]$$
$$y[n] = w[n] + 5 = 3\,x[n] + 5$$

Using the second configuration:

$$\bar{w}[n] = x[n] + 5$$
$$\bar{y}[n] = 3\,\bar{w}[n] = 3\,\big(x[n] + 5\big) = 3\,x[n] + 15$$

Input-output relationship of the system changes when the order of the two subsystems is changed.

3.5.

a. Since the system is linear

$$y[n] = \text{Sys}\left\{\delta[n] + \delta[n-1]\right\} = \text{Sys}\left\{\delta[n]\right\} + \text{Sys}\left\{\delta[n-1]\right\}$$

The system is also time-invariant, therefore

$$\text{Sys}\left\{\delta[n]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{2}, 1, -1 \} \qquad \Rightarrow \qquad \text{Sys}\left\{\delta[n-1]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 2, 1, -1 \}$$

and

$$y[n] = \{ \underset{\substack{\uparrow \\ n=0}}{2}, 3, 0, -1 \}$$

b. Since the system is linear

$$\begin{aligned} y[n] &= \text{Sys}\left\{\delta[n] - 2\,\delta[n-1] + \delta[n-2]\right\} \\ &= \text{Sys}\left\{\delta[n]\right\} - 2\ \text{Sys}\left\{\delta[n-1]\right\} + \text{Sys}\left\{\delta[n-2]\right\} \end{aligned}$$

The system is also time-invariant, therefore

$$\text{Sys}\left\{\delta[n-1]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 2, 1, -1 \}$$

and

$$\text{Sys}\left\{\delta[n-2]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 0, 2, 1, -1 \}$$

The response is

$$y[n] = \{ \underset{\substack{\uparrow \\ n=0}}{2}, -3, -1, 3, -1 \}$$

c.

$$u[n] - u[n-5] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

Using the linearity of the system we have

$$\begin{aligned} y[n] &= \text{Sys}\left\{\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]\right\} \\ &= \text{Sys}\left\{\delta[n]\right\} + \text{Sys}\left\{\delta[n-1]\right\} + \text{Sys}\left\{\delta[n-2]\right\} + \text{Sys}\left\{\delta[n-3]\right\} + \text{Sys}\left\{\delta[n-4]\right\} \end{aligned}$$

Since the system is also time-invariant, we have

$$\text{Sys}\left\{\delta[n-1]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 2, 1, -1 \}$$

$$\text{Sys}\left\{\delta[n-2]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 0, 2, 1, -1 \}$$

$$\text{Sys}\left\{\delta[n-3]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 0, 0, 2, 1, -1 \}$$

$$\text{Sys}\left\{\delta[n-4]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 0, 0, 0, 2, 1, -1 \}$$

The output signal is

$$y[n] = \{ \underset{\substack{\uparrow \\ n=0}}{2}, 3, 2, 2, 2, 0, -1 \}$$

d.

$$n\,\big(u[n] - u[n-5]\big) = \delta[n-1] + 2\,\delta[n-2] + 3\,\delta[n-3] + 4\,\delta[n-4]$$

Using the linearity of the system we have

$$\begin{aligned} y[n] &= \text{Sys}\left\{\delta[n-1] + 2\,\delta[n-2] + 3\,\delta[n-3] + 4\,\delta[n-4]\right\} \\ &= \text{Sys}\left\{\delta[n-1]\right\} + 2\ \text{Sys}\left\{\delta[n-2]\right\} + 3\ \text{Sys}\left\{\delta[n-3]\right\} + 4\ \text{Sys}\left\{\delta[n-4]\right\} \end{aligned}$$

Since the system is also time-invariant, we have

$$\text{Sys}\left\{\delta[n-1]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 2, 1, -1 \}$$

$$\text{Sys}\left\{\delta[n-2]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 0, 2, 1, -1 \}$$

$$\text{Sys}\left\{\delta[n-3]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 0, 0, 2, 1, -1 \}$$

$$\text{Sys}\left\{\delta[n-4]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 0, 0, 0, 2, 1, -1 \}$$

The output signal is

$$y[n] = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 2, 5, 7, 9, 1, -4 \}$$

3.6.

a. Since the system is linear

$$y[n] = \text{Sys}\left\{5\,\delta[n-1]\right\} = 5\ \text{Sys}\left\{\delta[n-1]\right\} = \{ \underset{\substack{\uparrow \\ n=0}}{5}, 10, 15 \}$$

b. Since the system is linear

$$\begin{aligned} y[n] &= \text{Sys}\left\{3\,\delta[n-1] + 2\,\delta[n-2]\right\} \\ &= 3\ \text{Sys}\left\{\delta[n-1]\right\} + 2\ \text{Sys}\left\{\delta[n-2]\right\} \\ &= \{ \underset{\substack{\uparrow \\ n=0}}{3}, 12, 15, 4 \} \end{aligned}$$

c. Since the system is linear

$$\begin{aligned} y[n] &= \text{Sys}\left\{\delta[n] - 2\,\delta[n-1] + 4\,\delta[n-2]\right\} \\ &= \text{Sys}\left\{\delta[n]\right\} - 2\ \text{Sys}\left\{\delta[n-1]\right\} + 4\ \text{Sys}\left\{\delta[n-2]\right\} \\ &= \{ \underset{\substack{\uparrow \\ n=0}}{1}, -3, 9, 4, 4 \} \end{aligned}$$

d. The input signal can be written as

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

Since the system is linear

$$\begin{aligned} y[n] &= \text{Sys}\left\{\delta[n] + \delta[n-1] + \delta[n-2]\right\} \\ &= \text{Sys}\left\{\delta[n]\right\} + \text{Sys}\left\{\delta[n-1]\right\} + \text{Sys}\left\{\delta[n-2]\right\} \\ &= \{ \underset{\substack{\uparrow \\ n=0}}{1}, 5, 9, 4, 1 \} \end{aligned}$$

e. The input signal can be written as

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$

The response of the system to $\delta[n-3]$ is not given, and can not be obtained from the information provided since the system is not time-invariant (we reach this conclusion by looking at the relationship between the responses to $\delta[n]$, $\delta[n-1]$ and $\delta[n-2]$). Therefore, the response cannot be computed in this case.

3.7.

$$\begin{aligned} y[0] &= \frac{x[0] + x[-1]}{2} = \frac{1.7 + 0}{2} = 0.85 \\ y[1] &= \frac{x[1] + x[0]}{2} = \frac{2.3 + 1.7}{2} = 2 \\ y[2] &= \frac{x[2] + x[1]}{2} = \frac{3.1 + 2.3}{2} = 2.7 \\ y[3] &= \frac{x[3] + x[2]}{2} = \frac{3.3 + 3.1}{2} = 3.2 \\ y[4] &= \frac{x[4] + x[3]}{2} = \frac{3.7 + 3.3}{2} = 3.5 \\ y[5] &= \frac{x[5] + x[4]}{2} = \frac{2.9 + 3.7}{2} = 3.3 \end{aligned}$$

$$\begin{aligned}
y[6] &= \frac{x[6]+x[5]}{2} = \frac{2.2+2.9}{2} = 2.55\\
y[7] &= \frac{x[7]+x[6]}{2} = \frac{1.4+2.2}{2} = 1.8\\
y[8] &= \frac{x[8]+x[7]}{2} = \frac{0.6+1.4}{2} = 1\\
y[9] &= \frac{x[9]+x[8]}{2} = \frac{-0.2+0.6}{2} = 0.2\\
y[10] &= \frac{x[10]+x[9]}{2} = \frac{0.4-0.2}{2} = 0.1
\end{aligned}$$

Therefore

$$y[n] = \{\underset{\substack{\uparrow\\ n=0}}{0.85},\, 2,\, 2.7,\, 3.2,\, 3.5,\, 3.3,\, 2.55,\, 1.8,\, 1,\, 0.2,\, 0.1\,\}$$

3.8.

$$\begin{aligned}
y[0] &= \frac{x[0]+x[-1]+x-2]+x[-3]}{4} = \frac{1.7+0+0+0}{4} = 0.425\\
y[1] &= \frac{x[1]+x[0]+x-1]+x[-2]}{4} = \frac{2.3+1.7+0+0}{4} = 1\\
y[2] &= \frac{x[2]+x[1]+x0]+x[-1]}{4} = \frac{3.1+2.3+1.7+0}{4} = 1.775\\
y[3] &= \frac{x[3]+x[2]+x1]+x[0]}{4} = \frac{3.3+3.1+2.3+1.7}{4} = 2.6\\
y[4] &= \frac{x[4]+x[3]+x2]+x[1]}{4} = \frac{3.7+3.3+3.1+2.3}{4} = 3.1\\
y[5] &= \frac{x[5]+x[4]+x3]+x[2]}{4} = \frac{2.9+3.7+3.3+3.1}{4} = 3.25\\
y[6] &= \frac{x[6]+x[5]+x4]+x[3]}{4} = \frac{2.2+2.9+3.7+3.3}{4} = 3.025\\
y[7] &= \frac{x[7]+x[6]+x5]+x[4]}{4} = \frac{1.4+2.2+2.9+3.7}{4} = 2.55\\
y[8] &= \frac{x[8]+x[7]+x6]+x[5]}{4} = \frac{0.6+1.4+2.2+2.9}{4} = 1.775\\
y[9] &= \frac{x[9]+x[8]+x7]+x[6]}{4} = \frac{-0.2+0.6+1.4+2.2}{4} = 1\\
y[10] &= \frac{x[10]+x[9]+x8]+x[7]}{4} = \frac{0.4-0.2+0.6+1.4}{4} = 0.55
\end{aligned}$$

Therefore

$$y[n] = \{\underset{\substack{\uparrow\\ n=0}}{0.425},\, 1,\, 1.775,\, 2.6,\, 3.1,\, 3.25,\, 3.025,\, 2.55,\, 1.775,\, 1,\, 0.55\,\}$$

3.9. The exponential smoother with $\alpha = 0.2$ has the difference equation

$$y[n] = 0.8\,y[n-1] + 0.2\,x[n]$$

Writing the output signal iteratively for $n = 0, \ldots, 10$ we obtain

$$
\begin{aligned}
y[0] =& 0.8\, y[-1] + 0.2\, x[0] = (0.8)\ (0) + (0.2)\ (1.7) = 0.34 \\
y[1] =& 0.8\, y[0] + 0.2\, x[1] = (0.8)\ (0.34) + (0.2)\ (2.3) = 0.8 \\
y[2] =& 0.8\, y[1] + 0.2\, x[2] = (0.8)\ (0.8) + (0.2)\ (3.1) = 1.42 \\
y[3] =& 0.8\, y[2] + 0.2\, x[3] = (0.8)\ (1.42) + (0.2)\ (3.3) = 2.08 \\
y[4] =& 0.8\, y[3] + 0.2\, x[4] = (0.8)\ (2.08) + (0.2)\ (3.7) = 2.82 \\
y[5] =& 0.8\, y[4] + 0.2\, x[5] = (0.8)\ (2.82) + (0.2)\ (2.9) = 3.4 \\
y[6] =& 0.8\, y[5] + 0.2\, x[6] = (0.8)\ (3.4) + (0.2)\ (2.2) = 3.84 \\
y[7] =& 0.8\, y[6] + 0.2\, x[7] = (0.8)\ (3.84) + (0.2)\ (1.4) = 4.12 \\
y[8] =& 0.8\, y[7] + 0.2\, x[8] = (0.8)\ (4.12) + (0.2)\ (0.6) = 4.24 \\
y[9] =& 0.8\, y[8] + 0.2\, x[9] = (0.8)\ (4.24) + (0.2)\ (-0.2) = 4.2 \\
y[10] =& 0.8\, y[9] + 0.2\, x[10] = (0.8)\ (4.2) + (0.2)\ (0.4) = 4.28
\end{aligned}
$$

Therefore

$$
y[n] = \{\underset{\substack{\uparrow \\ n=0}}{0.34}, 0.8, 1.42, 2.08, 2.82, 3.4, 3.84, 4.12, 4.24, 4.2, 4.28\}
$$

3.10. The difference equation is

$$
y[n] = 1.005\, y[n-1] - x[n]
$$

Writing the output signal iteratively for $n = 1, \ldots, 12$ we obtain

$$
\begin{aligned}
y[1] =& 1.005\, y[0] - x[1] = (1.005)\ (10000) - 250 = 9800 \\
y[2] =& 1.005\, y[1] - x[2] = (1.005)\ (9800) - 250 = 9599 \\
y[3] =& 1.005\, y[2] - x[3] = (1.005)\ (9599) - 250 = 9396.99 \\
y[4] =& 1.005\, y[3] - x[4] = (1.005)\ (9396.99) - 250 = 9193.98 \\
y[5] =& 1.005\, y[4] - x[5] = (1.005)\ (9193.98) - 250 = 8989.95 \\
y[6] =& 1.005\, y[5] - x[6] = (1.005)\ (8989.95) - 250 = 8784.9 \\
y[7] =& 1.005\, y[6] - x[7] = (1.005)\ (8784.9) - 250 = 8578.82 \\
y[8] =& 1.005\, y[7] - x[8] = (1.005)\ (8578.82) - 250 = 8371.72 \\
y[9] =& 1.005\, y[8] - x[9] = (1.005)\ (8371.72) - 250 = 8163.58 \\
y[10] =& 1.005\, y[9] - x[10] = (1.005)\ (8163.58) - 250 = 7954.39 \\
y[11] =& 1.005\, y[10] - x[11] = (1.005)\ (7954.39) - 250 = 7744.17 \\
y[12] =& 1.005\, y[11] - x[12] = (1.005)\ (7744.17) - 250 = 7532.89
\end{aligned}
$$

3.11. The difference equation given by Eqn. (3.25) is

$$y[n]=\frac{1}{2}\left(y[n-1]+\frac{A}{y[n-1]}\right)$$

a. With $A=5$ and $y[0]=1$ we have

$$y[1]=\frac{1}{2}\left(y[0]+\frac{A}{y[0]}\right)=\frac{1}{2}\left(1+\frac{5}{1}\right)=3$$

$$y[2]=\frac{1}{2}\left(y[1]+\frac{A}{y[1]}\right)=\frac{1}{2}\left(3+\frac{5}{3}\right)=2.33333$$

$$y[3]=\frac{1}{2}\left(y[2]+\frac{A}{y[2]}\right)=\frac{1}{2}\left(2.33333+\frac{5}{2.33333}\right)=2.2381$$

$$y[4]=\frac{1}{2}\left(y[3]+\frac{A}{y[3]}\right)=\frac{1}{2}\left(2.2381+\frac{5}{2.2381}\right)=2.23607$$

$$y[5]=\frac{1}{2}\left(y[4]+\frac{A}{y[4]}\right)=\frac{1}{2}\left(2.23607+\frac{5}{2.23607}\right)=2.23607$$

b. With $A=17$ and $y[0]=1$ we have

$$y[1]=\frac{1}{2}\left(y[0]+\frac{A}{y[0]}\right)=\frac{1}{2}\left(1+\frac{17}{1}\right)=9$$

$$y[2]=\frac{1}{2}\left(y[1]+\frac{A}{y[1]}\right)=\frac{1}{2}\left(9+\frac{17}{9}\right)=5.44444$$

$$y[3]=\frac{1}{2}\left(y[2]+\frac{A}{y[2]}\right)=\frac{1}{2}\left(5.44444+\frac{17}{5.44444}\right)=4.28345$$

$$y[4]=\frac{1}{2}\left(y[3]+\frac{A}{y[3]}\right)=\frac{1}{2}\left(4.28345+\frac{17}{4.28345}\right)=4.12611$$

$$y[5]=\frac{1}{2}\left(y[4]+\frac{A}{y[4]}\right)=\frac{1}{2}\left(4.12611+\frac{17}{4.12611}\right)=4.12311$$

$$y[6]=\frac{1}{2}\left(y[5]+\frac{A}{y[5]}\right)=\frac{1}{2}\left(4.12311+\frac{17}{4.12311}\right)=4.12311$$

c. With $A = 132$ and $y[0] = 1$ we have

$$y[1] = \frac{1}{2}\left(y[0] + \frac{A}{y[0]}\right) = \frac{1}{2}\left(1 + \frac{132}{1}\right) = 66.5$$

$$y[2] = \frac{1}{2}\left(y[1] + \frac{A}{y[1]}\right) = \frac{1}{2}\left(66.5 + \frac{132}{66.5}\right) = 34.2425$$

$$y[3] = \frac{1}{2}\left(y[2] + \frac{A}{y[2]}\right) = \frac{1}{2}\left(34.2425 + \frac{132}{34.2425}\right) = 19.0487$$

$$y[4] = \frac{1}{2}\left(y[3] + \frac{A}{y[3]}\right) = \frac{1}{2}\left(19.0487 + \frac{132}{19.0487}\right) = 12.9891$$

$$y[5] = \frac{1}{2}\left(y[4] + \frac{A}{y[4]}\right) = \frac{1}{2}\left(12.9891 + \frac{132}{12.9891}\right) = 11.5757$$

$$y[6] = \frac{1}{2}\left(y[5] + \frac{A}{y[5]}\right) = \frac{1}{2}\left(11.5757 + \frac{132}{11.5757}\right) = 11.4894$$

$$y[7] = \frac{1}{2}\left(y[6] + \frac{A}{y[6]}\right) = \frac{1}{2}\left(11.4894 + \frac{132}{11.4894}\right) = 11.4891$$

$$y[8] = \frac{1}{2}\left(y[7] + \frac{A}{y[7]}\right) = \frac{1}{2}\left(11.4891 + \frac{132}{11.4891}\right) = 11.4891$$

3.12.

a.

Characteristic equation:

$$z^2 + 0.2\,z - 0.63 = 0 \qquad \Rightarrow \qquad (z - 0.7)\,(z + 0.9) = 0$$

Homogeneous solution is in the form

$$y[n] = c_1\,(0.7)^n + c_2\,(-0.9)^n\ , \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$1.4286\,c_1 - 1.1111\,c_2 = 5$$

$$2.0408\,c_1 + 1.2346\,c_2 = -3$$

which can be solved to obtain $c_1 = 0.7044$ and $c_2 = -3.5944$. The homogeneous solution is

$$y[n] = 0.7044\,(0.7)^n - 3.5944\,(-0.9)^n\ , \qquad n \ge 0$$

b.

Characteristic equation:

$$z^2 + 1.3\,z + 0.4 = 0 \qquad \Rightarrow \qquad (z + 0.5)\,(z + 0.8) = 0$$

Homogeneous solution is in the form

$$y[n] = c_1\,(-0.5)^n + c_2\,(-0.8)^n\ , \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$-2\,c_1 - 1.25\,c_2 = 0$$

$$4\,c_1 + 1.5625\,c_2 = 5$$

which can be solved to obtain $c_1 = 3.3333$ and $c_2 = -5.3333$. The homogeneous solution is

$$y[n] = 3.3333\,(-0.5)^n - 5.3333\,(-0.8)^n\ , \qquad n \ge 0$$

c.

Characteristic equation:

$$z^2 - 1.7\,z + 0.72 = 0 \qquad \Rightarrow \qquad (z-0.8)\,(z-0.9) = 0$$

Homogeneous solution is in the form

$$y[n] = c_1\,(0.8)^n + c_2\,(0.9)^n\ , \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$1.25\,c_1 + 1.1111\,c_2 = 1$$

$$1.5625\,c_1 + 1.2346\,c_2 = 2$$

which can be solved to obtain $c_1 = 5.12$ and $c_2 = -4.86$. The homogeneous solution is

$$y[n] = 5.12\,(0.8)^n - 4.86\,(0.9)^n\ , \qquad n \ge 0$$

d.

Characteristic equation:

$$z^2 - 0.49 = 0 \qquad \Rightarrow \qquad (z-0.7)\,(z+0.7) = 0$$

Homogeneous solution is in the form

$$y[n] = c_1\,(0.7)^n + c_2\,(-0.7)^n\ , \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$1.4286\,c_1 - 1.4286\,c_2 = -3$$

$$2.0408\,c_1 + 2.0408\,c_2 = -1$$

which can be solved to obtain $c_1 = -1.295$ and $c_2 = 0.805$. The homogeneous solution is

$$y[n] = -1.295\,(0.7)^n + 0.805\,(-0.7)^n\ , \qquad n \ge 0$$

e. Characteristic equation:

$$z^3 + 0.6\,z^2 - 0.51\,z - 0.28 = 0; \qquad \Rightarrow \qquad (z+0.8)\,(z+0.5)\,(z-0.7) = 0$$

Homogeneous solution is in the form

$$y[n] = c_1\,(-0.8)^n + c_2\,(-0.5)^n + c_3\,(0.7)^n \;\;, \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$\begin{aligned} -1.25\,c_1 - 2\,c_2 + 1.4286\,c_3 &= 3 \\ 1.5625\,c_1 + 4\,c_2 + 2.0408\,c_3 &= 2 \\ -1.9531\,c_1 - 8\,c_2 + 2.9155\,c_3 &= 1 \end{aligned} \tag{3.1}$$

which can be solved to obtain $c_1 = -2.56$, $c_2 = 0.9167$ and $c_3 = 1.1433$. The homogeneous solution is

$$y[n] = -2.56\,(-0.8)^n + 0.9167\,(-0.5)^n + 1.1433\,(0.7)^n \;\;, \qquad n \ge 0$$

3.13.

a.

Characteristic equation:

$$z^2 - 1.4\,z + 0.85 = 0 \qquad \Rightarrow \qquad \big(z - 0.7 - j0.6\big)\,\big(z - 0.7 + j0.6\big) = 0$$

The roots are

$$z_{1,2} = 0.9220\,e^{\pm j\,0.7086}$$

Homogeneous solution is in the form

$$y[n] = d_1\,(0.9220)^n \cos(0.7086n) + d_2\,(0.9220)^n \sin(0.7086n) \;\;, \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$\begin{aligned} 0.6923\,d_1 - 0.5385\,d_2 &= 2 \\ 0.1893\,d_1 - 0.7456\,d_2 &= -2 \end{aligned}$$

which can be solved to obtain $d_1 = 4.5004$ and $d_2 = 2.4171$. The homogeneous solution is

$$y[n] = 4.5004\,(0.9220)^n \cos(0.7086n) + 2.4171\,(0.9220)^n \sin(0.7086n) \;\;, \qquad n \ge 0$$

b.

Characteristic equation:

$$z^2 - 1.6\,z + 1 = 0 \qquad \Rightarrow \qquad \big(z - 0.8 - j0.6\big)\,\big(z - 0.8 + j0.6\big) = 0$$

The roots are

$$z_{1,2} = 1, e^{\pm j0.6435}$$

Homogeneous solution is in the form

$$y[n] = d_1 \cos(0.6435n) + d_2 \sin(0.6435n) \ , \qquad n \geq 0$$

Imposing initial conditions leads to the equations

$$0.8\,d_1 - 0.6\,d_2 = 0$$

$$0.28\,d_1 - 0.96\,d_2 = 3$$

which can be solved to obtain $d_1 = -3$ and $d_2 = -4$. The homogeneous solution is

$$y[n] = -3 \cos(0.6435n) - 4 \sin(0.6435n) \ , \qquad n \geq 0$$

c.

Characteristic equation:

$$z^2 + 1 = 0 \qquad \Rightarrow \qquad \left(z - j1\right)\left(z + j1\right) = 0$$

The roots are

$$z_{1,2} = \pm j = 1\,e^{\pm j\pi/2}$$

Homogeneous solution is in the form

$$y[n] = d_1 \cos(\pi n/2) + d_2 \sin(\pi n/2) \ , \qquad n \geq 0$$

Imposing initial conditions leads to the equations

$$d_1 \cos(-\pi/2) + d_2 \sin(-\pi/2) = 3$$

$$d_1 \cos(-\pi) + d_2 \sin(-\pi) = 2$$

which can be solved to obtain $d_1 = -2$ and $d_2 = -3$. The homogeneous solution is

$$y[n] = -2 \cos(\pi n/2) - 3 \sin(\pi n/2) \ , \qquad n \geq 0$$

d.

Characteristic equation:

$$z^3 - 2.5\,z^2 + 2.44\,z - 0.9 = 0 \qquad \Rightarrow \qquad \left(z - 0.8 - j0.6\right)\left(z - 0.8 + j0.6\right)(z - 0.9) = 0$$

The roots are

$$z_{1,2} = 1, e^{\pm j0.6435} \qquad \text{and} \qquad z_3 = 0.9$$

Homogeneous solution is in the form

$$y[n] = d_1 \cos(0.6435n) + d_2 \sin(0.6435n) + c_3\,(0.9)^n \ , \qquad n \geq 0$$

Imposing initial conditions leads to the equations

$$0.8\,d_1 - 0.6\,d_2 + 1.1111\,c_3 = 1$$

$$0.28\,d_1 - 0.96\,d_2 + 1.2346\,c_3 = 2$$

$$-0.352\,d_1 - 0.936\,d_2 + 1.3717\,c_3 = 3$$

which can be solved to obtain $d_1 = -1.2562$, $d_2 = -0.4227$, and $c_3 = 1.5762$. The homogeneous solution is

$$y[n] = -1.2562 \cos(0.6435n) - 0.4227 \sin(0.6435n) + 1.5762\,(0.9)^n \ , \qquad n \geq 0$$

3.14.

a.

Characteristic equation:

$$z^2 - 1.4\,z + 0.49 = 0 \qquad \Rightarrow \qquad (z-0.7)^2 = 0$$

The roots are

$$z_{1,2} = 0.7$$

Homogeneous solution is in the form

$$y[n] = c_1\,(0.7)^n + c_2\,n\,(0.7)^n\ , \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$1.4286\,c_1 - 1.4286\,c_2 = 1$$

$$2.0408\,c_1 - 4.0816\,c_2 = 1$$

which can be solved to obtain $c_1 = 0.91$ and $c_2 = 0.21$. The homogeneous solution is

$$y[n] = 0.91\,(0.7)^n + 0.21\,n\,(0.7)^n\ , \qquad n \ge 0$$

b.

Characteristic equation:

$$z^2 + 1.8\,z + 0.81 = 0 \qquad \Rightarrow \qquad (z+0.9)^2 = 0$$

The roots are

$$z_{1,2} = -0.9$$

Homogeneous solution is in the form

$$y[n] = c_1\,(-0.9)^n + c_2\,n\,(-0.9)^n\ , \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$-1.1111\,c_1 + 1.1111\,c_2 = 0$$

$$1.2346\,c_1 - 2.4691\,c_2 = 2$$

which can be solved to obtain $c_1 = -1.62$ and $c_2 = -1.62$. The homogeneous solution is

$$y[n] = -1.62\,(-0.9)^n - 1.62\,n\,(-0.9)^n\ , \qquad n \ge 0$$

c.

Characteristic equation:

$$z^3 - 0.8\,z^2 - 0.64\,z + 0.512 = 0 \qquad \Rightarrow \qquad (z+0.8)\,(z-0.8)^2 = 0$$

The roots are

$$z_1 = -0.8 \qquad \text{and} \qquad z_{2,3} = 0.8$$

Homogeneous solution is in the form

$$y[n] = c_1\ (-0.8)^n + c_2\ (0.8)^n + c_3\, n\ (0.8)^n \ , \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$-1.25\, c_1 + 1.25\, c_2 - 1.25\, c_3 = 1$$

$$1.5625\, c_1 + 1.5625\, c_2 - 3.125\, c_3 = 1$$

$$-1.9531\, c_1 + 1.9531\, c_2 - 5.8594\, c_3 = 2$$

which can be solved to obtain $c_1 = -0.136$, $c_2 = 0.552$ and $c_3 = -0.112$. The homogeneous solution is

$$y[n] = -0.136\ (-0.8)^n + 0.552\ (0.8)^n - 0.112\, n\ (0.8)^n \ , \qquad n \ge 0$$

d.

Characteristic equation:

$$z^3 + 1.7\, z^2 + 0.4\, z - 0.3 = 0 \qquad \Rightarrow \qquad (z+1)^2\ (z-0.3)^2 = 0$$

The roots are

$$z_{1,2} = -1 \qquad \text{and} \qquad z_3 = 0.3$$

Homogeneous solution is in the form

$$y[n] = c_1\ (-1)^n + c_2\, n\ (-1)^n + c_3\ (0.3)^n \ , \qquad n \ge 0$$

Imposing initial conditions leads to the equations

$$-c_1 + c_2 + 3.3333\, c_3 = 1$$

$$c_1 - 2\, c_2 + 11.1111\, c_3 = 2$$

$$-c_1 + 3\, c_3 + 37.037\, c_3 = 1$$

which can be solved to obtain $c_1 = -2.2959$, $c_2 = -1.6154$ and $c_3 = 0.0959$. The homogeneous solution is

$$y[n] = -2.2959\ (-1)^n - 1.6154\, n\ (-1)^n + 0.0959\ (0.3)^n \ , \qquad n \ge 0$$

3.15.

a.

Characteristic equation is

$$z - 0.6 = 0$$

The homogeneous solution is in the form

$$y_h[n] = c_1\ (0.6)^n \ , \qquad n \ge 0$$

For the unit-step input the particular solution is

$$y_p = k_1$$

The particular solution must satisfy the difference equation, therefore

$$k_1 = 0.6\, k_1 + 1$$

which leads to $k_1 = 2.5$. The complete solution is in the form

$$y[n] = c_1\ (0.6)^n + 2.5$$

In order to satisfy the specified initial condition we need

$$y[-1] = c_1\ (0.6)^{-1} + 2.5 = 2 \qquad \Rightarrow \qquad c_1 = -0.3$$

The signal $y[n]$ is

$$y[n] = -0.3\ (0.6)^n + 2.5\ , \qquad n \ge 0$$

b.

Characteristic equation is

$$z - 0.8 = 0$$

The homogeneous solution is in the form

$$y_h[n] = c_1\ (0.8)^n\ , \qquad n \ge 0$$

For the unit-step input the particular solution is

$$y_p = k_1\ \sin(0.2n) + k_2\ \cos(0.2n)$$

The particular solution must satisfy the difference equation, therefore

$$k_1\ \sin(0.2n) + k_2\ \cos(0.2n) = 0.8\ \left[k_1\ \sin(0.2n - 0.2) + k_2\ \cos(0.2n - 0.2)\right] + 2\, sin\,(0.2n)$$

which leads to the equations

$$0.2159\, k_1 - 0.1589\, k_2 = -2$$

$$0.1589\, k_1 + 0.2159\, k_2 = 0$$

and yields the solutions $k_1 = -6.0087$ and $k_2 = 4.4224$. The forced solution of the difference equation is in the form

$$y[n] = c_1\ (0.8)^n - 6.0087\ \sin(0.2n) + 4.4224\ \cos(0.2n)$$

In order to satisfy the specified initial condition we need

$$y[-1] = c_1\ (0.8)^{-1} - 6.0087\ \sin(-0.2) + 4.4224\ \cos(-0.2) = 1 \qquad \Rightarrow \qquad c_1 = -3.6224$$

The signal $y[n]$ is

$$y[n] = -3.6224\ (0.8)^n - 6.0087\ \sin(0.2n) + 4.4224\ \cos(0.2n)\ , \qquad n \ge 0$$

c.

Characteristic equation is

$$z^2 - 0.2\, z - 0.63 = 0 \qquad \Rightarrow \qquad (z + 0.7)\ (z - 0.9) = 0$$

The homogeneous solution is in the form

$$y_h[n] = c_1\,(-0.7)^n + c_2\,(0.9)^n \qquad n \ge 0$$

For the exponential input signal the particular solution is

$$y_p = k_1\,e^{-0.2n}$$

The particular solution must satisfy the difference equation, therefore

$$k_1\,e^{-0.2n} - 0.2\,e^{-0.2(n-1)} - 0.63\,e^{-0.2(n-2)} = e^{-0.2n}$$

which leads to the equation

$$k_1\left(1 - 0.2\,e^{0.2} - 0.63\,e^{0.4}\right) = 1$$

and yields the solution $k_1 = -5.4309$. The forced solution of the difference equation is in the form

$$y[n] = c_1\,(-0.7)^n + c_2\,(0.9)^n - 5.4309\,e^{-0.2n}$$

In order to satisfy the specified initial conditions we need

$$y[-1] = c_1\,(-0.7)^{-1} + c_2\,(0.9)^{-1} - 5.4309\,e^{0.2} = 0$$

and

$$y[-2] = c_1\,(-0.7)^{-2} + c_2\,(0.9)^{-2} - 5.4309\,e^{0.4} = 3$$

The solutions are found as

$$c_1 = 1.0285 \qquad \text{and} \qquad c_2 = 7.2924$$

The signal $y[n]$ is

$$y[n] = 1.0285\,(-0.7)^n + 7.2924\,(0.9)^n - 5.4309\,e^{-0.2n}\ , \qquad n \ge 0$$

d.

Characteristic equation is

$$z^2 + 1,4\,z + 0.85 = 0 \qquad \Rightarrow \qquad \left(z + 0.7 + j0.6\right)\left(z + 0.7 - j0.6\right) = 0$$

The roots are

$$z_{1,2} = 0.922\,e^{\pm j\,2.433}$$

The homogeneous solution is in the form

$$y_h[n] = d_1\,(0.922)^n\,\cos(2.433n) + d_2\,(0.922)^n\,\sin(2.433n) \qquad n \ge 0$$

For the unit-step input signal the particular solution is

$$y_p = k_1$$

The particular solution must satisfy the difference equation, therefore

$$k_1 + 1.4\,k_1 + 0.85\,k_1 = 1 \qquad \Rightarrow \qquad k_1 = 0.3077$$

The forced solution of the difference equation is in the form

$$y[n] = d_1\ (0.922)^n \cos(2.433n) + d_2\ (0.922)^n \sin(2.433n) + 0.3077$$

In order to satisfy the specified initial conditions we need

$$y[-1] = d_1\ (0.922)^{-1} \cos(-2.433) + d_2\ (0.922)^{-1} \sin(-2.433) + 0.3077 = -2$$

and

$$y[-2] = d_1\ (0.922)^{-2} \cos(-4.866) + d_2\ (0.922)^{-2} \sin(-4.866) + 0.3077 = 0$$

The solutions for d_1 and d_2 are found as

$$d_1 = 3.4926 \qquad \text{and} \qquad d_2 = -0.8054$$

The signal $y[n]$ is

$$y[n] = 3.4926\ (0.922)^n \cos(2.433n) - 0.8054\ (0.922)^n \sin(2.433n) + 0.3077\ , \qquad n \ge 0$$

e.

Characteristic equation is

$$z^2 + 1.6\,z + 0.64 = 0 \qquad \Rightarrow \qquad (z+0.8)^2 = 0$$

The homogeneous solution is in the form

$$y_h[n] = c_1\ (-0.8)^n + c_2\,n\ (-0.8)^n \qquad n \ge 0$$

For the unit-step input signal the particular solution is

$$y_p = k_1$$

The particular solution must satisfy the difference equation, therefore

$$k_1 + 1.6\,k_1 + 0.64\,k_1 = 1 \qquad \Rightarrow \qquad k_1 = 0.3086$$

The forced solution of the difference equation is in the form

$$y[n] = c_1\ (-0.8)^n + c_2\,n\ (-0.8)^n + 0.3086$$

In order to satisfy the specified initial conditions we need

$$y[-1] = c_1\ (-0.8)^{-1} - c_2\ (-0.8)^{-1} + 0.3086 = 0$$

and

$$y[-2] = c_1\ (-0.8)^{-2} - 2\,c_2\ (-0.8)^{-2} + 0.3086 = 1$$

The solutions are found as

$$c_1 = 0.0514 \qquad \text{and} \qquad c_2 = -1.1956$$

The signal $y[n]$ is

$$y[n] = 0.0514\ (-0.8)^n - 1.1956\,n\ (-0.8)^n + 0.3086\ , \qquad n \ge 0$$

3.16.

a.

The characteristic equation is

$$z-(1-\alpha)=0$$

and the homogeneous solution is in the form

$$y_h[n]=c_1\,(1-\alpha)^n\;,\qquad n\ge 0$$

For a unit-step input signal the particular solution is in the form

$$y_p[n]=k_1$$

Imposing the particular solution into the difference equation yields

$$k_1=(1-\alpha)\;k_1+\alpha\;(1)$$

and subsequently $k_1=1$. Therefore, the complete solution is in the form

$$y[n]=c_1\,(1-\alpha)^n+1\;,\qquad n\ge 0$$

In order to satisfy the initial condition $y[-1]=0$ we need

$$c_1\,(1-\alpha)^{-1}+1=0\qquad\Rightarrow\qquad c_1=\alpha-1$$

The output signal is

$$\begin{aligned}y[n]&=(\alpha-1)\,(1-\alpha)^n+1\\&=1-(1-\alpha)^{n+1}\;,\qquad n\ge 0\end{aligned}$$

b.

For a unit-ramp input signal the particular solution is in the form

$$y_p[n]=k_1\,n+k_2$$

Imposing the particular solution into the difference equation yields

$$k_1\,n+k_2=(1-\alpha)\,\left[k_1\,(n-1)+k_2\right]+\alpha\,n$$

which requires $k_1=1$ and $k_2=(\alpha-1)\,/\,\alpha$. The particular solution is

$$y_p[n]=n+\frac{\alpha-1}{\alpha}$$

and the complete solution is in the form

$$y[n]=c_1\,(1-\alpha)^n+n+\frac{\alpha-1}{\alpha}\;,\qquad n\ge 0$$

In order to satisfy the initial condition $y[-1]=0$ we need

$$c_1\,(1-\alpha)^{-1}-1+\frac{\alpha-1}{\alpha}=0\qquad\Rightarrow\qquad c_1=\frac{1-\alpha}{\alpha}$$

The output signal is

$$
\begin{aligned}
y[n] &= \frac{1-\alpha}{\alpha}\,(1-\alpha)^n + n + \frac{\alpha-1}{\alpha} \\
&= \frac{(1-\alpha)^{n+1}}{\alpha} + n + \frac{\alpha-1}{\alpha}\,, \qquad n \ge 0
\end{aligned}
$$

3.17.

a.

$$\frac{y_a(t) - y_a(t-T)}{T} + A\,y_a(t) = A\,x_a(t)$$

Solving for $y_a(t)$ we get

$$y_a(t) = \frac{1}{1+AT}\,y_a(t-T) + \frac{AT}{1+AT}\,x_a(t)$$

b.

Let

$$t = nT \qquad \text{and} \qquad t - T = (n-1)\,T$$

Defining

$$
\begin{aligned}
x[n] &= x_a(nT) \\
y[n] &= y_a(nT) \\
y[n-1] &= y_a(nT-T)
\end{aligned}
$$

we obtain

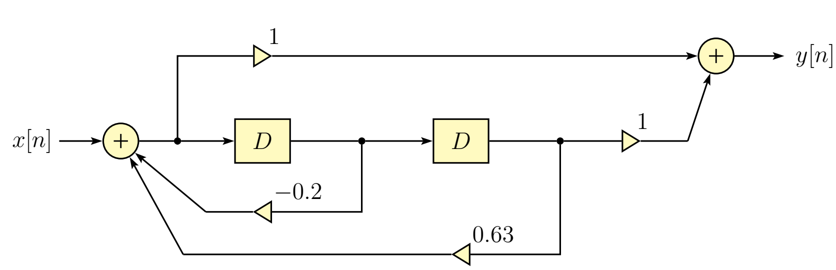
$$y[n] = \frac{1}{1+AT}\,y[n-1] + \frac{AT}{1+AT}\,x[n]$$

which corresponds to an exponential smoother with

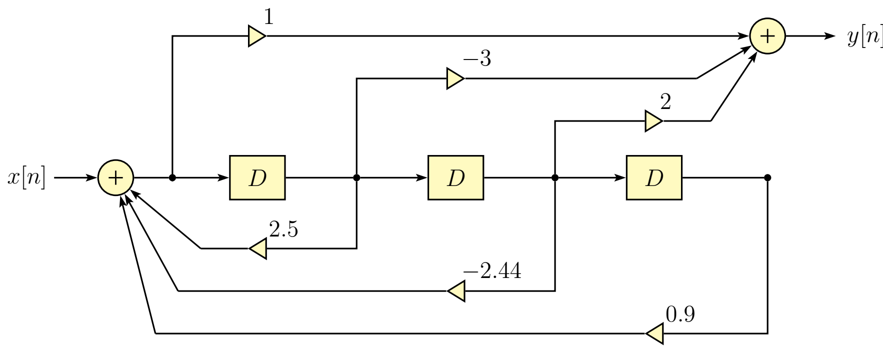
$$\alpha = \frac{AT}{1+AT}$$

3.18.

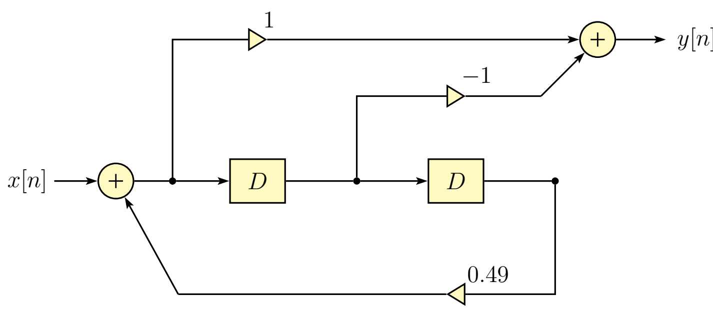
a.



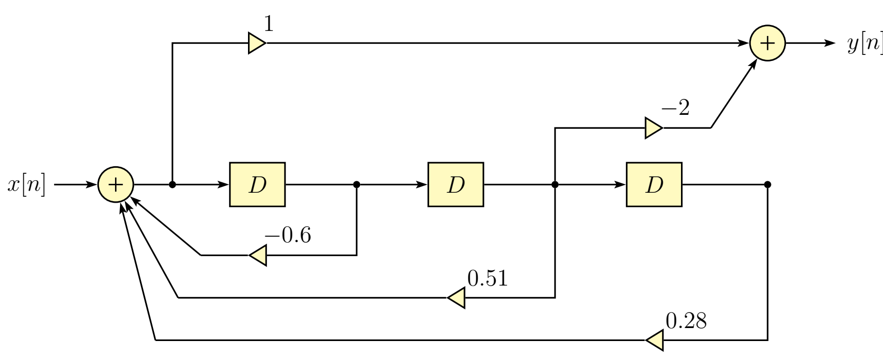
b.



c.



d.



3.19.

a.

$$w[n] = x[n] * h_1[n]$$

$$\begin{aligned} y[n] &= w[n] * h_2[n] \\ &= (x[n] * h_1[n]) * h_2[n] \\ &= x[n] * (h_1[n] * h_2[n]) = x[n] * h_{eq}[n] \end{aligned}$$

Therefore

$$h_{eq}[n] = h_1[n] * h_2[n]$$

b.

$$h_{eq}[n] = \sum_k h_1[k]\, h_2[n-k] = \sum_{k=0}^{4} (1)\ h_2[n-k]$$

We know that

$$h_2[n-k] = u[n-k] - u[n-k-5] = \begin{cases} 1\,, & -4+n < k < n \\ 0\,, & \text{otherwise} \end{cases}$$

Therefore

$$h_{eq}[n] = \{\ \underset{\substack{\uparrow \\ n=0}}{1}\ , 2, 3, 4, 5, 4, 3, 2, 1\}$$

c.

$$y[n] = \{\ \underset{\substack{\uparrow \\ n=0}}{1}\ , 3, 6, 10, 15, 19, 22, 24, 25, \ldots\}$$

3.20.

a. It can easily be seen that

$$y_1[n] = x[n] * h_1[n]$$

$$y_2[n] = x[n] * h_2[n]$$

Combining these two relationships we have

$$\begin{aligned} y[n] &= y_1[n] + y_2[n] \\ &= x[n] * h_1[n] + x[n] * h_2[n] \\ &= x[n] * (h_1[n] + h_2[n]) = x[n] * h_{eq}[n] \end{aligned}$$

Therefore

$$h_{eq}[n] = h_1[n] + h_2[n]$$

b.

$$h[n] = (0.9)^n\ u[n] + (-0.7)^n\ u[n]$$

c.

$$y_1[n] = \sum_{k=-\infty}^{\infty} h_1[k]\, u[n-k] = \sum_{k=0}^{\infty} h_1[k] = \frac{1-(0.9)^{n+1}}{1-0.9}\ , \qquad n \ge 0$$

Similarly

$$y_2[n] = \sum_{k=-\infty}^{\infty} h_2[k]\, u[n-k] = \sum_{k=0}^{\infty} h_2[k] = \frac{1-(-0.7)^{n+1}}{1+0.7}\ , \qquad n \ge 0$$

The output signal is

$$y[n] = y_1[n] + y_2[n] = \frac{1-(0.9)^{n+1}}{1-0.9} + \frac{1-(-0.7)^{n+1}}{1+0.7}\ , \qquad n \ge 0$$

3.21.

a.

$$y_1[n] = h_1[n] * x[n]$$
$$w[n] = h_2[n] * x[n]$$
$$y_3[n] = h_3[n] * w[n] = h_2[n] * h_3[n] * x[n]$$

The output signal is

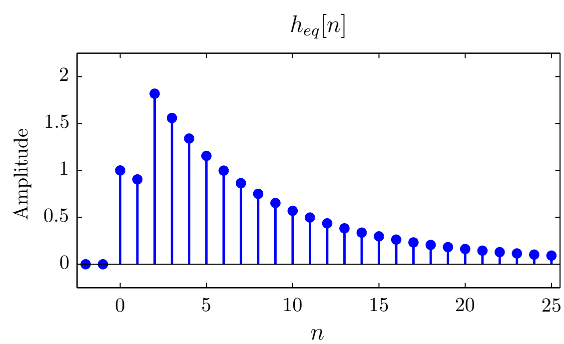
$$\begin{aligned} y[n] &= y_1[n] + y_2[n] \\ &= \big(h_1[n] + h_2[n] * h_3[n]\big) * x[n] \end{aligned}$$

and the equivalent impulse response is

$$h_{eq}[n] = h_1[n] + h_2[n] * h_3[n]$$

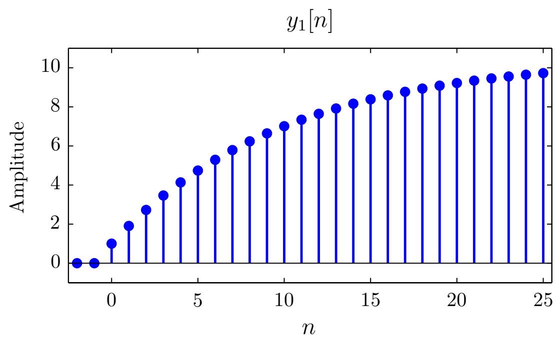
b.

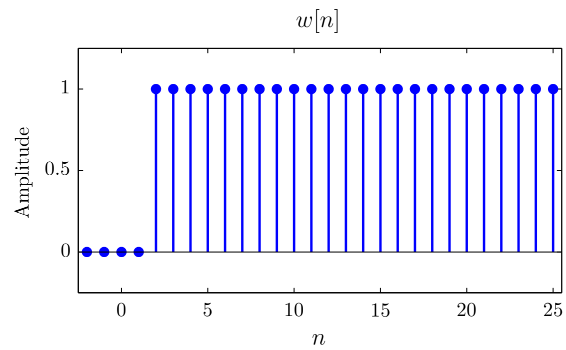
$$h_2[n] * h_3[n] = \delta[n-2] * e^{-0.2n}\, u[n] = e^{-0.2(n-2)}\, u[n-2]$$
$$h_{eq}[n] = e^{-0.1n}\, u[n] + e^{-0.2(n-2)}\, u[n-2]$$

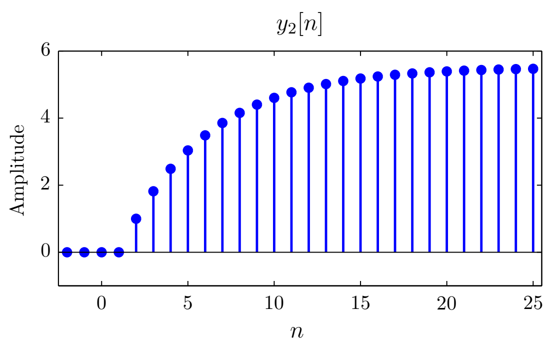


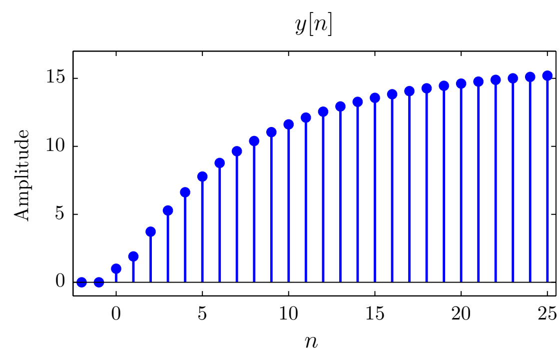
c.

$$y_1[n] = h_1[n] * u[n] = \sum_{k=-\infty}^{\infty} e^{-0.1k}\, u[k]\, u[n-k]$$
$$y_1[n] = \sum_{k=0}^{n} e^{-0.1k} = \frac{1-e^{-0.1(n+1)}}{1-e^{-0.1}} \;, \qquad n \ge 0$$
$$w[n] = h_2[n] * u[n] = \delta[n-2] * u[n] = u[n-2]$$
$$y_2[n] = h_3[n] * w[n] = \sum_{k=-\infty}^{\infty} e^{-0.2k}\, u[k]\, u[n-k-2]$$
$$y_2[n] = \sum_{k=0}^{n-2} e^{-0.2k} = \frac{1-e^{-0.2(n-1)}}{1-e^{-0.2}} \;, \qquad n \ge 2$$
$$y[n] = y_1[n] + y_2[n] = \frac{1-e^{-0.1(n+1)}}{1-e^{-0.1}}\, u[n] + \frac{1-e^{-0.2(n-1)}}{1-e^{-0.2}}\, u[n-2]$$









3.22.

a.

$$y_1[n] = h_1[n] * x[n]$$

$$w[n] = h_2[n] * x[n]$$

$$y_3[n] = h_3[n] * w[n] = h_3[n] * h_2[n] * x[n]$$

$$y_4[n] = h_4[n] * w[n] = h_4[n] * h_2[n] * x[n]$$

The output signal is

$$\begin{aligned} y[n] &= y_1[n] + y_3[n] + y_4[n] \\ &= h_1[n] * x[n] + h_3[n] * h_2[n] * x[n] + h_4[n] * h_2[n] * x[n] \\ &= \big(h_1[n] + h_3[n] * h_2[n] + h_4[n] * h_2[n]\big) * x[n] \end{aligned}$$

and the equivalent impulse response is

$$h_{eq}[n] = h_1[n] + h_3[n] * h_2[n] + h_4[n] * h_2[n]$$

b.

$$h_3[n] * h_2[n] = \big(u[n] - u[n-3]\big) * \big(u[n] - u[n-3]\big) =$$

Using the unit ramp function $r[n]$, the result can be written as

$$h_3[n] * h_2[n] = r[n] - 2\,r[n-3] + r[n-6]$$

$$h_4[n] * h_2[n] = \delta[n-2] * h_2[n] = h_2[n-2] = u[n-2] - u[n-5]$$

The impulse response of the equivalent system is

$$h_{eq}[n] = e^{-0.1n}\, u[n] + r[n] - 2\,r[n-3] + r[n-6] + u[n-2] - u[n-5]$$

c.

$$w[n] = h_2 n * u[n] = \sum_{k=0}^{n} h_2[k] = \{ \underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 3, 3, 3, 3, 3, 3, \ldots \}$$

$$y_3[n] = h_3[n] * w[n] = \{ \underset{\substack{\uparrow \\ n=0}}{1}, 3, 6, 8, 9, 9, 9, 9, 9, \ldots \}$$

$$y_4[n] = h_4[n] * w[n] = w[n-2] = \{ \underset{\substack{\uparrow \\ n=2}}{1}, 2, 3, 3, 3, 3, 3, 3, 3, \ldots \}$$

$$y[n] = y_1[n] + y_3[n] + y_4[n] = \{ \underset{\substack{\uparrow \\ n=0}}{2}, 5, 10, 13, 15, 15, 15, 15, 15, \ldots \}$$

3.23.

Convolution of the two signals $x[n]$ and $h[n]$ is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

The limits of the summation can be adjusted based on the term $h[n-k]$. Since

$$h[n-k] = 0 \;\text{ if } \begin{cases} n-k < N_{h1} & \Rightarrow \quad k > n - N_{h1} \;, \;\text{or} \\ n-k > N_{h2} & \Rightarrow \quad k < n - N_{h2} \end{cases}$$

the convolution sum can be written as

$$y[n] = \sum_{k=-n-N_{h2}}^{n-N_{h1}} x[k]\, h[n-k]$$

We know that $x[k] = 0$ if $k < N_{x1}$ or $k > N_{x2}$. If the lower limit of the summation is greater than N_{x2} all terms of the summation are zero. Similarly if the upper limit of the summation is less than N_{x1} then all terms of the summation are zero.

$$y[n] = 0 \;\text{ if } \begin{cases} n - N_{h2} > N_{x2} & \Rightarrow \quad n > N_{h2} + N_{x2} \;, \;\text{or} \\ n - N_{h1} < N_{x1} & \Rightarrow \quad n < N_{h1} + N_{x1} \end{cases}$$

Therefore, the nonzero samples of $y[n]$ are in the index range

$$N_{h1} + N_{x1} \le n \le N_{h2} + N_{x2}$$

3.24.

$$y[n] = x[n] * h[n] \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

If the signal $x[n]$ is time shifted by m samples, we have

$$x[n-m] * h[n] \sum_{k=-\infty}^{\infty} x[k-m]\, h[n-k]$$

Let us use the variable change $k-m=\bar{k}$.

$$x[n-m] * h[n] \sum_{\bar{k}=-\infty}^{\infty} x[\bar{k}]\, h[n-m-\bar{k}] = y[n-m]$$

If the signal $h[n]$ is time shifted instead, then we have

$$x[n] * h[n-m] \sum_{k=-\infty}^{\infty} x[k]\, h[n-m-k] = y[n-m]$$

3.25.

a.

$$\text{Sys}\left\{\delta[n]\right\} = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1\,, & n \ge 0 \\ 0\,, & n < 0 \end{cases}$$

Therefore, $h[n] = u[n]$.

b.

$$\text{Sys}\left\{\delta[n]\right\} = \sum_{-\infty}^{n} e^{-0.1(n-k)}\,\delta[k]$$

$$= e^{-0.1n} \sum_{-\infty}^{n} e^{0.1k}\,\delta[k]$$

$$= \begin{cases} e^{-0.1n}\,, & n \ge 0 \\ 0\,, & n < 0 \end{cases}$$

Therefore, $h[n] = e^{-0.1n}\,u[n]$.

c.

$$\text{Sys}\left\{\delta[n]\right\} = \sum_{k=0}^{n} \delta[k] = 1 \ \text{ for } \ n \ge 0$$

Therefore, $h[n] = u[n]$.

d.

$$\text{Sys}\left\{\delta[n]\right\} = \sum_{k=n-10}^{n} \delta[k] = \begin{cases} 1\,, & n-10 \le 0 \le n \quad \Rightarrow \quad 0 \le n \le 10 \\ 0\,, & \text{otherwise} \end{cases}$$

Therefore $h[n] = u[n] - u[n-11]$.

e.

$$\text{Sys}\left\{\delta[n]\right\} = \sum_{k=n-10}^{n+10} \delta[k] = \begin{cases} 1\,, & n-10 \le 0 \le n+10 \quad \Rightarrow \quad -10 \le n \le 10 \\ 0\,, & \text{otherwise} \end{cases}$$

Therefore $h[n] = u[n+10] - u[n-11]$.

3.26.

a.

```
n = [0:19];
ynm1 = 5;
ynm2 = -3;
y = [];
for i = 0:19,
   yn = -0.2*ynm1+0.63*ynm2;
   y = [y,yn];
   ynm2 = ynm1;
   ynm1 = yn;
end;
% Display the results
[n',y']
% Compute the output from analytical solution
y_anl = 0.7044*(0.7).^n-3.5944*(-0.9).^n;
[n',y_anl']
```

b.

```
n = [0:19];
ynm1 = 0;
ynm2 = 5;
y = [];
for i = 0:19,
   yn = -1.3*ynm1-0.4*ynm2;
   y = [y,yn];
   ynm2 = ynm1;
   ynm1 = yn;
end;
% Display the results
[n',y']
% Compute the output from analytical solution
y_anl = 3.3333*(-0.5).^n-5.3333*(-0.8).^n;
[n',y_anl']
```

c.

```
n = [0:19];
ynm1 = 1;
ynm2 = 2;
y = [];
for i = 0:19,
   yn = 1.7*ynm1-0.72*ynm2;
   y = [y,yn];
   ynm2 = ynm1;
   ynm1 = yn;
end;
% Display the results
[n',y']
```

```
% Compute the output from analytical solution
y_anl = 5.12*(0.8).^n-4.86*(0.9).^n;
[n',y_anl']
```

d.

```
n = [0:19];
ynm1 = -3;
ynm2 = -1;
y = [];
for i = 0:19,
  yn = 0.49*ynm2;
  y = [y,yn];
  ynm2 = ynm1;
  ynm1 = yn;
end;
% Display the results
[n',y']
% Compute the output from analytical solution
y_anl = -1.295*(0.7).^n+0.805*(-0.7).^n;
[n',y_anl']
```

e.

```
n = [0:19];
ynm1 = 3;
ynm2 = 2;
ynm3 = 1;
y = [];
for i = 0:19,
  yn = -0.6*ynm1+0.51*ynm2+0.28*ynm3;
  y = [y,yn];
  ynm3 = ynm2;
  ynm2 = ynm1;
  ynm1 = yn;
end;
% Display the results
[n',y']
% Compute the output from analytical solution
y_anl = -2.56*(-0.8).^n+0.9167*(-0.5).^n+1.1433*(0.7).^n;
[n',y_anl']
```

3.27.

a.

```
x = @(n) (0.95).^n.*cos(0.1*pi*n).*(n>=0);
n = [0:29];
```

b.

```
w = @(n) 3*x(n);
[n',w(n)']
y = @(n) w(n-2);
[n',y(n)']
```

c.

```
wtilde = @(n) x(n-2);
[n',wtilde(n)']
ytilde = @(n) 3*wtilde(n);
[n',ytilde(n)']
```

3.28.

a.

```
x = @(n) (0.95).^n.*cos(0.1*pi*n).*(n>=0);
n = [0:29];
w = @(n) 3*x(n);
[n',w(n)']
stem(n,w(n));
y = @(n) n.*w(n);
[n',y(n)']
stem(n,y(n));
wtilde = @(n) n.*x(n);
[n',wtilde(n)']
stem(n,wtilde(n));
ytilde = @(n) 3*wtilde(n);
[n',ytilde(n)']
stem(n,ytilde(n));
```

b.

```
w = @(n) 3*x(n);
[n',w(n)']
stem(n,w(n));
y = @(n) w(n)+5;
[n',y(n)']
stem(n,y(n));
wtilde = @(n) x(n)+5;
[n',wtilde(n)']
stem(n,wtilde(n));
ytilde = @(n) 3*wtilde(n);
[n',ytilde(n)']
stem(n,ytilde(n));
```

3.29.

a. Anonymous functions for the two signals can be defined as follows:

```
x1 = @(n) n.*exp(-0.2*n).*((n>=0)&(n<20));
x2 = @(n) cos(0.05*pi*n).*((n>=0)&(n<20));
```

b. Anonymous functions for the systems can be defined as follows:

```
sys1 = @(x,n) x.*(n>=0);
sys2 = @(x,n) 3*x+5;
sys3 = @(x,n) 3*x+5*(n>=0);
sys4 = @(x,n) n.*x;
sys5 = @(x,n) cos(0.2*pi*n).*x;
```

c. Find the responses of each system to $x_1[n]$, $x_2[n]$, and $x[n] = 5\,x_1[n] - 3\,x_2[n]$. The script below is for the first system. It may be edited to test the remaining systems.

```
y1 = sys1(x1(n),n);
y2 = sys1(x2(n),n);
y = sys1(5*x1(n)-3*x2(n),n);
clf;
subplot(2,1,1);
stem(n,y);
xlabel('n');
title('Sys_{1}\{ 5x_{1}[n]-3x_{2}[n] \}');
subplot(2,1,2);
stem(n,5*y1-3*y2);
xlabel('n');
title('5y_{1}[n]-3y_{2}[n]');
```

d. Find the responses of each system to $x_1[n-1]$. The script below is for the first system. It may be edited to test the remaining systems.

```
clf;
subplot(2,1,1);
stem(n,sys1(x1(n),n));
xlabel('n');
title('Sys_{1}\{ x_{1}[n] \}');
subplot(2,1,2);
stem(n,sys1(x1(n-1),n));
xlabel('n');
title('Sys_{1}\{ x_{1}[n-1] \}');
```

3.30.

a.

```
function bal = ss_lbal(A,B,c,n)
  bal = A;
  for k=1:n,
    bal = bal*(1+c)-B;
  end;
end
```

b.

```
A = 10000;
B = 150;
c = 0.01;
bal1 = ss_lbal(A,B,c,12)
bal2 = ss_lbal(A,B,c,24)
bal3 = ss_lbal(A,B,c,36)
```

3.31.

a.

```
function y = ss_sqrt(A,y_init,tol)
  ynm1 = 1;
  for k=1:100,  % Quit if it doesn't work in 100 iterations.
    yn = 0.5*(ynm1+A/ynm1);
    if abs(yn-ynm1)<tol,
      break;
    end;
    ynm1 = yn;
  end;
  y = yn;
```

b.

```
y1 = ss_sqrt(5,1,0.00001)
y2 = ss_sqrt(17,1,0.00001)
y3 = ss_sqrt(132,1,0.00001)
```

3.32.

a. Unit-step response of the exponential smoother:

```
yu = @(n,alpha) (1-(1-alpha).^(n+1)).*(n>=0);
```

b. Unit-ramp response of the exponential smoother:

```
yr = @(n,alpha) ((1-alpha).^(n+1)/alpha+n+(alpha-1)/alpha).*(n>=0);
```

c. The input signal is

$$x[n] = u[n] - 0.5\,r[n-3] + 0.8\,r[n-8] - 0.3\,r[n-18] - 1.5\,u[n-23]$$

which can be expressed in MATLAB and graphed as follows:

```
n = [-5:28];
x = ss_step(n)-0.5*ss_ramp(n-3)+0.8*ss_ramp(n-8)...
    -0.3*ss_ramp(n-18)-1.5*ss_step(n-23);
stem(n,x);
axis([-5.5,28.5,-2,2]);
xlabel('n');
title('x[n]');
```

d. The output of the exponential smoother is computed and graphed through the use of the following:

```
alpha = 0.1;
y = yu(n,alpha)-0.5*yr(n-3,alpha)+0.8*yr(n-8,alpha)...
    -0.3*yr(n-18,alpha)-1.5*yu(n-23,alpha);
stem(n,y);
axis([-5.5,28.5,-2,2]);
xlabel('n');
title('y[n] for \alpha=0.1');
```

Edit the first line to try for different values of α.

Chapter 4

Fourier Analysis for Continuous-Time Signals and Systems

4.1.

a. The approximation error is

$$\tilde{\varepsilon}_3(t) = \tilde{x}(t) - \tilde{x}^{(3)}(t) = \tilde{x}(t) - b_1 \sin(\omega_0 t) - b_2 \sin(2\omega_0 t) - b_3 \sin(3\omega_0 t)$$

and the cost function is

$$J = \int_0^{T_0} \left[\tilde{\varepsilon}_3(t)\right]^2 dt = \int_0^{T_0} \left[\tilde{x}(t) - b_1 \sin(\omega_0 t) - b_2 \sin(2\omega_0 t) - b_3 \sin(3\omega_0 t)\right]^2 dt$$

The fundamental frequency is $\omega_0 = 2\pi / T_0$.

b. For minimum J we need

$$\frac{dJ}{db_1} = 0 \qquad \Rightarrow \qquad \int_0^{T_0} 2\,\tilde{\varepsilon}_3(t) \left[\frac{d\tilde{\varepsilon}_3(t)}{db_1}\right] dt = 0$$

Using orthogonality, this leads to

$$b_1 = \frac{2}{T_0} \int_0^{T_0} \tilde{x}(t) \sin(\omega_0 t)\, dt = \frac{4A}{\pi}$$

Similarly, differentiating J with respect to b_2 yields

$$\frac{dJ}{db_2} = 0 \qquad \Rightarrow \qquad \int_0^{T_0} 2\,\tilde{\varepsilon}_3(t) \left[\frac{d\tilde{\varepsilon}_3(t)}{db_2}\right] dt = 0$$

and

$$b_2 = \frac{2}{T_0} \int_0^{T_0} \tilde{x}(t) \sin(2\omega_0 t)\, dt = 0$$

Finally, differentiating J with respect to b_3 we get

$$\frac{dJ}{db_3} = 0 \qquad \Rightarrow \qquad \int_0^{T_0} 2\,\tilde{\varepsilon}_3(t) \left[\frac{d\tilde{\varepsilon}_3(t)}{db_3}\right] dt = 0$$

and

$$b_3 = \frac{2}{T_0} \int_0^{T_0} \tilde{x}(t) \sin(3\omega_0 t)\, dt = \frac{4A}{3\pi}$$

4.2.

a. The fundamental period is $T_0 = 1$ second which corresponds to a fundamental frequency of $f_0 = 1$ Hz or $\omega_0 = 2\pi$ rad/s.

b. The approximation error is

$$\begin{aligned}\tilde{\varepsilon}_1(t) &= \tilde{x}(t) - \tilde{x}^{(1)}(t)\\ &= \tilde{x}(t) - a_0 - a_1\cos(2\pi t) - b_1\sin(2\pi t)\end{aligned}$$

and the cost function is

$$J = \int_0^1 \left[\tilde{\varepsilon}_1(t)\right]^2 dt = \int_0^1 \left[\tilde{x}(t) - a_0 - a_1\cos(2\pi t) - b_1\sin(2\pi t)\right]^2 dt$$

Conditions for minimizing the cost function are found by differentiating the cost function with respect to each unknown coefficient and setting the result equal to zero. The use of orthogonality properties leads to the following results:

$$\frac{dJ}{da_0} = 0 \quad\Rightarrow\quad \int_0^{T_0} 2\,\tilde{\varepsilon}_1(t)\left[\frac{d\tilde{\varepsilon}_1(t)}{da_0}\right] dt = 0 \quad\Rightarrow\quad a_0 = \int_0^1 \tilde{x}(t)\,dt = 0.3\,A$$

$$\frac{dJ}{da_1} = 0 \quad\Rightarrow\quad \int_0^{T_0} 2\,\tilde{\varepsilon}_1(t)\left[\frac{d\tilde{\varepsilon}_1(t)}{da_1}\right] dt = 0 \quad\Rightarrow\quad a_1 = 2\int_{-1}^1 \tilde{x}(t)\cos(2\pi t)\,dt = \frac{A\sin(0.6\pi)}{\pi}$$

$$\frac{dJ}{db_1} = 0 \quad\Rightarrow\quad \int_0^{T_0} 2\,\tilde{\varepsilon}_1(t)\left[\frac{d\tilde{\varepsilon}_1(t)}{db_1}\right] dt = 0 \quad\Rightarrow\quad b_1 = 2\int_0^1 \tilde{x}(t)\sin(2\pi t)\,dt = \frac{A\left[1-\cos(0.6\pi)\right]}{\pi}$$

4.3.

The approximation error is

$$\begin{aligned}\tilde{\varepsilon}_2(t) &= \tilde{x}(t) - \tilde{x}^{(2)}(t)\\ &= \tilde{x}(t) - a_0 - a_1\cos(2\pi t) - b_1\sin(2\pi t) - a_2\cos(4\pi t) - b_2\sin(4\pi t)\end{aligned}$$

and the cost function is

$$J = \int_0^1 \left[\tilde{\varepsilon}_2(t)\right]^2 dt = \int_0^1 \left[\tilde{x}(t) - a_0 - a_1\cos(2\pi t) - b_1\sin(2\pi t) - a_2\cos(4\pi t) - b_2\sin(4\pi t)\right]^2 dt$$

Conditions for minimizing the cost function are found by differentiating the cost function with respect to each unknown coefficient and setting the result equal to zero. The use of orthogonality properties leads to the following results:

$$\frac{dJ}{da_0} = 0 \quad\Rightarrow\quad \int_0^{T_0} 2\,\tilde{\varepsilon}_1(t)\left[\frac{d\tilde{\varepsilon}_1(t)}{da_0}\right] dt = 0 \quad\Rightarrow\quad a_0 = \int_0^1 \tilde{x}(t)\,dt = 0.3\,A$$

$$\frac{dJ}{da_1} = 0 \quad\Rightarrow\quad \int_0^{T_0} 2\,\tilde{\varepsilon}_1(t)\left[\frac{d\tilde{\varepsilon}_1(t)}{da_1}\right] dt = 0 \quad\Rightarrow\quad a_1 = 2\int_0^1 \tilde{x}(t)\cos(2\pi t)\,dt = \frac{A\sin(0.6\pi)}{\pi}$$

$$\frac{dJ}{db_1} = 0 \quad\Rightarrow\quad \int_0^{T_0} 2\,\tilde{\varepsilon}_1(t)\left[\frac{d\tilde{\varepsilon}_1(t)}{db_1}\right] dt = 0 \quad\Rightarrow\quad b_1 = 2\int_0^1 \tilde{x}(t)\sin(2\pi t)\,dt = \frac{A\left[1-\cos(0.6\pi)\right]}{\pi}$$

$$\frac{dJ}{da_2} = 0 \quad\Rightarrow\quad \int_0^{T_0} 2\,\tilde{\varepsilon}_1(t)\left[\frac{d\tilde{\varepsilon}_1(t)}{da_2}\right] dt = 0 \quad\Rightarrow\quad a_2 = 2\int_0^1 \tilde{x}(t)\cos(4\pi t)\,dt = \frac{A\sin(1.2\pi)}{2\pi}$$

$$\frac{dJ}{db_2} = 0 \quad\Rightarrow\quad \int_0^{T_0} 2\,\tilde{\varepsilon}_1(t)\left[\frac{d\tilde{\varepsilon}_1(t)}{db_2}\right] dt = 0 \quad\Rightarrow\quad b_2 = 2\int_0^1 \tilde{x}(t)\sin(4\pi t)\,dt = \frac{A\left[1-\cos(1.2\pi)\right]}{2\pi}$$

4.4.

The discontinuities of the signal $\tilde{x}(t)$ occur at time instants $t = nT_0/2$. The approximation using M terms can be evaluated at these time instants to yield

$$\tilde{x}^{(M)}\left(\frac{nT_0}{2}\right) = \sum_{k=1}^{M} b_k \sin\left(k\omega_0 \frac{nT_0}{2}\right)$$

Since $\omega_0 = 2\pi/T_0$ we have

$$\tilde{x}^{(M)}\left(\frac{nT_0}{2}\right) = \sum_{k=1}^{M} b_k \sin(kn\pi) = 0$$

The amplitude of the signal $\tilde{x}(t)$ is equal to $\pm A$ right before or right after a discontinuity, therefore

$$\tilde{\varepsilon}^{(M)}\left(\frac{nT_0}{2}\right) = \tilde{x}\left(\frac{nT_0}{2}\right) - \tilde{x}^{(M)}\left(\frac{nT_0}{2}\right) = \pm A$$

4.5.

a. The fundamental period is $T_0 = 2$ seconds which corresponds to a fundamental frequency of $f_0 = 1/2$ Hz or $\omega_0 = \pi$ rad/s.

b. The approximation error is

$$\begin{aligned}\tilde{\varepsilon}_2(t) &= \tilde{x}(t) - \tilde{x}^{(2)}(t)\\ &= \tilde{x}(t) - a_0 - a_1\cos(\pi t) - a_2\cos(2\pi t)\end{aligned}$$

and the cost function is

$$J = \int_{-1}^{1} \left[\tilde{\varepsilon}_2(t)\right]^2 dt = \int_{-1}^{1} \left[\tilde{x}(t) - a_0 - a_1\cos(\pi t) - a_2\cos(2\pi t)\right]^2 dt$$

Conditions for minimizing the cost function are found by differentiating the cost function with respect to each unknown coefficient and setting the result equal to zero. The use of orthogonality properties leads to the following results:

$$\begin{aligned}
\frac{dJ}{da_0} = 0 \quad &\Rightarrow \quad \int_{-1}^{1} 2\tilde{\varepsilon}_2(t)\left[\frac{d\tilde{\varepsilon}_2(t)}{da_0}\right] dt = 0 \quad \Rightarrow \quad a_0 = \frac{1}{2}\int_{-1}^{1} \tilde{x}(t)\, dt = \frac{1}{2}\\
\frac{dJ}{da_1} = 0 \quad &\Rightarrow \quad \int_{-1}^{1} 2\tilde{\varepsilon}_2(t)\left[\frac{d\tilde{\varepsilon}_2(t)}{da_1}\right] dt = 0 \quad \Rightarrow \quad a_1 = \int_{0}^{1} \tilde{x}(t)\cos(\pi t)\, dt = \frac{2}{\pi}\\
\frac{dJ}{da_2} = 0 \quad &\Rightarrow \quad \int_{-1}^{1} 2\tilde{\varepsilon}_1(t)\left[\frac{d\tilde{\varepsilon}_1(t)}{da_2}\right] dt = 0 \quad \Rightarrow \quad a_2 = \int_{-1}^{1} \tilde{x}(t)\cos(2\pi t)\, dt = 0
\end{aligned}$$

4.6.

First change summation indices in Eqn. (4.26) from k to m, then multiply both sides of it with $\sin(k\omega_0 t)$

and integrate over one full period:

$$\begin{aligned}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\, \sin(k\omega_0 t)\, dt =& \int_{t_0}^{t_0+T_0} a_0 \sin(k\omega_0 t)\, dt \\ &+ \int_{t_0}^{t_0+T_0} \left[\sum_{m=1}^{\infty} a_m \cos(m\omega_0 t)\right] \sin(k\omega_0 t)\, dt \\ &+ \int_{t_0}^{t_0+T_0} \left[\sum_{m=1}^{\infty} b_m \sin(m\omega_0 t)\right] \sin(k\omega_0 t)\, dt\end{aligned}$$

Swapping the order of integration and summation we obtain

$$\begin{aligned}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\, \sin(k\omega_0 t)\, dt =& a_0 \int_{t_0}^{t_0+T_0} \sin(k\omega_0 t)\, dt \\ &+ \sum_{m=1}^{\infty} a_m \left[\int_{t_0}^{t_0+T_0} \cos(m\omega_0 t)\sin(k\omega_0 t)\, dt\right] \\ &+ \sum_{m=1}^{\infty} b_m \left[\int_{t_0}^{t_0+T_0} \sin(m\omega_0 t)\sin(k\omega_0 t)\, dt\right]\end{aligned}$$

Let us consider the three terms on the right side of the equal sign:

1. For $k > 0$, the first term on the right side of the equal sign evaluates to zero since it includes, as a factor, the integral of a sine function over a full period.
2. In the second term we have a summation. Each term within the summation has a factor which is the integral of the product of a sine function and a cosine function over a span of T_0. Because of the orthogonality of basis functions, all terms of this summation disappear.
3. In the third term we have another summation. Each term within the summation has a factor which is the integral of the product of two sine functions over a span of T_0. All terms of this summation disappear with the exception of one term for which $m = k$.

Therefore, the last result simplifies to

$$\int_{t_0}^{t_0+T_0} \tilde{x}(t)\, \sin(k\omega_0 t)\, dt = b_k \int_{t_0}^{t_0+T_0} \sin^2(k\omega_0 t)\, dt = b_k\, \frac{T_0}{2}$$

It follows that

$$b_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t)\, \sin(k\omega_0 t)\, dt\,, \quad \text{for} \quad k = 1, \ldots, \infty$$

4.7.

The fundamental period is $T_0 = 3$ seconds which corresponds to a fundamental frequency of $f_0 = 1/3$ Hz or $\omega_0 = 2\pi/3$ rad/s.

$$a_0 = \frac{1}{3} \int_0^3 \tilde{x}(t)\, dt = \frac{1}{3} \left[\int_0^1 (2)\, dt \int_1^2 (1)\, dt\right] = 1$$

$$\begin{aligned}
a_k &= \frac{2}{3}\int_0^3 \tilde{x}(t)\cos\left(\frac{2\pi k t}{3}\right) dt \\
&= \frac{2}{3}\left[\int_0^1 (2)\cos\left(\frac{2\pi k t}{3}\right) dt + \int_1^2 (1)\cos\left(\frac{2\pi k t}{3}\right) dt\right] \\
&= \frac{1}{\pi k}\left[\sin\left(\frac{2\pi k}{3}\right) + \sin\left(\frac{4\pi k}{3}\right)\right] = 0
\end{aligned}$$

$$\begin{aligned}
b_k &= \frac{2}{3}\int_0^3 \tilde{x}(t)\sin\left(\frac{2\pi k t}{3}\right) dt \\
&= \frac{2}{3}\left[\int_0^1 (2)\sin\left(\frac{2\pi k t}{3}\right) dt + \int_1^2 (1)\sin\left(\frac{2\pi k t}{3}\right) dt\right] \\
&= \frac{1}{\pi k}\left[2 - \cos\left(\frac{2\pi k}{3}\right) - \cos\left(\frac{4\pi k}{3}\right)\right]
\end{aligned}$$

4.8.

The fundamental period is $T_0 = 1$ second which corresponds to a fundamental frequency of $f_0 = 1$ Hz or $\omega_0 = 2\pi$ rad/s.

$$a_0 = \int_0^1 e^{-2t}\, dt = 0.4323$$

The coefficients a_k are found through

$$a_k = 2\int_0^1 e^{-2t}\cos(2\pi k t)\, dt$$

Using the entry (B.22) of the integral table in Appendix B we obtain

$$a_k = \frac{3.4587}{4 + 4\pi^2 k^2}$$

Similarly, the coefficients b_k are found through

$$b_k = 2\int_0^1 e^{-2t}\sin(2\pi k t)\, dt$$

Using the entry (B.23) of the integral table in Appendix B we obtain

$$b_k = \frac{3.4587\pi k}{4 + 4\pi^2 k^2}$$

4.9.

a. Using Eqns. (4.55), (4.60) and (4.61) we obtain

$$c_0 = a_0 = 1$$

$$c_k = \frac{1}{2}\left(a_k - j\,b_k\right) = -j\,\frac{1}{2\pi k}\left[2-\cos\left(\frac{2\pi k}{3}\right)-\cos\left(\frac{4\pi k}{3}\right)\right]\,, \qquad k=1,\ldots,\infty$$

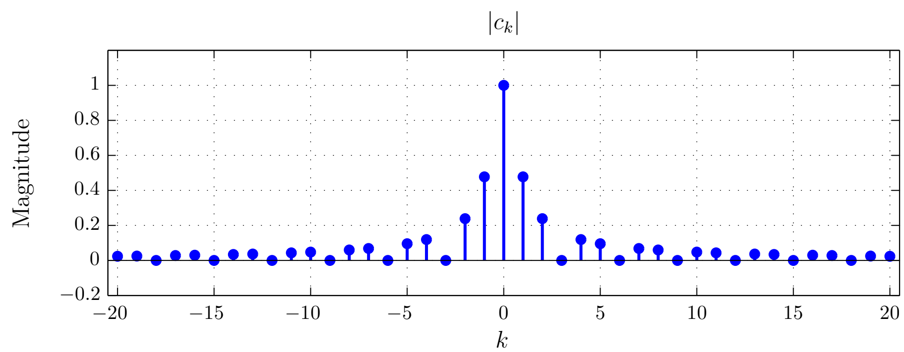
$$c_{-k} = \frac{1}{2}\left(a_k + j\,b_k\right) = j\,\frac{1}{2\pi k}\left[2-\cos\left(\frac{2\pi k}{3}\right)-\cos\left(\frac{4\pi k}{3}\right)\right]\,, \qquad k=1,\ldots,\infty$$

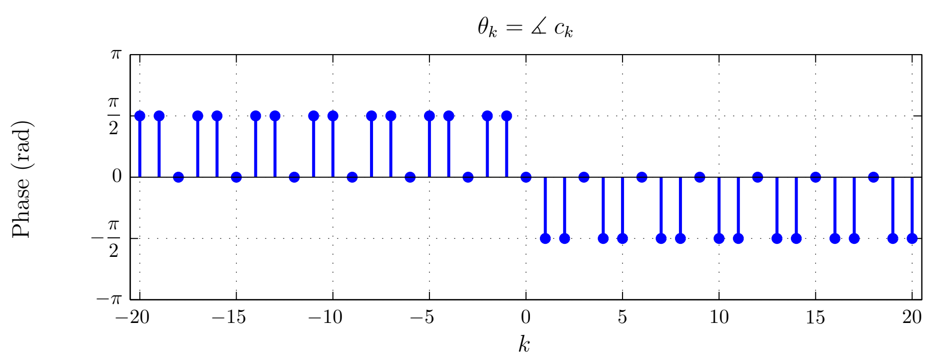
b.

$$\begin{aligned} c_k &= \frac{1}{3}\int_0^3 \tilde{x}(t)\, e^{-j2\pi k t/3}\,dt \\ &= \frac{1}{3}\left[\int_0^1 (2)\, e^{-j2\pi k t/3}\,dt + \int_1^2 (1)\, e^{-j2\pi k t/3}\,dt\right] \\ &= -j\,\frac{1}{2\pi k}\left[2-\cos\left(\frac{2\pi k}{3}\right)-\cos\left(\frac{4\pi k}{3}\right)\right]\,, \qquad \text{all } k \end{aligned}$$

For $k=0$ the use of L'Hospital's rule yields $c_0 = 1$.

c.





4.10.

a. The EFS coefficients are determined from the TFS coefficients as follows:

$$c_0 = a_0 = 0.4323$$

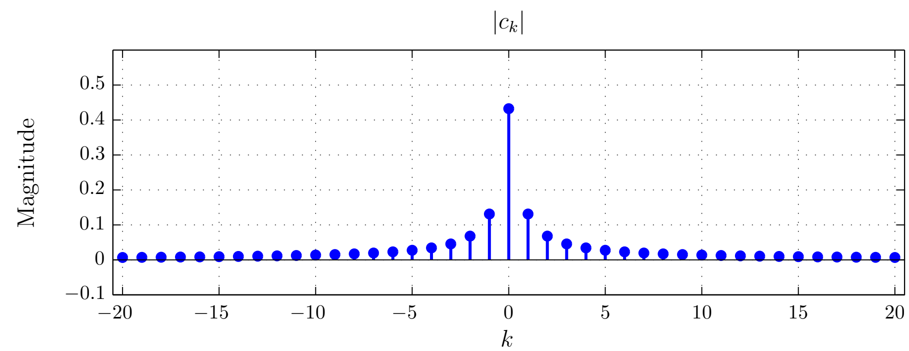
$$c_k = \frac{1}{2}\left(a_k - j\,b_k\right) = \frac{0.6647}{2 + j\,2\pi k}\,, \qquad k > 0$$

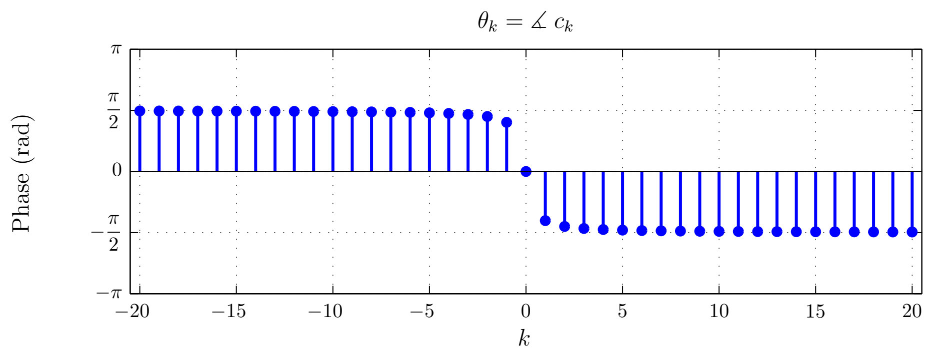
$$c_{-k} = \frac{1}{2}\left(a_k + j\,b_k\right) = \frac{0.6647}{2 - j\,2\pi k} \qquad k > 0$$

b. The EFS coefficients can be found by direct application of the EFS integral as

$$c_k = \int_0^1 e^{-2t}\, e^{-j2\pi k t}\, dt = \frac{0.8647}{2 + j\,2\pi k}\,, \qquad \text{all } k$$

c.





4.11.

a. The fundamental period is $T_0 = 1$ second which corresponds to a fundamental frequency of $f_0 = 1$ Hz or $\omega_0 = 2\pi$ rad/s. The EFS coefficients are computed as

$$\begin{aligned} c_k &= \int_0^1 (-a t + a)\, e^{-j2\pi k t}\, dt \\ &= -a \int_0^1 t\, e^{-j2\pi k t}\, dt + a \int_0^1 e^{-j2\pi k t}\, dt \end{aligned}$$

The first integral can be evaluated using entry (B.16) in Appendix B to yield

$$-a\int_0^1 t\,e^{-j2\pi kt}\,dt = \frac{a}{4\pi^2k^2}\left[1-(1+j2\pi k)\,e^{-j2\pi k}\right]$$

The second integral yields

$$a\int_0^1 e^{-j2\pi kt}\,dt = \frac{a}{j\,2\pi k}\left[1-e^{-j2\pi k}\right]$$

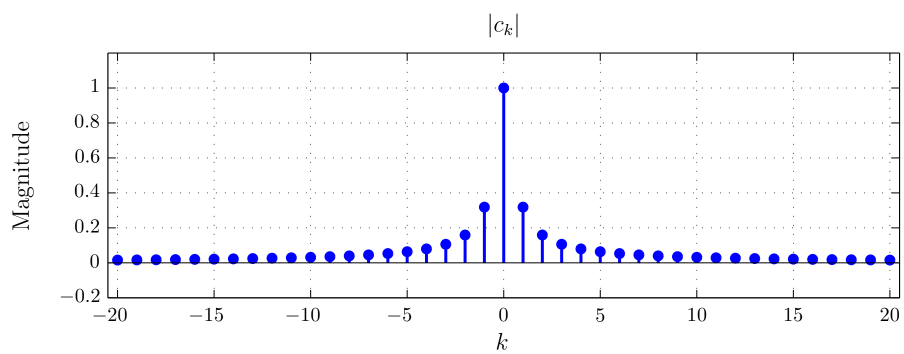
EFS coefficients are

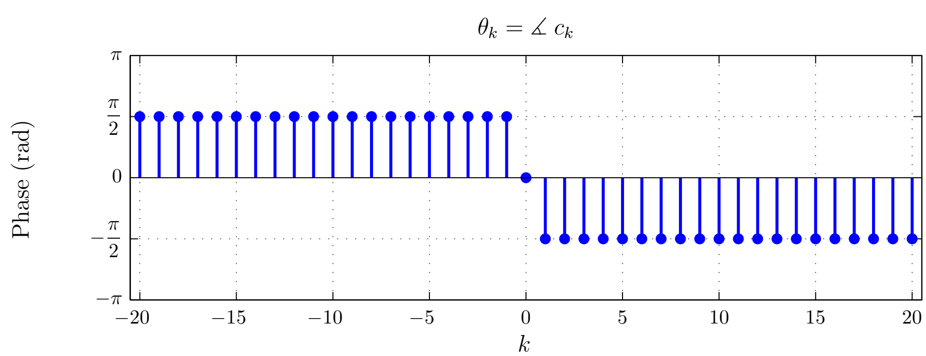
$$c_k = -j\,\frac{a}{2\pi k}\,, \qquad \text{all } k\neq 0$$

The center coefficient c_0 is found separately as

$$c_0 = \int_0^1 (-at+a)\,dt = \frac{a}{2}$$

b.





c. The TFS coefficients are computed from EFS coefficients as

$$\begin{aligned} a_0 &= c_0 \\ a_k &= c_k + c_{-k}\,, &\qquad k=1,\ldots,\infty \\ b_k &= j\,(c_k - c_{-k})\,, &\qquad k=1,\ldots,\infty \end{aligned}$$

Thus

$$a_0 = \frac{a}{2}$$

$$a_k = -j\,\frac{a}{2\pi\,k} - j\,\frac{a}{2\pi\,(-k)} = 0\,, \qquad k = 1,\ldots,\infty$$

$$b_k = j\left(-j\,\frac{a}{2\pi\,k} + j\,\frac{a}{2\pi\,(-k)}\right) = \frac{a}{\pi\,k}\,, \qquad k = 1,\ldots,\infty$$

4.12.

a. The signals $\tilde{x}(t)$ and $\tilde{g}(t)$ can be written in terms of their EFS coefficients as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{j2\pi k f_0 t}$$

and

$$\tilde{g}(t) \sum_{k=-\infty}^{\infty} d_k\, e^{j2\pi k f_0 t}$$

Using the relationship $\tilde{g}(t) = -\tilde{x}(t)$ we have

$$\sum_{k=-\infty}^{\infty} d_k\, e^{j2\pi k f_0 t} = \sum_{k=-\infty}^{\infty} c_k\, e^{-j2\pi k f_0 t}$$

which implies that

$$d_0 = c_0 \qquad \text{and} \qquad d_k = c_{-k}$$

b. For the signal $\tilde{x}(t)$ used in Example 4-8 the EFS coefficients were

$$c_0 = \frac{a}{2} \qquad \text{and} \qquad c_k = j\,\frac{a}{2\pi k} \quad \text{for } k \neq 0$$

The EFS coefficients for $\tilde{g}(t)$ are

$$d_0 = \frac{a}{2} \qquad \text{and} \qquad d_k = -j\,\frac{a}{2\pi k} \quad \text{for } k \neq 0$$

4.13.

Let $\tilde{x}(t)$ and $\tilde{g}(t)$ have the EFS coefficients c_k and d_k respectively, so that

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{j2\pi k f_0 t}$$

and

$$\tilde{g}(t) = \sum_{k=-\infty}^{\infty} d_k\, e^{j2\pi k f_0 t}$$

Adding the two signals yields

$$\tilde{x}(t) + \tilde{g}(t) = \sum_{k=-\infty}^{\infty} (c_k + d_k)\, e^{j2\pi k f_0 t} = a$$

Setting the coefficients of both sides equal to each other we get

$$\begin{aligned} c_0 + d_0 = a \qquad &\Rightarrow \qquad d_0 = a - c_0 \\ c_k + d_k = 0 \qquad &\Rightarrow \qquad d_k = -c_k\,, \quad k \neq 0 \end{aligned}$$

From Example 4-8

$$c_k = \begin{cases} \dfrac{a}{2}\,, & k = 0 \\ j\,\dfrac{a}{2\pi k}\,, & k \neq 0 \end{cases}$$

EFS coefficients d_k for the signal $\tilde{g}(t)$ are found as

$$d_k = \begin{cases} 1 - \dfrac{a}{2}\,, & k = 0 \\ -j\,\dfrac{a}{2\pi k}\,, & k \neq 0 \end{cases}$$

4.14.

The EFS coefficients are found as

$$\begin{aligned} c_k &= \frac{1}{T_0} \int_0^{T_0} \left|\sin(2\pi t/T_0)\right| e^{-j2\pi k t/T_0}\,dt \\ &= \frac{1}{T_0} \int_0^{T_0/2} \sin(2\pi t/T_0)\, e^{-j2\pi k t/T_0}\,dt - \frac{1}{T_0} \int_{T_0/2}^{T_0} \sin(2\pi t/T_0)\, e^{-j2\pi k t/T_0}\,dt \end{aligned}$$

The integrals can be evaluated using entry (B.23) of Appendix B. The result is

$$c_k = \begin{cases} \dfrac{2}{\pi\left(1 - k^2\right)}\,, & k \text{ even} \\ 0\,, & k \text{ odd} \end{cases}$$

4.15.

Let a periodic signal $\tilde{x}(t)$ be written in terms of its EFS coefficients as

$$\tilde{x}(t) = \sum_{-\infty}^{\infty} c_k\, e^{jk\omega_0 t}$$

Time shifting the signal by τ results in

$$\tilde{x}(t-\tau) = \sum_{-\infty}^{\infty} c_k\, e^{jk\omega_0 (t-\tau)} = \sum_{-\infty}^{\infty} c_k\, e^{-jk\omega_0 \tau}\, e^{jk\omega_0 t} = \sum_{-\infty}^{\infty} d_k\, e^{jk\omega_0 t}$$

It therefore follows that

$$d_k = c_k\, e^{-jk\omega_0 \tau}$$

4.16.

Let d_k be the coefficients of the half-wave sinusoidal signal $\tilde{g}(t)$ so that

$$\tilde{g}(t) = \sum_{-\infty}^{\infty} d_k \, e^{j2\pi k f_0 t}$$

It was determined in Example 4-10 that the EFS coefficients for $\tilde{g}(t)$ are

$$d_k = \begin{cases} 0\,, & k \text{ odd and } k \neq \mp 1 \\ -j/4\,, & k = 1 \\ j/4\,, & k = -1 \\ \dfrac{-1}{\pi\left(k^2-1\right)}\,, & k \text{ even} \end{cases}$$

The full-wave sinusoidal signal $\tilde{x}(t)$ can be written in terms of the half-wave rectified sinusoidal signal $\tilde{g}(t)$ as

$$\tilde{x}(t) = \tilde{g}(t) + \tilde{g}(t - T_0/2)$$

In terms of the EFS coefficients, the corresponding relationship is

$$\begin{aligned} c_k &= d_k + e^{-j2\pi k f_0 (T_0/2)} \, d_k \\ &= d_k + (-1)^k \, d_k \end{aligned}$$

which results in

$$c_k = \begin{cases} \dfrac{2}{\pi\left(1-k^2\right)}\,, & k \text{ even} \\ 0\,, & k \text{ odd} \end{cases}$$

4.17.

For the half-wave rectified sinusoidal signal $\tilde{x}(t)$ the EFS coefficients were found as

$$c_k = \begin{cases} 0\,, & k \text{ odd and } k \neq \mp 1 \\ -j/4\,, & k = 1 \\ j/4\,, & k = -1 \\ \dfrac{-1}{\pi\left(k^2-1\right)}\,, & k \text{ even} \end{cases}$$

The CFS representation of the signal is

$$\tilde{x}(t) = d_0 + \sum_{k=1}^{\infty} d_k \cos\left(2\pi k f_0 t + \phi_k\right)$$

with

$$d_k = 2\left|c_k\right| \qquad \text{and} \qquad \phi_k = \tan^{-1}\left(\frac{\mathrm{Im}\left\{c_k\right\}}{\mathrm{Re}\left\{c_k\right\}}\right) = \measuredangle\, c_k$$

$$d_k = \begin{cases} 0\,, & k \text{ odd and } k \neq 1 \\ 1/2\,, & k = 1 \\ \dfrac{2}{\pi\left(k^2-1\right)}\,, & k \text{ even} \end{cases} \qquad \phi_k = \begin{cases} 0\,, & k \text{ odd and } k \neq 1 \\ -\pi/2\,, & k = 1 \\ \pi\,, & k \text{ even} \end{cases}$$

4.18.

a.

$$X(f) = 3\,\text{sinc}(f)$$

b.

$$X(f) = 3\,\text{sinc}(f)\,e^{-j\pi f}$$

c.

$$X(f) = 8\,\text{sinc}(4f)$$

d.

$$X(f) = 4\,\text{sinc}(2f)\,e^{-j6\pi f}$$

4.19.

Using the Fourier transform integral we have

$$\begin{aligned}
X(f) &= \int_0^1 e^{-j2\pi f t}\,dt - \int_1^2 e^{-j2\pi f t}\,dt \\
&= \left.\frac{e^{-j2\pi f t}}{-j2\pi f}\right|_0^1 - \left.\frac{e^{-j2\pi f t}}{-j2\pi f}\right|_1^2 \\
&= \frac{1}{-j2\pi f}\left[e^{-j2\pi f} - 1 - e^{-j4\pi f} + e^{-j2\pi f}\right]
\end{aligned}$$

This result can be written as

$$\begin{aligned}
X(f) &= \frac{1}{-j2\pi f}\left[e^{-j2\pi f} - e^{-j4\pi f}\right] + \frac{1}{-j2\pi f}\left[-1 + e^{-j2\pi f}\right] \\
&= \frac{e^{-j3\pi f}}{-j2\pi f}\left[e^{j\pi f} - e^{-j\pi f}\right] + \frac{e^{-j\pi f}}{-j2\pi f}\left[-e^{j\pi f} + e^{-j\pi f}\right] \\
&= \text{sinc}(f)\left[e^{-j\pi f} - e^{-j3\pi f}\right]
\end{aligned}$$

4.20.

Using linearity of the Fourier transform we have

$$X(f) = \mathscr{F}\left\{e^{-at}\,u(t) - e^{at}\,u(-t)\right\} = \mathscr{F}\left\{e^{-at}\,u(t)\right\} - \mathscr{F}\left\{e^{at}\,u(-t)\right\}$$

Since

$$\mathscr{F}\left\{e^{-at}\,u(t)\right\} = \int_{-\infty}^{\infty} e^{-at}\,u(t)\,e^{-j2\pi f t}\,dt = \int_0^{\infty} e^{-at}\,e^{-j2\pi f t}\,dt = \frac{1}{a + j2\pi f}$$

and

$$\mathscr{F}\left\{e^{at}\,u(-t)\right\} = \int_{-\infty}^{\infty} e^{at}\,u(-t)\,e^{-j2\pi f t}\,dt = \int_{-\infty}^{0} e^{at}\,e^{-j2\pi f t}\,dt = \frac{1}{a - j2\pi f}$$

we obtain

$$X\left(f\right)=\frac{1}{a+j2\pi f}-\frac{1}{a-j2\pi f}=\frac{-j4\pi f}{a^2+4\pi^2 f^2}$$

4.21.

Using the unit-pulse function $\Pi\left(t\right)$ we have

$$\mathscr{F}\left\{\Pi\left(t-0.5\right)\right\}=\text{sinc}\left(f\right)\,e^{-j\pi f}$$

and

$$\mathscr{F}\left\{\Pi\left(t-1.5\right)\right\}=\text{sinc}\left(f\right)\,e^{-j3\pi f}$$

Utilizing linearity of the Fourier transform

$$\mathscr{F}\left\{\Pi\left(t-0.5\right)-\Pi\left(t-1.5\right)\right\}=\text{sinc}\left(f\right)\,\left[e^{-j\pi f}-e^{-j3\pi f}\right]$$

4.22.

Let the signal $g\left(t\right)$ be defined as

$$g\left(t\right)=e^{-at}\,u\left(t\right)$$

Its Fourier transform is

$$G\left(f\right)=\frac{1}{a+j2\pi f}$$

The signal $x\left(t\right)$ can be written as

$$x\left(t\right)=g\left(t\right)-g\left(-t\right)$$

Since

$$\mathscr{F}\left\{g\left(-t\right)\right\}=G\left(-f\right)=\frac{1}{a-j2\pi f}$$

$X\left(f\right)$ is found as

$$X\left(f\right)=G\left(f\right)+G\left(-f\right)=\frac{1}{a+j2\pi f}-\frac{1}{a-j2\pi f}=\frac{-j4\pi f}{a^2+4\pi^2 f^2}$$

4.23.

The signal $x\left(t\right)$ can be written as

$$x\left(t\right)=\Lambda\left(t-0.5\right)-\Lambda\left(t-1.5\right)$$

Using the time shifting property of the Fourier transform we obtain

$$\mathscr{F}\left\{\Lambda\left(t-0.5\right)\right\}=\text{sinc}^2\left(\frac{\omega}{2\pi}\right)\,e^{-j0.5\omega}$$

and

$$\mathscr{F}\left\{\Lambda\left(t-1.5\right)\right\}=\text{sinc}^2\left(\frac{\omega}{2\pi}\right)\,e^{-j1.5\omega}$$

Thus the transform of $x\left(t\right)$ is

$$X\left(\omega\right)=\text{sinc}^2\left(\frac{\omega}{2\pi}\right)\,\left[e^{-j0.5\omega}-e^{-j1.5\omega}\right]$$

4.24.

Using the duality property we have

$$X(t) \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$$

or equivalently

$$\frac{2a}{a^2+t^2} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\, e^{-a|-\omega|}$$

Multiplying both the numerator and the denominator of the time-domain signal by 4 yields

$$\frac{8a}{4\,a^2+4\,t^2} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\, e^{-a|\omega|}$$

Let us choose

$$4\,a^2 = 1 \qquad \Rightarrow \qquad a = \frac{1}{2}$$

so that

$$\frac{4}{1+4\,t^2} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\, e^{-|\omega|/2}$$

Scaling both sides of the transform relationship by $1/2$ we obtain the desired result:

$$\frac{2}{1+4\,t^2} \overset{\mathscr{F}}{\longleftrightarrow} \pi\, e^{-|\omega|/2}$$

4.25.

Starting with the Fourier transform integral

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and conjugating both sides we get

$$X^*(\omega) = \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right]^* = \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

Since $x(t)$ is real-valued we have $x^*(t) = x(t)$, and

$$X^*(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt = X(-\omega)$$

Using the variable change $\lambda = -t$ and $d\lambda = -dt$, the transform $X(-\omega)$ can be written as

$$X(-\omega) = \int_{\infty}^{-\infty} x(-\lambda)\, e^{-j\omega\lambda}\, (-d\lambda) = \int_{-\infty}^{\infty} x(-\lambda)\, e^{-j\omega\lambda}\, d\lambda$$

Since $x(t)$ is odd, we have $x(-\lambda) = -x(\lambda)$, and

$$X(-\omega) = -\int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, d\lambda = -X(\omega)$$

Therefore

$$X^*(\omega) = -X(\omega)$$

Since conjugating the transform causes it to be negated, the transform must be purely imaginary, that is

$$\mathrm{Re}\left\{X(\omega)\right\}=0$$

4.26.

a. Starting with

$$\Pi(t) \overset{\mathscr{F}}{\longleftrightarrow} \mathrm{sinc}(f)$$

we obtain

$$\Pi\left(\frac{t}{2}\right) \overset{\mathscr{F}}{\longleftrightarrow} 2\,\mathrm{sinc}(2f)$$

and

$$\Pi\left(\frac{t-1}{2}\right) \overset{\mathscr{F}}{\longleftrightarrow} 2\,\mathrm{sinc}(2f)\,e^{-j2\pi f}$$

Therefore, the transform $X(f)$ is

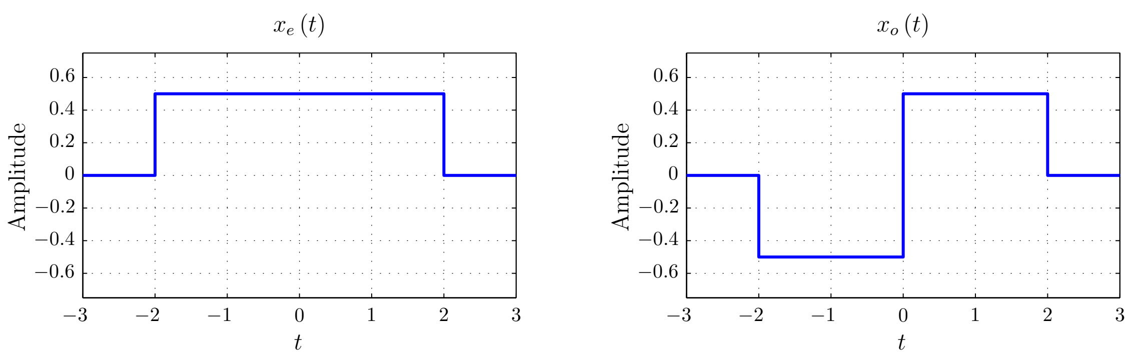
$$X(f)=2\,\mathrm{sinc}(2f)\,e^{-j2\pi f}$$

b. The even component of $x(t)$ is

$$x_e(t)=\frac{1}{2}\Pi\left(\frac{t-1}{2}\right)+\frac{1}{2}\Pi\left(\frac{-t-1}{2}\right)$$

and its odd component is

$$x_o(t)=\frac{1}{2}\Pi\left(\frac{t-1}{2}\right)-\frac{1}{2}\Pi\left(\frac{-t-1}{2}\right)$$



c. The transform of the even component is

$$X_e(f)=\mathscr{F}\left\{\frac{1}{2}\Pi\left(\frac{t}{4}\right)\right\}=2\,\mathrm{sinc}(4f)$$

and the transform of the odd component is

$$\begin{aligned}X_o(f)&=\mathscr{F}\left\{-\frac{1}{2}\Pi\left(\frac{t+1}{2}\right)+\frac{1}{2}\Pi\left(\frac{t-1}{2}\right)\right\}\\&=-\mathrm{sinc}(2f)\,e^{j2\pi f}+\mathrm{sinc}(2f)\,e^{-j2\pi f}\\&=-j\,2\,\mathrm{sinc}(2f)\,\sin(2\pi f)\end{aligned}$$

Using Euler's formula on the transform $X(f)$ we can write

$$\begin{aligned}\mathrm{Re}\{X(f)\} &= 2\,\mathrm{sinc}(2f)\,\cos(2\pi f)\\ &= 2\left[\frac{\sin(2\pi f)}{2\pi f}\right]\cos(2\pi f)\end{aligned}$$

Using the trigonometric identity $2\,sin(\alpha)\cos(\alpha)=\sin(2\alpha)$ we get

$$\mathrm{Re}\{X(f)\} = 2\,\mathrm{sinc}(4f)$$

Similarly, the imaginary part of $X(f)$ is

$$\mathrm{Im}\{X(f)\} = -2\,\mathrm{sinc}(2f)\,\sin(2\pi f)$$

We have

$$X_e(f) = \mathrm{Re}\{X(f)\} \qquad \text{and} \qquad X_o(f) = j\,\mathrm{Im}\{X(f)\}$$

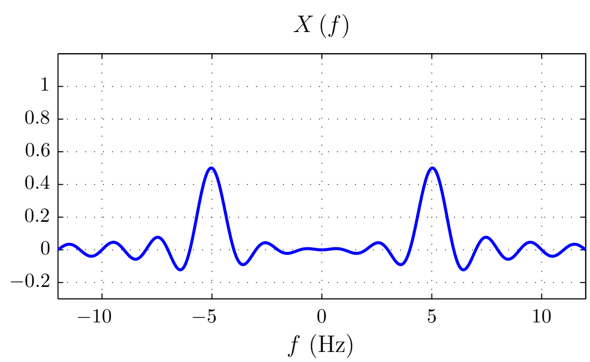
4.27.

a. Starting with

$$\mathscr{F}\{\Pi(t)\} = \mathrm{sinc}(f)$$

and using the modulation property, we obtain

$$X(f) = \frac{1}{2}\,\mathrm{sinc}(f+5) + \frac{1}{2}\,\mathrm{sinc}(f-5)$$

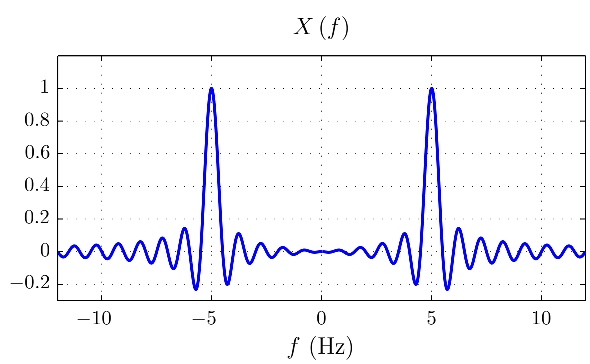


b. Since

$$\mathscr{F}\left\{\Pi\left(\frac{t}{2}\right)\right\} = 2\,\mathrm{sinc}(2f)$$

the use of the modulation property yields

$$X(f) = \mathrm{sinc}(2f+10) + \mathrm{sinc}(2f-10)$$

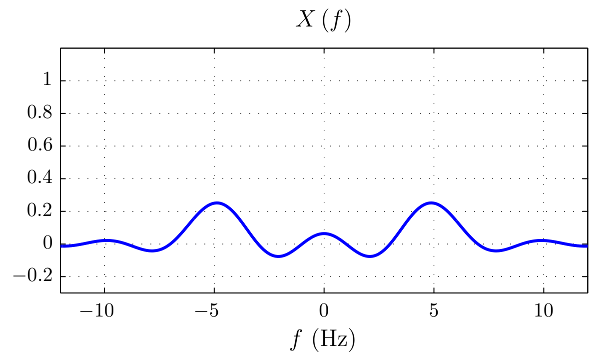


c. Since

$$\mathscr{F}\left\{\Pi\left(2t\right)\right\} = \frac{1}{2}\,\text{sinc}\left(\frac{f}{2}\right)$$

the use of the modulation property yields

$$X\left(f\right) = \frac{1}{4}\,\text{sinc}\left(\frac{f+5}{2}\right) + \frac{1}{4}\,\text{sinc}\left(\frac{f-5}{2}\right)$$

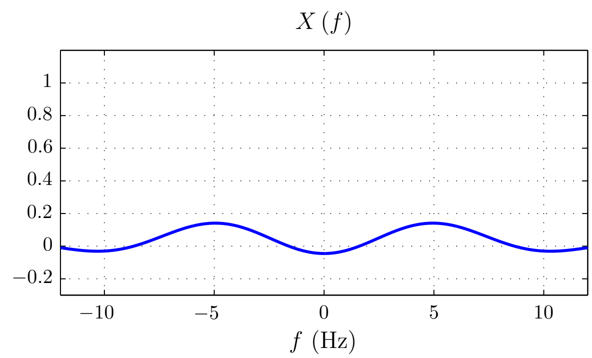


d. Since

$$\mathscr{F}\left\{\Pi\left(4t\right)\right\} = \frac{1}{4}\,\text{sinc}\left(\frac{f}{4}\right)$$

the use of the modulation property yields

$$X\left(f\right) = \frac{1}{8}\,\text{sinc}\left(\frac{f+5}{4}\right) + \frac{1}{8}\,\text{sinc}\left(\frac{f-5}{4}\right)$$



4.28.

a. Starting with the inverse Fourier transform integral

$$x\left(t\right) = \int_{-\infty}^{\infty} X\left(f\right)\,e^{j2\pi f t}\,df$$

and differentiating both sides with respect to t yields

$$\begin{aligned}\frac{dx\left(t\right)}{dt} &= \frac{d}{dt}\left[\int_{-\infty}^{\infty} X\left(f\right)\,e^{j2\pi f t}\,df\right]\\ &= \int_{-\infty}^{\infty} \frac{d}{dt}\left[X\left(f\right)\,e^{j2\pi f t}\right]\,df\\ &= \int_{-\infty}^{\infty} j2\pi f\,X\left(f\right)\,e^{j2\pi f t}\,df\end{aligned}$$

Therefore

$$\frac{dx\left(t\right)}{dt} = \mathscr{F}^{-1}\left\{j2\pi f\,X\left(f\right)\right\} \qquad \Rightarrow \qquad \mathscr{F}\left\{\frac{dx\left(t\right)}{dt}\right\} = j2\pi f\,X\left(f\right)$$

b. Given that

$$\mathscr{F}\left\{\frac{d^k x\left(t\right)}{dt^k}\right\} = \left(j2\pi f\right)^k X\left(f\right)$$

the use of the result in part (a) yields

$$\mathscr{F}\left\{\frac{d^{k+1}x(t)}{dt^{k+1}}\right\}=\mathscr{F}\left\{\frac{d}{dt}\left[\frac{d^{k}x(t)}{dt^{k}}\right]\right\}=j2\pi f\,\left(j2\pi f\right)^{k}\,X\left(f\right)=\left(j2\pi f\right)^{k+1}\,X\left(f\right)$$

4.29.

a. Starting with the Fourier transform integral

$$X\left(f\right)=\int_{-\infty}^{\infty}x(t)\,e^{-j2\pi ft}\,dt$$

and differentiating both sides with respect to f yields

$$\begin{aligned}\frac{dX\left(f\right)}{df}&=\frac{d}{df}\left[\int_{-\infty}^{\infty}x(t)\,e^{-j2\pi ft}\,dt\right]\\&=\int_{-\infty}^{\infty}\frac{d}{df}\left[x(t)\,e^{-j2\pi ft}\right]\,dt\\&=\int_{-\infty}^{\infty}\left(-j2\pi t\right)\,x(t)\,e^{-j2\pi ft}\,dt\end{aligned}$$

Therefore

$$\mathscr{F}\left\{\left(-j2\pi t\right)\,x(t)\right\}=\frac{dX\left(f\right)}{df}$$

b. Given that

$$\mathscr{F}\left\{\left(-j2\pi t\right)^{k}\,x(t)\right\}=\frac{d^{k}X\left(f\right)}{df^{k}}$$

the use of the result in part (a) yields

$$\mathscr{F}\left\{\left(-j2\pi t\right)\left(-j2\pi t\right)^{k}\,x(t)\right\}=\frac{d}{df}\left[\frac{d^{k}X\left(f\right)}{df^{k}}\right]$$

and therefore

$$\mathscr{F}\left\{\left(-j2\pi t\right)^{k+1}\,x(t)\right\}=\frac{d^{k+1}X\left(f\right)}{df^{k+1}}$$

4.30.

The differentiation in time property states that

$$x(t)\overset{\mathscr{F}}{\longleftrightarrow}X\left(f\right)\qquad\text{implies}\qquad\frac{d^{n}x(t)}{dt^{n}}\overset{\mathscr{F}}{\longleftrightarrow}\left(j2\pi f\right)^{n}\,X\left(f\right)$$

Applying the duality property of the Fourier transform to both transform pairs leads to the statement that

$$X(t)\overset{\mathscr{F}}{\longleftrightarrow}x\left(-f\right)\qquad\text{implies}\qquad\left(j2\pi t\right)^{n}\,X(t)\overset{\mathscr{F}}{\longleftrightarrow}\frac{d^{n}}{d\left(-f\right)^{n}}\left[x\left(-f\right)\right]$$

Defining two new functions as $\bar{x}(t) = X(t)$ and $\bar{X}(f) = x(-f)$, the statement above becomes

$$\bar{x}(t) \overset{\mathscr{F}}{\longleftrightarrow} \bar{X}(f) \qquad \text{implies} \qquad (j2\pi t)^n\, \bar{x}(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{d^n}{d(-f)^n}\left[\bar{X}(f)\right]$$

Multiplying both sides of the second transform pair by $(-1)^n$ we get

$$\bar{x}(t) \overset{\mathscr{F}}{\longleftrightarrow} \bar{X}(f) \qquad \text{implies} \qquad (-j2\pi t)^n\, \bar{x}(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{d^n}{df^n}\left[\bar{X}(f)\right]$$

4.31.

Let $w(t)$ be the derivative of the signal $x(t)$, that is

$$w(t) = \frac{dx(t)}{dt} = \Pi(t-0.5) - 0.5\,\Pi\left(\frac{t-4}{2}\right)$$

The transform $W(f)$ is

$$W(f) = \text{sinc}(f)\, e^{-j\pi f} - \text{sinc}(2f)\, e^{-j8\pi f}$$

Using the differentiation property of the Fourier transform

$$W(f) = j2\pi f\, X(f)$$

and

$$X(f) = \frac{X(f)}{j2\pi f} = \frac{1}{j2\pi f}\left[\text{sinc}(f)\, e^{-j\pi f} - \text{sinc}(2f)\, e^{-j8\pi f}\right]$$

4.32.

Using duality we have

$$X(t) \overset{\mathscr{F}}{\longleftrightarrow} x(-f)$$

Since the signal $x(t)$ is even, we have $x(-f) = x(f)$. Let

$$\tau = f_0\,(1+r) - f_0\,(1-r) = 2f_0\,r \qquad \text{and} \qquad \lambda = f_0$$

The inverse transform we seek is

$$X(t) = 2\,f_0\,\text{sinc}(2f_0\,r\,t)\,\text{sinc}(2f_0\,t)$$

4.33.

a. The modulation property states that

$$\mathscr{F}\left\{\sin(2\pi f_0 t)\; p(t)\right\} = \frac{1}{2}P(f-f_0)\, e^{-j\pi/2} + \frac{1}{2}P(f+f_0)\, e^{j\pi/2}$$

Let the signal $p(t)$ be defined as

$$p(t) = \Pi\left(t - \frac{1}{2}\right)$$

with the corresponding transform

$$P\left(f\right)=\text{sinc}\left(f\right)\,e^{-j\pi f}$$

The use of the modulation property with $f_0=0.5$ Hz yields

$$\begin{aligned}X\left(f\right)&=\frac{1}{2}P\left(f-0.5\right)\,e^{-j\pi/2}+\frac{1}{2}P\left(f+0.5\right)\,e^{j\pi/2}\\&=\frac{1}{2}\,\text{sinc}\left(f-0.5\right)\,e^{-j\pi\,(f-0.5)}\,e^{-j\pi/2}+\frac{1}{2}\,\text{sinc}\left(f+0.5\right)\,e^{-j\pi\,(f+0.5)}\,e^{j\pi/2}\\&=\frac{1}{2}\,\left[\text{sinc}\left(f-0.5\right)+\text{sinc}\left(f+0.5\right)\right]\,e^{-j\pi f}\end{aligned}$$

b. The multiplication property states that

$$\mathscr{F}\left\{q\,(t)\;p\,(t)\right\}=Q\left(f\right)*P\left(f\right)$$

Let the signals $q\,(t)$ and $p\,(t)$ be defined as

$$q\,(t)=\sin\left(\pi t\right)\qquad\text{and}\qquad p\,(t)=\Pi\left(t-\frac{1}{2}\right)$$

The transforms $Q\left(f\right)$ and $P\left(f\right)$ are

$$Q\left(f\right)=-j\,\frac{1}{2}\,\delta\left(f-0.5\right)+j\,\frac{1}{2}\,\delta\left(f+0.5\right)\qquad\Rightarrow\qquad P\left(f\right)=\text{sinc}\left(f\right)\,e^{-j\pi f}$$

The transform $X\left(f\right)$ is

$$\begin{aligned}X\left(f\right)&=Q\left(f\right)*P\left(f\right)\\&=\frac{1}{2}\,\left[\text{sinc}\left(f-0.5\right)+\text{sinc}\left(f+0.5\right)\right]\,e^{-j\pi f}\end{aligned}$$

4.34.

EFS coefficients for the signal $\tilde{x}\,(t)$ are

$$c_k=\frac{1}{3}\,\text{sinc}\left(\frac{k}{3}\right)\,e^{-j\pi k/3}$$

The Fourier transform of $\tilde{x}\,(t)$ is

$$X\left(f\right)=\sum_{k=-\infty}^{\infty}c_k\,\delta\left(f-kf_0\right)=\frac{1}{3}\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{k}{3}\right)\,e^{-j\pi k/3}\,\delta\left(f-k/3\right)$$

4.35.

a. The power in the pulse train is

$$P_x=\langle x^2\,(t)\rangle=\frac{1}{T_0}\int_{T_0/2}^{T_0/2}x^2\,(t)\;dt\frac{1}{T_0}\int_{\tau/2}^{\tau/2}(1)\;dt$$

Since $\tau = d\,T_0$

$$P_x = \frac{1}{T_0}\int_{d\,T_0/2}^{d\,T_0/2} (1)\;dt = d$$

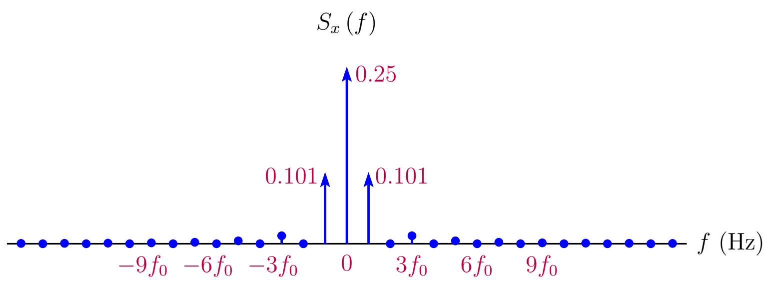
b. EFS coefficients for the pulse train are

$$c_k = d\,\text{sinc}\,(kd)\;, \qquad k = -\infty, \ldots, \infty$$

The power spectral density is

$$S_x\left(f\right) = \sum_{k=-\infty}^{\infty} |c_k|^2\,\delta\left(f - kf_0\right) = \sum_{k=-\infty}^{\infty} d^2\,\text{sinc}^2\,(kd)\,\delta\left(f - kf_0\right)$$

where the fundamental frequency is $f_0 = 1/T_0$. $S_x\left(f\right)$ is shown below for $d = 0.5$.



c. If EFS terms up to and including the M-th harmonic are retained, the normalized average power of the signal would be

$$P_x^{(M)} = \sum_{k=-M}^{M} |c_k|^2 = \sum_{k=-M}^{M} d^2\,\text{sinc}^2\,(kd)$$

and the percentage of this to the total average power in the signal $x\,(t)$ is

$$\eta = \frac{P_x^{(M)}}{P_x} = \frac{P_x^{(M)}}{d} = \sum_{k=-M}^{M} d\,\text{sinc}^2\,(kd)$$

It can be shown that, with $d = 0.5$ and $M = 1$ we get

$$\sum_{k=-1}^{1} (0.5)\,\text{sinc}^2\,(0.5\,k) = 0.9053$$

d. Frequencies up to the third harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-3}^{3} (0.5)\,\text{sinc}^2\,(0.5\,k) = 0.9503$$

e. Frequencies up to the 21-st harmonic are needed to retain 99 percent of the signal power since

$$\sum_{k=-21}^{21} (0.5)\,\text{sinc}^2\,(0.5\,k) = 0.9908$$

4.36.

c. If EFS terms up to and including the M-th harmonic are retained, the percentage of this to the total average power in the signal $x(t)$ is

$$\eta = \frac{P_x^{(M)}}{P_x} = \frac{P_x^{(M)}}{d} = \sum_{k=-M}^{M} d\,\text{sinc}^2(kd)$$

It can be shown that, with $d = 0.2$ and $M = 4$ we get

$$\sum_{k=-4}^{4} (0.2)\,\text{sinc}^2(0.2\,k) = 0.9029$$

d. Frequencies up to the 11-th harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-11}^{11} (0.2)\,\text{sinc}^2(0.2\,k) = 0.9528$$

e. Frequencies up to the 51-st harmonic are needed to retain 99 percent of the signal power since

$$\sum_{k=-51}^{51} (0.2)\,\text{sinc}^2(0.2\,k) = 0.9900$$

4.37.

Parseval's theorem states that

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt \int_{-\infty}^{\infty} |X(f)|^2\,dt$$

Let $x(t) = \Pi(t)$. The normalized average power of $x(t)$ is

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-1/2}^{1/2} (1)^2\,dt = 1$$

The Fourier transform of $x(t)$ is

$$X(f) = \mathscr{F}\{\Pi(t)\} = \text{sinc}(f)$$

Therefore

$$\int_{-\infty}^{\infty} |X(f)|^2\,dt = \int_{-\infty}^{\infty} |\text{sinc}(f)|^2\,df = 1$$

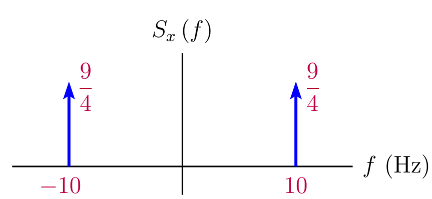
4.38.

a. For the signal $x(t)$ the fundamental frequency is $f_0 = 10$ Hz, and the EFS coefficients are

$$c_k = \begin{cases} \dfrac{3}{2}, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

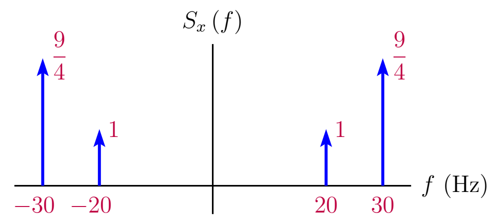
The power spectral density is

$$S_x(f) = \frac{9}{4}\,\delta(f+10) + \frac{9}{4}\,\delta(f-10)$$



b. For the signal $x(t)$ the fundamental frequency is $f_0 = 10$ Hz, and the EFS coefficients are

$$c_k = \begin{cases} 1\,, & k = \pm 2 \\ \dfrac{3}{2}\,, & k = \pm 3 \\ 0\,, & \text{otherwise} \end{cases}$$

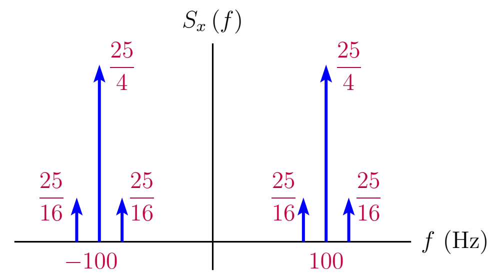


The power spectral density is

$$S_x(f) = \frac{9}{4}\,\delta(f+30) + \delta(f+20) + \delta(f-20) + \frac{9}{4}\,\delta(f-30)$$

c. For the signal $x(t)$ the fundamental frequency is $f_0 = 5$ Hz, and the EFS coefficients are

$$c_k = \begin{cases} \dfrac{5}{4}\,, & k = \pm 17 \\ \dfrac{5}{2}\,, & k = \pm 20 \\ \dfrac{5}{4}\,, & k = \pm 23 \\ 0\,, & \text{otherwise} \end{cases}$$



The power spectral density is

$$S_x(f) = \frac{25}{16}\,\delta(f+230) + \frac{25}{4}\,\delta(f+200) + \frac{25}{16}\,\delta(f+170) + \frac{25}{16}\,\delta(f-170) + \frac{25}{4}\,\delta(f-200) + \frac{25}{16}\,\delta(f-230)$$

4.39.

```
% Compute the signal x(t)
A = 1;
T0 = 1;
xp = @(t) A*(t<=T0/2)-A*(t>T0/2);  % One period
t = [-1.5:0.005:2.5];
x = xp(mod(t,T0));
% Compute the approximation
b1 = 4*A/pi;
b2 = 0;
b3 = 4*A/(3*pi);
f0 = 1/T0;
xhat = b1*sin(2*pi*f0*t)+b2*sin(4*pi*f0*t)+b3*sin(6*pi*f0*t);
% Graph the signal and the approximation
plot(t,x,t,xhat);
axis([-1.5,2.5,-1.6,2.4]);
title('The signal x(t) and its approximation');
xlabel('t (sec)');
legend('Original signal','Approximation');
grid;
```

4.40.

```
% Compute the signal x(t)
A = 1;
xp = @(t) A*(t<=0.3);  % One period
t = [-1.5:0.005:2.5];
x = xp(mod(t,1));
% Compute the approximation
a0 = 0.3*A;
a1 = A*sin(0.6*pi)/pi;
b1 = A*(1-cos(0.6*pi))/pi;
xhat = a0+a1*cos(2*pi*t)+b1*sin(2*pi*t);
% Graph the signal and the approximation
plot(t,x,t,xhat);
axis([-1.5,2.5,-0.5,1.8]);
title('The signal x(t) and its approximation');
xlabel('t (sec)');
legend('Original signal','Approximation');
grid;
```

4.41.

The code below computes and graphs the original signal, the approximation using $M = 3$ harmonics, and the approximation error.

```
1  % Compute the signal x(t)
2  xp = @(t) 2*(t<1)+1*((t>=1)&(t<=2));  % One period
3  t = [-4:0.005:6];
4  x = xp(mod(t,3));  % Periodic extension
5  % Compute the coefficients
6  k = [1:10];
7  a0 = 1;
8  a = (sin(2*pi*k/3)+sin(4*pi*k/3))./(pi*k);
9  b = (2-cos(2*pi*k/3)-cos(4*pi*k/3))./(pi*k);
10 % Approximation with M=3 harmonics
11 xhat = a0*ones(size(t));
12 for m=1:3,
13   xhat = xhat+a(m)*cos(2*pi*m*t/3)+b(m)*sin(2*pi*m*t/3);
14 end;
15 plot(t,x,t,xhat);
16 axis([-3,3,-1,3.5]);
17 xlabel('t');
18 legend('Original signal','Approx. with M=3');
19 grid;
20 % Approximation error for M=3
21 plot(t,x-xhat);
22 axis([-3,3,-1.2,1.2]);
23 xlabel('t');
24 grid;
```

To use $M = 4$ harmonics, modify line 12 as

```
12  for m=1:4,
```

To use $M = 5$ harmonics, modify it as

```
12  for m=1:5,
```

4.42. The code below computes and graphs the original signal, the approximation using $M = 3$ harmonics, and the approximation error.

```
1   % Compute the signal x(t)
2   xp = @(t) exp(-2*t);   % One period
3   t = [-1:0.005:2];
4   x = xp(mod(t,1));   % Periodic extension
5   % Compute the EFS coefficients
6   k = [1:10];
7   a0 = 0.4323;
8   a = 3.4587./(4+4*pi*pi*k.*k);
9   b = 3.4587*pi*k./(4+4*pi*pi*k.*k);
10  % Approximation with M=3 harmonics
11  xhat = a0*ones(size(t));
12  for m=1:3,
13    xhat = xhat+a(m)*cos(2*pi*m*t)+b(m)*sin(2*pi*m*t);
14  end;
15  plot(t,x,t,real(xhat));
16  axis([-1,2,-0.2,1.8]);
17  xlabel('t');
18  legend('Original signal','Approx. with M=3');
19  grid;
20  % Approximation error for M=3
21  plot(t,x-real(xhat));
22  axis([-1,2,-0.5,0.5]);
23  xlabel('t');
24  grid;
```

To use $M = 4$ harmonics, modify line 12 as

```
12  for m=1:4,
```

To use $M = 5$ harmonics, modify it as

```
12  for m=1:5,
```

4.43. The code below computes and graphs the EFS line spectrum for duty cycle $d = 0.4$.

```
1   % Compute and graph the line spectrum for d=0.4
2   k = [-20:20];
3   d = 0.4;
```

```
c = d*sinc(k*d);
stem(k,c);
axis([-20.5,20.5,-0.3,1.2]);
title('c_k');
xlabel('k');
grid;
```

To repeat with other values of the duty cycle, modify the line

```
d = 0.4;
```

As the duty cycle is increased, the line spectrum becomes more concentrated around $k = 0$.

4.44. The code below computes and graphs the original signal and its approximation using $M = 2$ harmonics.

```
1  % Compute the half-wave rectified sinusoid
2  t = [-0.25:0.005:2.25];
3  xp = @(t) sin(2*pi*t).*(t<=0.5);  % One period
4  x = xp(mod(t,1));
5  % Compute the CFS coefficients for k=0,...,10
6  k = [1:10];
7  dk = zeros(size(k));
8  d0 = 1/pi;  % d0 = c0
9  dk(1) = 0.5;
10 for m=2:2:10,
11   dk(m) = 2/(pi*(m*m-1));
12 end;
13 theta = zeros(size(k));
14 theta(1) = -pi/2;
15 for m = 2:2:10,
16   theta(m) = pi;
17 end;
18 % Compute and graph the approximation using M=2 harmonics
19 omg0 = 2*pi;
20 xhat = d0*ones(size(t));
21 for m=1:2,
22   xhat = xhat+dk(m)*cos(m*omg0*t+theta(m));
23 end;
24 plot(t,x,t,xhat,'r');
25 axis([-0.25,2.25,-0.2,1.8]);
26 xlabel('t');
27 legend('Original signal','Approx. for M=2');
28 grid;
```

To use $M = 4$ harmonics, modify line 21 as

```
21 for m=1:4,
```

To use $M = 6$ harmonics, modify it as

```
for m=1:6,
```

4.45. The code listed below defines an anonymous function and uses it to compute the transform of

$$\cos(2\pi f_0 t)\,\Pi\left(\frac{t}{\tau}\right)$$

with $\tau = 3$ seconds and $f_0 = 2$ Hz.

```
% Anonymous function to compute the transform of a pulse
P = @(f,tau) tau*sinc(f*tau);
% Compute and graph the transform of modulated pulse
tau = 3;
f0 = 2;
f = [-4:0.005:4];
X = 0.5*P(f+f0,tau)+0.5*P(f-f0,tau);
plot(f,X);
axis([-4,4,-0.6,2]);
title('X(f)');
xlabel('f (Hz)');
grid;
```

Parts (a) through (d) of Problem 4.27 can be verified using following code listings:

```
% Verify part (a) of Problem 4.27
f = [-12:0.02:12];
X1 = 0.5*sinc(f+5)+0.5*sinc(f-5);
plot(f,X1);
grid;
axis([-12,12,-0.3,1.2]);
title('X(f)');
xlabel('f (Hz)');
```

```
% Verify part (b) of Problem 4.27
X2 = sinc(2*f+10)+sinc(2*f-10);
plot(f,X2);
grid;
axis([-12,12,-0.3,1.2]);
title('X(f)');
xlabel('f (Hz)');
```

```
% Verify part (c) of Problem 4.27
X3 = 0.25*sinc((f+5)/2)+0.25*sinc((f-5)/2);
plot(f,X3);
grid;
axis([-12,12,-0.3,1.2]);
title('X(f)');
xlabel('f (Hz)');
```

```
% Verify part (d) of Problem 4.27
X4 = 0.125*sinc((f+5)/4)+0.125*sinc((f-5)/4);
plot(f,X4);
grid;
axis([-12,12,-0.3,1.2]);
title('X(f)');
xlabel('f (Hz)');
```

4.46. The script listed below computes the EFS coefficients, and then computes and graphs the signal with only the dc component and the fundamental frequency retained. Recall from Problem 4.35 that, for $d_0 = 0.5$, this corresponds to preserving 90 percent of the spectral power in the signal.

```
1  % Compute the EFS coefficients
2  d = 0.5;
3  k = [-25:25];
4  c = d*sinc(k*d);
5  offset = -k(1)+1;  % MATLAB index of center coefficient "c0"
6  % Set f0 = 1 Hz and create a time vector.
7  f0 = 1;
8  t = [-2:0.005:2];
9  % Compute the signal for M=1
10 m = 1;
11 xhat = zeros(size(t));
12 for i=-m:m,
13   xhat = xhat+c(i+offset)*exp(j*2*pi*i*f0*t);
14 end;
15 plot(t,real(xhat));
16 axis([-2,2,-0.2,1.2]);
17 grid;
```

It was found in Problem 4.35 that, in order to preserve 95 percent of the spectral power, frequencies up to and including the third harmonic are needed. The resulting signal can be computed and graphed by modifying line 10 of the script as

```
10 m = 3;
```

Preserving 99 percent of the spectral power requires frequencies up to and including the 21-st harmonic. The resulting signal can be computed and graphed by modifying line 10 of the script as

```
m = 21;
```

Chapter 5

Fourier Analysis for Discrete-Time Signals and Systems

5.1.

Angular frequency: $\Omega = 0.3\pi$

Normalized frequency: $F = \dfrac{\Omega}{2\pi} = 0.15$

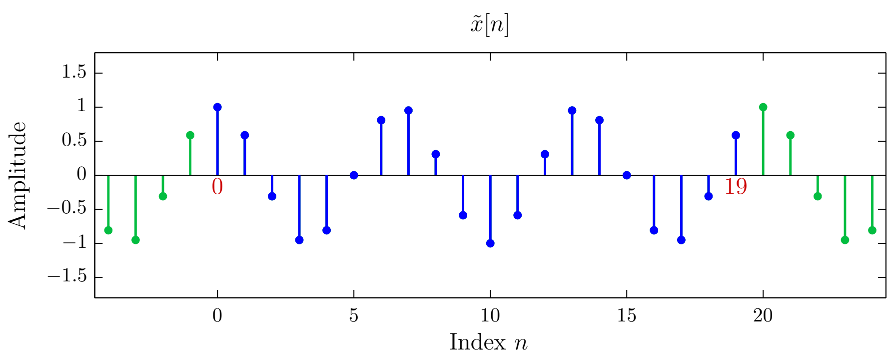
Period: $N = \dfrac{3}{0.15} = 20$ samples

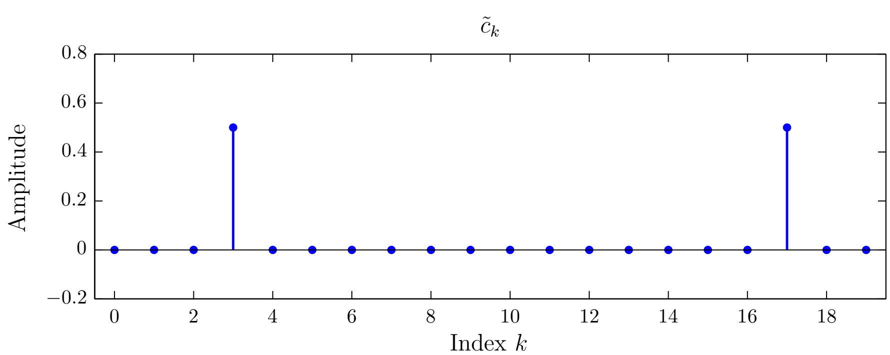
Normalized fundamental frequency: $F_0 = \dfrac{1}{N} = \dfrac{1}{20}$, $\Omega_0 = \dfrac{2\pi}{N} = \dfrac{2\pi}{20}$

$$\tilde{x}(n) = \frac{1}{2} e^{j(2\pi/20)3n} + \frac{1}{2} e^{-j(2\pi/20)3n}$$

Therefore

$$c_3 = \frac{1}{2}, \qquad c_{-3} = c_{17} = \frac{1}{2}, \qquad c_k = 0 \quad \text{for all other } k$$





5.2.

Angular frequencies of the two sinusoidal terms: $\Omega_1 = 0.24\pi \quad , \quad \Omega_2 = 0.56\pi$

Normalized frequencies: $F_1 = 12 \quad , \quad F_2 = 0.28$

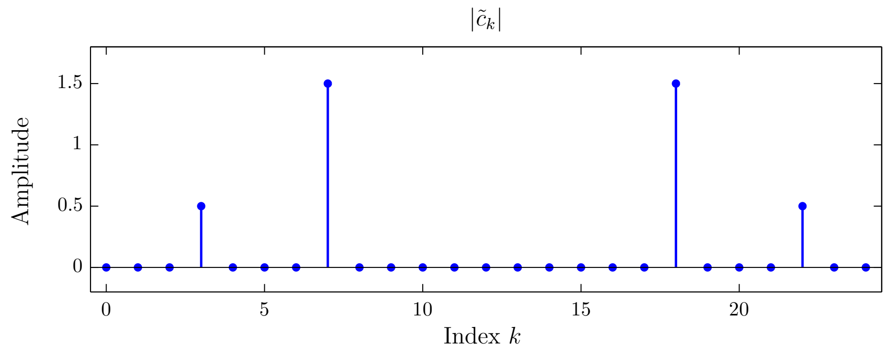
Normalized fundamental frequency: $F_0 = 0.04$

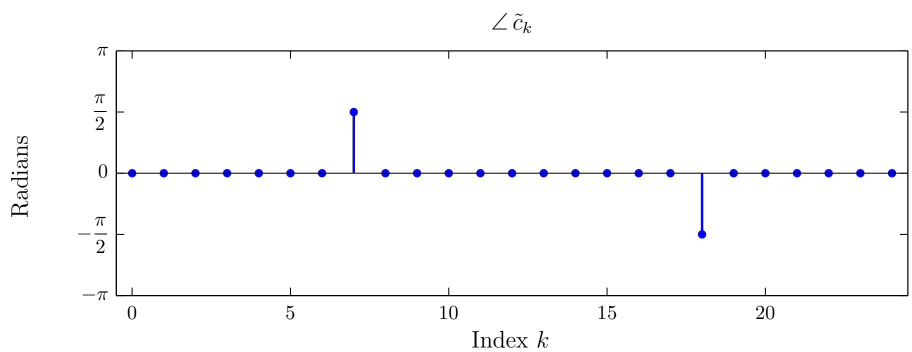
Period: $N = \dfrac{1}{0.04} = 25$ samples

$$\tilde{x}[n] = 1 + \frac{1}{2}\, e^{j(2\pi/25)\,3n} + \frac{1}{2}\, e^{-j(2\pi/25)\,3n} + \frac{3}{2j}\, e^{j(2\pi/25)\,7n} - \frac{3}{2j}\, e^{-j(2\pi/25)\,7n}$$

Therefore

$$c_3 = \frac{1}{2}\,, \qquad c_{-3} = c_{22} = \frac{1}{2}\,, \qquad c_7 = \frac{3}{2j} = -j\,\frac{3}{2}\,, \qquad c_{-7} = c_{18} = -\frac{3}{2j} = j\,\frac{3}{2}$$





5.3.

a.

$$\tilde{c}_k = 4 + 3\,e^{-j(2\pi/8)k} + 2\,e^{-j(2\pi/8)2k} + 1\,e^{-j(2\pi/8)3k} +$$
$$1\,e^{-j(2\pi/8)5k} + 2\,e^{-j(2\pi/8)6k} + 3\,e^{-j(2\pi/8)7k}\,, \qquad \text{for } k = 0,\ldots,7$$

Evaluating for $k = 0, \ldots, 7$ yields

$$\begin{aligned}
\tilde{c}_0 &= 2 \\
\tilde{c}_1 &= 0.8536 \\
\tilde{c}_2 &= 0 \\
\tilde{c}_3 &= 0.1464 \\
\tilde{c}_4 &= 0 \\
\tilde{c}_5 &= 0.1464 \\
\tilde{c}_6 &= 0 \\
\tilde{c}_7 &= 0.8536
\end{aligned}$$

b.

$$\tilde{c}_k = 1 + e^{-j(2\pi/8)k} + e^{-j(2\pi/8)2k} \qquad \text{for } k = 0, \ldots, 7$$

Evaluating for $k = 0, \ldots, 7$ yields

$$\begin{aligned}
\tilde{c}_0 &= 0.3750 \\
\tilde{c}_1 &= 0.2134 - j\,0.2134 \\
\tilde{c}_2 &= -j\,0.1250 \\
\tilde{c}_3 &= 0.0366 + j\,0.0366 \\
\tilde{c}_4 &= 0.1250 \\
\tilde{c}_5 &= 0.0366 - j\,0.0366 \\
\tilde{c}_6 &= j\,0.1250 \\
\tilde{c}_7 &= 0.2134 + j\,0.2134
\end{aligned}$$

c.

$$\tilde{c}_k = 1 + e^{-j(2\pi/8)k} + e^{-j(2\pi/8)2k} + e^{-j(2\pi/8)3k} + e^{-j(2\pi/8)4k} \qquad \text{for } k = 0, \ldots, 7$$

Evaluating for $k = 0, \ldots, 7$ yields

$$\begin{aligned}
\tilde{c}_0 &= 0.6250 \\
\tilde{c}_1 &= -j\,0.3018 \\
\tilde{c}_2 &= 0.1250 \\
\tilde{c}_3 &= -j\,0.0518 \\
\tilde{c}_4 &= 0.1250 \\
\tilde{c}_5 &= j\,0.0518 \\
\tilde{c}_6 &= 0.1250 \\
\tilde{c}_7 &= j\,0.3018
\end{aligned}$$

5.4.

a. Let

$$\tilde{x}[n] = \sum_{k=0}^{4} \tilde{c}_k \, e^{-j(2\pi/5)\,kn}$$

and

$$\tilde{g}[n] = \tilde{x}[n-1] = \sum_{k=0}^{4} \tilde{d}_k \, e^{-j(2\pi/5)\,kn}$$

Coefficients $\tilde{d}_k$ are found as

$$\begin{aligned}\tilde{d}_k &= 4 + e^{-j(2\pi/5)2k} + 2\,e^{-j(2\pi/5)3k} + 3\,e^{-j(2\pi/5)4k}\\ &= 4 + e^{-j4\pi k/5} + 2\,e^{-j6\pi k/5} + 3\,e^{-j8\pi k/5}\,, \qquad \text{for } k = 0,\ldots,4\end{aligned}$$

Evaluating for $k = 0,\ldots,4$ we get

$$\begin{aligned}\tilde{d}_0 &= 2\\ \tilde{d}_1 &= 0.5 + j\,0.6882\\ \tilde{d}_2 &= 0.5 + j\,0.1625\\ \tilde{d}_3 &= 0.5 - j\,0.1625\\ \tilde{d}_4 &= 0.5 - j\,0.6882\end{aligned}$$

b. Using the time shifting property of the DTFS, we have

$$\tilde{d}_k = e^{-j2\pi k/5}\,\tilde{c}_k$$

Table shown below illustrates the relationship between the coefficients $\tilde{c}_k$ and $\tilde{d}_k$:

k	$\tilde{c}_k$	$e^{-j2\pi k/5}$	$\tilde{d}_k$
0	$2.0000+j\,0.0000$	$1.0000+j\,0.0000$	$2.0000+j\,0.0000$
1	$-0.5000+j\,0.6882$	$0.3090-j\,0.9511$	$0.5000+j\,0.6882$
2	$-0.5000+j\,0.1625$	$-0.8090-j\,0.5878$	$0.5000+j\,0.1625$
3	$-0.5000-j\,0.1625$	$-0.8090+j\,0.5878$	$0.5000-j\,0.1625$
4	$-0.5000-j\,0.6882$	$0.3090+j\,0.9511$	$0.5000-j\,0.6882$

5.5. For each set the following relationships hold true:

$$\begin{aligned}\tilde{c}_1 &= \tilde{c}^*_{-1} = \tilde{c}^*_7\\ \tilde{c}_2 &= \tilde{c}^*_{-2} = \tilde{c}^*_6\\ \tilde{c}_3 &= \tilde{c}^*_{-3} = \tilde{c}^*_5\end{aligned}$$

Furthermore, both $\tilde{c}_0$ and $\tilde{c}_4$ are real for each set of coefficients so that

$$\tilde{c}_0 = \tilde{c}^*_0$$

and

$$\tilde{c}_4 = \tilde{c}^*_{-4} = \tilde{c}^*_4$$

5.6.

a.

$$\tilde{g}_e[n] = \frac{\tilde{g}[n] + \tilde{g}[-n]}{2}$$
$$\tilde{g}_o[n] = \frac{\tilde{g}[n] - \tilde{g}[-n]}{2}$$

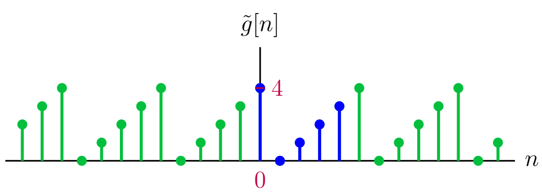
Tabular forms of the signals involved:

$$\tilde{g}[n] = \{\ldots,\ \underset{\substack{\uparrow \\ n=0}}{4},\ 0,\ 1,\ 2,\ 3,\ \ldots\}$$

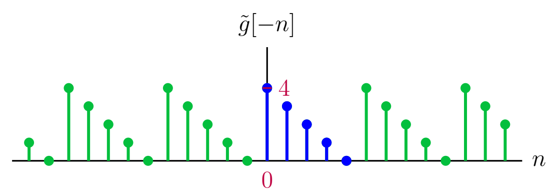
$$\tilde{g}[-n] = \{\ldots,\ \underset{\substack{\uparrow \\ n=0}}{4},\ 3,\ 2,\ 1,\ 0,\ \ldots\}$$

$$\tilde{g}_e[n] = \{\ldots,\ \underset{\substack{\uparrow \\ n=0}}{4},\ 1.5,\ 1.5,\ 1.5,\ 1.5,\ \ldots\}$$

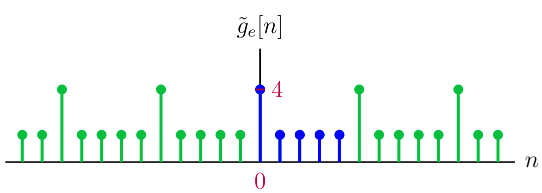
$$\tilde{g}_o[n] = \{\ldots,\ \underset{\substack{\uparrow \\ n=0}}{0},\ -1.5,\ -0.5,\ 0.5,\ 1.5,\ \ldots\}$$



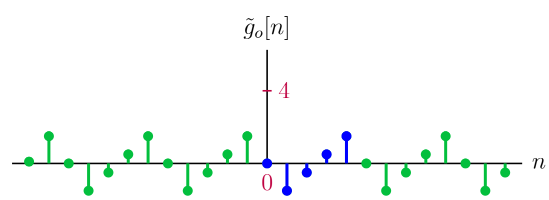
(e)



(f)



(g)



(h)

b.

Let

$$\tilde{g}_e[n] = \sum_{k=0}^{4} \tilde{c}_k\, e^{-j2\pi k/5}$$

and

$$\tilde{g}_o[n] = \sum_{k=0}^{4} \tilde{d}_k\, e^{-j2\pi k/5}$$

Using Eqn. (5.22) we obtain

$$\tilde{c}_0 = 2$$
$$\tilde{c}_1 = 0.5$$
$$\tilde{c}_2 = 0.5$$
$$\tilde{c}_3 = 0.5$$
$$\tilde{c}_4 = 0.5$$

and

$$\begin{aligned}
\tilde{d}_0 &= 0 \\
\tilde{d}_1 &= j\,0.6882 \\
\tilde{d}_2 &= j\,0.1625 \\
\tilde{d}_3 &= -j\,0.1625 \\
\tilde{d}_4 &= -j\,0.6882
\end{aligned}$$

5.7. Using the definition in Eqn. (5.47)

$$\tilde{y}[n] = \tilde{x}[n] \otimes \tilde{h}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\,\tilde{h}[n-k]\,, \quad \text{all } n$$

and evaluating $y[n+N]$ we get

$$\tilde{y}[n+N] = \sum_{k=0}^{N-1} \tilde{x}[k]\,\tilde{h}[n+N-k]\,, \quad \text{all } n$$

Since $h[n]$ is periodic with period N we have $h[n+N-k] = h[n-k]$, and

$$\tilde{y}[n+N] = \tilde{y}[n]$$

5.8. Tabular form of the periodic convolution result is

$$\tilde{y}[n] = \{\ldots,\ \underset{\substack{\uparrow \\ n=0}}{5},\ 7,\ 9,\ 6,\ 3,\ \ldots\}$$

If the DTFS cofficients for $\tilde{x}[n]$ and $\tilde{h}[n]$ are $\tilde{c}_k$ and $\tilde{d}_k$ respectively, the DTFS coefficients of $y[n]$ are

$$\tilde{y}[n] \stackrel{\text{DTFS}}{\longleftrightarrow} 5\,\tilde{c}_k\,\tilde{d}_k$$

The table below lists DTFS coefficients of the three signals:

k	Coeffs for $x[n]$	Coeffs for $h[n]$	Coeffs for $y[n]$
0	$2.0000+j\,0.0000$	$0.6000+j\,0.0000$	$6.0000+j\,0.0000$
1	$0.5000-j\,0.6882$	$0.1000-j\,0.3078$	$-0.8090-j\,1.1135$
2	$0.5000-j\,0.1625$	$0.1000+j\,0.0727$	$0.3090+j\,0.1004$
3	$0.5000+j\,0.1625$	$0.1000-j\,0.0727$	$0.3090-j\,0.1004$
4	$0.5000+j\,0.6882$	$0.1000+j\,0.3078$	$-0.8090+j\,1.1135$

5.9. Tabular form of the periodic convolution result is

$$\tilde{y}[n] = \{\ldots,\ \underset{\substack{\uparrow \\ n=0}}{1},\ 2,\ 3,\ 3,\ 3,\ 2,\ 1,\ 0,\ \ldots\}$$

If the DTFS cofficients for $\tilde{x}[n]$ and $\tilde{h}[n]$ are $\tilde{c}_k$ and $\tilde{d}_k$ respectively, the DTFS coefficients of $y[n]$ are

$$\tilde{y}[n] \overset{\text{DTFS}}{\longleftrightarrow} 8\,\tilde{c}_k\,\tilde{d}_k$$

The table below lists DTFS coefficients of the three signals:

k	Coeffs for $x[n]$	Coeffs for $h[n]$	Coeffs for $y[n]$
0	$0.3750+j\,0.0000$	$0.6250+j\,0.0000$	$1.8750+j\,0.0000$
1	$0.2134-j\,0.2134$	$0.0000-j\,0.3018$	$-0.5152-j\,0.5152$
2	$0.0000-j\,0.1250$	$0.1250+j\,0.0000$	$0.0000-j\,0.1250$
3	$0.0366+j\,0.0366$	$0.0000-j\,0.0518$	$0.0152-j\,0.0152$
4	$0.1250+j\,0.0000$	$0.1250+j\,0.0000$	$0.1250+j\,0.0000$
5	$0.0366-j\,0.0366$	$0.0000+j\,0.0518$	$0.0152+j\,0.0152$
6	$0.0000+j\,0.1250$	$0.1250+j\,0.0000$	$0.0000+j\,0.1250$
7	$0.2134+j\,0.2134$	$0.0000+j\,0.3018$	$-0.5152+j\,0.5152$

5.10.

a. The transform is

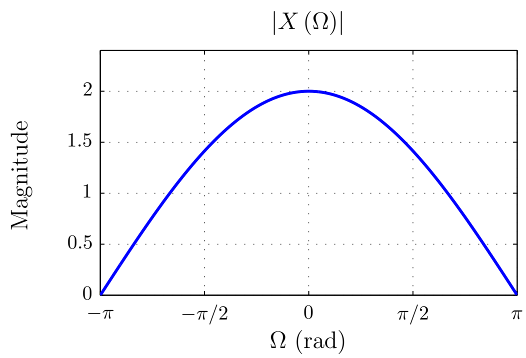
$$X(\Omega) = 1 + e^{-j\Omega} = 1 + \cos(\Omega) - j\,\sin(\Omega)$$

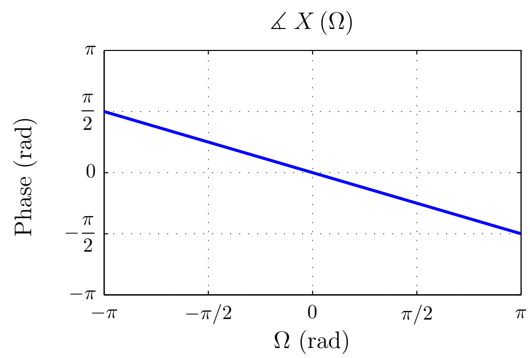
with magnitude

$$|X(\Omega)| = \sqrt{[1+\cos(\Omega)]^2 + \sin^2(\Omega)} = \sqrt{2 + 2\,\cos(\Omega)}$$

and phase

$$\measuredangle X(\Omega) = -\tan^{-1}\left[\frac{\sin(\Omega)}{1+\cos(\Omega)}\right]$$





b. The transform is

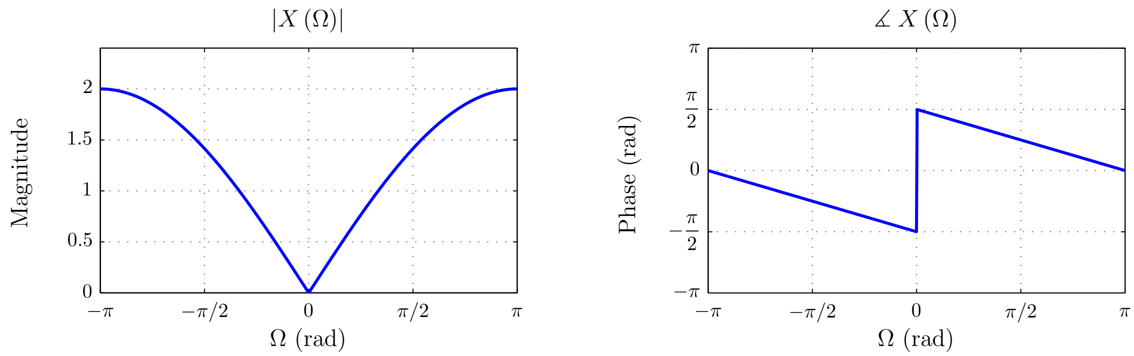
$$X(\Omega) = 1 - e^{-j\Omega} = 1 - \cos(\Omega) + j\,\sin(\Omega)$$

with magnitude

$$|X(\Omega)| = \sqrt{[1-\cos(\Omega)]^2 + \sin^2(\Omega)} = \sqrt{2 - 2\,\cos(\Omega)}$$

and phase

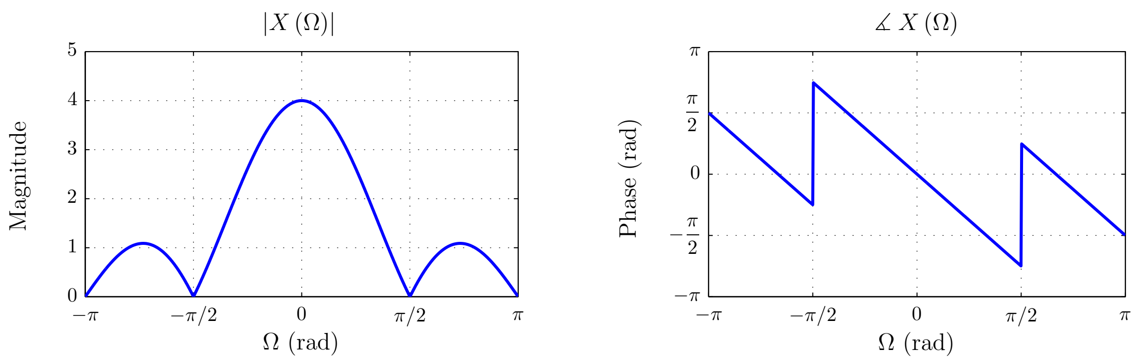
$$\measuredangle X(\Omega) = \tan^{-1}\left[\frac{\sin(\Omega)}{1-\cos(\Omega)}\right]$$



c. The transform is

$$
\begin{aligned}
X(\Omega) &= 1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega} \\
&= \left[e^{j3\Omega/2} + e^{j\Omega/2} + e^{-j\Omega/2} + e^{-j3\Omega/2} \right] e^{-j3\Omega/2} \\
&= \left[2\cos(\Omega/2) + 2\cos(3\Omega/2) \right] e^{-j3\Omega/2}
\end{aligned}
$$

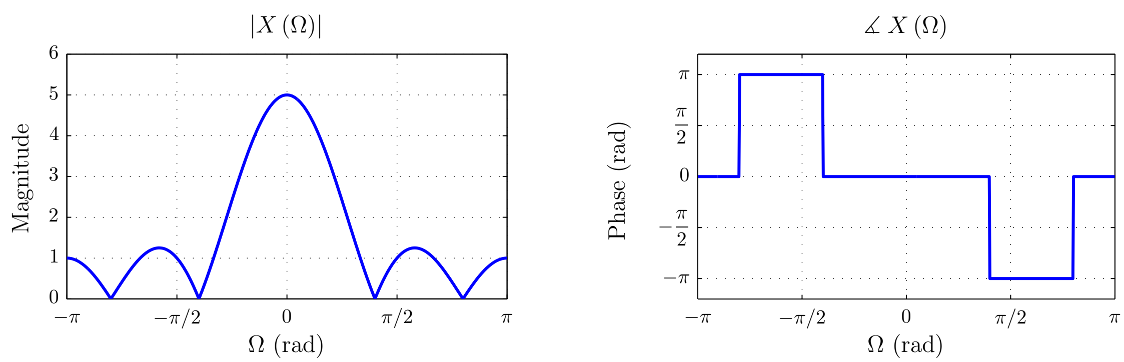
Magnitude and phase are shown below:



d. The transform is

$$
\begin{aligned}
X(\Omega) &= e^{j2\Omega} + e^{j\Omega} + 1 + e^{-j\Omega} + e^{-j2\Omega} \\
&= 1 + 2\cos(\Omega) + 2\cos(2\Omega)
\end{aligned}
$$

Magnitude and phase are shown below:



e. The transform is computed as

$$X(\Omega) = \sum_{n=0}^{\infty} (0.7)^n \, e^{-j\Omega n} = \frac{1}{1 - 0.7\, e^{-j\Omega}}$$

Using Euler's formula:

$$X(\Omega) = \frac{1}{1 - 0.7\cos(\Omega) + j0.7\sin(\Omega)}$$

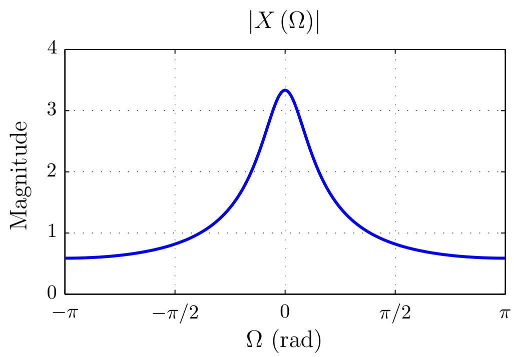
The magnitude of the transform is

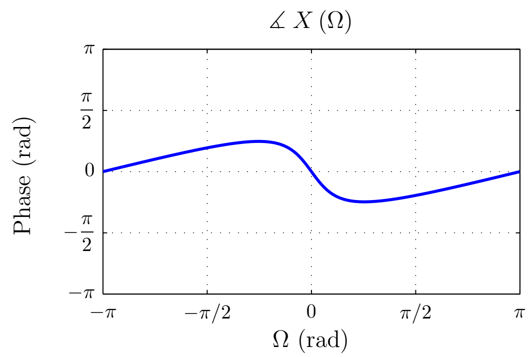
$$\left|X(\Omega)\right| = \frac{1}{\sqrt{[1 - 0.7\cos(\Omega)]^2 + [0.7\sin(\Omega)]^2}} = \frac{1}{\sqrt{1.49 - 1.4\cos(\Omega)}}$$

and the phase of the transform is

$$\measuredangle X(\Omega) = -\tan^{-1}\left(\frac{0.7\sin(\Omega)}{1 - 0.7\cos(\Omega)}\right)$$

Magnitude and phase are shown below:

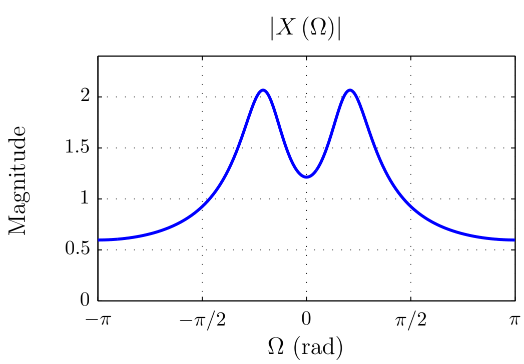


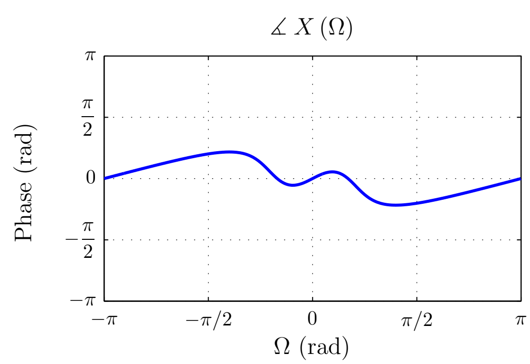


f. The transform is computed as

$$\begin{aligned} X(\Omega) &= \sum_{n=0}^{\infty} (0.7)^n \cos(\Omega n) \, e^{-j\Omega n} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (0.7)^n \, e^{j0.2\pi n} \, e^{-j\Omega n} + \frac{1}{2} \sum_{n=0}^{\infty} (0.7)^n \, e^{-j0.2\pi n} \, e^{-j\Omega n} \\ &= \frac{1 - 0.7\cos(0.2\pi)\, e^{-j\Omega}}{1 - 1.4\cos(0.2\pi)\, e^{-j\Omega} + 0.49\, e^{-j2\Omega}} \end{aligned}$$

Magnitude and phase are shown below:





5.11.

a. Using the inverse DTFT relationship:

$$\begin{aligned}
x[n] =& \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega \\
=& \frac{1}{2\pi} \int_{-0.2\pi}^{0.2\pi} (1)\, e^{j\Omega n}\, d\Omega \\
=& \left.\frac{e^{j\Omega n}}{j2\pi n}\right|_{-0.2\pi}^{0.2\pi} = \frac{\sin(0.2\pi n)}{\pi n} = 0.2\, \text{sinc}(0.2n)
\end{aligned}$$

b.

$$\begin{aligned}
x[n] =& \frac{1}{2\pi} \int_{-0.4\pi}^{0.4\pi} (1)\, e^{j\Omega n}\, d\Omega \\
=& \left.\frac{e^{j\Omega n}}{j2\pi n}\right|_{-0.4\pi}^{0.4\pi} = \frac{\sin(0.4\pi n)}{\pi n} = 0.4\, \text{sinc}(0.4n)
\end{aligned}$$

c.

$$\begin{aligned}
x[n] =& \frac{1}{2\pi} \int_{-\pi}^{-0.2\pi} (1)\, e^{j\Omega n}\, d\Omega + \frac{1}{2\pi} \int_{0.2\pi}^{\pi} (1)\, e^{j\Omega n}\, d\Omega \\
=& \left.\frac{e^{j\Omega n}}{j2\pi n}\right|_{-\pi}^{0.2\pi} + \left.\frac{e^{j\Omega n}}{j2\pi n}\right|_{0.2\pi}^{\pi} = \frac{\sin(\pi n)}{\pi n} - \frac{\sin(0.2\pi n)}{\pi n}
\end{aligned}$$

Using L'Hospital's rule it can be shown that

$$\frac{\sin(\pi n)}{\pi n} = \delta[n]$$

and therefore

$$x[n] = \delta[n] - \frac{\sin(0.2\pi n)}{\pi n} = \delta[n] - 0.2\, \text{sinc}(0.2n)$$

d.

$$\begin{aligned}
x[n] =& \frac{1}{2\pi} \int_{-0.2\pi}^{-0.1\pi} (1)\, e^{j\Omega n}\, d\Omega + \frac{1}{2\pi} \int_{0.1\pi}^{0.2\pi} (1)\, e^{j\Omega n}\, d\Omega \\
=& \left.\frac{e^{j\Omega n}}{j2\pi n}\right|_{-0.2\pi}^{-0.1\pi} + \left.\frac{e^{j\Omega n}}{j2\pi n}\right|_{0.1\pi}^{0.2\pi} = \frac{\sin(0.2\pi n)}{\pi n} - \frac{\sin(0.1\pi n)}{\pi n} \\
=& 0.2\, \text{sinc}(0.2n) - 0.1\, \text{sinc}(0.1n)
\end{aligned}$$

5.12.

a. We know that

$$(0.5)^n\, u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-0.5\,e^{-j\Omega}}$$

Using time shifting

$$(0.5)^{n-2}\, u[n-2] \overset{\mathcal{F}}{\longleftrightarrow} \frac{e^{-j2\Omega}}{1-0.5\,e^{-j\Omega}}$$

Using linearity, and scaling both sides

$$(0.5)^n\, u[n-2] \overset{\mathcal{F}}{\longleftrightarrow} \frac{0.25\,e^{-j2\Omega}}{1-0.5\,e^{-j\Omega}}$$

b. Since

$$(0.8)^n\, u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-0.8\,e^{-j\Omega}}$$

through the use of time shifting we obtain

$$(0.8)^{n-10}\, u[n-10] \overset{\mathcal{F}}{\longleftrightarrow} \frac{e^{-j10\Omega}}{1-0.8\,e^{-j\Omega}}$$

Linearity of the DTFT allows both sides to be scaled:

$$(0.8)^n\, u[n-10] \overset{\mathcal{F}}{\longleftrightarrow} \frac{(0.8)^{10}\, e^{-j10\Omega}}{1-0.8\,e^{-j\Omega}}$$

Therefore

$$(0.8)^n\, \big(u[n]-u[n-10]\big) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1-(0.8)^{10}\, e^{-j10\Omega}}{1-0.8\,e^{-j\Omega}}$$

c. We know that

$$(0.8)^n\, u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-0.8\,e^{-j\Omega}}$$

Shifting the signal to the left by 5 samples and using the time shifting property of the DTFT we obtain

$$(0.8)^{n+5}\, u[n+5] \overset{\mathcal{F}}{\longleftrightarrow} \frac{e^{j5\Omega}}{1-0.8\,e^{-j\Omega}}$$

Scaling both sides yields

$$(0.8)^n\, u[n+5] \overset{\mathcal{F}}{\longleftrightarrow} \frac{(0.8)^{-5}\, e^{j5\Omega}}{1-0.8\,e^{-j\Omega}}$$

Shifting the signal to the right by 5 samples yields

$$(0.8)^{n-5}\, u[n-5] \overset{\mathcal{F}}{\longleftrightarrow} \frac{e^{-j5\Omega}}{1-0.8\,e^{-j\Omega}}$$

and with scaling we get

$$(0.8)^n\, u[n-5] \overset{\mathcal{F}}{\longleftrightarrow} \frac{(0.8)^{5}\, e^{-j5\Omega}}{1-0.8\,e^{-j\Omega}}$$

Therefore

$$(0.8)^n\, \big(u[n+5]-u[n-5]\big) \overset{\mathcal{F}}{\longleftrightarrow} \frac{(0.8)^{-5}\, e^{j5\Omega}-(0.8)^{5}\, e^{-j5\Omega}}{1-0.8\,e^{-j\Omega}}$$

5.13.

a. We know that

$$\left(\frac{1}{2}\right)^n u[n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}\,e^{-j\Omega}}$$

Applying the time reversal property

$$\left(\frac{1}{2}\right)^{-n} u[-n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}\,e^{j\Omega}}$$

or equivalently

$$(2)^n\, u[-n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}\,e^{j\Omega}}$$

Using the time shifting property

$$(2)^{n+1}\, u[-n-1] \overset{\mathscr{F}}{\longleftrightarrow} \frac{e^{j\Omega}}{1-\frac{1}{2}\,e^{j\Omega}}$$

and scaling both sides we get

$$(2)^n\, u[-n-1] \overset{\mathscr{F}}{\longleftrightarrow} \frac{\frac{1}{2}\,e^{j\Omega}}{1-\frac{1}{2}\,e^{j\Omega}}$$

b. We know that

$$(0.8)^n\, u[n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-0.8\,e^{-j\Omega}}$$

Applying the time reversal property

$$(0.8)^{-n}\, u[-n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-0.8\,e^{j\Omega}}$$

or equivalently

$$(1.25)^n\, u[-n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-0.8\,e^{j\Omega}}$$

c. The signal $x[n]$ can be written as

$$x[n] = (0.8)^{-n}\, u[-n-1] + (0.8)^n\, u[n]$$

We know that

$$(0.8)^n\, u[n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-0.8\,e^{-j\Omega}}$$

Using time reversal property

$$(0.8)^{-n}\, u[-n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-0.8\,e^{j\Omega}}$$

and using time shifting

$$(0.8)^{-n-1}\, u[-n-1] \overset{\mathscr{F}}{\longleftrightarrow} \frac{e^{j\Omega}}{1-0.8\,e^{j\Omega}}$$

Scaling both sides of the last relationship yields

$$(0.8)^{-n}\, u[-n-1] \overset{\mathscr{F}}{\longleftrightarrow} \frac{0.8\,e^{j\Omega}}{1-0.8\,e^{j\Omega}}$$

The desired transform is

$$X(\Omega) = \frac{0.8\,e^{j\Omega}}{1-0.8\,e^{j\Omega}} + \frac{1}{1-0.8\,e^{-j\Omega}} = \frac{0.36}{1.64-1.6\cos(\Omega)}$$

d. The signal $x[n]$ can be written as

$$x[n] = \begin{cases} (0.8)^{-n}\,, & n=-5,\ldots,-1 \\ (0.8)^{n}\,, & n=0,\ldots,4 \end{cases}$$

Let

$$x_1[n] = (0.8)^n \left(u[n]-u[n-5]\right)$$

and

$$\begin{aligned} x_2[n] &= (0.8)^{-n}\left(u[-n-1]-u[-n-6]\right) \\ &= 0.8\,x_1[-n-1] \end{aligned}$$

so that

$$x[n] = x_1[n] + x_2[n]$$

The transform of $x_1[n]$ is

$$X_1(\Omega) = \frac{1-(0.8)^5\,e^{-j5\Omega}}{1-0.8\,e^{-j\Omega}} = \frac{1-0.3277\,e^{-j5\Omega}}{1-0.8\,e^{-j\Omega}} =$$

Using the time shifting and time reversal properties we obtain

$$X_2(\Omega) = e^{j\Omega}\,X_1(-\Omega) = \frac{e^{j\Omega}\left(1-0.3277\,e^{j5\Omega}\right)}{1-0.8\,e^{j\Omega}}$$

and

$$X(\Omega) = X_1(\Omega) + X_2(\Omega) = \frac{1-0.3277\,e^{-j5\Omega}}{1-0.8\,e^{-j\Omega}} + \frac{e^{j\Omega}\left(1-0.3277\,e^{j5\Omega}\right)}{1-0.8\,e^{j\Omega}}$$

5.14.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\Omega n}$$

Conjugating both sides yields

$$X^*(\Omega) = \left[\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\Omega n}\right]^* = \sum_{n=-\infty}^{\infty} x^*[n]\,e^{j\Omega n}$$

Since $x[n]$ is real-valued we have $x^*[n] = x[n]$, and therefore

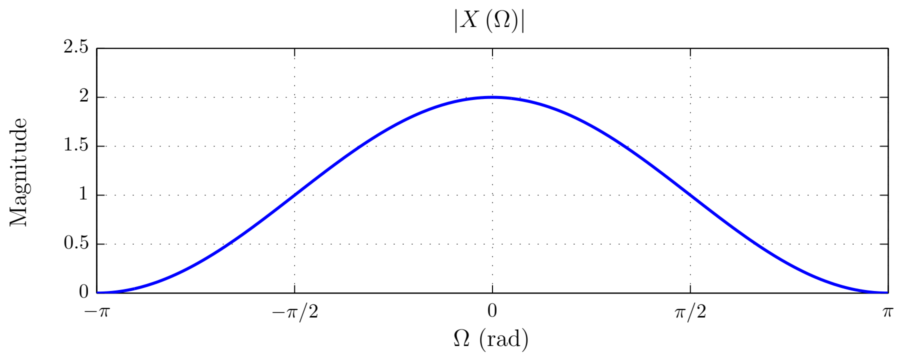
$$X^*(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\,e^{j\Omega n} = X(-\Omega)$$

5.15.

a.

$$X(\Omega) = 1 + \frac{1}{2}\,e^{j\Omega} + \frac{1}{2}\,e^{-j\Omega} = 1 + \cos(\Omega)$$

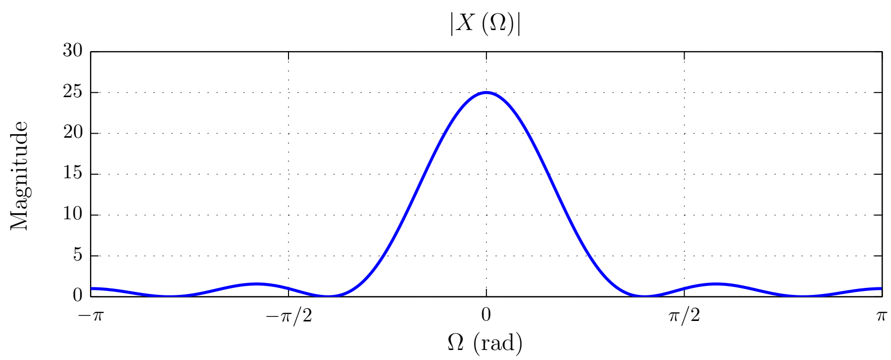
Magnitude of the transform is shown below:



b.

$$\begin{aligned} X(\Omega) =& 5 + 4\left[e^{j\Omega} + e^{-j\Omega}\right] + 3\left[e^{j2\Omega} + e^{-j2\Omega}\right] + 2\left[e^{j3\Omega} + e^{-j3\Omega}\right] + \left[e^{j4\Omega} + e^{-j4\Omega}\right] \\ =& 5 + 8\cos(\Omega) + 6\cos(2\Omega) + 4\cos(3\Omega) + 2\cos(4\Omega) \end{aligned}$$

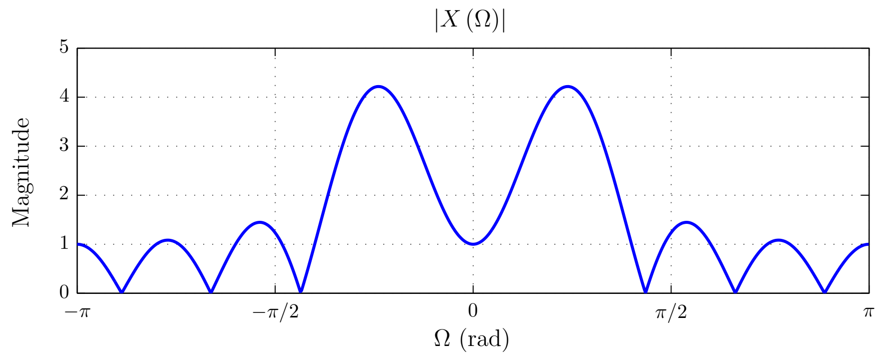
Magnitude of the transform is shown below:



c.

$$\begin{aligned} X(\Omega) =& \cos(-0.8\pi)\,e^{j4\Omega} + \cos(-0.6\pi)\,e^{j3\Omega} + \cos(-0.4\pi)\,e^{j2\Omega} + \cos(-0.2\pi)\,e^{j\Omega} \\ &+ 1 + \cos(0.2\pi)\,e^{-j\Omega} + \cos(0.4\pi)\,e^{-j2\Omega} + \cos(0.6\pi)\,e^{-j3\Omega} + \cos(0.8\pi)\,e^{-j4\Omega} \\ =& 1 + 2\cos(0.2\pi)\cos(\Omega) + 2\cos(0.4\pi)\cos(2\Omega) + 2\cos(0.6\pi)\cos(3\Omega) + 2\cos(0.8\pi)\cos(4\Omega) \end{aligned}$$

Magnitude of the transform is shown below:

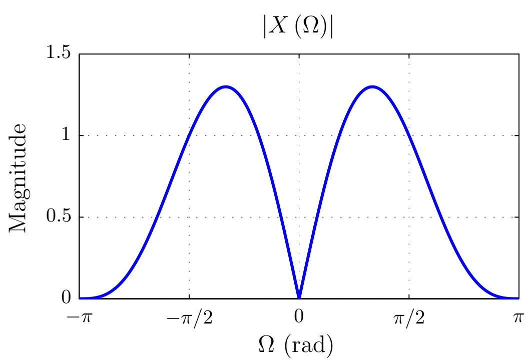


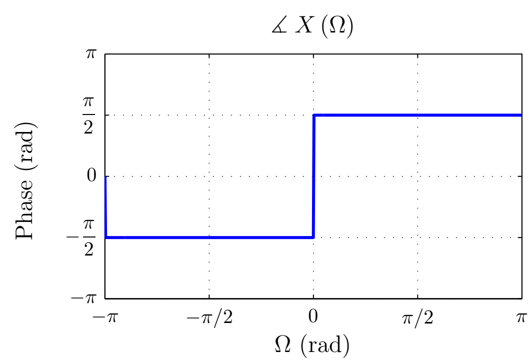
5.16.

a.

$$X(\Omega) = \frac{1}{2}\left[e^{j\Omega} - e^{-j\Omega}\right] + \frac{1}{4}\left[e^{j2\Omega} - e^{-j2\Omega}\right]$$
$$= j\,\sin(\Omega) + j\,\frac{1}{2}\,\sin(2\Omega)$$

Magnitude and phase of the transform are shown below:

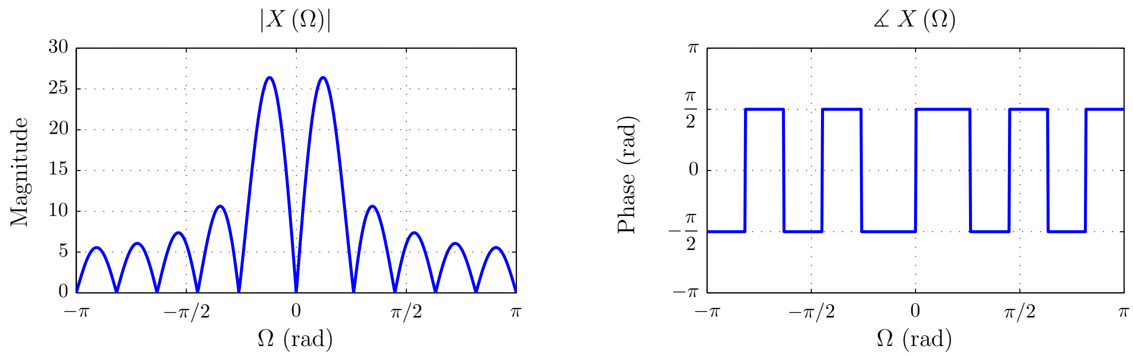




b.

$$X(\Omega) = \left[e^{j\Omega} - e^{-j\Omega}\right] + 2\left[e^{j2\Omega} - e^{-j2\Omega}\right] + 3\left[e^{j3\Omega} - e^{-j3\Omega}\right] + 4\left[e^{j4\Omega} - e^{-j4\Omega}\right] + 5\left[e^{j5\Omega} - e^{-j5\Omega}\right]$$
$$= j\,2\,\sin(\Omega) + j\,4\,\sin(2\Omega) + j\,6\,\sin(3\Omega) + j\,8\,\sin(4\Omega) + j\,10\,\sin(5\Omega)$$

Magnitude and phase of the transform are shown below:

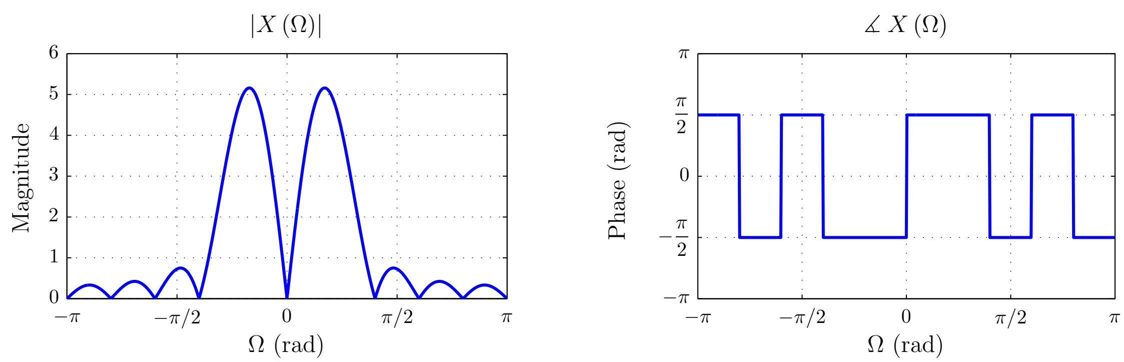


c.

$$X(\Omega) = \sin(0.2\pi)\left[e^{j\Omega} - e^{-j\Omega}\right] + \sin(0.4\pi)\left[e^{j2\Omega} - e^{-j2\Omega}\right] + \sin(0.6\pi)\left[e^{j3\Omega} - e^{-j3\Omega}\right] + \sin(0.8\pi)\left[e^{j4\Omega} - e^{-j4\Omega}\right]$$

$$= j\,2\sin(0.2\pi)\sin(\Omega) + j\,2\sin(0.4\pi)\sin(2\Omega) + j\,2\sin(0.6\pi)\sin(3\Omega) + j\,2\sin(0.8\pi)\sin(4\Omega)$$

Magnitude and phase of the transform are shown below:



5.17.

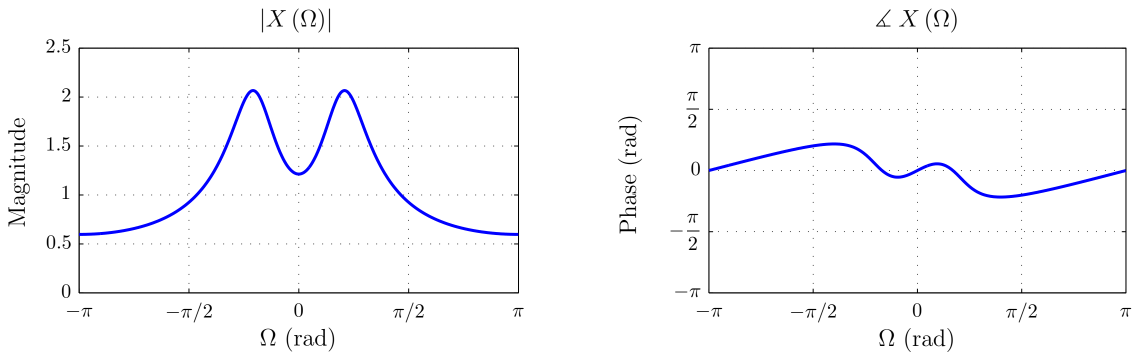
a. Starting with

$$(0.7)^n\, u[n] \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-0.7\,e^{-j\Omega}}$$

and using the modulation property we obtain

$$X(\Omega) = \frac{1/2}{1-0.7\,e^{-j(\Omega-0.2\pi)}} + \frac{1/2}{1-0.7\,e^{-j(\Omega+0.2\pi)}}$$

Magnitude and phase characteristics are shown below.



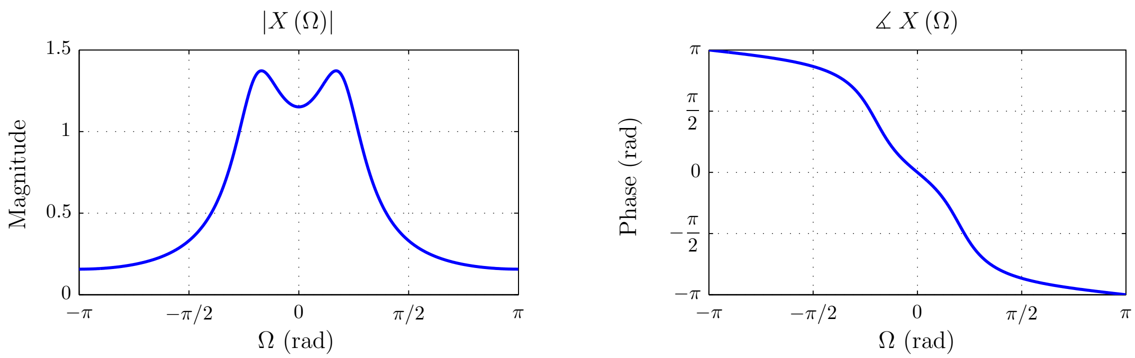
b. Starting with

$$(0.7)^n \, u[n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-0.7\,e^{-j\Omega}}$$

and using the modulation property we obtain

$$X(\Omega) = \frac{1/2\,e^{-j\pi/2}}{1-0.7\,e^{-j(\Omega-0.2\pi)}} + \frac{1/2\,e^{j\pi/2}}{1-0.7\,e^{-j(\Omega+0.2\pi)}}$$

Magnitude and phase characteristics are shown below.



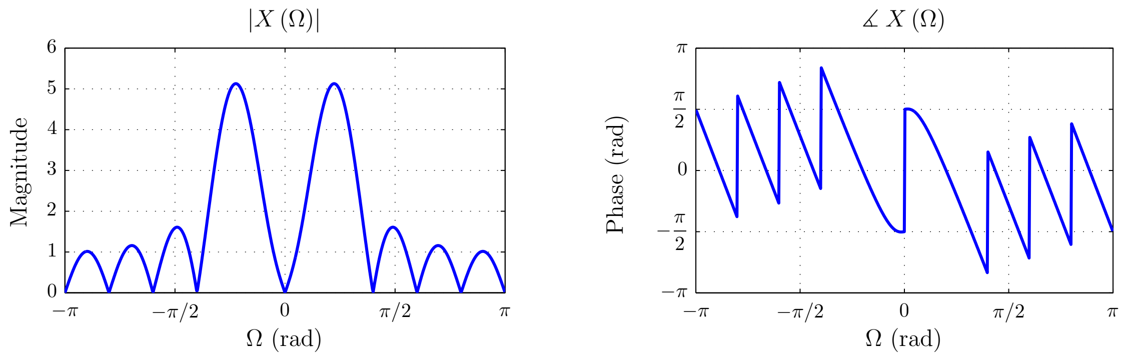
c. Starting with

$$u[n]-u[n-10] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1-e^{-j10\Omega}}{1-e^{-j\Omega}}$$

and applying the modulation property yields

$$X(\Omega) = \left(\frac{1}{2}\right)\frac{1-e^{-j10(\Omega-\pi/5)}}{1-e^{-j(\Omega-\pi/5)}} + \left(\frac{1}{2}\right)\frac{1-e^{-j10(\Omega+\pi/5)}}{1-e^{-j(\Omega+\pi/5)}}$$

Magnitude and phase characteristics are shown below.



5.18.

a. Starting with

$$(0.7)^n\, u[n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-0.7\,e^{-j\Omega}}$$

the use of the differentiation property yields

$$n\,(0.7)^n\, u[n] \overset{\mathscr{F}}{\longleftrightarrow} j\,\frac{d}{d\Omega}\left[\frac{1}{1-0.7\,e^{-j\Omega}}\right]$$

$$\begin{aligned}\frac{d}{d\Omega}\left[\frac{1}{1-0.7\,e^{-j\Omega}}\right] &= \frac{d}{d\Omega}\left[1-0.7\,e^{-j\Omega}\right]^{-1}\\ &= -2\left[1-0.7\,e^{-j\Omega}\right]\left[j\,0.7\,e^{-j\Omega}\right]\\ &= \frac{-j\,1.4\,e^{-j\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^2}\end{aligned}$$

The transform of $x[n]$ is

$$X(\Omega) = \frac{1.4\,e^{-j\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^2}$$

b. The signal $x[n]$ can be written as

$$x[n] = n^2\,(0.7)^n\,u[n] + n\,(0.7)^n\,u[n]$$

From part (a) we have

$$n\,(0.7)^n\,u[n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1.4\,e^{-j\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^2}$$

Using the differentiation property

$$n^2\,(0.7)^n\,u[n] \overset{\mathscr{F}}{\longleftrightarrow} j\,\frac{d}{d\Omega}\left[\frac{1.4\,e^{-j\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^2}\right]$$

$$\frac{d}{d\Omega}\left[\frac{1.4\,e^{-j\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^2}\right] = \frac{-j\left(1.4\,e^{-j\Omega}-0.98\,e^{-j2\Omega}\right)}{\left(1-0.7\,e^{-j\Omega}\right)^3}$$

and

$$n^2\,(0.7)^n\,u[n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1.4\,e^{-j\Omega}-0.98\,e^{-j2\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^3}$$

Combining the results

$$X(\Omega)=\frac{1.4\,e^{-j\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^2}+\frac{1.4\,e^{-j\Omega}-0.98\,e^{-j2\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^3}$$

c. Let the signal $x_1[n]$ be defined as

$$x_1[n]=(0.7)^n\,\big(u[n]-u[n-10]\big)$$

The transform $X_1(\Omega)$ is

$$X_1(\Omega)=\sum_{n=0}^{9}(0.7)^n\,e^{-j\Omega n}=\frac{1-\left(0.7\,e^{-j\Omega^{10}}\right)}{1-0.7\,e^{-j\Omega}}=\frac{1-0.0282\,e^{-j\,10\Omega}}{1-0.7\,e^{-j\Omega}}$$

$$\frac{dX_1(\Omega)}{d\Omega}=\frac{j\,0.282\,e^{-j\,10\Omega}-j\,0.7e^{-j\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^2}$$

$$X(\Omega)=j\,\frac{dX_1(\Omega)}{d\Omega}=\frac{0.7e^{-j\Omega}-0.282\,e^{-j\,10\Omega}}{\left(1-0.7\,e^{-j\Omega}\right)^2}$$

5.19.

a. The DTFT of $x[n]$ are $h[n]$ are

$$X(\Omega)=\frac{1-e^{-j\,10\Omega}}{1-e^{-j\Omega}}$$

and

$$H(\Omega)=\frac{1}{1-0.8\,e^{-j\Omega}}$$

b.

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]\,h[n-k]=\sum_{k=0}^{9}(0.8)^{n-k}\,u[n-k]$$

Since

$$u[n-k]=\begin{cases}1\,, & n\ge k\\ 0\,, & n<k\end{cases}$$

the convolution sum must be evaluated for three distinct possibilities:

<u>$n<0$</u>: $\quad y[n]=0$

<u>$0\le n\le 9$</u>:

$$\begin{aligned}y[n]&=\sum_{k=0}^{n}(0.8)^{n-k}\\&=(0.8)^n\sum_{k=0}^{n}(0.8)^{-k}\\&=(0.8)^n\left[\frac{1-(0.8)^{-(n+1)}}{1-(0.8)^{-1}}\right]\\&=5-4\,(0.8)^n\end{aligned}$$

$n > 9$:

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{9} (0.8)^{n-k} \\
&= (0.8)^n \sum_{k=0}^{9} (0.8)^{-k} \\
&= (0.8)^n \left(\frac{1-(0.8)^{-(10)}}{1-(0.8)^{-1}} \right) \\
&= 4\left((1.25)^{10} - 1 \right) (0.8)^n
\end{aligned}
$$

Combining the results obtained, the signal $y[n]$ is

$$
\begin{aligned}
y[n] =& \left(5 - 4\,(0.8)^n\right) \left(u[n] - u[n-10]\right) + 4\left((1.25)^{10} - 1\right)(0.8)^n\, u[n-10] \\
=& 5\left(u[n] - u[n-10]\right) - 4\,(0.8)^n\, u[n] + 4\,(1.25)^{10}\,(0.8)^n\, u[n-10]
\end{aligned}
$$

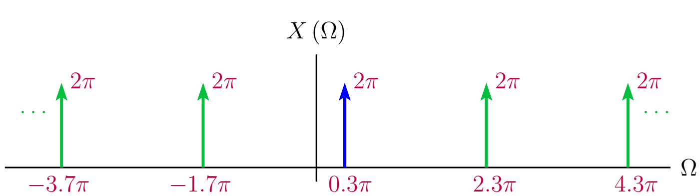
c. The DTFT of $y[n]$ is

$$
\begin{aligned}
Y(\Omega) =& 5\left(\frac{1-e^{-j10\Omega}}{1-e^{-j\Omega}}\right) - 4\left(\frac{1}{1-0.8\,e^{-j\Omega}}\right) + 4\,(1.25)^{10}\left(\frac{(0.8)^{10}\,e^{-j10\Omega}}{1-0.8\,e^{-j\Omega}}\right) \\
=& 5\left(\frac{1-e^{-j10\Omega}}{1-e^{-j\Omega}}\right) - 4\left(\frac{1-e^{-j\Omega}}{1-0.8\,e^{-j\Omega}}\right) \\
=& \frac{1-e^{-j10\Omega}}{\left(1-e^{-j\Omega}\right)\left(1-0.8\,e^{-j\Omega}\right)} \\
=& X(\Omega)\, H(\Omega)
\end{aligned}
$$

5.20.

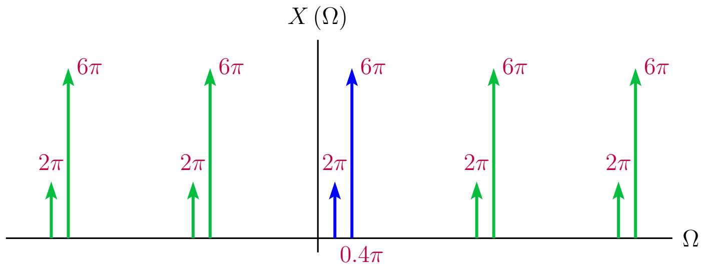
a.

$$
X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 0.3\pi - 2\pi m)
$$



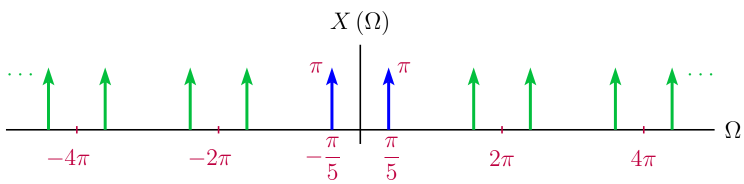
b.

$$
X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \left(\delta(\Omega - 0.2\pi - 2\pi m) + 3\,\delta(\Omega - 0.4\pi - 2\pi m)\right)
$$



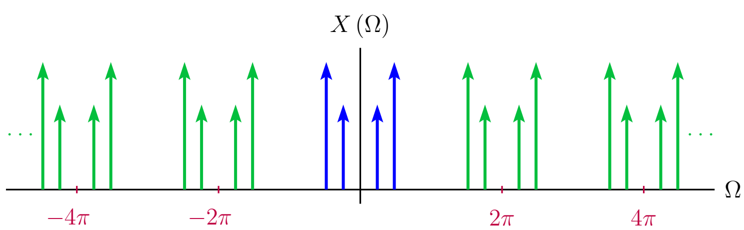
c.

$$X(\Omega)=\pi\sum_{m=-\infty}^{\infty}\big(\delta(\Omega-\pi/5-2\pi m)+\delta(\Omega+\pi/5-2\pi m)\big)$$



d.

$$X(\Omega)=\pi\sum_{m=-\infty}^{\infty}\big(2\delta(\Omega-\pi/5-2\pi m)+2\delta(\Omega+\pi/5-2\pi m)$$
$$+3\delta(\Omega-2\pi/5-2\pi m)+3\delta(\Omega+2\pi/5-2\pi m)\big)$$



5.21.

a. Using Eqn. (5.181) with the DTFS coefficients found in Problem 5.3 we obtain

$$S_x(\Omega)=\sum_{m=-\infty}^{\infty}\big[25.1327\,\delta(\Omega-2\pi m)+4.5776\,\delta(\Omega-2\pi/8-2\pi m)+0.1348\,\delta(\Omega-6\pi/8-2\pi m)$$
$$+0.1348\,\delta(\Omega-10\pi/8-2\pi m)+4.5776\,\delta(\Omega-14\pi/8-2\pi m)\big]$$

b. Using Eqn. (5.181) with the DTFS coefficients found in Problem 5.3 we obtain

$$S_x(\Omega) = \sum_{m=-\infty}^{\infty} \Big[0.8836\,\delta(\Omega-2\pi m)+0.5722,\delta(\Omega-2\pi/8-2\pi m)+0.0982\,\delta(\Omega-4\pi/8-2\pi m)$$
$$+0.0168\,\delta(\Omega-6\pi/8-2\pi m)+0.0982\,\delta(\Omega-\pi-2\pi m)+0.0168\,\delta(\Omega-10\pi/8-2\pi m)$$
$$+0.0982\,\delta(\Omega-12\pi/8-2\pi m)+0.5722\,\delta(\Omega-14\pi/8-2\pi m)\Big]$$

c. Using Eqn. (5.181) with the DTFS coefficients found in Problem 5.3 we obtain

$$S_x(\Omega) = \sum_{m=-\infty}^{\infty} \Big[2.4544\,\delta(\Omega-2\pi m)+0.5722,\delta(\Omega-2\pi/8-2\pi m)+0.0982\,\delta(\Omega-4\pi/8-2\pi m)$$
$$+0.0168\,\delta(\Omega-6\pi/8-2\pi m)+0.0982\,\delta(\Omega-\pi-2\pi m)+0.0168\,\delta(\Omega-10\pi/8-2\pi m)$$
$$+0.0982\,\delta(\Omega-12\pi/8-2\pi m)+0.5722\,\delta(\Omega-14\pi/8-2\pi m)\Big]$$

5.22.

a. The DTFS coefficients are

$$\tilde{c}_k = \frac{\sin\left(\frac{\pi k}{N}(2L+1)\right)}{N\sin\left(\frac{\pi k}{N}\right)}$$

and the power spectral density is

$$S_x(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi\,|\tilde{c}_k|^2\,\delta(\Omega-k\Omega_0)$$
$$= \sum_{k=-\infty}^{\infty} 2\pi \left|\frac{\sin\left(\frac{\pi k}{N}(2L+1)\right)}{N\sin\left(\frac{\pi k}{N}\right)}\right|^2 \delta\left(\Omega-k\frac{2\pi}{N}\right)$$

b. The average power in the signal $x[n]$ is

$$P_x = \frac{1}{40}\sum_{n=0}^{39}\big|x[n]\big|^2 = \frac{7}{4} = 0.1750$$

The DTFS coefficients up to the third harmonic are

$$\tilde{c}_0 = 0.1750$$
$$\tilde{c}_1 = \tilde{c}_{-1} = 0.1665$$
$$\tilde{c}_2 = \tilde{c}_{-2} = 0.1424$$
$$\tilde{c}_3 = \tilde{c}_{-3} = 0.0114$$

The normalized average power in the first three harmonics is

$$P_{x,3\text{ harmonics}} = (0.1750)^2 + 2\,(0.1665)^2 + 2\,(0.1424)^2 + 2\,(0.0114)^2 = 0.1494$$

which is $0.1494/0.1750 = 0.8538$, or 85.38 percent of the total normalized average power.

c. The average power in the signal $x[n]$ is

$$P_x = \frac{1}{40}\sum_{n=0}^{39}\left|x[n]\right|^2 = \frac{13}{4} = 0.3250$$

The DTFS coefficients up to the third harmonic are

$$\begin{aligned}\tilde{c}_0 &= 0.3250\\ \tilde{c}_1 = \tilde{c}_{-1} &= 0.2717\\ \tilde{c}_2 = \tilde{c}_{-2} &= 0.1424\\ \tilde{c}_3 = \tilde{c}_{-3} &= 0.0084\end{aligned}$$

The normalized average power in the first three harmonics is

$$P_{x,3\text{ harmonics}} = (0.3250)^2 + 2\,(0.2717)^2 + 2\,(0.1424)^2 + 2\,(0.0084)^2 = 0.2939$$

which is $0.2939/0.3250 = 0.9044$, or 90.44 percent of the total normalized average power.

5.23.

a. The value of the system function at $\Omega_0 = 0.2\pi$ is

$$H(0.2\pi) = 1.1159 + j\,0.2039 = 1.1344\,e^{j\,0.1807}$$

Therefore the steady-state response of the system is

$$\begin{aligned}y[n] &= 1.1344\,e^{j\,0.1807}\,x[n]\\ &= 1.1344\,e^{j(0.2\pi n + 0.1807)}\end{aligned}$$

b. Using $H(0.2\pi)$, the steady-state response of the system is

$$y[n] = 1.1344\,\cos(0.2\pi n + 0.1807)$$

c. The value of the system function at $\Omega_0 = 0.3\pi$ is

$$H(0.3\pi) = 1.2399 + j\,0.3310 = 1.2833\,e^{j\,0.2609}$$

Therefore the steady-state response of the system is

$$\begin{aligned}y[n] &= (2)\;(1.2833)\,\sin(0.3\pi n + 0.2609)\\ &= 2.5666\,\sin(0.3\pi n + 0.2609)\end{aligned}$$

d. The value of the system function at $\Omega_0 = 0.1\pi$ is

$$H(0.1\pi) = 1.0561 + j\,0.0976 = 1.0606\,e^{j\,0.0922}$$

Using this result with $H(0.2\pi)$ and from previous parts, the steady-state response of the system is

$$\begin{aligned} y[n] &= (3)\;(1.0606)\;\cos(0.1\pi n + 0.0922) - (5)\;(1.1344)\;\sin(0.2\pi n + 0.1807) \\ &= 3.1818\,\cos(0.1\pi n + 0.0922) - 5.6720\,\sin(0.2\pi n + 0.1807) \end{aligned}$$

5.24. The system function of a length-4 moving average filter is

$$H(\Omega) = \frac{1}{4}\left[1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega}\right]$$

a. The value of the system function at $\Omega_0 = 0.2\pi$ is

$$H(0.2\pi) = 0.4523 - j\,0.6225 = 0.7694\,e^{-j\,0.9425}$$

Therefore the steady-state response of the system is

$$\begin{aligned} y[n] &= 0.7694\,e^{-j\,0.9425}\,x[n] \\ &= 0.7694\,e^{j(0.2\pi n - 0.9425)} \end{aligned}$$

b. Using $H(0.2\pi)$, the steady-state response of the system is

$$y[n] = 0.7694\,\cos(0.2\pi n - 0.9425)$$

c. The value of the system function at $\Omega_0 = 0.3\pi$ is

$$H(0.3\pi) = 0.0819 - j\,0.5173 = 0.5237\,e^{-j\,1.4137}$$

Therefore the steady-state response of the system is

$$\begin{aligned} y[n] &= (2)\;(0.5237)\;\sin(0.3\pi n - 1.4137) \\ &= 1.0474\,\sin(0.3\pi n - 1.4137) \end{aligned}$$

d. The value of the system function at $\Omega_0 = 0.1\pi$ is

$$H(0.1\pi) = 0.8370 - j\,0.4265 = 0.9393\,e^{-j\,0.4712}$$

Using this result with $H(0.2\pi)$ and from previous parts, the steady-state response of the system is

$$\begin{aligned} y[n] &= (3)\;(0.9393)\;\cos(0.1\pi n - 0.4712) - (5)\;(0.7694)\;\sin(0.2\pi n - 0.9425) \\ &= 2.8179\,\cos(0.1\pi n - 0.4712) - 3.8470\,\sin(0.2\pi n - 0.9425) \end{aligned}$$

5.25.

The system function is

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1 + 2\,e^{-j\Omega}}{1 + e^{-j\Omega} + 0.89\,e^{-j2\Omega}}$$

All three periodic signals of Problem 5.3 have period $N = 8$ and the corresponding fundamental angular frequency $\Omega_0 = 2\pi/8$ radians. Therefore DTFS coefficients $\tilde{d}_k$ of the output signal $\tilde{y}[n]$ are related to the DTFS $\tilde{c}_k$ coefficients of the input signal $\tilde{x}[n]$ by

$$\tilde{d}_k = H\left(\frac{2\pi k}{8}\right)\tilde{c}_k$$

and the output signal is computed as

$$\tilde{y}[n] = \sum_{k=0}^{7} \tilde{d}_k\, e^{(j2\pi/8)kn}$$

a.

k	$\tilde{c}_k$	$H(2\pi k/8)$	$\tilde{d}_k$
0	$2.0000+j\,0.0000$	$1.0381+j\,0.0000$	$2.0761+j\,0.0000$
1	$0.8536+j\,0.0000$	$1.1674+j\,0.2638$	$0.9965+j\,0.2251$
2	$0.0000+j\,0.0000$	$2.0848+j\,0.7707$	$-0.0000+j\,0.0000$
3	$0.1464+j\,0.0000$	$-3.1867-j\,2.8385$	$-0.4667-j\,0.4157$
4	$0.0000+j\,0.0000$	$-1.1236+j\,0.0000$	$-0.0000+j\,0.0000$
5	$0.1464+j\,0.0000$	$-3.1867+j\,2.8385$	$-0.4667+j\,0.4157$
6	$0.0000+j\,0.0000$	$2.0848-j\,0.7707$	$-0.0000+j\,0.0000$
7	$0.8536-j\,0.0000$	$1.1674-j\,0.2638$	$0.9965-j\,0.2251$

The output signal is (one period shown)

$$\tilde{y}[n] = \{\ldots,\ \underset{\substack{\uparrow\\ n=0}}{3.1357},\ 4.4148,\ 0.7944,\ 0.2764,\ 1.0166,\ -0.2625,\ 3.3578,\ 3.8759,\ \ldots\}$$

b.

k	$\tilde{c}_k$	$H(2\pi k/8)$	$\tilde{d}_k$
0	$0.3750+j\,0.0000$	$1.0381+j\,0.0000$	$0.3893+j\,0.0000$
1	$0.2134-j\,0.2134$	$1.1674+j\,0.2638$	$0.3054-j\,0.1928$
2	$0.0000-j\,0.1250$	$2.0848+j\,0.7707$	$0.0963-j\,0.2606$
3	$0.0366+j\,0.0366$	$-3.1867-j\,2.8385$	$-0.0127-j\,0.2206$
4	$0.1250+j\,0.0000$	$-1.1236+j\,0.0000$	$-0.1404+j\,0.0000$
5	$0.0366-j\,0.0366$	$-3.1867+j\,2.8385$	$-0.0127+j\,0.2206$
6	$0.0000+j\,0.1250$	$2.0848-j\,0.7707$	$0.0963+j\,0.2606$
7	$0.2134+j\,0.2134$	$1.1674-j\,0.2638$	$0.3054+j\,0.1928$

The output signal is (one period shown)

$$\tilde{y}[n] = \{\ldots,\ \underset{\substack{\uparrow\\ n=0}}{1.0268},\ 2.0855,\ 0.0006,\ 0.1433,\ -0.1438,\ 0.0163,\ 0.1117,\ -0.1262,\ \ldots\}$$

c.

k	$\tilde{c}_k$	$H(2\pi k/8)$	$\tilde{d}_k$
0	$0.6250+j\,0.0000$	$1.0381+j\,0.0000$	$0.6488+j\,0.0000$
1	$0.0000-j\,0.3018$	$1.1674+j\,0.2638$	$0.0796-j\,0.3523$
2	$0.1250+j\,0.0000$	$2.0848+j\,0.7707$	$0.2606+j\,0.0963$
3	$0.0000-j\,0.0518$	$-3.1867-j\,2.8385$	$-0.1470+j\,0.1650$
4	$0.1250+j\,0.0000$	$-1.1236+j\,0.0000$	$-0.1404+j\,0.0000$
5	$0.0000+j\,0.0518$	$-3.1867+j\,2.8385$	$-0.1470-j\,0.1650$
6	$0.1250+j\,0.0000$	$2.0848-j\,0.7707$	$0.2606-j\,0.0963$
7	$0.0000+j\,0.3018$	$1.1674-j\,0.2638$	$0.0796+j\,0.3523$

The output signal is (one period shown)

$$\tilde{y}[n] = \{\ldots,\ \underset{\substack{\uparrow\\ n=0}}{0.8948},\ 1.1819,\ 1.0217,\ 0.9264,\ 1.1643,\ 0.0113,\ -1.0475,\ 1.0374,\ \ldots\}$$

5.26.

a.

$$X(k) = 1 + e^{-j2\pi k/3} + e^{-j4\pi k/3}\,, \quad k = 0,1,2$$

$$\begin{aligned} X(0) &= 3 + j0 \\ X(1) &= 0 + j0 \\ X(2) &= 0 + j0 \end{aligned}$$

b.

$$X(k) = 1 + e^{-j2\pi k/5} + e^{-j4\pi k/5}\,, \quad k = 0,\ldots,4$$

$$\begin{aligned} X(0) &= 3 + j\,0 \\ X(1) &= 0.5 - j\,1.5388 \\ X(2) &= 0.5 + j\,0.3633 \\ X(3) &= 0.5 - j\,0.3633 \\ X(4) &= 0.5 + j\,1.5388 \end{aligned}$$

c.

$$X(k) = 1 + e^{-j2\pi k/7} + e^{-j4\pi k/7}\,, \quad k = 0,\ldots,6$$

$$\begin{aligned}
X(0) &= 3 + j\,0 \\
X(1) &= 1.401 - j\,1.7568 \\
X(2) &= -0.1235 - j\,0.5410 \\
X(3) &= 0.7225 + j\,0.3479 \\
X(4) &= 0.7225 - j\,0.3479 \\
X(5) &= -0.1235 + j\,0.5410 \\
X(6) &= 1.401 + j\,1.7568
\end{aligned}$$

5.27.

a.

$$X(k) = 1 + e^{-j2\pi k/8} + e^{-j14\pi k/8}\,, \qquad k = 0,\ldots,7$$

Since $e^{-j14\pi k/8} = e^{j2\pi k/8}$ and $e^{-j2\pi k/8} + e^{j2\pi k/8} = 2\cos(2\pi k/8)$

$$\begin{aligned}
X(k) &= 1 + 2\cos\left(\frac{2\pi k}{8}\right)\,, \qquad k = 0,\ldots,7 \\
&= \{\, \underset{\substack{\uparrow \\ k=0}}{3},\ 2.4142,\ 1,\ -0.4142,\ -1,\ -0.4142,\ 1,\ 2.4142 \,\}
\end{aligned}$$

b.

$$X(k) = 1 + e^{-j2\pi k/8} + e^{-j4\pi k/8} + e^{-j12\pi k/8} + e^{-j14\pi k/8}\,, \qquad k = 0,\ldots,7$$

Since $e^{-j12\pi k/8} = e^{j4\pi k/8}$ and $e^{-j14\pi k/8} = e^{j2\pi k/8}$

$$\begin{aligned}
X(k) &= 1 + 2\cos\left(\frac{2\pi k}{8}\right) + 2\cos\left(\frac{4\pi k}{8}\right)\,, \qquad k = 0,\ldots,7 \\
&= \{\, \underset{\substack{\uparrow \\ k=0}}{5},\ 2.4142,\ -1,\ -0.4142,\ 1,\ -0.4142,\ -1,\ 2.4142 \,\}
\end{aligned}$$

c.

$$X(k) = 1 + e^{-j2\pi k/8} + e^{-j4\pi k/8} + + e^{-j6\pi k/8} + e^{-j10\pi k/8} + e^{-j12\pi k/8} + e^{-j14\pi k/8}\,, \qquad k = 0,\ldots,7$$

Since $e^{-j10\pi k/8} = e^{j6\pi k/8}$, $\quad e^{-j12\pi k/8} = e^{j4\pi k/8}$ and $\quad e^{-j14\pi k/8} = e^{j2\pi k/8}$

$$\begin{aligned}
X(k) &= 1 + 2\cos\left(\frac{2\pi k}{8}\right) + 2\cos\left(\frac{4\pi k}{8}\right) + 2\cos\left(\frac{6\pi k}{8}\right)\,, \qquad k = 0,\ldots,7 \\
&= \{\, \underset{\substack{\uparrow \\ k=0}}{7},\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1 \,\}
\end{aligned}$$

5.28.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi nk/N}\,, \qquad k = 0,\ldots,N-1$$

The transform sample $X[k]$ represents element $i = k+1$ of the vector $\mathbf{X}$. The coefficient matrix is

$$\mathbf{W} = \left[w_{ij}\right]_{N\times N}\,, \qquad w_{ij} = e^{2\pi(i-1)(j-1)/N}$$

which yields

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{-j2\pi/N} & \dots & e^{-j2\pi(N-1)/N} \\ \vdots & & & \\ 1 & e^{-j2\pi(N-1)/N} & \dots & e^{-j2\pi(N-1)(N-1)/N} \end{bmatrix}$$

5.29.

a.

$$x[n-2]_{\text{mod}\,4} = \{\underset{\substack{\uparrow\\ n=0}}{2}, 1, 4, 3\}$$

b.

$$x[n-4]_{\text{mod}\,5} = \{\underset{\substack{\uparrow\\ n=0}}{1}, 1, 0, 0, 1\}$$

c.

$$x[-n]_{\text{mod}\,8} = \{\underset{\substack{\uparrow\\ n=0}}{1}, 1, -3, -2, 1, 3, 2, 4\}$$

d.

$$x[-n+2]_{\text{mod}\,8} = \{\underset{\substack{\uparrow\\ n=0}}{2}, 4, 1, 1, -3, -2, 1, 3\}$$

5.30.

a.

$$x_E[n] = \frac{x[n] + x^*[-n]_{\text{mod}\,4}}{2} = \{\underset{\substack{\uparrow\\ n=0}}{4}, 2, 2, 2\}$$

$$x_O[n] = \frac{x[n] - x^*[-n]_{\text{mod}\,4}}{2} = \{\underset{\substack{\uparrow\\ n=0}}{0}, 1, 0, -1\}$$

b.

$$x_E[n] = \frac{x[n] + x^*[-n]_{\text{mod}\,5}}{2} = \{\underset{\substack{\uparrow\\ n=0}}{1}, 0.5, 0.5, 0.5, 0.5\}$$

$$x_O[n] = \frac{x[n] - x^*[-n]_{\text{mod}\,5}}{2} = \{\underset{\substack{\uparrow\\ n=0}}{0}, 0.5, 0.5, -0.5, -0.5\}$$

c.

$$x_E[n] = \frac{x[n] + x^*[-n]_{\text{mod}\,8}}{2} = \{\underset{\substack{\uparrow\\ n=0}}{1}, 2.5, -0.5, 0.5, 1, 0.5, -0.5, 2.5\}$$

$$x_O[n] = \frac{x[n] - x^*[-n]_{\text{mod}\,8}}{2} = \{\underset{\substack{\uparrow\\ n=0}}{0}, 1.5, 2.5, 2.5, 0, -2.5, -2.5, -1.5\}$$

5.31.

a.

$$\begin{aligned} X[k] &= 1 + e^{-j2\pi k/5} + e^{-j4\pi k/5} \\ &= \{ \underset{\substack{\uparrow \\ k=0}}{3}, (0.5 - j\,1.5388), (0.5 + j\,0.3633), (0.5 - j\,0.3633), (0.5 + j\,1.5388) \} \end{aligned}$$

b.

$$\begin{aligned} R[k] &= e^{-j2\pi/5}\, X[k] \\ &= e^{-j2\pi k/5} + e^{-j4\pi k/5} + e^{-j6\pi k/5} \\ &= \{ \underset{\substack{\uparrow \\ k=0}}{3}, (-1.3090 - j\,0.9511), (-0.1910 - j\,0.5878), (-0.1910 + j\,0.5878), (-1.3090 + j\,0.9511) \} \end{aligned}$$

c.

$$r[n] = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 1, 1, 1, 0 \}$$

d.

$$\begin{aligned} S[k] &= e^{-j4\pi/5}\, X[k] \\ &= e^{-j4\pi k/5} + e^{-j6\pi k/5} + e^{-j8\pi k/5} \\ &= \{ \underset{\substack{\uparrow \\ k=0}}{3}, (-1.3090 + j\,0.9511), (-0.1910 + j\,0.5878), (-0.1910 - j\,0.5878), (-1.3090 - j\,0.9511) \} \end{aligned}$$

$$s[n] = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 0, 1, 1, 1 \}$$

e.

Using the time shifting property

$$\begin{aligned} r[n] &= x[n-1]_{\text{mod } 5} \\ s[n] &= x[n-2]_{\text{mod } 5} \end{aligned}$$

5.32.

a.

$$\begin{aligned} X[k] &= 5 + 4\,e^{-j2\pi k/5} + 3\,e^{-j4\pi k/5} + 2\,e^{-j6\pi k/5} + e^{-j8\pi k/5} \\ &= \{ \underset{\substack{\uparrow \\ k=0}}{15}, (2.5 - j\,3.4410), (2.5 - j\,0.8123), (2.5 + j\,0.8123), (2.5 + j\,3.4410) \} \end{aligned}$$

b.

$$\begin{aligned} R[k] &= 5 + 4\,e^{j2\pi k/5} + 3\,e^{j4\pi k/5} + 2\,e^{j6\pi k/5} + e^{j8\pi k/5} \\ &= 5 + 4\,e^{-j8\pi k/5} + 3\,e^{-j6\pi k/5} + 2\,e^{-j4\pi k/5} + e^{-j2\pi k/5} \\ &= \{ \underset{\substack{\uparrow \\ k=0}}{15},\ (2.5 + j\,3.4410),\ (2.5 + j\,0.8123),\ (2.5 - j\,0.8123),\ (2.5 - j\,3.4410) \} \end{aligned}$$

c.

$$r[n] = \{ \underset{\substack{\uparrow \\ n=0}}{5},\ 1,\ 2,\ 3,\ 4 \}$$

It follows that $r[n] = x[-n]_{\text{mod}\,5}$ which is consistent with the time reversal property of the DFT.

5.33.

a. Let the transform $X[k]$ be written as

$$X[k] = X_E[k] + X_O[k]$$

where $X_E[k]$ and $X_O[k]$ are the conjugate symmetric and conjugate antisymmetric components of $X[k]$. Using Eqns. (5.285) and (5.286) we have

$$x_r[n] \stackrel{\text{DFT}}{\longleftrightarrow} X_E[k]$$

$$j\,x_i[n] \stackrel{\text{DFT}}{\longleftrightarrow} X_O[k]$$

Details of computing $X_E[k]$ and $X_O[k]$ are in the table below:

k	$X[k]$	$X^*[-k]_{\text{mod}\,6}$	$X_E[k]$	$X_O[k]$
0	$2+j3$	$2-j3$	$2+j0$	$0+j3$
1	$1+j5$	$3-j1$	$2+j2$	$-1+j3$
2	$-2+j4$	$2+j0$	$0+j2$	$-2+j2$
3	$-1-j3$	$-1+j3$	$-1+j0$	$0-j3$
4	$2+j0$	$-2-j4$	$0-j2$	$2+j2$
5	$3+j1$	$1-j5$	$2-j2$	$1+j3$

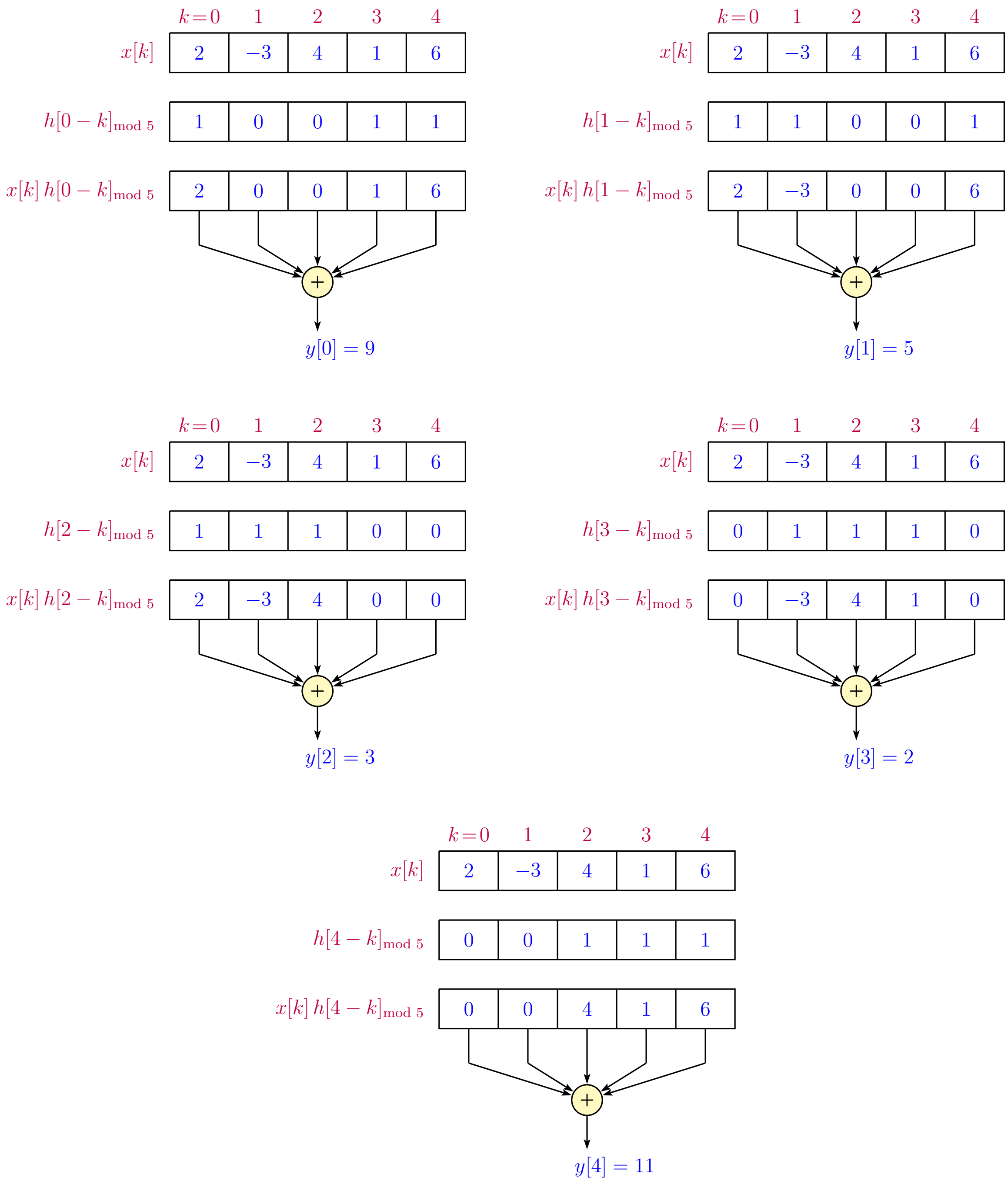
$$\mathscr{F}\left\{x_r[n]\right\} = X_E[k] = \{ \underset{\substack{\uparrow \\ k=0}}{2},(2+j2),\ j2,\ -1,\ -j2,\ (2-j2) \}$$

b.

$$\mathscr{F}\left\{x_i[n]\right\} = -j\,X_O[k] = \{ \underset{\substack{\uparrow \\ k=0}}{3},(3+j1),\ (2+j2),\ -3,\ (2-j2),\ (3-j1) \}$$

5.34.

a.



b.

The two transforms are

$$X[k] = 2 - 3\,e^{-j2\pi k/5} + 4\,e^{-j4\pi k/5} + e^{-j6\pi k/5} + 6\,e^{-j8\pi k/5}$$

and

$$H[k] = 1 + e^{-j2\pi k/5} + e^{-j4\pi k/5}$$

c. The product of the two transforms is

$$\begin{aligned}Y[k] =& X[k]\,H[k]\\ =& 2 - e^{-j2\pi k/5} + 3\,e^{-j4\pi k/5} + 2\,e^{-j6\pi k/5} + 11\,e^{-j8\pi k/5} + 7\,e^{-j10\pi k/5} + 6\,e^{-j12\pi k/5}\end{aligned}$$

Realizing that $e^{-j10\pi k/5} = 1$ and $e^{-j12\pi k/5} = e^{-j2\pi k/5}$ the transform $Y[k]$ is

$$\begin{aligned}Y[k] =& 2 - e^{-j2\pi k/5} + 3\,e^{-j4\pi k/5} + 2\,e^{-j6\pi k/5} + 11\,e^{-j8\pi k/5} + 7 + 6\,e^{-j2\pi k/5}\\ =& 9 + 5\,e^{-j2\pi k/5} + 3\,e^{-j4\pi k/5} + 2\,e^{-j6\pi k/5} + 11\,e^{-j8\pi k/5}\end{aligned}$$

which is the DFT of the signal

$$y[n] = \{\underset{\substack{\uparrow\\ n=0}}{9}, 5, 3, 2, 11\}$$

5.35.

The signal $x[n]$ has $N_x = 5$ samples, and the signal $h[n]$ has $N_h = 3$ samples (its last two samples have zero amplitudes). Therefore, the length of the linear convolution result is $N_y = N_x + N_h - 1 = 7$ samples. In other words, the significant samples of the linear convolution result is in the index range $n = 0, \ldots, 6$.

The 7-point circular convolution of $x[n]$ and $h[n]$ would match the linear convolution of the two signals.

$$y[n] = \{\underset{\substack{\uparrow\\ n=0}}{2}, -1, 3, 2, 11, 7, 6\}$$

5.36.

a. The DTFT of the signal $x[n]$ is

$$X(\Omega) = \sum_{n=0}^{11} e^{-j\Omega n}$$

Therefore

$$S[k] = X(\Omega)\Big|_{\Omega=2\pi k/10} = \sum_{n=0}^{11} e^{-j(2\pi k/10)n}$$

Splitting the summation into two parts

$$S[k] = \sum_{n=0}^{9} e^{-j(2\pi k/10)n} + \sum_{n=10}^{11} e^{-j(2\pi k/10)n}$$

Changing the variable of the second summation through $m = n - 10$

$$S[k] = \sum_{n=0}^{9} e^{-j(2\pi k/10)n} + \sum_{m=0}^{1} e^{-j(2\pi k/10)(m+10)}$$

and recognizing that

$$e^{-j(2\pi k/10)(m+10)} = e^{-j(2\pi k/10)m}$$

we obtain

$$S[k] = \sum_{n=0}^{9} e^{-j(2\pi k/10)n} + \sum_{m=0}^{1} e^{-j(2\pi k/10)m}$$

which can be written as

$$S[k] = 2 + 2\,e^{-j2\pi k/10} + \sum_{n=2}^{9} e^{-j2\pi kn/10}$$

which is the 10-point DFT of the signal

$$s[n] = \{\underset{\substack{\uparrow \\ n=0}}{2},\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\}$$

b. The DTFT evaluated at N equally-spaced frequencies is

$$S[k] = X(\Omega)\Big|_{\Omega=2\pi k/N} = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j(2\pi k/N)n}$$

The infinite summation can be broken down into length-N summations as

$$S[k] = \sum_{r=-\infty}^{\infty} \left[\sum_{n=rN}^{rN+N-1} x[n]\,e^{-j(2\pi k/N)n} \right]$$

Applying the variable change $m = n - rN$ to the inner summation yields

$$S[k] = \sum_{r=-\infty}^{\infty} \left[\sum_{m=0}^{N-1} x[m+rN]\,e^{-j(2\pi k/N)(m+rN)} \right]$$

Changing the order of summations and recognizing that

$$e^{-j(2\pi k/N)(m+rN)} = e^{-j(2\pi k/N)m}$$

we obtain

$$S[k] = \sum_{m=0}^{N-1} \left[\sum_{r=-\infty}^{\infty} x[m+rN] \right] e^{-j(2\pi k/N)m}$$

This result is the N-point DFT of the signal

$$s[n] = \sum_{r=-\infty}^{\infty} x[n+rN]$$

5.37.

a. The DTFS coefficients $\tilde{c}_k$ of the signal $\tilde{x}[n]$ are computed using the following:

```
>> k = [0:7];
>> xn = [0,1,2,3,4,5,6,7];
>> ck = ss_dtfs(xn,k)
```

b. The DTFS coefficients $\tilde{d}_k$ of the signal $\tilde{y}[n]$ are computed using the following:

```
>> gn = [5,6,7,0,1,2,3,4];
>> dk = ss_dtfs(gn,k)
```

c. The relationship $\tilde{d}_k = e^{-j6\pi k/8}\,\tilde{c}_k$ can be verified using the following:

```
>> ek = exp(-j*6*pi*k/8);
>> ck
>> ek
>> dk
```

5.38. Print the coefficients $\tilde{c}_k$ and $\tilde{d}_k$ and in the range $k = -7, \ldots, 7$ and check symmetry properties using the following code:

```
k = [-7:7];
xn = [0,1,2,3,4,5,6,7];
ck = ss_dtfs(xn,k)
gn = [5,6,7,0,1,2,3,4];
dk = ss_dtfs(gn,k)
```

5.39. The script "`pr_05_39.m`" listed below computes the DTFS coefficients of the periodic pulse train:

```
k = [0:N-1];
k(1) = eps;  % Avoid division by zero
ck = sin(pi*k/N*(2*L+1))./(N*sin(pi*k/N));  % DTFS coefficients
subplot(211)
stem(k,abs(ck));
subplot(212)
stem(k,angle(ck));
```

The graphs can be obtained by using the script as shown below:

a.

```
>> N = 30;
>> L = 5;
>> pr_05_39;
```

b.

```
>> N = 30;
>> L = 8;
>> pr_05_39;
```

c.

```
>> N = 40;
>> L = 10;
>> pr_05_39;
```

d.

```
>> N = 40;
>> L = 15;
>> pr_05_39;
```

5.40.

The script "`pr_05_40.m`" listed below computes the DTFS coefficients of the periodic pulse train. It is a slightly modified version of the script used in Problem 5.39.

```
k = [-M:M]+eps;
N = 2*M+1;
Omg = 2*pi*k/N;
ck = sin(pi*k/N*(2*L+1))./(N*sin(pi*k/N));   % DTFS coefficients
stem(Omg, ck);
```

The graphs can be obtained by using the script as shown below:

a.

```
>> M = 20;
>> L = 3;
>> pr_05_40;
```

b.

```
>> M = 35;
>> L = 3;
>> pr_05_40;
```

c.

```
>> M = 60;
>> L = 3;
>> pr_05_40;
```

5.41.

a.

```
k = [-3:3]+eps;
n = [-50:50];
ck = sin(7*pi*k/40)./(40*sin(pi*k/40));   % DTFS coefficients
sum = 0;
for kk = 1:7,
  sum = sum+ck(kk)*exp(j*2*pi/40*k(kk)*n);
end;
sum = real(sum);        % Clean up imaginary part that is
                        % due to roundoff error.
stem(n,sum);
```

b.

```
k = [-3:3]+eps;
n = [-50:50];
ck = sin(13*pi*k/40)./(40*sin(pi*k/40));   % DTFS coefficients
sum = 0;
for kk = 1:7,
  sum = sum+ck(kk)*exp(j*2*pi/40*k(kk)*n);
end;
sum = real(sum);      % Clean up imaginary part that is
                      % due to roundoff error.
stem(n,sum);
```

5.42.

a.

The output of the length-4 moving average filter can be computed using the following script:

```
>>  clear all                % Clear any persistent variables
>>  n = [0:49];
>>  inp = exp(j*0.2*pi*n);   % Input stream
>>  out = [];                % Output stream
>>  for nn = 1:50,
>>    xn = inp(nn);
>>    yn = ss_movavg4(xn);
>>    out = [out,yn];
>>  end;
```

The steady-state response was found in Problem 5.24 and is computed using the following statement:

```
>>  yss = 0.7694*exp(j*(0.2*pi*n-0.9425));   % Steady state response
```

The two responses may be printed on the screen for comparison using

```
>>  out
>>  yss
```

b.

```
clear all                % Clear any persistent variables
n = [0:49];
inp = cos(0.2*pi*n);     % Input stream
out = [];                % Output stream
for nn = 1:50,
  xn = inp(nn);
  yn = ss_movavg4(xn);
  out = [out,yn];
end;
yss = 0.7694*cos(0.2*pi*n-0.9425);
% Compare the results
[n',out',yss']
```

c.

```
clear all                  % Clear any persistent variables
n = [0:49];
inp = 2*sin(0.3*pi*n); % Input stream
out = [];                  % Output stream
for nn = 1:50,
  xn = inp(nn);
  yn = ss_movavg4(xn);
  out = [out,yn];
end;
yss = 1.0474*sin(0.3*pi*n-1.4137);
% Compare the results
[n',out',yss']
```

d.

```
clear all                  % Clear any persistent variables
n = [0:49];
inp = 3*cos(0.1*pi*n)-5*sin(0.2*pi*n); % Input stream
out = [];                  % Output stream
for nn = 1:50,
  xn = inp(nn);
  yn = ss_movavg4(xn);
  out = [out,yn];
end;
yss = 2.8179*cos(0.1*pi*n-0.9425)-3.8470*sin(0.2*pi*n-0.9425);
% Compare the results
[n',out',yss']
```

5.43.

a. The script "`pr_5_43a.m`" listed below computes the output by iterating through the difference equation for $n = 0, \ldots, 49$.

```
n = [0:49];
inp = ss_per(xper,n);
out = [];
ynm1 = 0;
ynm2 = 0;
xnm1 = 0;
for nn=1:50,
  xn = inp(nn);
  yn = xn+2*xnm1-ynm1-0.89*ynm2;
  xnm1 = xn;
  ynm2 = ynm1;
  ynm1 = yn;
  out = [out,yn];
end;
```

The variable "`xper`" should hold samples of one period prior to using this script. The variable "`out`" holds the response of the system after the script completes.

b. Compute the output of the system for the input signal in part (a) of Problem 5.3 using the following statements:

```
>> xper = [4,3,2,1,0,1,2,3];  % One period of the signal.
>> pr_5_43a;                   % Use the script from part (a).
>> out1 = out(26:50);          % Discard the first 25 samples.
```

c. Compute the steady-state output using DTFS coefficients. Us an anonymous function to represent $H(\Omega)$ and then evaluate it at the frequencies $\Omega_k = 2\pi k/8$. Compute the DTFS coefficients of the input and the output signals and construct the output signal.

```
>> H = @(Omg) (1+2*exp(-j*Omg))./(1+exp(-j*Omg)+0.89*exp(-j*2*Omg));
>> k = [0:7];
>> Hk = H(2*pi*k/8);
>> n = [25:49];
>> ck = ss_dtfs(xper,k);
>> dk = Hk.*ck;
>> out2 = real(ss_invdtfs(dk,n));  % Clean imaginary part that is
                                   % due to roundoff error
```

The output computed from the difference equation and the correct steady-state output computed through DTFS may be graphed simultaneously using the following statements:

```
>> stem(n, out1);
>> hold on;
>> plot(n,out2,'ro');
>> hold off;
```

5.44.

a. The function **ss_dftmat**(..) is listed below.

```
function W = ss_dftmat(N)
  W = zeros(N,N);
  for k=0:N-1,
    for n=0:N-1,
      W(k+1,n+1) = exp(j*2*pi/N*n*k);
    end;
  end;
```

b. Using **ss_dftmat**(..) the 10-point DFT of $x[n]$ is computed as

```
>> W = ss_dftmat(10);
>> xn = [1,1,1,1,1,1,1,1,1,1]';
>> Xk=W*xn;
```

c. Using **ss_dftmat**(..) the 20-point DFT of $x[n]$ is computed as

```
>> W = ss_dftmat(20);
>> xn = [ones(1,10),zeros(1,10)]';
>> Xk=W*xn;
```

5.45.

a. One period of the signal $\tilde{x}(t)$ is

$$\tilde{x}(t) = \begin{cases} \sin(2\pi t)\,, & 0 \le t < 0.5 \\ 0\,, & 0.5 \le t < 1 \end{cases}$$

The following script computes estimated EFS coefficients using the function **ss_efsapprox**(..):

```
t = [0:99]/100;
x = sin(2*pi*t);
x = x.*(x>=0);
k = [-15:15];
ck_est = ss_efsapprox(x,k)
```

b. Actual EFS coefficients were determined in Example 4-10 of Chapter 4 as

$$c_k = \begin{cases} 0\,, & k \text{ odd and } k \ne \mp 1 \\ -j/4\,, & k = 1 \\ j/4\,, & k = -1 \\ \dfrac{-1}{\pi\left(k^2-1\right)}\,, & k \text{ even} \end{cases}$$

They can be computed with the following script:

```
>> kk = k+eps;    % Avoid division by zero.
>> ck_act = -0.25*j*(k==1)+0.25*j*(k==-1)-1./(pi*(kk.*kk-1)).*(mod(k,2)==0)
```

c. The statement below allows the actual and estimated coefficients to be printed side by side for comparison:

```
>> conj([k',ck_act',ck_est'])
```

Note: In the last MATLAB statement, vectors "`ck_act`" and "`ck_est`" are transposed so that they can be tabulated on the screen in column format. However, it must be remembered that they are complex vectors. The transpose operation in MATLAB has an additional effect of conjugating complex vectors. The **conj**(..) function is used for counteracting that.

5.46.

a. The scriupt listed below defined an anonymous function to return the DTFT of the signal $x[n] = n\left(u[n] - u[n-12]\right)$.

```
Q = @(n,Omg) n*exp(-j*n*Omg);
X = @(Omg) Q(1,Omg)+Q(2,Omg)+Q(3,Omg)+Q(4,Omg)+Q(5,Omg)+Q(6,Omg)+...
           Q(7,Omg)+Q(8,Omg)+Q(9,Omg)+Q(10,Omg)+Q(11,Omg);
```

b. The transform $X(\Omega)$ may be evaluated at 10 equally spaced frequencies and the corresponding inverse transform may be computed with the following code.

```
>> k = [0:9];
>> Sk = X(2*pi*k/10);
>> sn = ifft(Sk)
```

c. The transform $X(\Omega)$ may be evaluated at 8 equally spaced frequencies and the corresponding inverse transform may be computed with the following code.

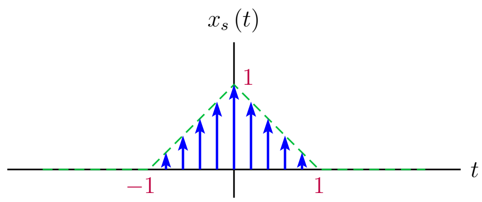
```
>> k = [0:7];
>> Sk = X(2*pi*k/8);
>> sn = ifft(Sk)
```

Chapter 6

Sampling and Reconstruction

6.1.

a.



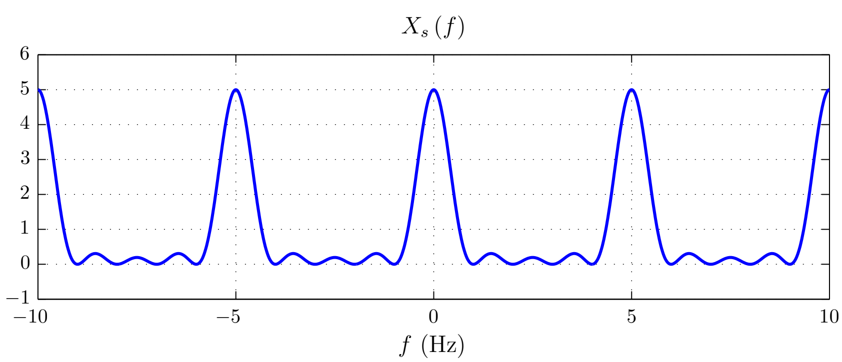
b. Using $A = 1$ and $\tau = 1$ the transform of the original signal $x_a(t)$ is

$$X_a() = \text{sinc}^2(f)$$

The transform of the impulse sampled signal $x_s(t)$ is

$$\begin{aligned} X_s() &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - k f_s) \\ &= 5 \sum_{k=-\infty}^{\infty} \text{sinc}^2(f - 5k) \end{aligned}$$

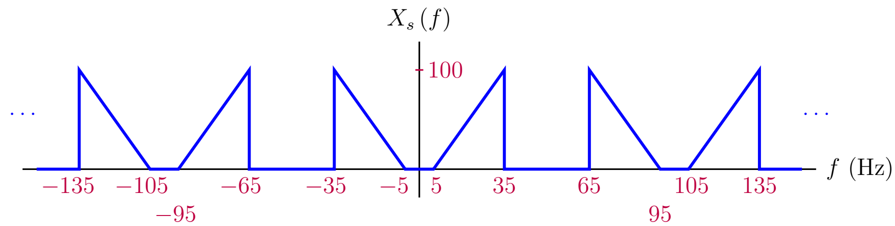
c.



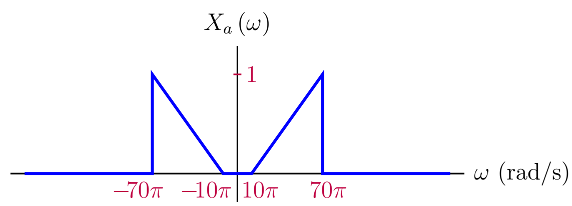
6.2.

$$X_s\left(\right) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(f - kf_s\right)$$

$$= 100 \sum_{k=-\infty}^{\infty} X_a\left(f - 100k\right)$$



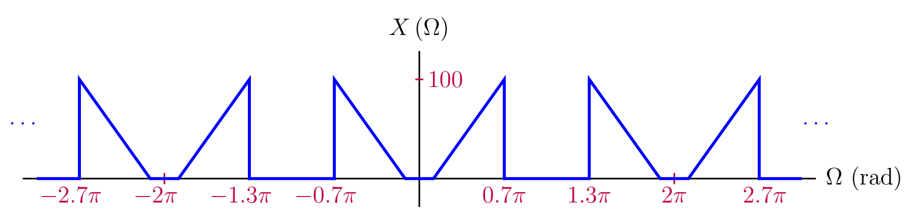
6.3. Start with $X_a(\omega)$ shown below.



The spectrum of the discrete-time signal $x[n]$ is

$$X(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(\frac{\Omega - 2\pi k}{T_s}\right)$$

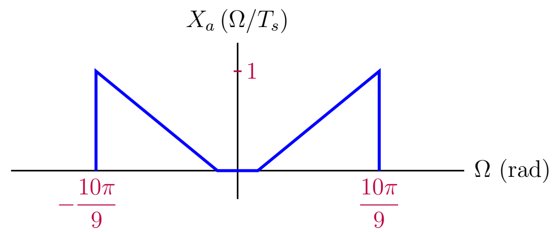
$$= 100 \sum_{k=-\infty}^{\infty} X_a\left(\frac{\Omega - 2\pi k}{0.01}\right)$$

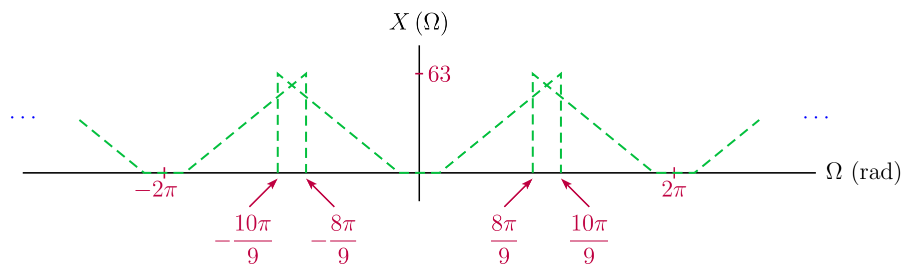


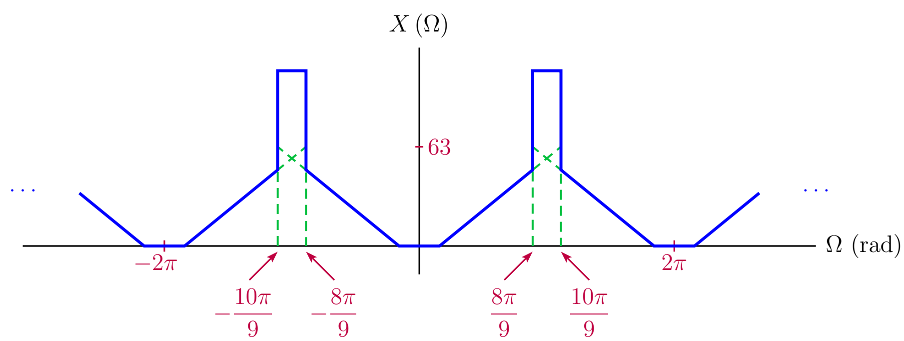
6.4. If aliasing is to be avoided, the minimum sampling rate required is $f_{s,min} = 70$ Hz. Let the sampling rate be 90 percent of the required minimum, that is, $f_s = 63$ Hz.

$$X(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(\frac{\Omega - 2\pi k}{T_s}\right)$$

$$= 63 \sum_{k=-\infty}^{\infty} X_a\left(\frac{\Omega - 2\pi k}{1/63}\right)$$







6.5.

a. The continuous-time signal is

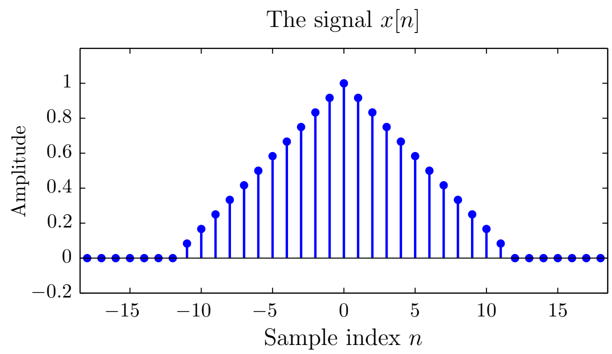
$$x_a(t) = \begin{cases} 1 - \dfrac{A\,|t|}{\tau}\,, & -\tau < t < \tau \\ 0\,, & \text{otherwise} \end{cases}$$

Using $A = 1$ and $\tau = 1$ sec, we get

$$x_a(t) = \begin{cases} 1 - |t|\,, & -\tau < t < \tau \\ 0\,, & \text{otherwise} \end{cases}$$

The sampling interval is $T_s = 1/f_s = 1/12$ sec. The sampled signal $x[n] = x_a(nT_s)$ is obtained as

$$x[n] = \begin{cases} 1 - \dfrac{|n|}{12}\,, & -12 \le n \le 12 \\ 0\,, & \text{otherwise} \end{cases}$$



b. Let us use Eqn. (6.25). The spectrum of the continuous-time signal $x_a(t)$ is

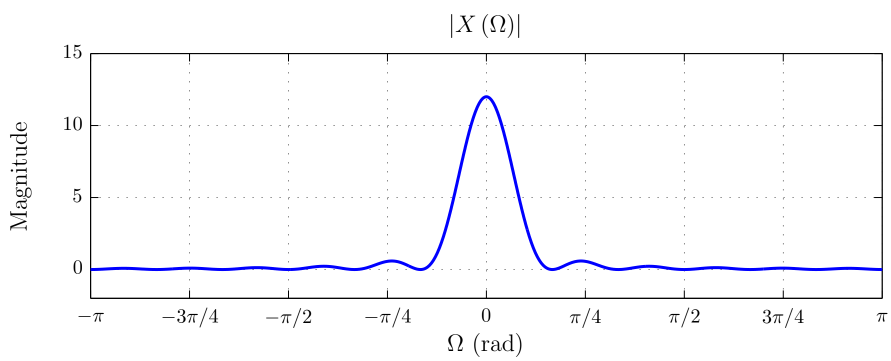
$$X_a(\omega) = A\tau \, \text{sinc}^2\left(\frac{\omega\tau}{2\pi}\right) = \text{sinc}^2\left(\frac{\omega}{2\pi}\right)$$

and $X(\Omega)$ is computed using Eqn. (6.25) as

$$X(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(\frac{\Omega - 2\pi k}{T_s}\right) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{\Omega - 2\pi k}{2\pi\, T_s}\right)$$

Using $T_s = 1/12$ s, we get

$$X(\Omega) = 12 \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{\Omega - 2\pi k}{2\pi\,(1/12)}\right) = 12 \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{6\Omega}{\pi} - 12k\right)$$



6.6.

The discrete-time signal is

$$x[n] = \begin{cases} \sin\left(\dfrac{\pi n}{15}\right), & n = 0, \ldots, 14 \\ 0\,, & \text{otherwise} \end{cases}$$

or, equivalently

$$x[n] = \sin\left(\frac{\pi n}{15}\right)\left(u[n] - u[n-15]\right)$$

Let

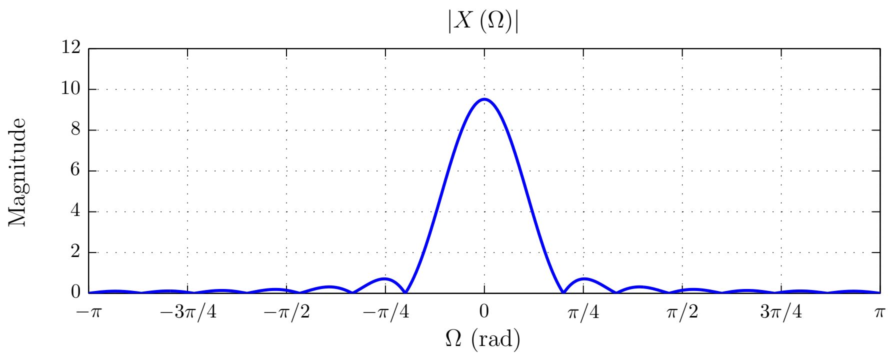
$$p[n] = u[n] - u[n-15]$$

so that

$$P(\Omega) = \sum_{n=0}^{14} e^{-j\Omega n} = \frac{\sin(15\Omega/2)}{\sin(\Omega/2)}\, e^{-j7\Omega}$$

Using the modulation property, $X(\Omega)$ is

$$X(\Omega) = -j\,\frac{1}{2}\,P(\Omega - \pi/15) + j\,\frac{1}{2}\,P(\Omega + \pi/15)$$



6.7.

a. Not bandlimited. Cannot be sampled without loss of information.

b. Not bandlimited. Cannot be sampled without loss of information.

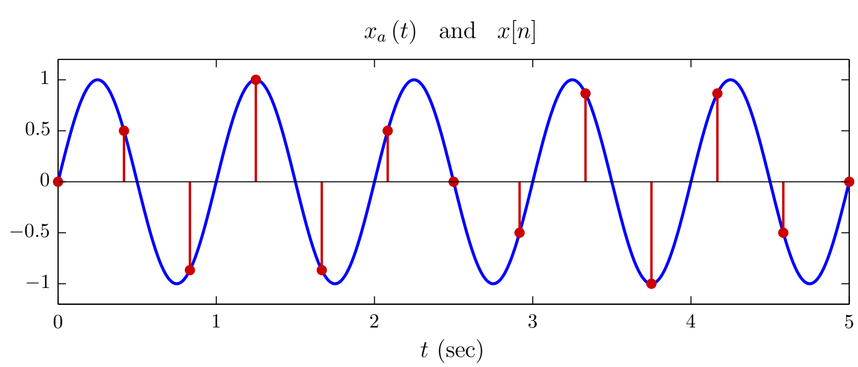
c. Bandlimited. $f_{max} = 75$ Hz, $f_s \geq 150$ Hz.

d. Bandlimited. $f_{max} = 175$ Hz, $f_s \geq 350$ Hz.

e. Not bandlimited. Cannot be sampled without loss of information.

6.8.

a, b.



c.

$$x[n] = \sin\left(2\pi f_a t\right)\Big|_{t=n/f_s} = \sin\left(\frac{2\pi n}{2.4}\right) = \sin\left(\frac{2\pi n}{2.4} + 2\pi r n\right) \qquad r,n\,:\,\text{Integer}$$

The normalized frequency is

$$F = \frac{f_a}{f_s} + r$$

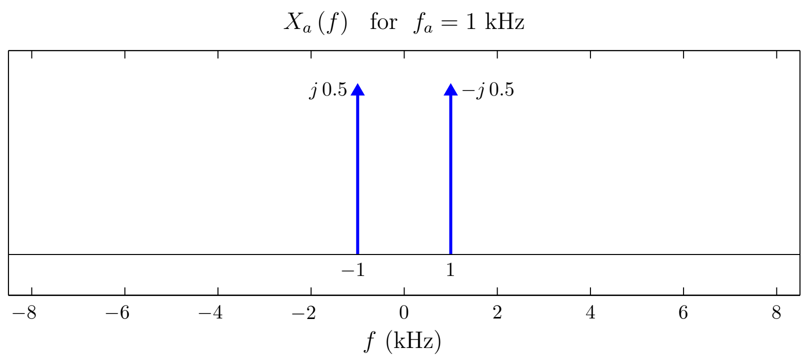
For $r = 1$: $\quad F = 1.4167 \quad \Rightarrow \quad f_a = 3400$ Hz

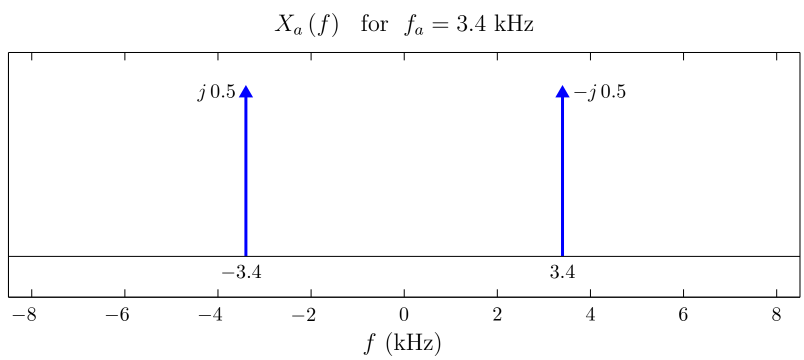
For $r = 2$: $\quad F = 2.4167 \quad \Rightarrow \quad f_a = 5800$ Hz

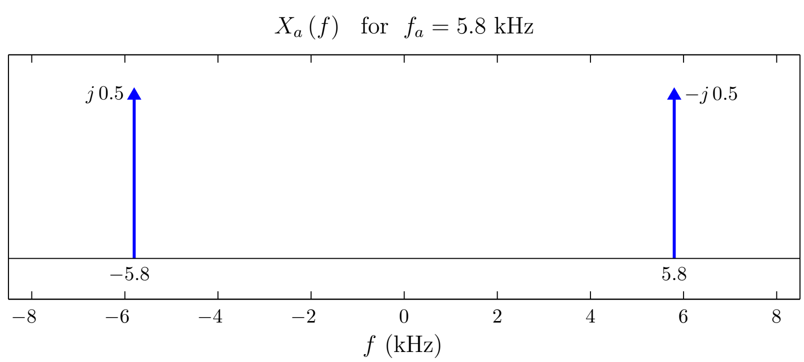
For $r = 3$: $\quad F = 3.4167 \quad \Rightarrow \quad f_a = 8200$ Hz

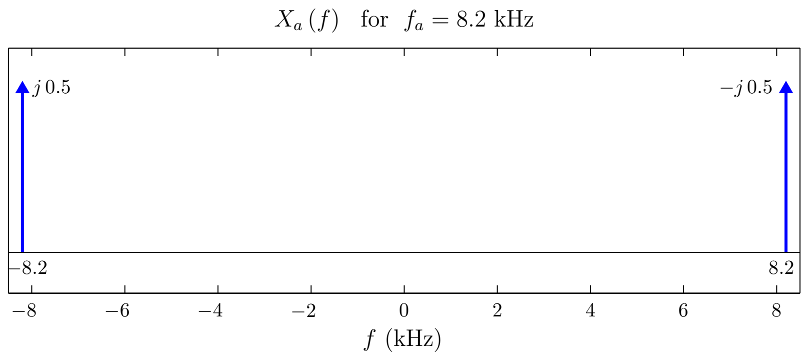
6.9.

a.





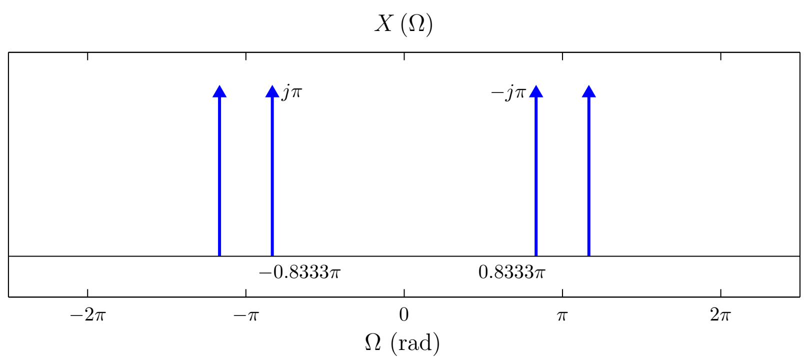




b. Since all four continuous-time signals lead to the same discrete-time signal when sampled; they all have the same DTFT spectrum.

$$x[n] = \sin\left(\frac{2\pi n}{2.4}\right) = \sin(0.8333\pi n)$$

$$X(\Omega) = \frac{\pi}{j}\sum_{m=-\infty}^{\infty}\left[\delta(\Omega - 0.8333\pi - 2\pi m) - \delta(\Omega + 0.8333\pi - 2\pi m)\right]$$



6.10.

a.

$$x[n] = x_a(t)\Big|_{t=n/f_s} = \sin\left(\frac{500\pi n}{f_s}\right) = \sin(0.4\pi n + 2\pi r n)\ , \qquad r, n : \text{Integer}$$

The normalized frequency is

$$F = \frac{f_a}{f_s} = \frac{250}{f_s} = 0.2 + r$$

Using $r = 0$ for proper sampling, the sampling rate must be

$$f_s = \frac{250}{0.2} = 1250 \text{ Hz.}$$

b.

For $r = 1$: $\quad \dfrac{250}{f_s} = 1.2 \quad \Rightarrow \quad f_s = 208.33$ Hz

For $r=2$: $\quad \dfrac{250}{f_s}=2.2 \quad \Rightarrow \quad f_s=113.64$ Hz

c. Let the new signal frequency be $\bar{f}_a$.

$$\frac{\bar{f}_a}{f_s}=0.2+r$$

For $r=1$: $\quad \dfrac{\bar{f}_a}{1250}=1.2 \quad \Rightarrow \quad \bar{f}_a=1500$ Hz

For $r=2$: $\quad \dfrac{\bar{f}_a}{1250}=2.2 \quad \Rightarrow \quad \bar{f}_a=2750$ Hz

6.11.

a.

$$\begin{aligned} x[n] &= 3\cos(\pi n)+5\sin(2.5\pi n) \\ &= 3\cos(\pi n)+5\sin(0.5\pi n) \end{aligned}$$

$$\mathscr{F}\left\{3\cos(\pi n)\right\}=6\pi\sum_{m=-\infty}^{\infty}\delta(\Omega-\pi-2\pi m)$$

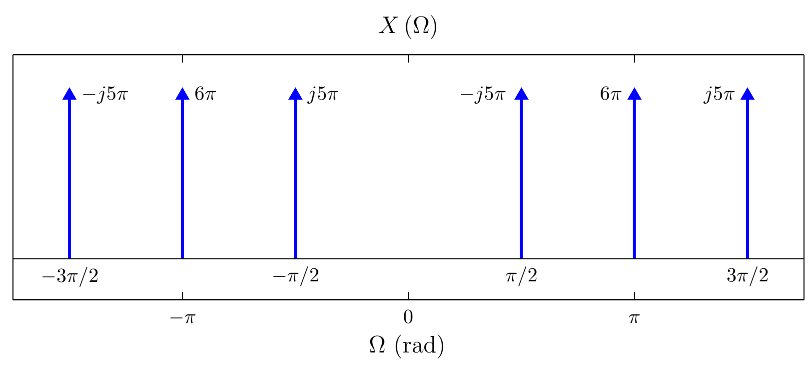
$$\mathscr{F}\left\{5\sin(\pi n)\right\}=\frac{5\pi}{j}\sum_{m=-\infty}^{\infty}\left[\delta(\Omega-\pi/2-2\pi m)-\delta(\Omega+\pi/2-2\pi m)\right]$$

Let

$$\tilde{X}(\Omega)=3\pi\,\delta(\Omega-\pi)+3\pi\,\delta(\Omega+\pi)-j\,5\pi\,\delta(\Omega-\pi/2)+j\,5\pi\,\delta(\Omega+\pi/2)$$

so that $X(\Omega)$ can be written as

$$X(\Omega)=\sum_{m=-\infty}^{\infty}\tilde{X}(\Omega-2\pi m)$$



b.

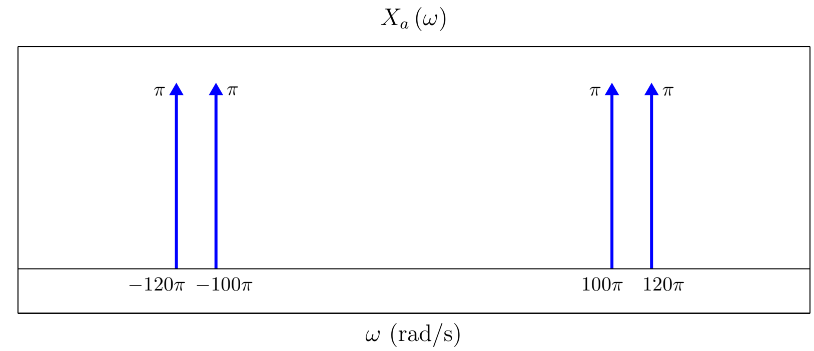
The recovered signal would be

$$\hat{x}(t)=3\cos(100\pi t)+5\sin(50\pi t)$$

6.12.

a.

$$X_a(\omega) = \pi\,\delta\left(f - 100\pi\right) + \pi\,\delta\left(f + 100\pi\right) + \pi\,\delta\left(f - 120\pi\right) + \pi\,\delta\left(f + 120\pi\right)$$

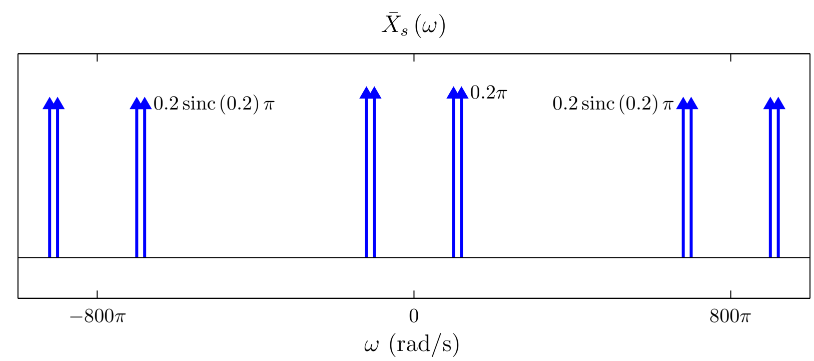


b. The period is $T_s = 1/f_s = 1/400 = 0.0025$ s, or equivalently $T_s = 2.5$ ms. This corresponds to a duty cycle of $d = \tau/T_s = 0.2$.

$$\bar{X}_s(\omega) = d \sum_{k=-\infty}^{\infty} \text{sinc}\,(kd)\; X_a(\omega - k\omega_s)$$

$$= 0.2 \sum_{k=-\infty}^{\infty} \text{sinc}\,(0.2k)\; X_a(\omega - 800\pi k)$$

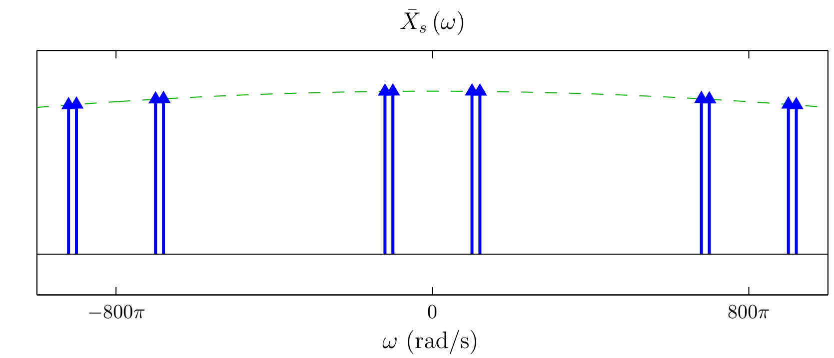
c.



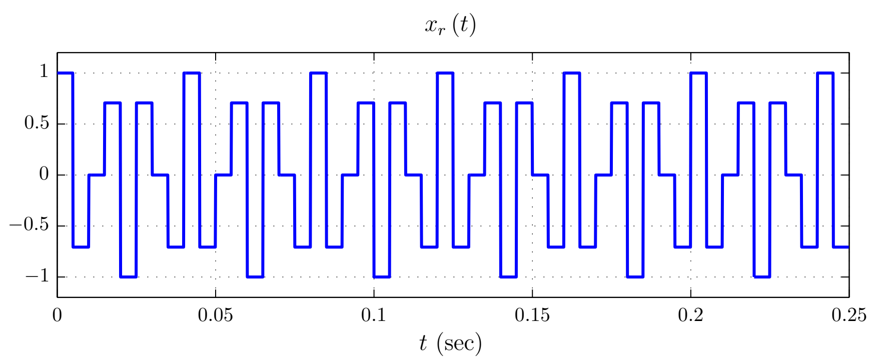
6.13. The spectrum of the zero-order hold sampled signal is

$$\bar{X}_s(\omega) = d\;\text{sinc}\left(\frac{\omega d T_s}{2\pi}\right) e^{-j\omega d T_s/2} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

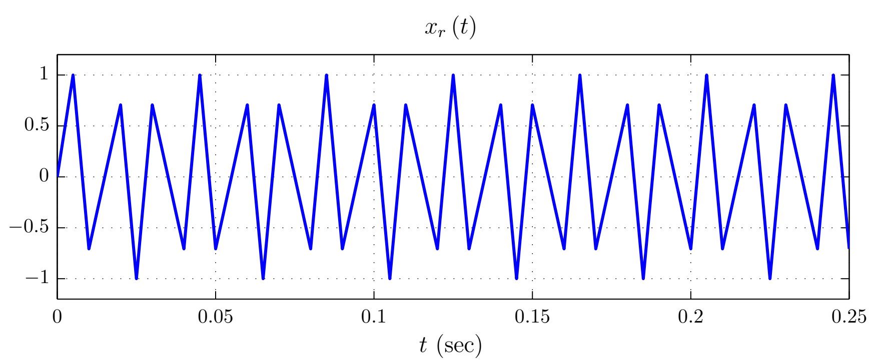
$$= 0.2\;\text{sinc}\left(\frac{0.2\omega}{800\pi}\right) e^{-j0.1\omega/800} \sum_{k=-\infty}^{\infty} X_a(\omega - 800\pi k)$$



6.14.



6.15.



6.16.

Using the geometric series formula

$$w[m] = \frac{1}{D}\frac{1-e^{j2\pi m}}{1-e^{j2\pi m/D}}$$

For $m \neq nD$ then the numerator is equal to zero, and the denominator is nonzero, resulting in $w[m] = 0$. If $m = nD$ then both the numerator and the denominator are equal to zero, requiring the use of

L'Hospital's rule:

$$w[nD] = \frac{1}{D}\; \frac{j2\pi\, e^{j2\pi m}}{j\,(2\pi/D)\; e^{j2\pi m/D}}\bigg|_{m=nD} = 1$$

6.17. Using the inverse DTFT relationship

$$h_r[n] = \frac{1}{2\pi}\int_{-\pi/L}^{\pi/L} (L)\; e^{j\Omega n}\, d\Omega = \frac{L}{2\pi}\;\frac{e^{j\Omega n}}{jn}\bigg|_{-\pi/L}^{\pi/L}$$

The result can be simplified to

$$h_r[n] = \frac{L}{\pi n}\,\sin\left(\frac{\pi n}{L}\right) = \text{sinc}\,(n/L)$$

6.18.

a. Not bandlimited. Cannot be downsampled without loss of information.

b. Not bandlimited. Cannot be downsampled without loss of information.

c. $\Omega_{max} = \pi/3$. May be downsampled with $D = 3$.

d. $\Omega_{max} = 2\pi/7$. May be downsampled with $D = 3$.

6.19.

a.

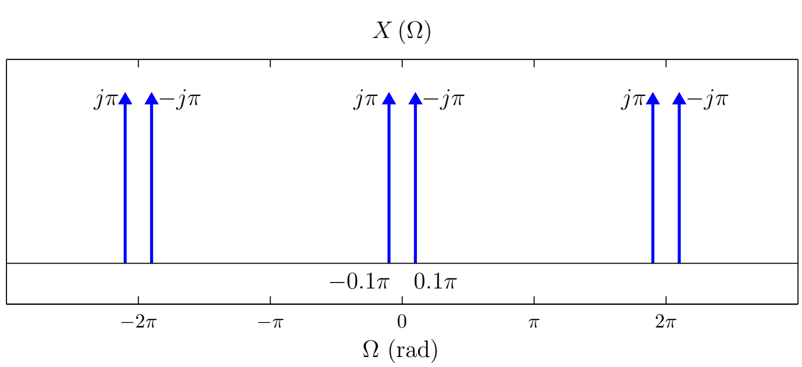
$$H_{zoh}\,(\Omega) = \sum_{n=0}^{2} e^{-j\Omega n} = 1 + e^{-j\Omega} + e^{-j2\Omega} = \left[1 + 2\,\cos\,(\Omega)\right]\, e^{-j\Omega}$$

b. Using Euler's formula

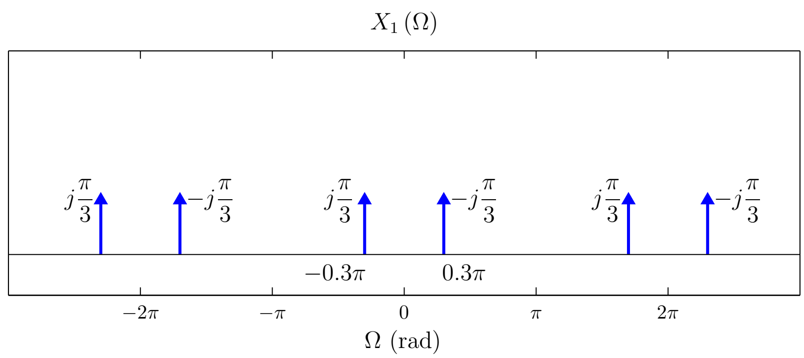
$$x[n] = \frac{1}{2j}\left[e^{j0.1\pi n} - e^{-j0.1\pi n}\right]$$

The transform of the input signal is

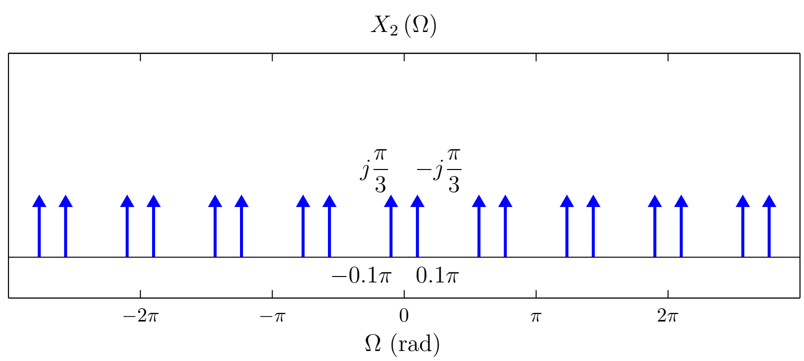
$$X\,(\Omega) = \frac{\pi}{j}\sum_{m=-\infty}^{\infty}\left[\delta\,(\Omega - 0.1\pi - 2\pi m) - \delta\,(\Omega + 0.1\pi - 2\pi m)\right]$$



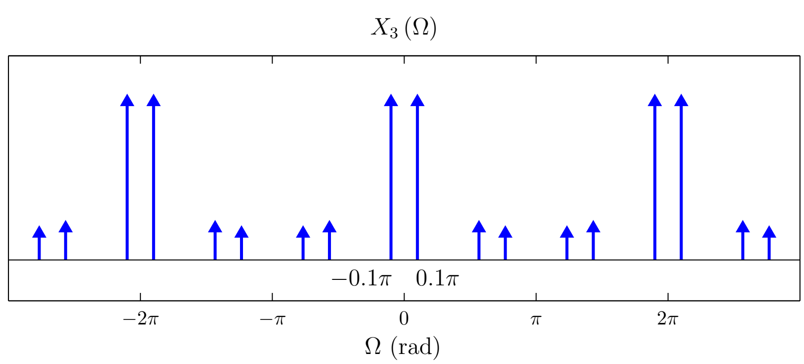
$$X_1 = \frac{1}{3}\sum_{k=0}^{2} X\left(\frac{\Omega - 2\pi k}{3}\right)$$



$$X_2(\Omega) = X_1(3\Omega)$$



$$X_3(\Omega) = H_{zoh}(\Omega)\,X_2(\Omega)$$



6.20.

The impulse response of the first-order hold interpolation filter is

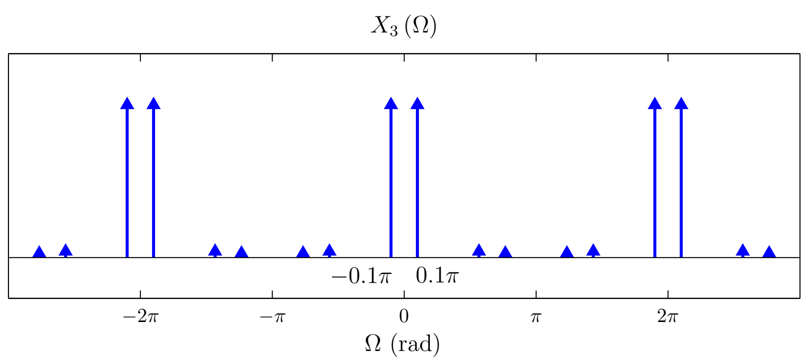
$$h_{foh}[n] = \{\underset{\substack{\uparrow \\ n=-2}}{1/3}, 2/3,\ 1,\ 2/3,\ 1/3\}$$

and its transform is

$$\begin{aligned} H_{foh}(\Omega) &= \frac{1}{3}e^{j2\Omega} + \frac{2}{3}e^{j\Omega} + 1 + \frac{2}{3}e^{-j\Omega} + \frac{1}{3}e^{-j2\Omega} \\ &= 1 + \frac{4}{3}\cos(\Omega) + \frac{2}{3}\cos(2\Omega) \end{aligned}$$

The spectra $X(\Omega)$, $X_1(\Omega)$ and $X_2(\Omega)$ are as shown in the solution to Problem 6.19.

$$X_3(\Omega) = H_{foh}(\Omega)\ X_2(\Omega)$$



6.21.

a.

```
Xa = @(f) 2./(1+4*pi*pi*f.*f);   % Original spectrum
f = [-7:0.01:7];
fs = 3;
Ts = 1/fs;
Xs = zeros(size(f));
for k=-5:5,
  Xs = Xs+fs*Xa(f-k*fs);
end;
plot(f,Xs);
axis([-7,7,-1,10]);
title('X_{s}(f)');
xlabel('f (Hz)');
grid;
```

b.

```
Xa = @(f) 2./(1+4*pi*pi*f.*f);  % Original spectrum
f = [-7:0.01:7];
fs = 4;
Ts = 1/fs;
Xs = zeros(size(f));
for k=-5:5,
  Xs = Xs+fs*Xa(f-k*fs);
end;
plot(f,Xs);
axis([-7,7,-1,10]);
title('X_{s}(f)');
xlabel('f (Hz)');
grid;
```

c. Aliasing effect is present in both cases, but is less pronounced for $f_s = 4$ Hz.

6.22.

Compute and graph the spectrum $X_a(f)$:

```
Xa = @(f,A,tau) A*tau*sinc(f*tau).*sinc(f*tau);
f = [-10:0.02:10];
plot(f,Xa(f,1,1));
axis([-10,10,-0.25,1.25]);
xlabel('f (Hz)');
title('X_{a}(f)');
grid;
```

Compute and graph the spectrum $X_s(s)$ for the sampling rate $f_s = 5$ Hz.

```
fs = 5;     % Edit this for other sampling rates.
Ts = 1/fs;
Xs = zeros(size(f));
for k=-15:15,
  Xs = Xs+1/Ts*Xa(f-k*fs,1,1);
end;
plot(f,Xs);
axis([-10,10,-1,6]);
xlabel('f (Hz)');
title('X_{s}(f)');
grid;
```

6.23.

```
A = 1;
tau = 1;
Xa = @(omg) A*tau*(sinc(omg*tau/(2*pi))).^2;
```

```
fs = 12;
Omg = [-1:0.001:1]*pi;
% Use Eqn. (6.25)
X = zeros(size(Omg));
for k=-10:10,
  X = X+12*Xa((Omg-2*pi*k)*fs);
end;
p1 = plot(Omg,X);
axis([-pi,pi,-2,15]);
xlabel('\Omega (rad)');
ylabel('Magnitude');
title('|X(\Omega)|$');
axis([-pi,pi,0,15]);
grid;
```

6.24.

```
% Compute the transform of p[n].
Omg = [-1:0.002:1]*pi+eps;
P = @(Omg) sin(7.5*Omg)./sin(0.5*Omg).*exp(-j*7*Omg);
% Use the modulation property of the DTFT.
X = -0.5*j*P(Omg-pi/15)+0.5*j*P(Omg+pi/15);
plot(Omg,abs(X));
axis([-pi,pi,0,12]);
xlabel('\Omega (rad)');
title('|X(\Omega)|');
grid;
```

6.25. a.

```
xa = @(t) sin(2*pi*1000*t);
t = [0:5e-6:5e-3];
fs = 2400;
Ts = 1/fs;
n = [0:14];
plot(1000*t,xa(t));
axis([0,5,-1.2,1.2]);
hold on;
stem(1000*n*Ts,xa(n*Ts),'r');
hold off;
grid;
```

b.

```
xa = @(t) sin(2*pi*1000*t);
xb = @(t) sin(2*pi*3400*t);
xc = @(t) sin(2*pi*5800*t);
```

```
xd = @(t) sin(2*pi*8200*t);
t = [0:5e-6:1e-3];
fs = 2400;
Ts = 1/fs;
n = [0:14];
plot(1000*t,xa(t),'b',1000*t,xb(t),'b--',1000*t,xc(t),'g--',1000*t,xd(t),'k--');
axis([0,1,-1.2,1.2]);
hold on;
stem(1000*n*Ts,xa(n*Ts),'r');
hold off;
grid;
```

6.26.

a. Lines 2 and 3 create a naturally sampled version of the signal rather then a zero-order hold version. The vector "t1" contains 0s and 1s depending on whether the corresponding time instant in vector "t" is within an active pulse of the signal $\tilde{p}(t)$ or not (refer to Fig. 6.18).

The loop between lines 5 and 15 searches through this naturally sampled signal. When the left edge of a pulse is encountered, its value is saved in the variable "value", and the amplitude is adjusted to the saved level for the duration of the pulse.

b. A modified version of the function **ss_zohsamp**(..) is listed below as **ss_zohsamp2**(..). This function is only for the case $d = 1$.

The function **ss_zohsamp2**(..) is listed below:

```
function xzoh = ss_zohsamp2(xa,Ts,t)
  delt = t(2)-t(1);      % Time increment used in vector "t"
  L = floor(Ts/delt);   % Number of samples per pulse (must be integer!)
  pulse = ones(1,L);
  xzoh = downsample(xa(t),L);
  xzoh = kron(xzoh,pulse);
```

The script below can be used for testing.

```
x = @(t) exp(-abs(t));
t = [-4:0.001:3.999];
xzoh = ss_zohsamp2(x,0.2,t);
plot(t,xzoh);
axis([-4,4,-0.2,1.2]);
```

6.27.

a.

```
xa = @(t) exp(-abs(t));
t = [-4:0.001:4];
xzoh = ss_zohsamp(xa,0.2,0.90,t);
plot(t,xzoh);
```

b.

```
% System with a=3
sys = tf([3],[1,3]);
y = lsim(sys,xzoh,t);
plot(t,xzoh,t,y);
```

```
% System with a=2
sys = tf([2],[1,2]);
y = lsim(sys,xzoh,t);
plot(t,xzoh,t,y);
```

```
% System with a=1
sys = tf([1],[1,1]);
y = lsim(sys,xzoh,t);
plot(t,xzoh,t,y);
```

6.28.

The system can be simulated with the following code:

```
n = [0:199];
xn = sin(0.1*pi*n);  % Signal x[n].
x1 = downsample(xn,3);
x2 = upsample(x1,3);
hzoh = ones(1,3);  % Impulse response of interpolation filter.
x3 = conv(x2,hzoh);
```

Individual signals can be graphed using following code segments:

```
stem(n,xn);
axis([-0.5,199.5,-1.2,1.2]);
xlabel('n');
title('x[n]');
```

```
stem([0:66],x1);
axis([-0.5,66.5,-1.2,1.2]);
xlabel('n');
title('x_{1}[n]');
```

```
stem([0:200],x2);
axis([-0.5,200.5,-1.2,1.2]);
xlabel('n');
title('x_{2}[n]');
```

```
stem([0:200],x3(1:201));
axis([-0.5,200.5,-1.2,1.2]);
xlabel('n');
title('x_{3}[n]');
```

6.29.

The system can be simulated with the following code:

```
n = [0:199];
xn = sin(0.1*pi*n);   % Signal x[n].
x1 = downsample(xn,3);
x2 = upsample(x1,3);
hfoh = [1/3,2/3,1,2/3,1/3];
x3 = conv(x2,hfoh);
```

Individual signals can be graphed using following code segments:

```
stem(n,xn);
axis([-0.5,199.5,-1.2,1.2]);
xlabel('n');
title('x[n]');
```

```
stem([0:66],x1);
axis([-0.5,66.5,-1.2,1.2]);
xlabel('n');
title('x_{1}[n]');
```

```
stem([0:200],x2);
axis([-0.5,200.5,-1.2,1.2]);
xlabel('n');
title('x_{2}[n]');
```

```
stem([0:200],x3(1:201));
axis([-0.5,200.5,-1.2,1.2]);
xlabel('n');
title('x_{3}[n]');
```

Chapter 7

Laplace Transform for Continuous-Time Signals and Systems

7.1.

a.

Using Laplace transform definition

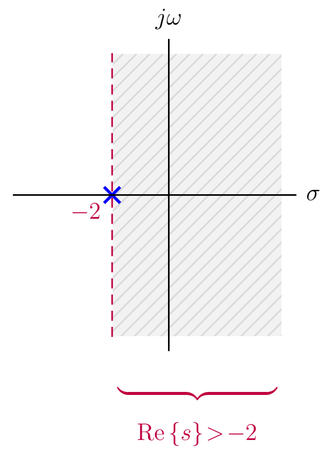
$$X(s) = \int_{-\infty}^{\infty} e^{-2t}\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-2t}\, e^{-st}\, dt = \left.\frac{e^{-(s+2)t}}{-(s+2)}\right|_{0}^{\infty}$$

$$= \frac{1}{s+2}$$

For convergence of the integral, we need

$$\text{Re}\{s+2\} > 0 \qquad \Rightarrow \qquad \text{Re}\{s\} > -2$$



b.

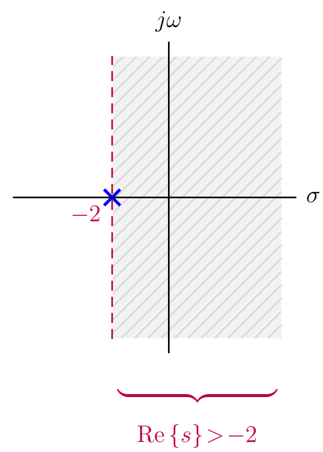
$$X(s) = \int_{-\infty}^{\infty} e^{-2t}\, u(t)\, e^{-st}\, dt$$

$$= \int_{1}^{\infty} e^{-2t}\, e^{-st}\, dt$$

$$= \left.\frac{e^{-(s+2)t}}{-(s+2)}\right|_{1}^{\infty} = \frac{e^{-2}\, e^{-s}}{s+2}$$

For convergence of the integral, we need

$$\text{Re}\{s+2\} > 0 \qquad \Rightarrow \qquad \text{Re}\{s\} > -2$$

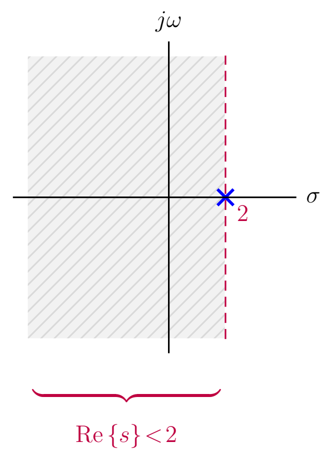


c.

$$X(s) = \int_{-\infty}^{\infty} e^{2t}\, u(-t)\, e^{-st}\, dt$$
$$= \int_{-\infty}^{0} e^{2t}\, e^{-st}\, dt = \left. \frac{e^{(2-s)t}}{(2-s)} \right|_{-\infty}^{0} = \frac{1}{2-s}$$

For convergence of the integral, we need

$$\mathrm{Re}\{2-s\} > 0 \qquad \Rightarrow \qquad \mathrm{Re}\{s\} < 2$$

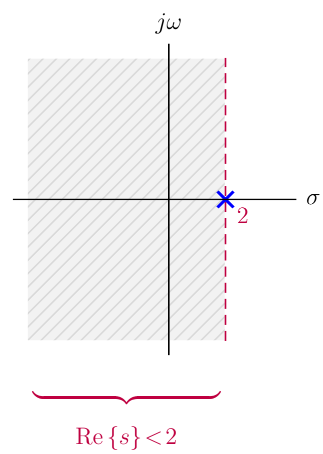


d.

$$X(s) = \int_{-\infty}^{\infty} e^{2t}\, u(-t+1)\, e^{-st}\, dt$$
$$= \int_{-\infty}^{1} e^{2t}\, e^{-st}\, dt = \left. \frac{e^{(2-s)t}}{(2-s)} \right|_{-\infty}^{1} = \frac{e^{2}\, e^{-s}}{2-s}$$

For convergence of the integral, we need

$$\mathrm{Re}\{2-s\} > 0 \qquad \Rightarrow \qquad \mathrm{Re}\{s\} < 2$$



e.

$$X(s) = \int_{0}^{1} (1)\, e^{-st}\, dt + \int_{1}^{2} (-1)\, e^{-st}\, dt$$
$$= \left. \frac{e^{-st}}{-s} \right|_{0}^{1} - \left. \frac{e^{-st}}{-s} \right|_{1}^{2} = \frac{1}{s}\left[1 - 2e^{-s} + e^{-2s}\right]$$

The ROC is the entire s-plane with the exception of $\mathrm{Re}\{s\} \to -\infty$.

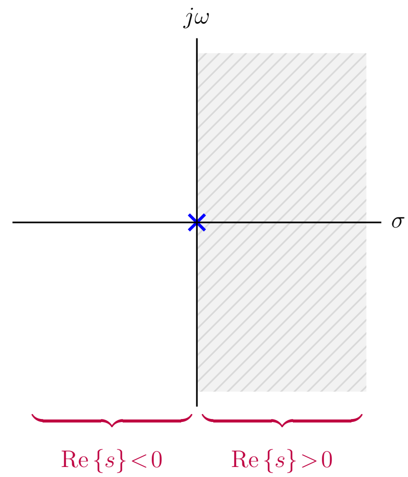
f.

$$X(s) = \int_0^1 (1)\, e^{-st}\, dt + \int_1^\infty (-1)\, e^{-st}\, dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_0^1 + \left.\frac{e^{-st}}{s}\right|_1^\infty = \frac{a}{s}\left[1 - 2\, e^{-s}\right]$$

For convergence of the second integral, we need

$$\text{Re}\{s\} > 0$$



7.2.

Using the definition of the Laplace transform given by Eqn. (7.1)

$$X(s) = \int_{-\infty}^{\infty} \left[\sum_{n=0}^{\infty} e^{-anT}\, \delta(t - nT)\right] e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} e^{-anT} \int_{-\infty}^{\infty} \delta(t - nT)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} e^{-anT}\, e^{-snT} = \sum_{n=0}^{\infty} e^{-(s+a)Tn} = \frac{1}{1 - e^{-(s+a)T}}$$

The closed form expression is valid only if

$$\left|e^{-(s+a)T}\right| < 1 \qquad \Rightarrow \qquad \text{Re}\left\{-(s+a)\right\} < 0 \qquad \Rightarrow \qquad \text{Re}\left\{s\right\} > -\text{Re}\left\{a\right\}$$

which establishes the ROC. The poles of $X(s)$ are found by solving

$$e^{-(s+a)T} = 1$$

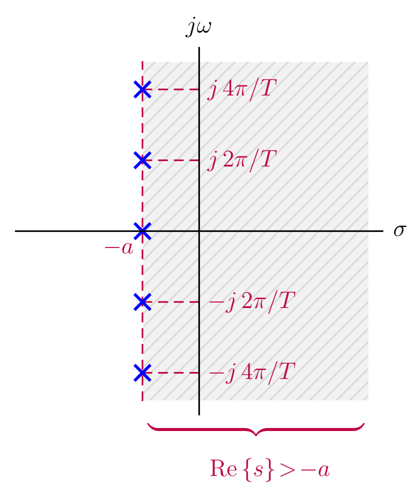
which, using the identity $e^{j2\pi k} = 1$, can also be written as

$$e^{-(s+a)T} = e^{j2\pi k}$$

The poles are at

$$s_k = -a - j\,\frac{2\pi k}{T}\,, \qquad \text{all integer } k$$

as shown in the pole-zero diagram. (We are assuming that a is real-valued.)



7.3.

For the Fourier transform to exist, the ROC needs to include the $j\omega$-axis.

a. ROC: $-1 < \text{Re}\{s\} < 1$

b. ROC: $\text{Re}\{s\} > -1$

c. ROC: $\text{Re}\{s\} < 1$

d. ROC: $\text{Re}\{s\} > -1$

7.4.

For the signal to be causal the ROC needs to be to the right of a vertical line in the s-plane and include $s \to \infty$.

a. ROC: $\text{Re}\{s\} > 1$

b. ROC: $\text{Re}\{s\} > -1$

c. ROC: $\text{Re}\{s\} > 2$

d. ROC: $\text{Re}\{s\} > -1$

7.5.

The ROC of the transform an anti-causal signal is to the left of a vertical line in the s-plane.

a. ROC: $\text{Re}\{s\} < -1$

b. ROC: $\text{Re}\{s\} < -2$

c. ROC: $\text{Re}\{s\} < 1$

d. ROC: $\text{Re}\{s\} < -1$

7.6.

a.

$$\mathscr{F}\{x(t)\} = X(s)\Big|_{s=j\omega}$$

Since $s = j\omega$ is included in the ROC the Fourier transform exists, and is found as

$$\mathscr{F}\{x(t)\} = \frac{j\omega - 1}{(j\omega)^2 - j\omega - 2} = \frac{1 - j\omega}{(2 + \omega^2) + j\omega}$$

b. The desired Fourier transform is related to $X(s)$ by

$$\mathscr{F}\{x(t)\, e^{t}\} = X(s)\Big|_{s=-1+j\omega}$$

Since the trajectory $s = -1 + j\omega$ is not included in the ROC, the Fourier transform of $x(t)\, e^{t}$ does not exist.

c. The desired Fourier transform is related to $X(s)$ by

$$\mathscr{F}\left\{x(t)\, e^{-3t}\right\} = X(s)\Big|_{s=3+j\omega}$$

Since the trajectory $s = 3 + j\omega$ is not included in the ROC, the Fourier transform of $x(t)\, e^{-3t}$ does not exist.

7.7.

a.

$$\mathscr{F}\left\{x(t)\right\} = X(s)\Big|_{s=j\omega} = \frac{1}{(j\omega)^2 + j2\omega + 5} = \frac{1}{(5-\omega^2) + j2\omega}$$

b.

$$\mathscr{F}\left\{x(t)\, e^{-t}\right\} = X(s)\Big|_{s=1+j\omega} = \frac{1}{(1+j\omega)^2 + 2\,(1+j\omega) + 5} = \frac{1}{(8-\omega^2) + j4\omega}$$

c. The desired Fourier transform is related to $X(s)$ by

$$\mathscr{F}\left\{x(t)\, e^{t}\right\} = X(s)\Big|_{s=-1+j\omega}$$

The trajectory $s = -1 + j\omega$ is not included in the ROC. Therefore the Fourier transform of $x(t)\, e^{t}$ does not exist.

d. The desired Fourier transform is related to $X(s)$ by

$$\mathscr{F}\left\{x(t)\, e^{3t}\right\} = X(s)\Big|_{s=-3+j\omega}$$

The trajectory $s = -3 + j\omega$ is not included in the ROC. Therefore the Fourier transform of $x(t)\, e^{3t}$ does not exist.

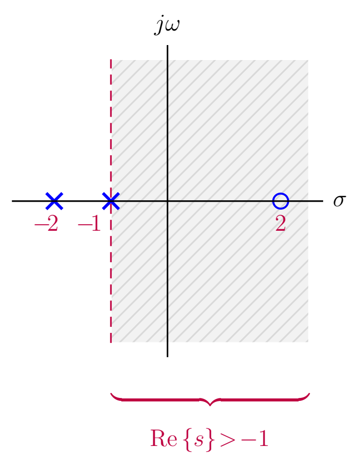
7.8.

a. The transform has a zero at $s = 2$ and poles at $s = -1, -2$. Since $x(t)$ is causal, the ROC is

$$\text{Re}\left\{s\right\} > -1$$

The Fourier transform $X(\omega)$ is

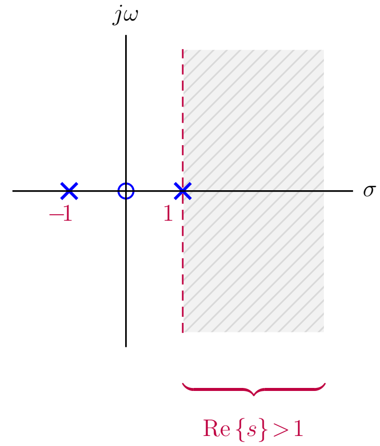
$$X(\omega) = X(s)\Big|_{s=j\omega} = \frac{j\omega - 2}{(j\omega)^2 + j3\omega + 2} = \frac{j\omega - 2}{(2-\omega^2) + j3\omega}$$



b. The transform has a zero at $s = 0$ and poles at $s = \mp 1$. Since $x(t)$ is causal, the ROC is

$$\text{Re}\{s\} > 1$$

The ROC does not include the $j\omega$-axis of the s-plane. Therefore the Fourier transform does not exist.

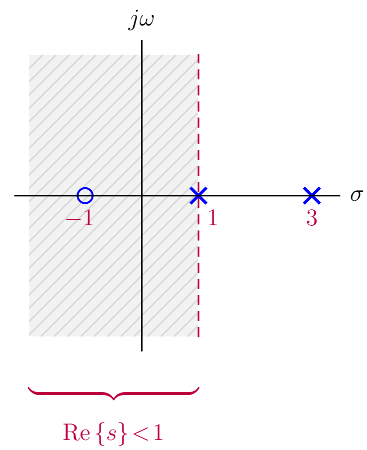


c. The transform has a zero at $s = -1$ and poles at $s = 1, 3$. Since $x(t)$ is specified to be anti-causal, the ROC is

$$\text{Re}\{s\} < 1$$

The ROC includes the $j\omega$-axis. The Fourier transform $X(\omega)$ is

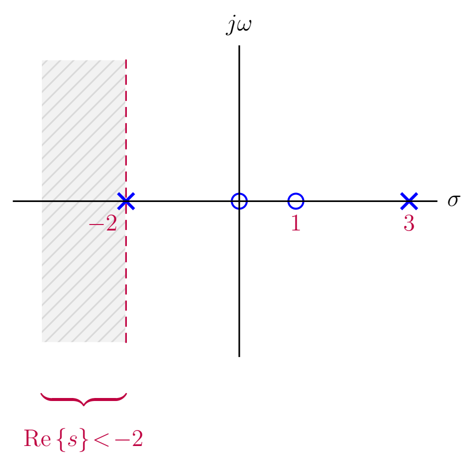
$$X(\omega) = X(s)\Big|_{s=j\omega} = \frac{j\omega + 1}{(j\omega)^2 - j4\omega + 3} = \frac{j\omega + 1}{(3 - \omega^2) - j4\omega}$$



d. The transform has zeros at $s = 0, 1$ and poles at $s = -2, 3$. Since $x(t)$ is specified to be anti-causal, the ROC must be

$$\text{Re}\{s\} < -2$$

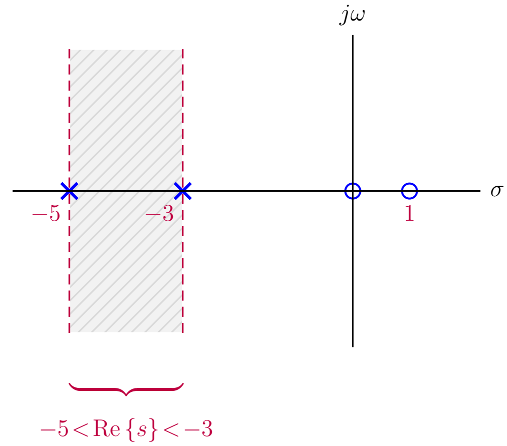
The ROC does not include the $j\omega$-axis. Consequently, the Fourier transform $X(\omega)$ does not exist.



e. The transform has zeros at $s=0,1$ and poles at $s=-3,-5$. Since the Fourier transform of $x(t)\,e^{4t}$ exists, the ROC must include the trajectory $s=-4+j\omega$. Therefore the ROC is

$$-5<\text{Re}\{s\}<-3$$

The ROC does not include the $j\omega$-axis. Consequently, the Fourier transform $X(\omega)$ does not exist.



7.9.

a.

$$\mathcal{L}\left\{e^{-t}\,u(t)\right\}=\frac{1}{s+1}\,,\qquad \text{Re}\{s\}>-1$$

$$\mathcal{L}\left\{e^{-3t}\,u(t)\right\}=\frac{1}{s+3}\,,\qquad \text{Re}\{s\}>-3$$

Using the linearity of the Laplace transform we obtain

$$X(s)=\frac{3}{s+1}-\frac{5}{s+3}=\frac{-2\,(s-2)}{(s+1)\,(s+3)}\,,\qquad \text{Re}\{s\}>-1$$

b.

$$\mathcal{L}\left\{e^{-t}\,u(t)\right\}=\frac{1}{s+1}\,,\qquad \text{Re}\{s\}>-1$$

$$\mathcal{L}\left\{e^{3t}\,u(-t)\right\}=\frac{-1}{s-3}\,,\qquad \text{Re}\{s\}<3$$

Using the linearity of the Laplace transform we obtain

$$X(s)=\frac{3}{s+1}-\frac{2}{s-3}=\frac{s-11}{(s+1)\,(s-3)}\,,\qquad -1<\text{Re}\{s\}<3$$

c.

$$\mathcal{L}\left\{\delta(t)\right\}=1\,,\qquad \text{all } s$$

$$\mathcal{L}\left\{e^{-t}\,u(t)\right\}=\frac{1}{s+1}\,,\qquad \text{Re}\{s\}>-1$$

Using the linearity of the Laplace transform we obtain

$$X(s)=1+\frac{2}{s+1}=\frac{s+3}{s+1}\,,\qquad \text{Re}\{s\}>-1$$

d.

$$\mathcal{L}\left\{u(t)\right\}=\frac{1}{s}\,,\qquad \text{Re}\{s\}>0$$

$$\mathscr{L}\left\{e^{-t}\,u(t)\right\} = \frac{1}{s+1}\,, \qquad \text{Re}\left\{s\right\} > -1$$

Using the linearity of the Laplace transform we obtain

$$X(s) = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s\,(s+1)}\,, \qquad \text{Re}\left\{s\right\} > 0$$

e.

$$\mathscr{L}\left\{\cos(2t)\;u(t)\right\} = \frac{s}{s^2+4}\,, \qquad \text{Re}\left\{s\right\} > 0$$

$$\mathscr{L}\left\{\sin(3t)\;u(t)\right\} = \frac{3}{s^2+9}\,, \qquad \text{Re}\left\{s\right\} > 0$$

Using the linearity of the Laplace transform we obtain

$$X(s) = \frac{s^3+6\,s^2+9\,s+24}{\left(s^2+4\right)\left(s^2+9\right)}\,, \qquad \text{Re}\left\{s\right\} > 0$$

f. The signal $x(t)$ can be written as

$$\begin{aligned} x(t) =& e^{-2t}\left[\frac{1}{2}\,e^{j3t}+\frac{1}{2}\,e^{-j3t}\right]\,u(t) \\ =& \frac{1}{2}\,e^{(-2+j3)\,t}\,u(t)+\frac{1}{2}\,e^{(-2-j3)\,t}\,u(t) \end{aligned}$$

$$\mathscr{L}\left\{e^{(-2+j3)\,t}\,u(t)\right\} = \frac{1}{s+2-j3}\,, \qquad \text{Re}\left\{s\right\} > -2$$

$$\mathscr{L}\left\{e^{(-2+j3)\,t}\,u(t)\right\} = \frac{1}{s+2+j3}\,, \qquad \text{Re}\left\{s\right\} > -2$$

Using the linearity of the Laplace transform we obtain

$$X(s) = \frac{1/2}{s+2-j3} + \frac{1/2}{s+2+j3} = \frac{s+2}{s^2+4\,s+13}\,, \qquad \text{Re}\left\{s\right\} > -2$$

7.10.

a.

$$\mathscr{L}\left\{e^{-2t}\,u(t)\right\} = \frac{1}{s+2}\,, \qquad \text{Re}\left\{s\right\} > -2$$

Using the time shifting property of the Laplace transform we get

$$X(s) = \mathscr{L}\left\{e^{-2(t-1)}\,u(t-1)\right\} = \frac{e^{-s}}{s+2}\,, \qquad \text{Re}\left\{s\right\} > -2$$

b. Scaling the transform pair found in part (a) we obtain

$$\mathscr{L}\left\{e^{-2t}\,u(t-1)\right\} = e^{-2}\,\mathscr{L}\left\{e^{-2(t-1)}\,u(t-1)\right\} = \frac{e^{-(s+2)}}{s+2}\,, \qquad \text{Re}\left\{s\right\} > -2$$

c. Starting with the transform pair

$$\mathcal{L}\left\{e^{-2t}\,u(t)\right\}=\frac{1}{s+2}\,,\qquad \mathrm{Re}\{s\}>-2$$

and using the time scaling property of the Laplace transform we get

$$\mathcal{L}\left\{e^{2t}\,u(-t)\right\}=\frac{1}{-s+2}-\frac{-1}{s-2}\,,\qquad \mathrm{Re}\{s\}<2$$

Finally, using the time shifting property on this result leads to

$$X(s)=\mathcal{L}\left\{e^{2(t+1)}\,u(-t-1)\right\}=\frac{-e^{s}}{s-2}\,,\qquad \mathrm{Re}\{s\}<2$$

d. Using the transform pair found in part (c) with the linearity property of the Laplace transform we obtain

$$X(s)=\mathcal{L}\left\{e^{2t}\,u(-t-1)\right\}=e^{-2}\,\mathcal{L}\left\{e^{2(t+1)}\,u(-t-1)\right\}=\frac{-e^{(s-2)}}{s-2}\,,\qquad \mathrm{Re}\{s\}<2$$

e. From earlier parts of the problem we have

$$\mathcal{L}\left\{e^{2t}\,u(-t)\right\}=\frac{-1}{s-2}\,,\qquad \mathrm{Re}\{s\}<2$$

Applying the time shifting property of the Laplace transform we get

$$\mathcal{L}\left\{e^{2(t-1)}\,u(-t+1)\right\}=\frac{-e^{-s}}{s-2}\,,\qquad \mathrm{Re}\{s\}<2$$

$$e^{-2}\,\mathcal{L}\left\{e^{2t}\,u(-t+1)\right\}=\frac{-e^{-s}}{s-2}\,,\qquad \mathrm{Re}\{s\}<2$$

and therefore

$$X(s)=\mathcal{L}\left\{e^{2t}\,u(-t+1)\right\}=\frac{-e^{-(s-2)}}{s-2}\,,\qquad \mathrm{Re}\{s\}<2$$

7.11.

a. Using the appropriate trigonometric identity, $x(t)$ is

$$x(t)=\cos(3t)\,\cos(\pi/6)\,u(t)-\sin(3t)\,\sin(\pi/6)\,u(t)$$

Transforms of individual terms are

$$\mathcal{L}\left\{\cos(3t)\,u(t)\right\}=\frac{s}{s^2+9}\,,\qquad \mathrm{Re}\{s\}>0$$

$$\mathcal{L}\left\{\sin(3t)\,u(t)\right\}=\frac{3}{s^2+9}\,,\qquad \mathrm{Re}\{s\}>0$$

Using the linearity of the Laplace transform we get

$$X(s)=\frac{\cos(\pi/6)\,s}{s^2+9}-\frac{3\sin(\pi/6)}{s^2+9}=\frac{0.866\,s-1.5}{s^2+9}$$

b. Using Euler's formula, $x(t)$ is

$$x(t) = \frac{1}{2}\,e^{j\pi/6}\,e^{j3t}\,u(t) + \frac{1}{2}\,e^{-j\pi/6}\,e^{-j3t}\,u(t)$$

Transforms of individual terms are

$$\mathcal{L}\left\{e^{j3t}\,u(t)\right\} = \frac{1}{s-j3}\,, \qquad \mathrm{Re}\left\{s\right\} > 0$$

$$\mathcal{L}\left\{e^{-j3t}\,u(t)\right\} = \frac{1}{s+j3}\,, \qquad \mathrm{Re}\left\{s\right\} > 0$$

Using the linearity of the Laplace transform we get

$$\begin{aligned} X(s) &= \frac{\frac{1}{2}\,e^{j\pi/6}}{s-j3} + \frac{\frac{1}{2}\,e^{-j\pi/6}}{s+j3} \\ &= \frac{\frac{1}{2}\,e^{j\pi/6}\left(s+j3\right) + \frac{1}{2}\,e^{-j\pi/6}\left(s-j3\right)}{s^2+9} \\ &= \frac{\cos(\pi/6)\;s - 3\,\sin(\pi/6)}{s^2+9} \\ &= \frac{0.866\,s - 1.5}{s^2+9}\,, \qquad \mathrm{Re}\left\{s\right\} > 0 \end{aligned}$$

7.12.

a. Using the relationships

$$\mathcal{L}\left\{u(t)\right\} = \frac{1}{s}\,, \qquad \mathrm{Re}\left\{s\right\} > 0$$

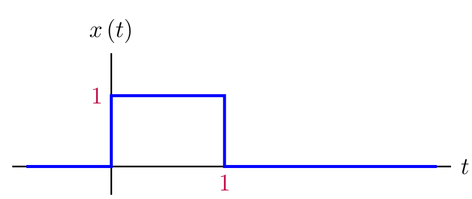
and

$$\mathcal{L}\left\{u(t-1)\right\} = \frac{e^{-s}}{s}\,, \qquad \mathrm{Re}\left\{s\right\} > 0$$

the transform is found through the use of the linearity property as

$$X(s) = \frac{1-e^{-s}}{s}\,, \qquad \mathrm{Re}\left\{s\right\} > -\infty$$

The ROC is the entire s-plane with the exception of $\mathrm{Re}\left\{s\right\} \to -\infty$ since the pole at $s=0$ is canceled when the two terms are added. This is evident from the fact that $X\left(0+j\,0\right) = 1$ and that $x(t)$ is a finite length signal.



b. Using the relationships

$$\mathcal{L}\left\{u(t)\right\} = \frac{1}{s}\,, \qquad \mathrm{Re}\left\{s\right\} > 0$$

$$\mathscr{L}\left\{u\left(t-1\right)\right\}=\frac{e^{-s}}{s}\,,\qquad \text{Re}\left\{s\right\}>0$$

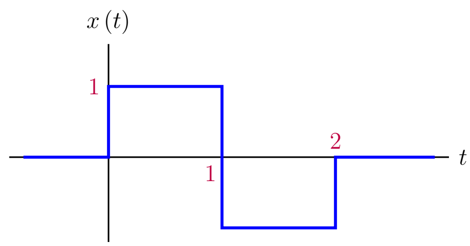
and

$$\mathscr{L}\left\{u\left(t-2\right)\right\}=\frac{e^{-2s}}{s}\,,\qquad \text{Re}\left\{s\right\}>0$$

the transform is found through the use of the linearity property as

$$X\left(s\right)=\frac{1-2\,e^{-s}+e^{-2s}}{s}\,,\qquad \text{Re}\left\{s\right\}>-\infty$$

The ROC is the entire s-plane with the exception of $\text{Re}\left\{s\right\}\to-\infty$ since the pole at $s=0$ is canceled when the two terms are added. This is evident from the fact that $X\left(0+j\,0\right)=0$ and that $x\left(t\right)$ is a finite length signal.



c. Using the relationships

$$\mathscr{L}\left\{u\left(t\right)\right\}=\frac{1}{s}\,,\qquad \text{Re}\left\{s\right\}>0$$

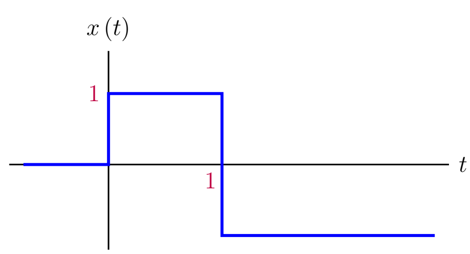
and

$$\mathscr{L}\left\{u\left(t-1\right)\right\}=\frac{e^{-s}}{s}\,,\qquad \text{Re}\left\{s\right\}>0$$

the transform is found through the use of the linearity property as

$$X\left(s\right)=\frac{1-2\,e^{-s}}{s}\,,\qquad \text{Re}\left\{s\right\}>0$$

The ROC is the same as the ROCs of the two individual transforms that were added. Note that the pole at $s=0$ is not canceled in this case (the numerator of $X\left(s\right)$ is nonzero for $s=0$), and the signal $x\left(t\right)$ is of infinite length.



7.13.

a. The signal $x(t)$ can be written as

$$x(t) = u(t) - 2\,u(t-1) + u(t-2)$$

which leads to the transform

$$X(s) = \frac{1}{s}\left(1 - 2\,e^{-s} + e^{-2s}\right), \qquad \mathrm{Re}\{s\} > -\infty$$

b. An expression for $x(t)$ using unit step functions is

$$x(t) = u(t) - 1.5\,u(t-1) + u(t-2) - 0.5\,u(t-3)$$

It leads to the transform

$$X(s) = \frac{1}{s}\left(1 - 1.5\,e^{-s} + e^{-2s} - 0.5\,e^{-3s}\right), \qquad \mathrm{Re}\{s\} > -\infty$$

7.14.

The Laplace transform of the unit ramp function $r(t) = t\,u(t)$ was found in Example 7-18 as

$$R(s) = \mathcal{L}\{r(t)\} = \frac{1}{s^2}, \qquad \mathrm{Re}\{s\} > 0$$

The transform of the unit step signal is

$$U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}, \qquad \mathrm{Re}\{s\} > 0$$

Writing a signal using these two functions allows computation of the Laplace transform in terms of the corresponding transforms.

a.

Signal: $x(t) = r(t) - r(t-1) - r(t-2) + r(t-3)$

Transform: $X(s) = \dfrac{1}{s^2}\left(1 - e^{-s} - e^{-2s} + e^{-3s}\right)$

b.

Signal: $x(t) = r(t) - u(t-1) - 2\,r(t-1) + u(t-2) + r(t-2)$

Transform: $X(s) = \dfrac{1}{s^2}\left(1 - 2\,e^{-s} + e^{-2s}\right) + \dfrac{1}{s}\left(-e^{-s} + e^{-2s}\right)$

c.

Signal: $x(t) = r(t) - 2\,r(t-1) + r(t-2)$

Transform: $X(s) = \dfrac{1}{s^2}\left(1 - 2\,e^{-s} + e^{-2s}\right)$

7.15.

a. Using Euler's formula the signal $x(t)$ can be written as

$$\begin{aligned}
x(t) &= \sin(\pi t)\,\left[u(t) - u(t-2)\right] \\
&= \frac{1}{2j}\left(e^{j\pi t} - e^{-j\pi t}\right)\left[u(t) - u(t-2)\right] \\
&= \frac{1}{2j}\,e^{j\pi t}\,u(t) - \frac{1}{2j}\,e^{j\pi t}\,u(t-2) - \frac{1}{2j}\,e^{-j\pi t}\,u(t)\,\frac{1}{2j}\,e^{-j\pi t}\,u(t-2)
\end{aligned}$$

The transforms needed are

$$\mathcal{L}\left\{e^{j\pi t}\,u(t)\right\} = \frac{1}{s-j\pi}$$

$$\mathcal{L}\left\{e^{j\pi t}\,u(t-2)\right\} = e^{j2\pi}\,\mathcal{L}\left\{e^{j\pi(t-2)}\,u(t-2)\right\} = \frac{e^{-2s}}{s-j\pi}$$

$$\mathcal{L}\left\{e^{-j\pi t}\,u(t)\right\} = \frac{1}{s+j\pi}$$

$$\mathcal{L}\left\{e^{-j\pi t}\,u(t-2)\right\} = e^{-j2\pi}\,\mathcal{L}\left\{e^{-j\pi(t-2)}\,u(t-2)\right\} = \frac{e^{-2s}}{s+j\pi}$$

In the relationships above we have recognized that $e^{\pm j2\pi} = 0$. The transform $X(s)$ is

$$X(s) = \frac{1}{2j}\left[\frac{1-e^{-2s}}{s-j\pi} - \frac{1-e^{-2s}}{s+j\pi}\right] = \frac{\pi\left(1-e^{-2s}\right)}{s^2+\pi^2}$$

b. Let us write $x(t)$ as

$$x(t) = \sin(\pi t)\;u(t) - \sin(\pi(t-2))\;u(t-2)$$

The transforms needed are

$$\mathcal{L}\left\{\sin(\pi t)\;u(t)\right\} = \frac{\pi}{s^2+\pi^2}$$

and

$$\mathcal{L}\left\{\sin(\pi(t-2))\;u(t-2)\right\} = \frac{e^{-2s}\,\pi}{s^2+\pi^2}$$

Using the linearity of the Laplace transform we obtain

$$X(s) = \frac{\pi}{s^2+\pi^2} - \frac{e^{-2s}\,\pi}{s^2+\pi^2} = \frac{\pi\left(1-e^{-2s}\right)}{s^2+\pi^2}$$

7.16.

a. Given that

$$\mathcal{L}\left\{e^{at}\,u(t)\right\} = \frac{1}{s-a}$$

through the use of the time differentiation property we get

$$\mathcal{L}\left\{t\,e^{at}\,u(t)\right\} = \frac{1}{(s-a)^2}$$

Setting $a = -2$ yields

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = t\,e^{-2t}\,u(t)$$

b. From the transform table in Appendix B.3

$$\mathcal{L}\left\{e^{at}\cos(\omega_0 t)\,u(t)\right\} = \frac{s-a}{(s-a)^2+\omega_0^2}$$

Setting $a = -3$ and $\omega_0 = \sqrt{2}$ we have

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2}\right\} = e^{-3t}\cos\left(\sqrt{2}\,t\right)\,u(t)$$

c. From the transform table in Appendix B.3

$$\mathcal{L}\left\{e^{at}\sin(\omega_0 t)\,u(t)\right\} = \frac{\omega_0}{(s-a)^2+\omega_0^2}$$

Setting $a = -1$ and $\omega_0 = \sqrt{6}$ we have

$$\mathcal{L}\left\{e^{-t}\sin\left(\sqrt{6}\,t\right)\,u(t)\right\} = \frac{\sqrt{6}}{(s+1)^2+6}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+6}\right\} = \frac{1}{\sqrt{6}}\,e^{-t}\sin\left(\sqrt{6}\,t\right)\,u(t)$$

7.17.

a. The transform $X(s)$ has the partial fraction expansion

$$X(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

with residues

$$k_1 = \left.\frac{1}{s+2}\right|_{s=-1} = 1$$

and

$$k_2 = \left.\frac{1}{s+1}\right|_{s=-2} = -1$$

The partial fraction expansion for $X(s)$ is

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Since the ROC is $\mathrm{Re}\{s\} > -1$ both terms in the partial fraction expansion correspond to causal signal components. Therefore

$$x(t) = e^{-t}\,u(t) - e^{-2t}\,u(t)$$

b. The partial fraction expansion is the same as in part (a). Since the ROC is $\text{Re}\{s\} < -2$ both terms in the partial fraction expansion correspond to anti-causal signal components. Therefore

$$x(t) = -e^{-t}\,u(-t) + e^{-2t}\,u(-t)$$

c. The partial fraction expansion is the same as in part (a). Since the ROC is $-2 < \text{Re}\{s\} < -1$ the first term in the partial fraction expansion corresponds to an anti-causal signal component, and teh second term corresponds to a causal signal component, that is

$$X(s) = \underbrace{\frac{1}{s+1}}_{\text{anti-causal}} - \underbrace{\frac{1}{s+2}}_{\text{causal}}$$

Therefore

$$x(t) = -e^{-t}\,u(-t) - e^{-2t}\,u(t)$$

d. The transform $X(s)$ has the partial fraction expansion

$$X(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{s+3}$$

with residues

$$k_1 = \left.\frac{(s-1)\,(s-2)}{(s+2)\,(s+3)}\right|_{s=-1} = 3$$

$$k_2 = \left.\frac{(s-1)\,(s-2)}{(s+1)\,(s+3)}\right|_{s=-2} = -12$$

and

$$k_3 = \left.\frac{(s-1)\,(s-2)}{(s+1)\,(s+2)}\right|_{s=-3} = 10$$

The partial fraction expansion for $X(s)$ is

$$X(s) = \frac{3}{s+1} - \frac{12}{s+2} + \frac{10}{s+3}$$

Using the causal signal component for each term we have

$$x(t) = 3\,e^{-t}\,u(t) - 12\,e^{-2t}\,u(t) + 10\,e^{-3t}\,u(t)$$

e. The transform $X(s)$ has the partial fraction expansion

$$X(s) = \frac{k_1}{s-1} + \frac{k_2}{s+2} + \frac{k_3}{s+3}$$

with residues

$$k_1 = \left.\frac{(s+1)\,(s-2)}{(s+2)\,(s+3)}\right|_{s=1} = -\frac{1}{6}$$

$$k_2 = \left.\frac{(s+1)\,(s-2)}{(s-1)\,(s+3)}\right|_{s=-2} = -\frac{4}{3}$$

and

$$k_3 = \left.\frac{(s+1)\,(s-2)}{(s-1)\,(s+2)}\right|_{s=-3} = \frac{5}{2}$$

The partial fraction expansion for $X(s)$ is

$$X(s) = \underbrace{-\frac{1/6}{s-1}}_{\text{anti-causal}} \underbrace{-\frac{4/3}{s+2} + \frac{5/2}{s+3}}_{\text{causal}}$$

Since the Fourier transform of $x(t)$ exists, the ROC of $X(s)$ must include the $j\omega$-axis of the s-plane. Therefore, the term with the pole at $s=1$ must correspond to an anti-causal signal component, and the other two terms must correspond to causal signal components.

$$x(t) = \frac{1}{6}e^{t}\,u(-t) - \frac{4}{3}e^{-2t}\,u(t) + \frac{5}{2}e^{-3t}\,u(t)$$

7.18.

a. Since the numerator order is the same as the denominator order, an constant term must be extracted from $X(s)$ to write it in the form

$$X(s) = 2 - \frac{1}{s+2}$$

The ROC indicates a causal signal. Therefore

$$x(t) = 2\delta(t) - e^{-2t}\,u(t)$$

b. The transform $X(s)$ has the partial fraction expansion

$$X(s) = \frac{k_1}{s+2} + \frac{k_2}{s+3}$$

with residues

$$k_1 = \left.\frac{s(s+1)}{s+3}\right|_{s=-2} = 2$$

and

$$k_2 = \left.\frac{s(s+1)}{s+2}\right|_{s=-3} = -6$$

The partial fraction expansion for $X(s)$ is

$$X(s) = \frac{2}{s+2} - \frac{6}{s+3}$$

The ROC for the transform is $\mathrm{Re}\{s\} < -3$, and it indicates an anti-causal signal. Therefore

$$x(t) = -2\,e^{-2t}\,u(-t) + 6\,e^{-3t}\,u(-t)$$

c. The transform $X(s)$ has the partial fraction expansion

$$X(s) = \frac{k_1}{s+j2} + \frac{k_2}{s-j2}$$

with residues

$$k_1 = \left.\frac{s+5}{s-j2}\right|_{s=-j2} = \frac{1}{2} + j\frac{5}{4}$$

and

$$k_2 = k_1^* = \frac{1}{2} - j\frac{5}{4}$$

The partial fraction expansion for $X(s)$ is

$$X(s) = \left(\frac{1}{2} + j\frac{5}{4}\right)\frac{1}{s+j2} + \left(\frac{1}{2} - j\frac{5}{4}\right)\frac{1}{s-j2}$$

The ROC indicates a causal signal. Therefore

$$\begin{aligned} x(t) &= \left(\frac{1}{2} + j\frac{5}{4}\right) e^{-j2t}\,u(t) + \left(\frac{1}{2} - j\frac{5}{4}\right) e^{j2t}\,u(t) \\ &= \frac{1}{2}\left(e^{-j2t} + e^{j2t}\right) u(t) + j\frac{5}{4}\left(e^{-j2t} - e^{j2t}\right) u(t) \\ &= \cos(2t)\;u(t) + \frac{5}{2}\sin(2t)\;u(t) \end{aligned}$$

d. Factored form of $X(s)$ is

$$X(s) = \frac{s+6}{(s+1+j2)\,(s+1-j2)}$$

and leads to the partial fraction form

$$X(s) = \frac{k_1}{s+1+j2} + \frac{k_2}{s+1-j2}$$

The residues are

$$k_1 = \left.\frac{s+6}{s+1-j2}\right|_{s=-1-j2} = \frac{1}{2} + j\frac{5}{4}$$

and

$$k_2 = k_1^* = \frac{1}{2} - j\frac{5}{4}$$

The ROC indicates a causal signal. Therefore

$$\begin{aligned} x(t) &= \left(\frac{1}{2} + j\frac{5}{4}\right) e^{(-1-j2)t}\,u(t) + \left(\frac{1}{2} - j\frac{5}{4}\right) e^{(-1+j2)t}\,u(t) \\ &= \frac{1}{2}e^{-t}\left(e^{-j2t} + e^{j2t}\right) u(t) + j\frac{5}{4}e^{-t}\left(e^{-j2t} - e^{j2t}\right) u(t) \\ &= e^{-t}\cos(2t)\;u(t) + \frac{5}{2}e^{-t}\sin(2t)\;u(t) \end{aligned}$$

e. Partial fraction form of $X(s)$ is

$$X(s) = \frac{k_{1,1}}{s+1} + \frac{k_{1,2}}{(s+1)^2} + \frac{k_2}{s+2}$$

The residues are

$$k_{1,2} = \left.\frac{s\,(s-1)}{s+2}\right|_{s=-1} = 2$$

$$k_2 = \left.\frac{s\,(s-1)}{(s+1)^2}\right|_{s=-2} = 6$$

and

$$k_{1,1} = \frac{d}{ds}\left[\frac{s\,(s-1)}{s+2}\right]\Bigg|_{s=-1} = \left.\frac{s^2+4s-2}{(s+2)^2}\right|_{s=-1} = -5$$

The partial fraction expansion for $X(s)$ is

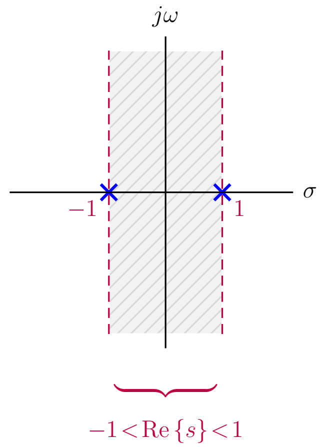
$$X(s) = -\frac{5}{s+1} + \frac{2}{(s+1)^2} + \frac{6}{s+2}$$

The ROC indicates a causal signal. Therefore

$$x(t) = -5\,e^{-t}\,u(t) + 2\,t\,e^{-t}\,u(t) + 6\,e^{-2t}\,u(t)$$

7.19.

a. Pole-zero diagram is shown below:



b. Possible choices for the ROC are:

1. $\mathrm{Re}\{s\} < -2$
2. $-2 < \mathrm{Re}\{s\} < -1$
3. $-1 < \mathrm{Re}\{s\} < 1$
4. $\mathrm{Re}\{s\} > 1$

c. The partial fraction form of $X(s)$ is

$$X(s) = \frac{k_1}{s-1} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

The residues are

$$k_1 = \left.\frac{s\,(s-3)}{(s+1)\,(s+2)}\right|_{s=1} = -\frac{1}{3}$$

$$k_2 = \left. \frac{s\,(s-3)}{(s-1)\,(s+2)} \right|_{s=-1} = -2$$

and

$$k_3 = \left. \frac{s\,(s-3)}{(s-1)\,(s+1)} \right|_{s=-2} = \frac{10}{3}$$

The partial fraction expansion for $X(s)$ is

$$X(s) = -\frac{1/3}{s-1} - \frac{2}{s+1} + \frac{10/3}{s+2}$$

The inverse transform is shown below for each choice of the ROC:

$$\begin{aligned}
\mathrm{Re}\left\{s\right\} < -2 \quad &: \quad x(t) = \frac{1}{3}\,e^{t}\,u(-t) + 2\,e^{-t}\,u(-t) - \frac{10}{3}\,e^{-2t}\,u(-t) \\
-2 < \mathrm{Re}\left\{s\right\} < -1 \quad &: \quad x(t) = \frac{1}{3}\,e^{t}\,u(-t) + 2\,e^{-t}\,u(-t) + \frac{10}{3}\,e^{-2t}\,u(t) \\
-1 < \mathrm{Re}\left\{s\right\} < 1 \quad &: \quad x(t) = \frac{1}{3}\,e^{t}\,u(-t) - 2\,e^{-t}\,u(t) + \frac{10}{3}\,e^{-2t}\,u(t) \\
\mathrm{Re}\left\{s\right\} > 1 \quad &: \quad x(t) = -\frac{1}{3}\,e^{t}\,u(t) - 2\,e^{-t}\,u(t) + \frac{10}{3}\,e^{-2t}\,u(t)
\end{aligned}$$

The inverse transform $x(t)$ is square integrable if the ROC includes the $j\omega$-axis. The only choice for ROC that satisfies this condition is $-1 < \mathrm{Re}\{s\} < 1$.

7.20.

a. The transform can be written as

$$X(s) = -\frac{1}{s+2} + \frac{2}{s+3}$$

Possible choices for the ROC and the corresponding signals $x(t)$ are as follows:

$$\begin{aligned}
\mathrm{Re}\left\{s\right\} < -3 \quad &: \quad x(t) = e^{-2t}\,u(-t) - 2\,e^{-3t}\,u(-t) \\
-3 < \mathrm{Re}\left\{s\right\} < -2 \quad &: \quad x(t) = e^{-2t}\,u(-t) + 2\,e^{-3t}\,u(t) \\
\mathrm{Re}\left\{s\right\} > -2 \quad &: \quad x(t) = -\,e^{-2t}\,u(t) + 2\,e^{-3t}\,u(t)
\end{aligned}$$

b. The transform can be written as

$$X(s) = \frac{2}{s+2} - \frac{1}{(s+2)^2} - \frac{2}{s+3}$$

Possible choices for the ROC and the corresponding signals $x(t)$ are as follows:

$$\begin{aligned}
\mathrm{Re}\left\{s\right\} < -3 \quad &: \quad x(t) = -2\,e^{-2t}\,u(-t) + t\,e^{-2t}\,u(-t) + 2\,e^{-3t}\,u(-t) \\
-3 < \mathrm{Re}\left\{s\right\} < -2 \quad &: \quad x(t) = -2\,e^{-2t}\,u(-t) + t\,e^{-2t}\,u(-t) - 2\,e^{-3t}\,u(t) \\
\mathrm{Re}\left\{s\right\} > -2 \quad &: \quad x(t) = 2\,e^{-2t}\,u(t) - t\,e^{-2t}\,u(t) - 2\,e^{-3t}\,u(t)
\end{aligned}$$

c. The transform can be written as

$$X(s) = 1 - \frac{7}{s+3} + \frac{12}{(s+3)^2}$$

Possible choices for the ROC and the corresponding signals $x(t)$ are as follows:

$$\begin{aligned} \operatorname{Re}\{s\} < -3 \quad &: \quad x(t) = \delta(t) + 7\,e^{-3t}\,u(-t) - 12\,t\,e^{-3t}\,u(-t) \\ \operatorname{Re}\{s\} > -3 \quad &: \quad x(t) = \delta(t) - 7\,e^{-3t}\,u(t) + 12\,t\,e^{-3t}\,u(t) \end{aligned}$$

d. The transform can be written as

$$X(s) = \frac{\frac{1}{2} - j\frac{1}{3}}{s+2+j3} + \frac{\frac{1}{2} + j\frac{1}{3}}{s+2-j3}$$

Possible choices for the ROC and the corresponding signals $x(t)$ are as follows:

$$\begin{aligned} \operatorname{Re}\{s\} < -2 \quad : \quad x(t) =& \; -\left(\frac{1}{2} - j\frac{1}{3}\right) e^{(-2-j3)t}\,u(-t) - \left(\frac{1}{2} + j\frac{1}{3}\right) e^{(-2+j3)t}\,u(-t) \\ =& \; -e^{-2t}\cos(3t)\,u(-t) + \frac{2}{3}\,e^{-2t}\sin(3t)\,u(-t) \\ \operatorname{Re}\{s\} > -2 \quad : \quad x(t) =& \; \left(\frac{1}{2} - j\frac{1}{3}\right) e^{(-2-j3)t}\,u(t) + \left(\frac{1}{2} + j\frac{1}{3}\right) e^{(-2+j3)t}\,u(t) \\ =& \; e^{-2t}\cos(3t)\,u(t) - \frac{2}{3}\,e^{-2t}\sin(3t)\,u(t) \end{aligned}$$

7.21.

a. Let the transform $X_1(s)$ be defined as

$$X_1(s) = \frac{1}{s+1}$$

so that

$$x_1(t) = e^{-t}\,u(t)$$

Since

$$X(s) = \left(1 - e^{-s}\right) X_1(s)$$

we conclude that

$$\begin{aligned} x(t) =& x_1(t) - x_1(t-1) \\ =& e^{-t}\,u(t) - e^{-(t-1)}\,u(t-1) \end{aligned}$$

b. Let the transform $X_1(s)$ be defined as

$$X_1(s) = \frac{s}{s+1} = 1 - \frac{1}{s+1}$$

so that

$$x_1(t) = \delta(t) - e^{-t}\,u(t)$$

Since

$$X(s) = \left(1 - e^{-s}\right) X_1(s)$$

we conclude that

$$\begin{aligned} x(t) =& x_1(t) - x_1(t-1) \\ =& \delta(t) - e^{-t}\, u(t) - \delta(t-1) + e^{-(t-1)}\, u(t-1) \end{aligned}$$

c. Let the transform $X_1(s)$ be defined as

$$X_1(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

so that

$$x_1(t) = u(t) - e^{-t}\, u(t)$$

Since

$$X(s) = \left(1 - e^{-s}\right) X_1(s)$$

we conclude that

$$\begin{aligned} x(t) =& x_1(t) - x_1(t-1) \\ =& u(t) - e^{-t}\, u(t) - u(t-1) + e^{-(t-1)}\, u(t-1) \end{aligned}$$

c. Let the transform $X_1(s)$ be defined as

$$X_1(s) = \frac{1}{s}$$

so that

$$x_1(t) = u(t)$$

Since

$$X(s) = \left(1 - e^{-s} + e^{-2s} - e^{-3s}\right) X_1(s)$$

we conclude that

$$x(t) = u(t) - u(t-1) + u(t-2) - u(t-3)$$

7.22. Taking the Laplace transform of each side yields the following:

a.

$$3s\, Y(s) + 2\, Y(s) = 7\, X(s) \qquad \Rightarrow \qquad H(s) = \frac{Y(s)}{X(s)} = \frac{7}{3s+2}$$

b.

$$s^2\, Y(s) + 4s\, Y(s) + 3\, Y(s) = s\, X(s) + X(s) \qquad \Rightarrow \qquad H(s) = \frac{Y(s)}{X(s)} = \frac{s+1}{s^2+4s+3}$$

c.

$$s^2\, Y(s) + 4\, Y(s) = s^2\, X(s) + s\, X(s) + 3\, X(s) \qquad \Rightarrow \qquad H(s) = \frac{Y(s)}{X(s)} = \frac{s^2+s+3}{s^2+4}$$

7.23.

a.

$$H(s)=\frac{1}{s+4}=\frac{Y(s)}{X(s)}\qquad\Rightarrow\qquad s\,Y(s)+4\,Y(s)=X(s)$$

$$\frac{dy(t)}{dt}+4\,y(t)=x(t)$$

b.

$$H(s)=\frac{s}{s+4}=\frac{Y(s)}{X(s)}\qquad\Rightarrow\qquad s\,Y(s)+4\,Y(s)=X(s)$$

$$\frac{dy(t)}{dt}+4\,y(t)=x(t)$$

c.

$$H(s)=\frac{s+1}{s^2+5\,s+6}=\frac{Y(s)}{X(s)}$$

$$s^2\,Y(s)+5s\,Y(s)+6\,Y(s)=s\,X(s)+X(s)$$

$$\frac{d^2y(t)}{dt^2}+5\,\frac{dy(t)}{dt}+6\,y(t)=\frac{dx(t)}{dt}+x(t)$$

d.

$$H(s)=\frac{s+1}{s^2+5\,s+6}=\frac{Y(s)}{X(s)}$$

$$s^2\,Y(s)+5s\,Y(s)+6\,Y(s)=s\,X(s)+X(s)$$

$$\frac{d^2y(t)}{dt^2}+5\,\frac{dy(t)}{dt}+6\,y(t)=\frac{dx(t)}{dt}+x(t)$$

e.

$$H(s)=\frac{s^2-s}{s^3+5\,s^2+8\,s+6}=\frac{Y(s)}{X(s)}$$

$$s^3\,Y(s)+5s^2\,Y(s)+8s\,Y(s)+6\,Y(s)=s^2\,X(s)-s\,X(s)$$

$$\frac{d^3y(t)}{dt^3}+5\,\frac{d^2y(t)}{dt^2}+8\,\frac{dy(t)}{dt}+6\,y(t)=\frac{d^2x(t)}{dt^2}-\frac{dx(t)}{dt}$$

f.

$$H(s)=\frac{s^2+s-2}{s^3+7\,s^2+19\,s+13}=\frac{Y(s)}{X(s)}$$

$$s^3\,Y(s)+7s^2\,Y(s)+19s\,Y(s)+13\,Y(s)=s^2\,X(s)+s\,X(s)-2\,X(s)$$

$$\frac{d^3y(t)}{dt^3}+7\,\frac{d^2y(t)}{dt^2}+19\,\frac{dy(t)}{dt}+13\,y(t)=\frac{d^2x(t)}{dt^2}+\frac{dx(t)}{dt}-2\,x(t)$$

7.24.

Let the system function be $H(s)$. For a unit-step input signal we have $X(s) = 1/s$, and the Laplace transform of the unit-step response of the system is

$$Y_u(s) = \frac{H(s)}{s}$$

The solution method is based on determining $H(s)$ from this relationship, and then finding a differential equation from the knowledge of $H(s)$.

a.

$$Y_u(s) = \frac{1}{s+1} \qquad \Rightarrow \qquad H(s) = s\,Y_u(s) = \frac{s}{s+1}$$

$$(s+1)\;Y(s) = s\,X(s)$$

$$\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt}$$

b.

$$Y_u(s) = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s\,(s+1)} \qquad \Rightarrow \qquad H(s) = s\,Y_u(s) = \frac{1}{s+1}$$

$$(s+1)\;Y(s) = X(s)$$

$$\frac{dy(t)}{dt} + y(t) = x(t)$$

c.

$$Y_u(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)\,(s+2)} \qquad \Rightarrow \qquad H(s) = s\,Y_u(s) = \frac{s}{(s+1)\,(s+2)}$$

$$\left(s^2+3\,s+2\right)\;Y(s) = s\,X(s)$$

$$\frac{d^2y(t)}{dt^2} + 3\,\frac{dy(t)}{dt} + 2\,y(t) = \frac{dx(t)}{dt}$$

d.

$$Y_u(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{2}{s+2} = \frac{2s^2+3s+2}{s^3+3s^2+2s} \qquad \Rightarrow \qquad H(s) = s\,Y_u(s) = \frac{2s^2+3s+2}{s^2+3s+2}$$

$$\left(s^2+3s+2\right)\;Y(s) = \left(2s^2+3s+2\right)\;X(s)$$

$$\frac{d^2y(t)}{dt^2} + 3\,\frac{dy(t)}{dt} + 2\,y(t) = 2\,\frac{d^2x(t)}{dt^2} + 3\,\frac{dx(t)}{dt} + 2\,x(t)$$

e.

$$Y_u(s) = \frac{1}{s} - \frac{0.3\,(s+1)}{(s+1)^2+4} = \frac{0.7\,s^2+1.7\,s+5}{s\left(s^2+2\,s+5\right)} \qquad \Rightarrow \qquad H(s) = s\,Y_u(s) = \frac{0.7\,s^2+1.7\,s+5}{s^2+2\,s+5}$$

$$\left(s^2+2\,s+5\right)\,Y\,(s)=\left(0.7\,s^2+1.7\,s+5\right)\,X\,(s)$$

$$\frac{d^2y\,(t)}{dt^2}+2\,\frac{dy\,(t)}{dt}+5\,y\,(t)=0.7\,\frac{d^2x\,(t)}{dt^2}+1.7\,\frac{dx\,(t)}{dt}+5\,x\,(t)$$

7.25.

Let the system function be $H\,(s)$. For a unit-ramp input signal we have $X\,(s)=1/s^2$, and the Laplace transform of the unit-step response of the system is

$$Y_u\,(s)=\frac{H\,(s)}{s^2}$$

The solution method is based on determining $H\,(s)$ from this relationship, and then finding a differential equation from the knowledge of $H\,(s)$.

a.

$$Y_r\,(s)=\frac{1}{s}-\frac{1}{s+1}=\frac{1}{s\,(s+1)}\qquad\Rightarrow\qquad H\,(s)=s^2\,Y_r\,(s)=\frac{s}{s+1}=\frac{Y\,(s)}{X\,(s)}$$

$$(s+1)\;Y\,(s)=s\,X\,(s)$$

$$\frac{dy\,(t)}{dt}+y\,(t)=\frac{dx\,(t)}{dt}$$

b.

$$Y_r\,(s)=\frac{1}{s+1}+\frac{2}{s+2}-\frac{4}{s+3}=\frac{-s^2+s+4}{(s+1)\,(s+2)\,(s+3)}$$

$$H\,(s)=s^2\,Y_r\,(s)=\frac{-s^4+s^3+4\,s^2}{s^3+6\,s^2+11\,s+6}=\frac{Y\,(s)}{X\,(s)}$$

$$\left(s^3+6\,s^2+11\,s+6\right)\,Y\,(s)=\left(-s^4+s^3+4\,s^2\right)\,X\,(s)$$

$$\frac{d^3y\,(t)}{dt^3}+6\,\frac{d^2y\,(t)}{dt^2}+11\,\frac{dy\,(t)}{dt}+6\,y\,(t)=-\frac{d^4x\,(t)}{dt^4}+\frac{d^3x\,(t)}{dt^3}+4\,\frac{d^2x\,(t)}{dt^2}$$

c.

$$Y_r\,(s)=\frac{1}{s}-\frac{0.3\,(s+1)}{(s+1)^2+4}=\frac{0.7\,s^2+1.7\,s+5}{s\left(s^2+2\,s+5\right)}$$

$$H\,(s)=s^2\,Y_r\,(s)=\frac{0.7\,s^3+1.7\,s^2+5\,s}{s^2+2\,s+5}=\frac{Y\,(s)}{X\,(s)}$$

$$\left(s^2+2\,s+5\right)\,Y\,(s)=\left(0.7\,s^3+1.7\,s^2+5\,s\right)\,X\,(s)$$

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+5\,y(t)=0.7\frac{d^3x(t)}{dt^3}+1.7\frac{d^2x(t)}{dt^2}+5\frac{dx(t)}{dt}$$

7.26.

The response of the CTLTI system with system function $H(s)$ to the signal $x(t)=e^{s_0 t}$ is

$$\text{Sys}\left\{e^{s_0 t}\right\}=e^{s_0 t}\,H(s_0)$$

a.

$$s_0=-0.5\,,\qquad H(-0.5)=3.3333+j0$$
$$=3.3333\,e^{j0}$$
$$y(t)=3.3333\,e^{-0.5t}$$

b.

$$s_0=-0.5+j2\,,\qquad H\left(-0.5+j2\right)=-0.0047-j\,0.6212$$
$$=0.6212\,e^{-j\,1.5783}$$
$$y(t)=0.6212\,e^{-0.5t-j\,1.5783}$$

c.

$$s_0=j3\,,\qquad H\left(j3\right)=0.0462-j\,0.3692$$
$$=0.3721\,e^{-j\,1.4464}$$
$$y(t)=0.3721\,e^{j\,(3t-1.4464)}$$

d.

The signal $x(t)$ is in the form

$$x(t)=e^{s_0 t}+e^{s_1 t}$$

with $s_0=-j3$ and $s_1=j3$.

$$s_0=-j3\,,\qquad H\left(-j3\right)=0.0462+j\,0.3692$$
$$=0.3721\,e^{j\,1.4464}$$
$$s_1=j3\,,\qquad H\left(j3\right)=0.0462-j\,0.3692$$
$$=0.3721\,e^{-j\,1.4464}$$

$$y(t)=0.3721\,e^{j\,(-3t+1.4464)}+0.3721\,e^{j\,(3t-1.4464)}$$
$$=0.7442\,\cos(3t-1.4464)$$

e. The signal $x(t)$ is in the form

$$x(t) = e^{s_0 t} + e^{s_1 t}$$

with $s_0 = j2$ and $s_1 = j3$.

$$s_0 = j2\,, \qquad H\left(j2\right) = 0.1500 - j\,0.5500$$
$$= 0.5701\,e^{-j\,1.3045}$$

$$s_1 = j3\,, \qquad H\left(j3\right) = 0.0462 - j\,0.3692$$
$$= 0.3721\,e^{-j\,1.4464}$$

$$y(t) = 0.5701\,e^{j\,(2t-1.3045)} + 0.3721\,e^{j\,(3t-1.4464)}$$

7.27.

Using the Laplace transform definition, $H(s_0)$ is found as

$$H(s_0) = \int_{-\infty}^{\infty} h(t)\, e^{-s_0 t}\, dt$$

Conjugating both sides yields

$$H^*(s_0) = \left[\int_{-\infty}^{\infty} h(t)\, e^{-s_0 t}\, dt\right]^*$$
$$= \int_{-\infty}^{\infty} h^*(t)\, e^{-s_0^* t}\, dt$$

Since $h(t)$ is real-valued we have $h^*(t) = h(t)$ and

$$H^*(s_0) = \int_{-\infty}^{\infty} h(t)\, e^{-s_0^* t}\, dt = H\left(s_0^*\right)$$

Therefore

$$H\left(s_0^*\right) = H^*(s_0) = H_0\, e^{-j\Theta_0}$$

7.28.

a.

$$s_0 = j2\,, \qquad H\left(j2\right) = -0.1379 + 0.3448$$
$$= 0.3714\,e^{j\,1.9513}$$

$$y(t) = 0.3714\,\cos(2t + 1.9513)$$

b.

$$s_0 = -0.5 + j2\,, \qquad H\left(-0.5 + j2\right) = -0.3479 + 0.2879$$
$$= 0.4516\,e^{j\,2.4503}$$

$$y(t) = 0.4516\,e^{-0.5t}\cos(2t + 2.4503)$$

c.

$$s_0 = -1 + j2\,, \qquad H\left(-1+j2\right) = -0.6154 + j\,0.0769$$
$$= 0.6202\,e^{j\,3.0172}$$

$$y(t) = 0.6202\,e^{-t}\sin(2t + 3.0172)$$

7.29.

The system function is

$$H(s) = \mathscr{L}\left\{h(t)\right\} = \frac{1}{s+1}$$

a. Laplace transform of the input signal is

$$X(s) = \mathscr{L}\left\{\cos(2t)\;u(t)\right\} = \frac{s}{s^2+4}$$

Laplace transform of the output signal is found as

$$Y(s) = H(s)\;X(s) = \frac{s}{(s+1)\left(s^2+4\right)} = \frac{s}{(s+1)\left(s+j2\right)\left(s-j2\right)}$$

Partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+j2} + \frac{k_3}{s-j2}$$

with the residues

$$k_1 = \left.\frac{s}{s^2+4}\right|_{s=-1} = -0.2$$

$$k_2 = \left.\frac{s}{(s+1)\left(s-j2\right)}\right|_{s=-j2} = 0.1 + j0.2$$

and

$$k_3 = k_2^* = 0.1 - j0.2$$

The output signal is found as

$$y(t) = -0.2\,e^{-t}\,u(t) + \left(0.1+j0.2\right)\,e^{-j2t}\,u(t) + \left(0.1-j0.2\right)\,e^{j2t}\,u(t)$$
$$= -0.2\,e^{-t}\,u(t) + 0.1\left(e^{-j2t}+e^{j2t}\right)u(t) + j0.2\left(e^{-j2t}-e^{j2t}\right)u(t)$$
$$= \underbrace{-0.2\,e^{-t}\,u(t)}_{\text{transient}} + \underbrace{0.2\cos(2t)\;u(t) + 0.4\sin(2t)\;u(t)}_{\text{steady-state}}$$

The transient component of the response is

$$y_t(t) = -0.2\,e^{-t}\,u(t)$$

and the steady-state component is

$$y_{ss}(t) = 0.2\cos(2t) + 0.4\sin(2t)$$

b. Laplace transform of the input signal is

$$X(s) = \mathscr{L}\left\{\sin(3t)\,u(t)\right\} = \frac{3}{s^2+9}$$

Laplace transform of the output signal is found as

$$Y(s) = H(s)\,X(s) = \frac{3}{(s+1)\left(s^2+9\right)} = \frac{3}{(s+1)\left(s+j3\right)\left(s-j3\right)}$$

Partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+j3} + \frac{k_3}{s-j3}$$

with the residues

$$k_1 = \left.\frac{3}{\left(s^2+9\right)}\right|_{s=-1} = 0.3$$

$$k_2 = \left.\frac{3}{(s+1)\left(s-j3\right)}\right|_{s=-j3} = -0.15+j0.05$$

and

$$k_3 = k_2^* = -0.15-j0.05$$

The output signal is found as

$$\begin{aligned}
y(t) =& 0.3\,e^{-t}\,u(t) + \left(-0.15+j0.05\right)\,e^{-j3t}\,u(t) + \left(-0.15-j0.05\right)\,e^{j3t}\,u(t) \\
=& 0.3\,e^{-t}\,u(t) - 0.15\left(e^{-j3t}+e^{+j3t}\right)u(t) + j0.05\left(e^{-j3t}-e^{j3t}\right)u(t) \\
=& \underbrace{0.3\,e^{-t}\,u(t)}_{\text{transient}} - \underbrace{0.3\cos(3t)\,u(t) + 0.1\sin(3t)\,u(t)}_{\text{steady-state}}
\end{aligned}$$

The transient component of the response is

$$y_t(t) = 0.3\,e^{-t}\,u(t)$$

and the steady-state component is

$$y_{ss}(t) = -0.3\cos(3t) + 0.1\sin(3t)$$

7.30.

a.

$$H(s) = 1 - \frac{2}{s+3} \qquad \Rightarrow \qquad h(t) = \delta(t) - 2\,e^{-3t}\,u(t)$$

b.

$$H(s) = 1 + \frac{-3s-1}{s^2+3s+2} = 1 + \frac{-3s-1}{(s+1)\,(s+2)}$$

Let

$$H_1(s) = \frac{-3s-1}{(s+1)\,(s+2)}$$

so that

$$H_1(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

The residues are

$$k_1 = \left.\frac{-3s-1}{s+2}\right|_{s=-1} = 2$$

and

$$k_2 = \left.\frac{-3s-1}{s+1}\right|_{s=-2} = -5$$

The system function is

$$H(s) = 1 + H_1(s) = 1 + \frac{2}{s+1} - \frac{5}{s+2}$$

and the impulse response is

$$h(t) = \delta(t) + 2\,e^{-t}\,u(t) - 5\,e^{-2t}\,u(t)$$

c.

$$\begin{aligned} H(s) =& \frac{s^2-1}{(s+2)\,\left(s+1+j\right)\,\left(s+1-j\right)} \\ =& \frac{k_1}{s+2} + \frac{k_2}{s+1+j} + \frac{k_3}{s+1-j} \end{aligned}$$

The residues are

$$k_1 = \left.\frac{s^2-1}{s^2+s+2}\right|_{s=-2} = 1.5$$

$$k_2 = \left.\frac{s^2-1}{(s+2)\,\left(s+1-j\right)}\right|_{s=-1-j} = -0.25 - j\,0.75$$

and

$$k_3 = k_2^* = -0.25 + j\,0.75$$

The impulse response is

$$\begin{aligned} h(t) =& 1.5\,e^{-2t}\,u(t) + \left(-0.25 - j\,0.75\right)\,e^{(-1-j)t}\,u(t) + \left(-0.25 + j\,0.75\right)\,e^{(-1+j)t}\,u(t) \\ =& 1.5\,e^{-2t}\,u(t) - 0.25\,e^{-t}\left(e^{-jt} + e^{jt}\right)\,u(t) - j\,0.75\,e^{-t}\left(e^{-jt} - e^{jt}\right)\,u(t) \\ =& 1.5\,e^{-2t}\,u(t) - 0.5\,e^{-t}\cos(t)\,u(t) - 1.5\,e^{-t}\sin(t)\,u(t) \end{aligned}$$

d.

$$H(s) = \frac{-4}{(s+1)^2} + \frac{1}{s+1}$$

The corresponding impulse response is

$$h(t) = -4t\,e^{-t}\,u(t) + e^{-t}\,u(t) = (-4t+1)\,e^{-t}\,u(t)$$

7.31.

The system function is

$$H(s) = \frac{1}{s+1}$$

The input signal can be written as

$$x(t) = \frac{1}{2}\,e^{j2t} + \frac{1}{2}\,e^{-j2t}$$

Let $s_0 = j2$. The system function evaluated at $s = s_0 = j2$ is

$$H(s_0) = \frac{1}{j2+1} = 0.2 - j\,0.4$$

The system function evaluated at $s = s_0^* = -j2$ is

$$H\left(s_0^*\right) = \frac{1}{-j2+1} = 0.2 + j\,0.4$$

The response of the system is

$$\begin{aligned}
y(t) =& \operatorname{Sys}\left\{\cos(2t)\right\} \\
=& \frac{1}{2}\operatorname{Sys}\left\{e^{j2t}\right\} + \frac{1}{2}\operatorname{Sys}\left\{e^{-j2t}\right\} \\
=& \frac{1}{2}\left(0.2 - j\,0.4\right)e^{j2t} + \frac{1}{2}\left(0.2 + j\,0.4\right)e^{-j2t} \\
=& \frac{1}{2}\,(0.2)\left(e^{j2t} + e^{-j2t}\right) - \frac{1}{2}\left(j\,0.4\right)\left(e^{j2t} - e^{-j2t}\right) \\
=& 0.2\cos(2t) + 0.4\sin(2t)
\end{aligned}$$

The result found matches the steady-state component of the result found in part (a) of Problem 7.29.

7.32.

The system function is found as

$$\begin{aligned}
H(s) &= \mathscr{L}\left\{e^{-t}\,u(t) - e^{-t}\,u(t-2)\right\} \\
&= \mathscr{L}\left\{e^{-t}\,u(t)\right\} - \mathscr{L}\left\{e^{-t}\,u(t-2)\right\} \\
&= \mathscr{L}\left\{e^{-t}\,u(t)\right\} - e^{-2}\,\mathscr{L}\left\{e^{-(t-2)}\,u(t-2)\right\} \\
&= \left[1 - e^{-2(s+1)}\right]\frac{1}{s+1}
\end{aligned}$$

Laplace transform of the input signal is

$$X(s) = \frac{s+0.5}{(s+0.5)^2+4} = \frac{s+0.5}{s^2+s+4.25}$$

and the Laplace transform of the output signal is

$$\begin{aligned} Y(s) &= H(s)\, X(s) \\ &= \left[1 - e^{-2(s+1)}\right] \frac{s+0.5}{(s+1)\left(s^2+s+4.25\right)} \end{aligned}$$

Let

$$Y_1(s) = \frac{s+0.5}{(s+1)\left(s^2+s+4.25\right)}$$

so that

$$Y(s) = \left[1 - e^{-2(s+1)}\right] Y_1(s)$$

The transform $Y_1(s)$ can be expanded into partial fractions as

$$Y_1(s) = \frac{k_1}{s+1} + \frac{k_2}{s+0.5+j2} + \frac{k_3}{s+0.5-j2}$$

with the residues

$$k_1 = \left.\frac{s+0.5}{s^2+s+4.25}\right|_{s=-1} = -0.1176$$

$$k_2 = \left.\frac{s+0.5}{(s+1)\left(s+0.5-j2\right)}\right|_{s=-0.5-j2} = 0.0588 + j\,0.2353$$

and

$$k_3 = k_2^* = 0.0588 - j\,0.2353$$

Using the residues, $y_1(t)$ is found as

$$\begin{aligned} y_1(t) &= -0.1176\, e^{-t}\, u(t) + \left(0.0588 + j\,0.2353\right) e^{(-0.5-j2)t}\, u(t) + \left(0.0588 - j\,0.2353\right) e^{(-0.5+j2)t}\, u(t) \\ &= -0.1176\, e^{-t}\, u(t) + 0.0588\, e^{-0.5t} \left(e^{-j2t} + e^{j2t}\right) u(t) + j\,0.2353\, e^{-0.5t} \left(e^{-j2t} - e^{j2t}\right) u(t) \\ &= -0.1176\, e^{-t}\, u(t) + 0.1176\, e^{-0.5t} \cos(2t)\, u(t) + 0.4706\, e^{-0.5t} \sin(2t)\, u(t) \end{aligned}$$

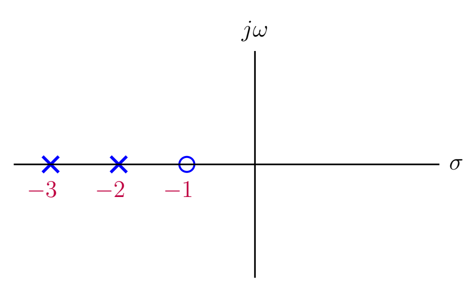
The signal $y(t)$ is

$$\begin{aligned} y(t) &= y_1(t) - e^{-2}\, y_1(t-2) \\ &= \left[-0.1176\, e^{-t} + 0.1176\, e^{-0.5t} \cos(2t) + 0.4706\, e^{-0.5t} \sin(2t)\right] u(t) \\ &\quad + \left[0.1176\, e^{-t} - 0.1176\, e^{-0.5t-1} \cos(2t-4) - 0.4706\, e^{-0.5t-1} \sin(2t-4)\right] u(t-2) \end{aligned}$$

7.33.

a.

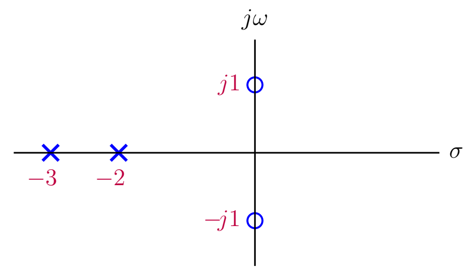
The system function has a zero at $s = -1$ and poles at $s = -2, -3$.



b.

$$H(s) = \frac{(s-j)\,(s+j)}{(s+2)\,(s+3)}$$

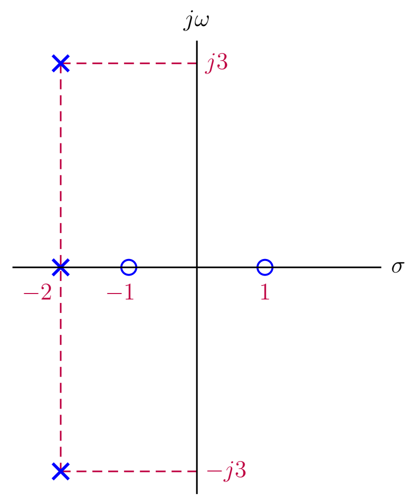
The system function has zeros at $s = \mp j1$ and poles at $s = -2, -3$.



c.

$$H(s) = \frac{(s-1)\,(s+1)}{(s+2)\,(s+2+j3)\,(s+2-j3)}$$

The system function has zeros at $s = \mp 1$ and poles at $s = -2$ and $s = -2 \pm j3$.



7.34.

a.

$$H(s) = K\,\frac{(s+j)\,(s-j)}{(s+1)\,(s+2)} = K\,\frac{s^2+1}{s^2+3s+2}$$

$$H(0) = K\left(\frac{1}{2}\right) = 1 \qquad \Rightarrow \qquad K = 2$$

$$H(s) = \frac{2\,(s^2+1)}{s^2+3s+2}$$

b.

$$H(s) = K\,\frac{(s-1.5)\,(s+1.5)}{(s+1-j1.5)\,(s+1+j1.5)} = K\,\frac{s^2-2.25}{s^2+2s+3.25}$$

$$H(0) = K\left(\frac{-2.25}{3.25}\right) = 1 \qquad \Rightarrow \qquad K = -\frac{13}{9}$$

$$H(s) = -\left(\frac{13}{9}\right)\,\frac{s^2-2.25}{s^2+2s+3.25}$$

c.

$$H(s) = K \frac{(s-j)(s+j)}{(s-1)(s+1)} = K \frac{s^2+1}{s^2-1}$$

$$H(0) = -K = 1 \qquad \Rightarrow \qquad K = -1$$

$$H(s) = \frac{-\left(s^2+1\right)}{s^2-1}$$

d.

$$H(s) = K \frac{s-1.5}{(s+1.5)(s+1-j1.5)(s+1+j1.5)} = K \frac{s-1.5}{s^3+3.5\,s^2+6.25\,s+4.875}$$

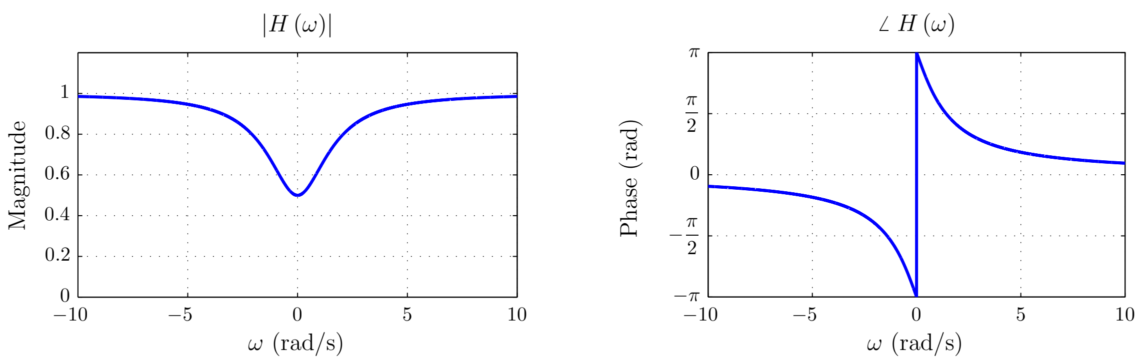
$$H(0) = K \left(\frac{-1.5}{4.875}\right) = 1 \qquad \Rightarrow \qquad K = -3.25$$

$$H(s) = \frac{-3.25\,(s-1.5)}{s^3+3.5\,s^2+6.25\,s+4.875}$$

7.35.

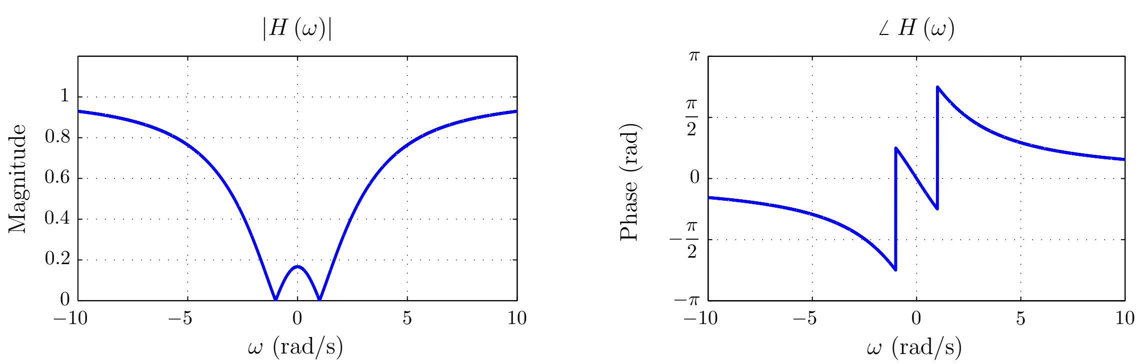
a.

$$H(\omega) = \frac{j\omega-1}{j\omega+2}$$



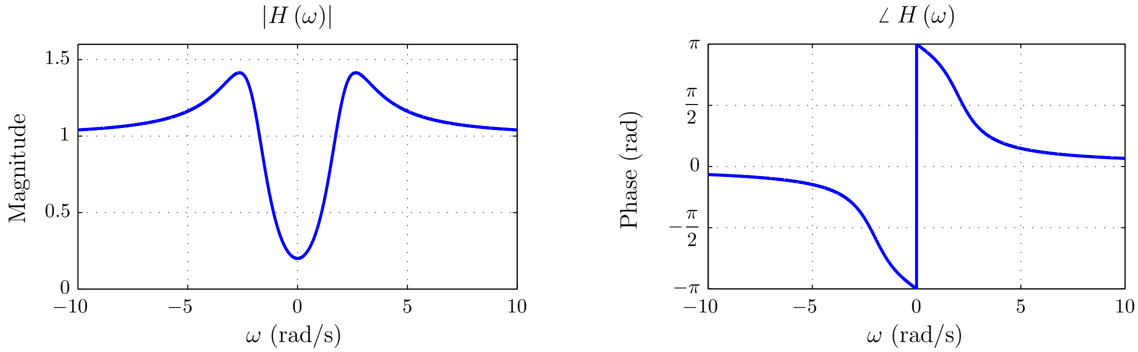
b.

$$H(\omega) = \frac{(j\omega-j1)(j\omega+j1)}{(j\omega+2)(j\omega+3)}$$



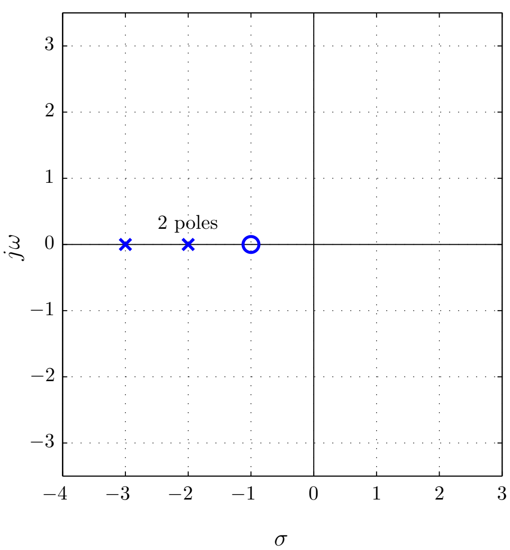
c.

$$H(\omega) = \frac{\left(j\omega-1\right)\left(j\omega+1\right)}{\left(j\omega+1+j2\right)\left(j\omega+1-j2\right)}$$



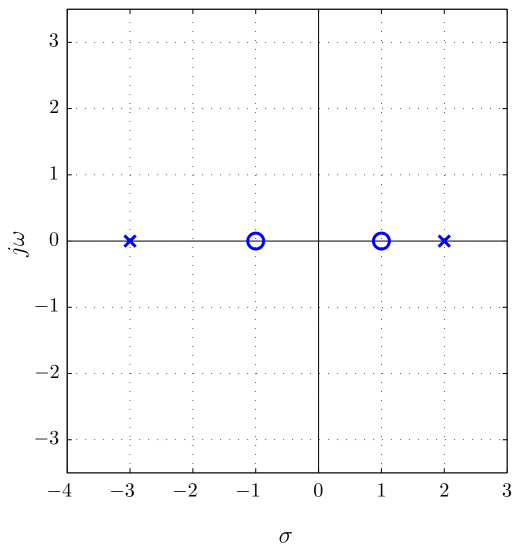
7.36.

a.



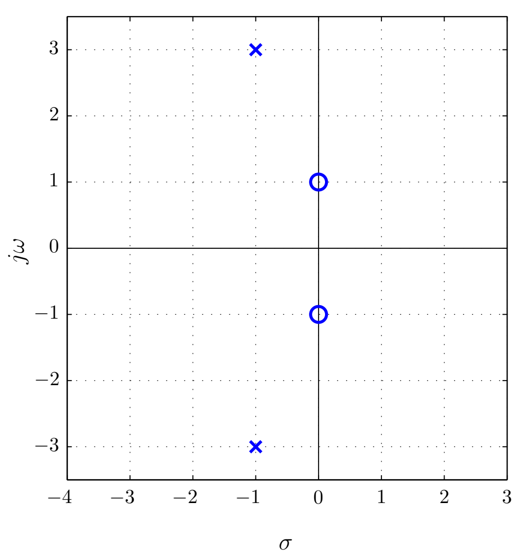
System is stable.

b.



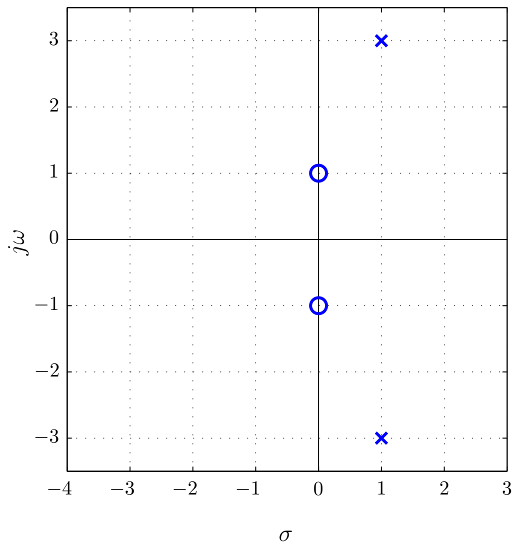
System is unstable.

c.



System is stable.

d.



System is unstable.

7.37.

For a system function to represent a system that is both causal and stable, the following conditions must be met:

- The order of the numerator of the system function must not be greater than the order of the denominator.
- All poles of the system function must be in the left half s-plane. We need $\text{Re}\{s\} < 0$ for all poles.

a. The system function can be written as

$$H(s) = \frac{s-1}{(s+1)\,(s+2)}$$

Poles are at $s = -1$ and $s = -2$. This could be the system function of a causal and stable system.

b. The system function can be written as

$$H(s) = \frac{s\,(s+1)}{\left(s+1-j2\right)\left(s+1+j2\right)}$$

Poles of the system function are at $s = -1 \pm j\,2$. This could be the system function of a causal and stable system.

c. The system function can be written as

$$H(s) = \frac{s^2+1}{s\,(s+2)\,(s+3)}$$

Poles of the system function are at $s = 0$, $s = -2$ and $s = -3$. This could not be the system function of a causal and stable system since there is a pole on the $j\omega$-axis.

d. The system function can be written as

$$H(s) = \frac{s+3}{\left(s-3-j1\right)\left(s-3-j1\right)}$$

Poles of the system function are at $s = 3 + j1$ and $s = 3 - j1$. This could not be the system function of a causal and stable system since the poles are in the right half of the s-plane.

7.38.

a. Taking the Laplace transform of both sides of the differential equation yields

$$s^2\, Y(s) = -2s\, Y(s) - a\, Y(s) + X(s)$$

which can be rearranged as

$$\left(s^2 + 2s + a\right) Y(s) = X(s)$$

and the system function is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 2s + a}$$

b. For the system to be causal and stable, the poles of the system function must be in the left half of the s-plane. The poles of $H(s)$ are found as

$$s^2 + 2s + a = 0 \qquad \Rightarrow \qquad s_{1,2} = -1 \pm \sqrt{1-a}$$

If $a > 1$, the poles are complex-valued, and they are at

$$s_{1,2} = -1 \pm j\sqrt{a-1}$$

If $a < 1$, both poles are real-valued. The system cannot be stable if at least one is in the right half s plane or on the $j\omega$-axis.

$$-1 + \sqrt{1-a} \ge 0 \qquad \Rightarrow \qquad a \le 0$$

Therefore, for stability we need $a > 0$.

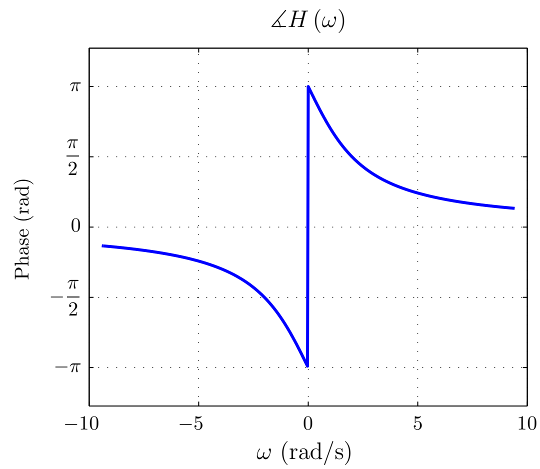
7.39.

a. Evaluating $H(s)$ for $s = j\omega$ yields

$$H(\omega) = H(s)\Big|_{s=j\omega} = \frac{j\omega - 2}{j\omega + 2}$$

The phase of the system function is

$$\measuredangle\, H(\omega) = \tan(-\omega/2) - \tan(\omega/2) = -2\tan(\omega/2)$$



b. The transform of the input signal is

$$X(s) = \frac{1}{s+1}$$

and the transform of the corresponding output signal is

$$Y(s) = H(s)\, X(s) = \frac{s-2}{(s+1)\,(s+2)}$$

Using partial fraction expansion on $Y(s)$ we get

$$Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

The residues are

$$k_1 = \left.\frac{s-2}{s+2}\right|_{s=-1} = -3$$

and

$$k_2 = \left.\frac{s-2}{s+1}\right|_{s=-2} = 4$$

and the response of the system is

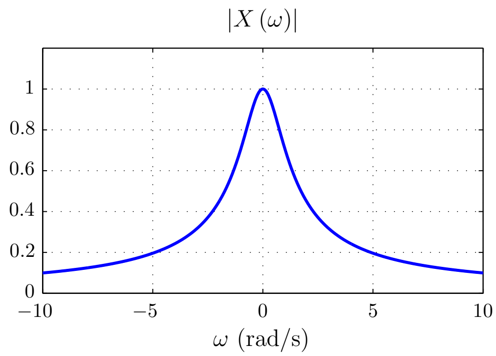
$$y(t) = -3\,e^{-t}\,u(t) + 4\,e^{-2t}\,u(t)$$

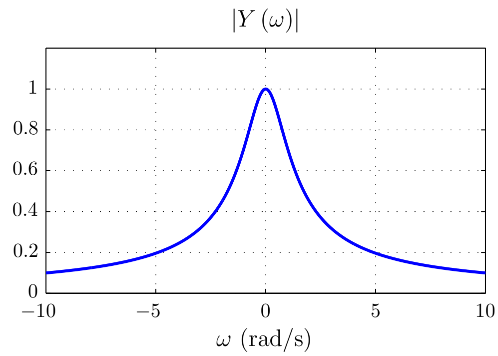
c. For the input signal we obtain

$$X(\omega) = \frac{1}{1+j\omega} \qquad \Rightarrow \qquad \left|X(\omega)\right| = \sqrt{\frac{1}{1+\omega^2}}$$

and for the output signal we obtain

$$Y(\omega) = \frac{-2+j\omega}{2-\omega^2+j3\omega} \qquad \Rightarrow \qquad \left|Y(\omega)\right| = \sqrt{\frac{4+\omega^2}{\left(2-\omega^2\right)^2+9\omega^2}}$$





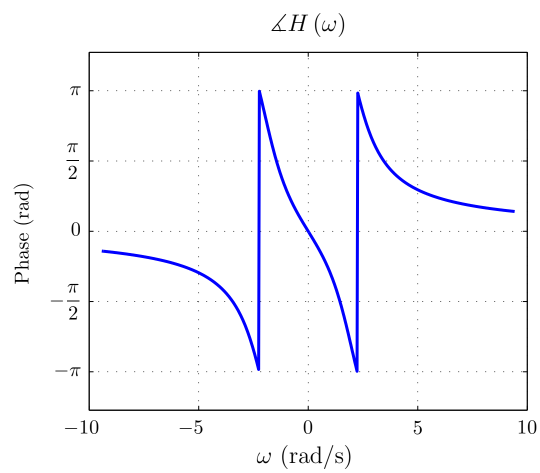
7.40.

a. The frequency response of the system is computed by evaluating $H(s)$ for $s = j\omega$:

$$H(\omega) = \left. H(s)\right|_{s=j\omega} = \frac{-\omega^2 + 5 - j\,2\omega}{-\omega^2 + 5 + j\,2\omega}$$

The phase characteristic is

$$\measuredangle H(\omega) = \tan^{-1}\left(\frac{-2\omega}{5-\omega^2}\right) - \tan^{-1}\left(\frac{2\omega}{5-\omega^2}\right) = -2\tan^{-1}\left(\frac{2\omega}{5-\omega^2}\right)$$



b. The transform of the input signal is

$$X(s) = \frac{1}{s+1}$$

and the transform of the output signal is

$$Y(s) = \frac{s^2 - 2\,s + 5}{(s+1)\left(s^2 + 2\,s + 5\right)} = \frac{s^2 - 2\,s + 5}{s^3 + 3\,s^2 + 7\,s + 5}$$

which can be expressed in partial fraction form as

$$Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+1+j2} + \frac{k_3}{s+1-j2}$$

The residues are

$$k_1 = \left.\frac{s^2-2s+5}{s^2+2s+5}\right|_{s=-1} = 2$$

$$k_2 = \left.\frac{s^2-2s+5}{(s+1)\,(s+1-j2)}\right|_{s=-1-j2} = -\frac{1}{2} - j1$$

and

$$k_3 = k_2^* = -\frac{1}{2} + j1$$

The output signal is

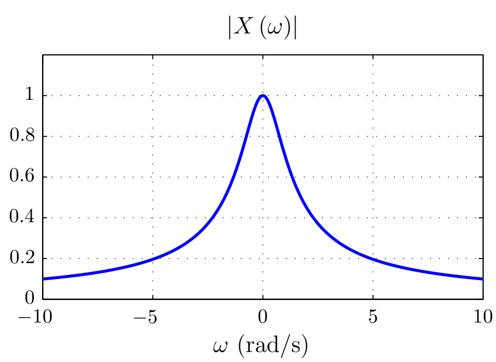
$$\begin{aligned} y(t) =& 2e^{-t}\,u(t) + \left(-\frac{1}{2}-j1\right)e^{\left(-\frac{1}{2}+j1\right)t}\,u(t) + \left(-\frac{1}{2}+j1\right)e^{\left(-\frac{1}{2}-j1\right)t}\,u(t) \\ =& 2e^{-t}\,u(t) - e^{-\frac{1}{2}t}\cos(t)\,u(t) - 2e^{-\frac{1}{2}t}\sin(t)\,u(t) \end{aligned}$$

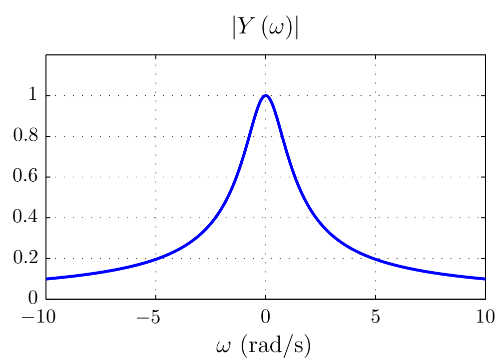
c. For the input signal

$$X(\omega) = \frac{1}{1+j\omega} \qquad \Rightarrow \qquad \left|X(\omega)\right| = \sqrt{\frac{1}{1+\omega^2}}$$

and for the output signal

$$Y(\omega) = \frac{5-\omega^2-j2\omega}{5-3\omega^2+j\omega\left(7-\omega^2\right)} \qquad \Rightarrow \qquad \left|Y(\omega)\right| = \sqrt{\frac{\left(5-\omega^2\right)^2+4\omega^2}{\left(5-3\omega^2\right)^2+\omega^2\left(7-\omega^2\right)^2}}$$





7.41.

a. The inverse of the system is

$$H^{-1}(s) = \frac{(s+3)\,(s+4)}{(s+1)\,(s-2)}$$

The inverse system is causal but not stable since there is a pole at $s = 2$ in the right half s-plane.

b. Let us express $H(s)$ as

$$H(s) = \frac{(s+1)(s-2)}{(s+3)(s+4)} \left(\frac{s+2}{s+2}\right)$$

Let

$$H_1(s) = \frac{(s+1)\,(s+2)}{(s+3)\,(s+4)}$$

and

$$H_2(s) = \frac{s-2}{s+2}$$

so that $H(s) = H_1(s)\,H_2(s)$ and the subsystem $H_2(s)$ is an all pass system.

c. The inverse system $H_1^{-1}(s)$ is

$$H_1^{-1}(s) = \frac{(s+3)\,(s+4)}{(s+1)\,(s+2)}$$

It is causal and stable.

d.

$$H(s)\,H_1^{-1}(s) = \frac{s-2}{s+2} = H_2(s)$$

7.42.

a. Taking the Laplace transform of each side we obtain

$$(s+5)\;Y(s) = (2s+3)\;X(s)$$

which leads to the system function

$$H(s) = \frac{Y(s)}{X(s)} = \left(\frac{1}{2}\right)\frac{(s+5)}{s+3/2}$$

The inverse system has the system function

$$H_i(s) = H^{-1}(s) = \frac{2\,(s+3/2)}{s+5}$$

The inverse system is both causal and stable. Its differential equation is

$$\frac{dy(t)}{dt} + 5\,y(t) = 2\,\frac{dx(t)}{dt} + 3\,x(t)$$

b. Taking the Laplace transform of each side we obtain

$$\left(s^2+7s+12\right)\,Y(s) = \left(s^2+s\right)\,X(s)$$

which leads to the system function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2+s}{s^2+7s+12} = \frac{s\,(s+1)}{(s+3)\,(s+4)}$$

The system function for the inverse system is

$$H_i(s) = H^{-1}(s) = \frac{(s+3)\,(s+4)}{s\,(s+1)}$$

Since there is a pole at $s = 0$ the inverse system cannot be stable.

c. Taking the Laplace transform of each side we obtain

$$\left(s^2+3\right)\,Y(s) = (s+5)\,X(s)$$

which leads to the system function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+5}{s^2+3}$$

The system function for the inverse system is

$$H_i(s) = H^{-1}(s) = \frac{s^2+3}{s+5}$$

The numerator order for the inverse system is 2, and the denominator order is 1. Therefore the inverse system cannot be causal.

d. Taking the Laplace transform of each side we obtain

$$(s-3)\,Y(s) = \left(s^2+2s+1\right)\,X(s)$$

which leads to the system function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2+2s+1}{s-3}$$

The numerator order is 2, and the denominator order is 1. Furthermore, there is a pole in the right half of the s-plane. The original system cannot be causal and stable. The system function for the inverse system is

$$H_i(s) = H^{-1}(s) = \frac{s-3}{s^2+2s+1} = \frac{s-3}{(s+1)^2}$$

The inverse system is causal and stable. Its differential equation is

$$\frac{d^2y(t)}{dt^2} + 2\,\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} - 3\,x(t)$$

7.43. Asymptotically we have

$$20\,\log_{10}|H(\omega)| = 20\,\log_{10}|H_1(\omega)| + 20\,\log_{10}|H_2(\omega)| + 20\,\log_{10}|H_3(\omega)| + 20\,\log_{10}|H_4(\omega)|$$

At $\omega = 5$ rad/s

$$20\,\log_{10}|H_1(5)| = 20\,\log_{10}(5) = 13.98\ \text{dB}$$

and

$$20\,\log_{10}|H_2(\omega)| = 20\,\log_{10}|H_3(\omega)| = 20\,\log_{10}|H_4(\omega)| = 0$$

resulting in

$$20\,\log_{10}|H(5)| = 13.98\ \text{dB}$$

At $\omega = 300$ rad/s we have

$$20\log_{10}|H(300)| = 20\log_{10}(300/1) - 20\log_{10}(300/5) - 20\log_{10}(300/40) + 20\log_{10}(300/300) = -3.52 \text{ dB}$$

7.44.

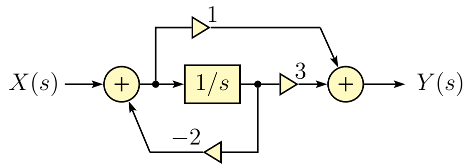
a.

$$H(s) = \frac{1+3s^{-1}}{1+2s^{-1}}$$

$$\left(1+2s^{-1}\right) Y(s) = \left(1+3s^{-1}\right) X(s)$$

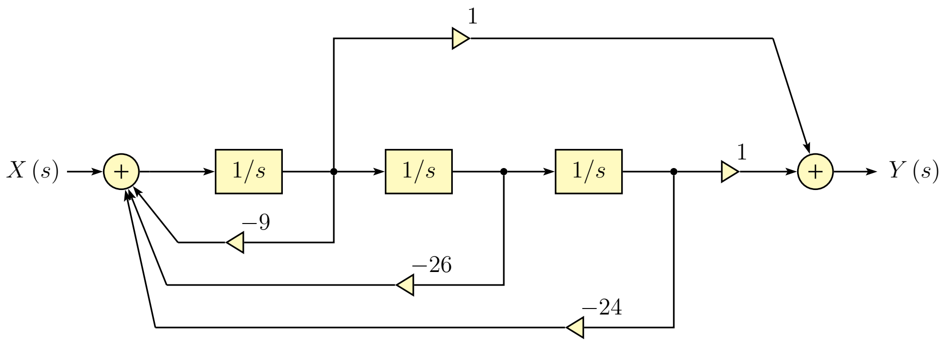
$$H_1(s) = \frac{W(s)}{X(s)} = \frac{1}{1+2s^{-1}}$$

$$H_2(s) = \frac{Y(s)}{W(s)} = 1+3s^{-1}$$



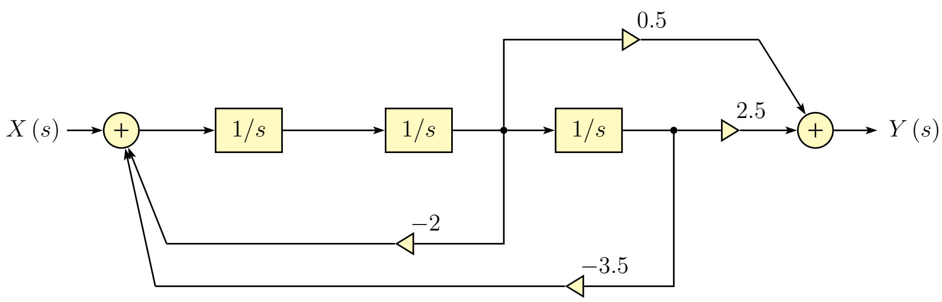
b.

$$H(s) = \frac{s^{-1}+s^{-3}}{1+9s^{-1}+26s^{-2}+24s^{-3}}$$



c.

$$H(s) = \frac{0.5s+2.5}{s^3+2s+3.5} = \frac{0.5s^{-2}+2.5s^{-3}}{1+2s^{-2}+3.5s^{-3}}$$



7.45.

a.

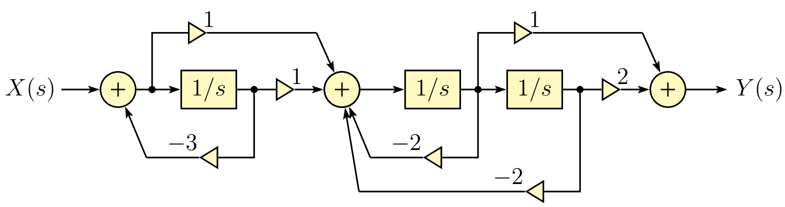
$$H(s)=\frac{(s+1)\,(s+2)}{(s+3)\left(s^2+2s+2\right)}$$

Let

$$H_1(s)=\frac{s+1}{s+3}\qquad\text{and}\qquad H_2(s)=\frac{s+2}{s^2+2s+2}$$

so that

$$H(s)=H_1(s)\,H_2(s)$$



b.

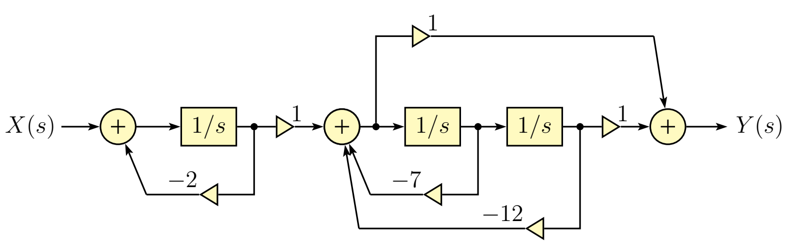
$$H(s)=\frac{s^2+1}{(s+2)\,(s+3)\,(s+4)}$$

Let

$$H_1(s)=\frac{1}{s+2}\qquad\text{and}\qquad H_2(s)=\frac{s^2+1}{(s+3)\,(s+4)}=\frac{s^2+1}{s^2+7s+12}$$

so that

$$H(s)=H_1(s)\,H_2(s)$$



c.

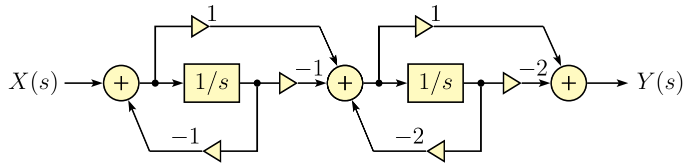
$$H(s)=\frac{(s-1)\,(s-2)}{(s+1)\,(s+2)}$$

Let

$$H_1(s)=\frac{s-1}{s+1}\qquad\text{and}\qquad H_2(s)=\frac{s-2}{s+2}$$

so that

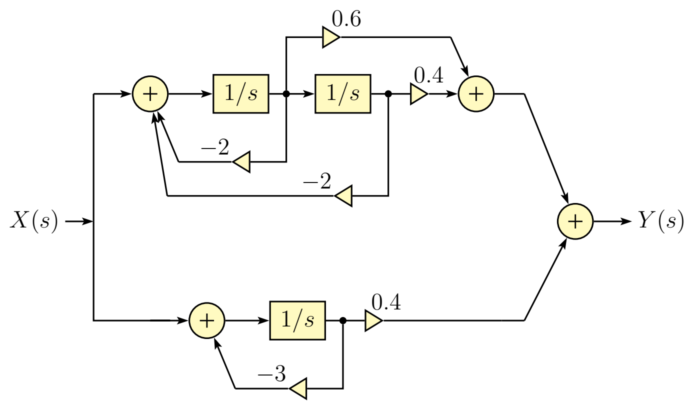
$$H(s)=H_1(s)\,H_2(s)$$



7.46.

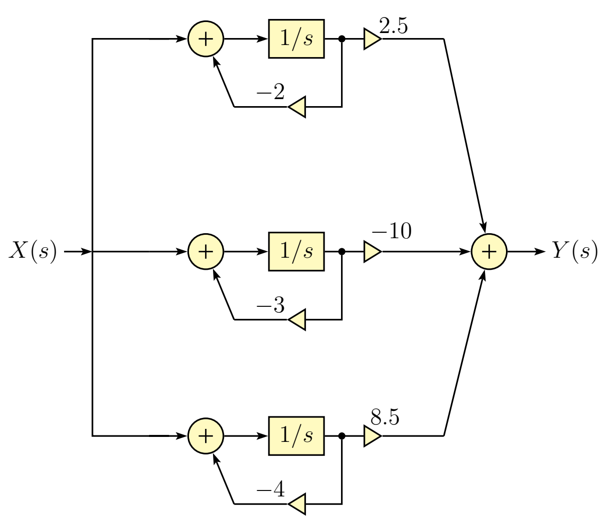
a. The system function can be written as

$$H(s) = \frac{0.4}{s+3} + \frac{0.6\,s + 0.4}{s^2 + 2s + 2}$$



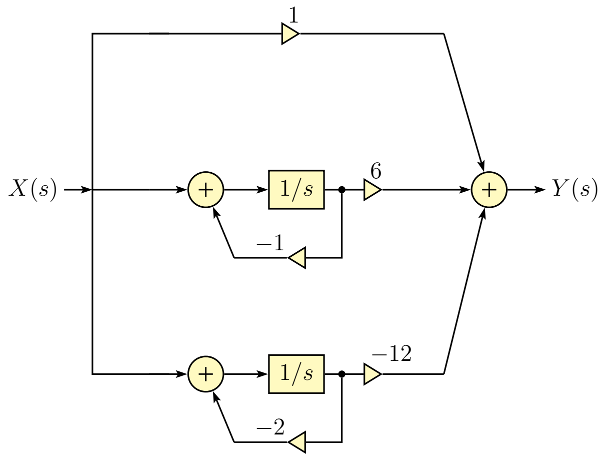
b. The system function can be written as

$$H(s) = \frac{2.5}{s+2} - \frac{10}{s+3} + \frac{8.5}{s+4}$$



c. The system function can be written as

$$H(s) = 1 + \frac{6}{s+1} - \frac{12}{s+2}$$



7.47.

a.

$$G_u(s) = \int_0^\infty x(t-1)\, e^{-st}\, dt$$

Using the variable change $t-1=\lambda$ we obtain

$$G_u(s) = \int_{-1}^\infty x(\lambda)\, e^{-s(\lambda+1)}\, d\lambda = e^{-s} \int_{-1}^\infty x(\lambda)\, e^{-s\lambda}\, d\lambda =$$

If $x(t)=0$ for $-1<t<0$, then

$$x(t-1)\, u(t) = x(t-1)\, u(t-1)$$

and

$$G_u(s) = e^{-s} \int_0^\infty x(\lambda)\, e^{-s\lambda}\, d\lambda = e^{-s} X_u(s)$$

b.

$$G_u(s) = \int_0^\infty x(t+2)\, e^{-st}\, dt$$

Using the variable change $t+2=\lambda$ we obtain

$$G_u(s) = \int_2^\infty x(\lambda)\, e^{-s(\lambda-2)}\, d\lambda = e^{2s} \int_2^\infty x(\lambda)\, e^{-s\lambda}\, d\lambda =$$

If $x(t)=0$ for $0<t<2$, then

$$x(t+2)\, u(t) = x(t+2)\, u(t+2)$$

and

$$G_u(s) = e^{2s} \int_0^\infty x(\lambda)\, e^{-s\lambda}\, d\lambda = e^{2s}\, X_u(s)$$

c.

$$G_u(s) = \int_0^\infty x(2t)\, e^{-st}\, dt$$

Using the variable change $2t = \lambda$ we obtain

$$G_u(s) = \int_0^\infty x(\lambda)\, e^{-s\lambda/2}\, \frac{1}{2}\, d\lambda = \frac{1}{2} \int_2^\infty x(\lambda)\, e^{-s\lambda/2}\, d\lambda = \frac{1}{2}\, X_u\left(\frac{s}{2}\right)$$

d.

$$G_u(s) = \int_0^\infty e^{-2t}\, x(t)\, e^{-st}\, dt = \int_0^\infty x(t)\, e^{-(s+2)t}\, dt = X_u(s+2)$$

e.

$$X_u(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

$$\frac{dX_u(s)}{ds} = \int_0^\infty -t\, x(t)\, e^{-st}\, dt = -G_u(s)$$

Therefore

$$G_u(s) = -\frac{dX_u(s)}{ds}$$

7.48.

a. Writing KVL around the loop we obtain

$$x(t) = R\, i(t) + L\, \frac{di(t)}{dt} + y(t)$$

Recall that

$$i(t) = C\, \frac{dy(t)}{dt}$$

and

$$\frac{di(t)}{dt} = C\, \frac{d^2 y(t)}{dt^2}$$

Substituting these two relationships into the differential equation yields

$$x(t) = RC\, \frac{dy(t)}{dt} + LC\, \frac{d^2 y(t)}{dt^2} + y(t)$$

Rearranging terms we obtain

$$\frac{d^2 y(t)}{dt^2} + \left(\frac{R}{L}\right) \frac{dy(t)}{dt} + \left(\frac{1}{LC}\right) y(t) = \left(\frac{1}{LC}\right) x(t)$$

Finally, substituting numerical values yields

$$\frac{d^2 y(t)}{dt^2} + 2\, \frac{dy(t)}{dt} + 8\, y(t) = 8\, x(t)$$

b. Taking the Laplace transform of both sides of the differential equation using zero initial conditions (a requirement for the system function) we get

$$\left(s^2+2\,s+8\right)\,Y\,(s)=8\,X\,(s)$$

and

$$H\,(s)=\frac{Y\,(s)}{X\,(s)}=\frac{8}{s^2+2\,s+8}$$

c. We can not use the system function found in part (b) to determine the transform of the output signal as $Y\,(s)=H\,(s)\,X\,(s)$ since initial conditions are specified. Recall that the system function is only meaningful when the system is CTLTI which requires all initial conditions to be zero. In this case we need to use the unilateral Laplace transform.

$$i\,(0)=C\,\frac{dy\,(t)}{dt}\bigg|_{t=0}\qquad\Rightarrow\qquad\frac{dy\,(t)}{dt}\bigg|_{t=0}=\frac{i\,(0)}{C}=\frac{0.5}{1/8}=4$$

$$\begin{aligned}\mathscr{L}\left\{\frac{d^2y\,(t)}{dt^2}\right\}=&s^2\,Y\,(s)-s\,y\,(0)\;-\frac{dy\,(t)}{dt}\bigg|_{t=0}\\=&s^2\,Y\,(s)-2s-4\end{aligned}$$

$$\frac{dy\,(t)}{dt}=s\,Y\,(s)-y\,(0)=s\,Y\,(s)-2$$

Computing the unilateral Laplace transform of both sides of the differential equation leads to

$$s^2\,Y\,(s)-2s-4+2\left[s\,Y\,(s)-2\right]+8\,Y\,(s)=8\,X\,(s)$$

and

$$\begin{aligned}\left(s^2+2\,s+8\right)\,Y\,(s)=&8\,X\,(s)+2s+8\\=&\frac{8}{s}+2s+8\end{aligned}$$

The transform $Y\,(s)$ is

$$Y\,(s)=\frac{2\,s^2+8\,s+8}{s\,\left(s^2+2\,s+8\right)}$$

Its partial fraction expansion is

$$Y\,(s)=\frac{k_1}{s}+\frac{k_2}{s+1+j\sqrt{7}}+\frac{k_3}{s+1-j\sqrt{7}}$$

with residues

$$k_1=1\,,\qquad k_2=0.5+j\,0.945\,,\qquad k_3=0.5-j\,0.945$$

The output signal is

$$\begin{aligned}y\,(t)=&u\,(t)+\left(0.5+j\,0.945\right)\,e^{\left(-1-j\sqrt{7}\right)t}\,u\,(t)+\left(0.5-j\,0.945\right)\,e^{\left(-1+j\sqrt{7}\right)t}\,u\,(t)\\=&u\,(t)+e^{-t}\,\cos\left(\sqrt{7}t\right)\,u\,(t)+1.89\,e^{-t}\,\sin\left(\sqrt{7}t\right)\,u\,(t)\end{aligned}$$

d. Taking the unilateral Laplace transform of the differential equation leads to

$$\begin{aligned}\left(s^2+2s+8\right) Y(s) =& 8X(s)+2s+8\\ =& \frac{8}{s+2}+2s+8\end{aligned}$$

The transform $Y(s)$ is

$$Y(s)=\frac{2s^2+12s+24}{(s+2)\left(s^2+2s+8\right)}$$

Its partial fraction expansion is

$$Y(s)=\frac{k_1}{s+2}+\frac{k_2}{s+1+j\sqrt{7}}+\frac{k_3}{s+1-j\sqrt{7}}$$

with residues

$$k_1=1\,,\qquad k_2=0.5+j\,1.323\,,\qquad k_3=0.5-j\,1.323$$

The output signal is

$$\begin{aligned}y(t) =& u(t)+\left(0.5+j\,1.323\right)e^{(-1-j\sqrt{7})t}\,u(t)+\left(0.5-j\,1.323\right)e^{(-1+j\sqrt{7})t}\,u(t)\\ =& e^{-2t}\,u(t)+e^{-t}\cos\left(\sqrt{7}t\right)u(t)+2.646\,e^{-t}\sin\left(\sqrt{7}t\right)u(t)\end{aligned}$$

7.49.

a. The output voltage is

$$y(t)=R_2\,i_2(t)=R_2C_2\,\frac{d}{dt}\left[v_0(t)-y(t)\right]$$

which leads to the relationship

$$\frac{dv_0(t)}{dt}=\frac{1}{R_2C_2}\,y(t)+\frac{dy(t)}{dt} \tag{P.7.49.1}$$

Additionally, writing the KCL at the center node we get

$$\frac{v_0(t)-x(t)}{R_1}+C_1\,\frac{dv_0}{dt}+C_2\,\frac{d}{dt}\left[v_0(t)-y(t)\right]=0$$

which can be simplified to

$$v_0(t)+R_1\,(C_1+C_2)\,\frac{dv_0(t)}{dt}=x(t)+R_1C_2\,\frac{dy(t)}{dt} \tag{P.7.49.2}$$

Substituting Eqn. (P.7.49.1) into Eqn. (P.7.49.2) and rearranging terms yields

$$v_0(t)=-R_1C_1\,\frac{dy(t)}{dt}-\frac{R_1\,(C_1+C_2)}{R_2C_2}\,y(t)+x(t) \tag{P.7.49.3}$$

Differentiating both sides of Eqn. (P.7.49.3)

$$\frac{dv_0(t)}{dt}=-R_1C_1\,\frac{d^2y(t)}{dt^2}-\frac{R_1\,(C_1+C_2)}{R_2C_2}\,\frac{dy(t)}{dt}+\frac{dx(t)}{dt} \tag{P.7.49.4}$$

Equating the right sides of Eqns. (P.7.49.1) and (P.7.49.4) leads to the desired differential equation:

$$R_1 C_1 \frac{d^2 y(t)}{dt^2} + \left(\frac{R_1 C_1 + R_1 C_2 + R_2 C_2}{R_2 C_2}\right) \frac{dy(t)}{dt} + \frac{1}{R_2 C_2}\, y(t) = \frac{dx(t)}{dt}$$

Substituting numerical values into the differential equation we obtain

$$\frac{d^2 y(t)}{dt^2} + 3\,\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt}$$

b. Taking the Laplace transform of both sides of the differential equation using zero initial conditions (a requirement for the system function) we get

$$\left(s^2 + 3\,s + 1\right) Y(s) = s\,X(s)$$

and

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^2 + 3\,s + 1}$$

c.

$$y(0) = R_2\, i_2(0) = 2 \qquad \Rightarrow \qquad i_2(0) = \frac{y(0)}{R_2} = 2$$

Since

$$i_2(t) = C_2 \frac{d}{dt}\left[v_0(t) - y(t)\right]$$

it follows that

$$i_2(0) = C_2 \left.\frac{dv_0(t)}{dt}\right|_{t=0} - C_2 \left.\frac{dy(t)}{dt}\right|_{t=0}$$

$$2 = \left.\frac{dv_0(t)}{dt}\right|_{t=0} - \left.\frac{dy(t)}{dt}\right|_{t=0}$$

The current of the capacitor C_1 is

$$C_1 \frac{dv_0(t)}{dt} = \frac{x(t) - v_0(t)}{R_1} - i_2(t)$$

At time $t = 0$ we have

$$C_1 \left.\frac{dv_0(t)}{dt}\right|_{t=0} = \frac{x(0) - v_0(0)}{R_1} - i_2(0) = \frac{1-3}{1} - 2 = -4 \qquad \Rightarrow \qquad \left.\frac{dv_0(t)}{dt}\right|_{t=0} = -4$$

and we obtain

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = -6$$

We are now ready to solve the differential equation.

$$\begin{aligned} \mathscr{L}\left\{\frac{d^2 y(t)}{dt^2}\right\} &= s^2\, Y(s) - s\, y(0) - \left.\frac{dy(t)}{dt}\right|_{t=0} \\ &= s^2\, Y(s) - 2s + 6 \end{aligned}$$

$$\frac{dy(t)}{dt} = s\,Y(s) - y(0) = s\,Y(s) - 2$$

Computing the unilateral Laplace transform of both sides of the differential equation leads to

$$\left[s^2\, Y(s) - 2s + 6\right] + 3\,\left[s\, Y(s) - 2\right] + Y(s) = s\, X(s)$$

and

$$\begin{aligned}\left(s^2 + 3\,s + 1\right)\, Y(s) &= s\, X(s) + 2s\\ &= 1 + 2s\end{aligned}$$

The transform $Y(s)$ is

$$Y(s) = \frac{2\,s + 1}{s^2 + 3\,s + 1}$$

Its partial fraction expansion is

$$Y(s) = +\frac{k_1}{s + 0.3820} + \frac{k_2}{s + 2.6180}$$

with residues

$$k_1 = 0.1056\,, \qquad k_2 = 1.8944$$

The output signal is

$$y(t) = 0.1056\, e^{-0.3820\,t}\, u(t) + 1.8944\, e^{-2.6180\,t}\, u(t)$$

d. Taking the unilateral Laplace transform of the differential equation leads to

$$\begin{aligned}\left(s^2 + 3\,s + 1\right)\, Y(s) &= s\, X(s) + 2s\\ &= \frac{s}{s+2} + 2s = \frac{2\,s^2 + 5\,s}{s+2}\end{aligned}$$

The transform $Y(s)$ is

$$Y(s) = \frac{2\,s^2 + 5\,s}{(s+2)\,\left(s^2 + 3\,s + 1\right)}$$

Its partial fraction expansion is

$$Y(s) = +\frac{k_1}{s+2} + \frac{k_1}{s + 0.3820} + \frac{k_2}{s + 2.6180}$$

with residues

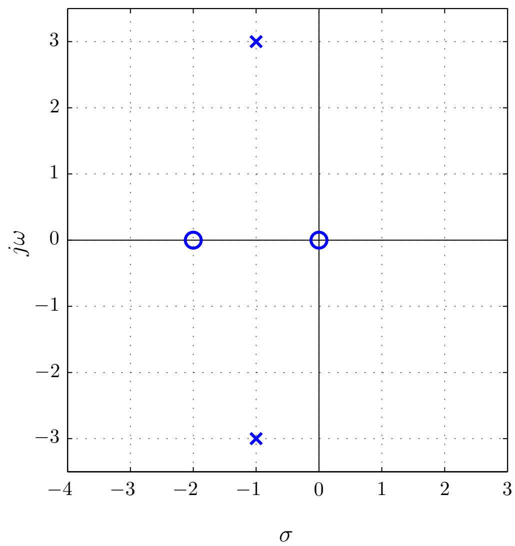
$$k_1 = 2\,, \qquad k_2 = -0.4472\,, \qquad k_3 = 0.4472$$

The output signal is

$$y(t) = e^{-2t}\, u(t) - 0.4472\, e^{-0.3820\,t}\, u(t) + 0.4472\, e^{-2.6180\,t}\, u(t)$$

7.50.

a.



b. MATLAB script to evaluate the magnitude of $X(s)$ at a grid of complex points in the s-plane:

```
[sr,si] = meshgrid([-5:0.1:5],[-5:0.1:5]);
s = sr+j*si;
Xs = @(s) s.*(s+2)./((s+1).^2+9);
XsMag = abs(Xs(s));
XsMag = XsMag.*(XsMag<=2)+2.*(XsMag>2);
```

c. Script to produce a three dimensional mesh plot of $|X(s)|$:

```
shading interp;      % Shading method: Interpolated
colormap copper;     % Specify the color map used.
m1 = mesh(sr,si,XsMag);
axis([-5,5,-5,5]);
% Adjust transparency of surface lines.
set(m1,'EdgeAlpha',0.6','FaceAlpha',0.6);
% Specify x,y,z axis labels.
xlabel('\sigma');
ylabel('j\omega');
zlabel('|X(s)|');
% Specify viewing angles.
view(gca,[23.5,38]);
```

d. Modify the script in part (c) to also evaluate the Laplace transform for $s = j\omega$ and draw it over the three dimensional mesh plot:

```
% Define the trajectory s=j*omega
```

```
omega = [-5:0.01:5];
tr = j*omega;
% Produce a mesh plot and hold it.
shading interp;
colormap copper;
m1 = mesh(sr,si,XsMag);
hold on;
% Superimpose a plot of X(s) magnitude values evaluated on the
% trajectory using 'plot3' function.
m2 = plot3(real(tr),imag(tr),abs(Xs(tr)),'b-','LineWidth',1.5);
hold off;
axis([-5,5,-5,5]);
% Adjust transparency of surface lines.
set(m1,'EdgeAlpha',0.6','FaceAlpha',0.6);
% Specify x,y,z axis labels.
xlabel('\sigma');
ylabel('j\omega');
zlabel('|X(s)|');
% Specify viewing angles.
view(gca,[23.5,38]);
```

7.51.

a.

```
num = [1,-2];
den = [1,3,2];
pls = roots(den)
zrs = roots(num)
plot([-10,10],[0,0],'k-',[0,0],[-10,10],'k',...
  real(zrs),imag(zrs),'bo',real(pls),imag(pls),'bx');
axis([-4,3,-3.5,3.5]);
xlabel('\sigma');
ylabel('j\omega');
grid;
```

b.

```
num = [1,0];
den = [1,0,-1];
pls = roots(den)
zrs = roots(num)
plot([-10,10],[0,0],'k-',[0,0],[-10,10],'k',...
  real(zrs),imag(zrs),'bo',real(pls),imag(pls),'bx');
axis([-4,3,-3.5,3.5]);
xlabel('\sigma');
ylabel('j\omega');
grid;
```

c.

```
num = [1,1];
den = [1,-4,3];
pls = roots(den)
zrs = roots(num)
plot([-10,10],[0,0],'k-',[0,0],[-10,10],'k',...
   real(zrs),imag(zrs),'bo',real(pls),imag(pls),'bx');
axis([-4,3,-3.5,3.5]);
xlabel('\sigma');
ylabel('j\omega');
grid;
```

d.

```
num = [1,-1,0];
den = [1,-1,-6];
pls = roots(den)
zrs = roots(num)
plot([-10,10],[0,0],'k-',[0,0],[-10,10],'k',...
   real(zrs),imag(zrs),'bo',real(pls),imag(pls),'bx');
axis([-3,4,-3.5,3.5]);
xlabel('\sigma');
ylabel('j\omega');
grid;
```

e.

```
num = [1,-1,0];
den = [1,8,15];
pls = roots(den)
zrs = roots(num)
plot([-10,10],[0,0],'k-',[0,0],[-10,10],'k',...
   real(zrs),imag(zrs),'bo',real(pls),imag(pls),'bx');
axis([-6,1,-3.5,3.5]);
xlabel('\sigma');
ylabel('j\omega');
grid;
```

7.52.

a.

```
%%
X = @(s) (s-2)./(s.^2+3*s+2);
omg = [-10:0.02:10];
Xomg = X(j*omg);
plot(omg,abs(Xomg));
axis([-10,10,0,1.2]);
xlabel('\omega (rad/s)');
```

```
ylabel('Magnitude');
title('|H(\omega)|');
grid;
%%
plot(omg, angle(Xomg));
axis([-10,10,-pi,pi]);
xlabel('\omega (rad/s)');
ylabel('Phase (rad)');
title('\angle H(\omega)');
grid;
```

c.

```
%%
X = @(s) (s+1)./(s.^2-4*s+3);
omg = [-10:0.02:10];
Xomg = X(j*omg);
plot(omg, abs(Xomg));
axis([-10,10,0,0.4]);
xlabel('\omega (rad/s)');
ylabel('Magnitude');
title('|H(\omega)|');
grid;
%%
plot(omg, angle(Xomg));
axis([-10,10,-pi,pi]);
xlabel('\omega (rad/s)');
ylabel('Phase (rad)');
title('\angle H(\omega)');
grid;
```

7.53.

a.

```
syms s t
Xs = 1/(s^2+3*s+2);
xt = ilaplace(Xs)
```

b.

```
syms s t
Xs = (s-1)*(s-2)/((s+1)*(s+2)*(s+3));
xt = ilaplace(Xs)
```

c.

```
syms s t
Xs = s*(s-1)/((s+1)^2*(s+2))
xt = ilaplace(Xs)
```

7.54.

a.

```
sys = tf([1,1,0],[1,4,13])
```

b.

```
t = [-1:0.01:10];
x = exp(-0.2*t).*(t>=0);
plot(t,x);
axis([-1,10,-0.25,1.25]);
xlabel('t (sec)');
title('x(t)');
grid;
```

c.

```
y = lsim(sys,x,t);
plot(t,y);
axis([-1,10,-0.25,1.25]);
xlabel('t (sec)');
title('y(t)');
grid;
```

7.55.

The code

```
% Enter numerator and denominator polynomials
num = [1,-9,30,-42,20];
den = [1,12,59,152,200,96];
% Compute the poles and the residues
[r,p,k] = residue(num,den)
```

results in the response

```
r =
   62.5000 + 0.0000i
  -74.0000 + 0.0000i
    4.5500 - 7.1500i
    4.5500 + 7.1500i
    3.4000 + 0.0000i

p =
   -4.0000 + 0.0000i
   -3.0000 + 0.0000i
   -2.0000 + 2.0000i
```

```
    -2.0000 - 2.0000i
    -1.0000 + 0.0000i

k =
     []
```

Correspondingly, the partial fraction expansion is

$$X(s) = \frac{3.4}{s+1} + \frac{4.55+j7.15}{s+2+j2} + \frac{4.55-j7.15}{s+2-j2} + \frac{-74}{s+3} + \frac{62.5}{s+4}$$

7.56.

a.

```
num1 = [1,1];
den1 = [1,5,6];
sys1 = tf(num1,den1);
bode(sys1);
grid;
```

b.

```
num2 = [1,-1,0];
den2 = [1,5,8,6];
sys2 = tf(num2,den2);
bode(sys2);
grid;
```

c.

```
num3 = conv([1,-1],[1,2]);  % Use conv() for polynomial multiplication.
den3 = conv([1,1],[1,6,13]);
sys3 = tf(num3,den3);
bode(sys3);
grid;
```

7.57. Use the following script to define $H(s)$ as a function of parameters ζ and ω_0:

```
numH = @(zeta,omg0) [omg0*omg0];
denH = @(zeta,omg0) [1,2*zeta*omg0,omg0*omg0];
```

Try with $\zeta = 0.01$ and $\omega_0 = 5$ rad/s:

```
sys1 = tf(numH(0.01,5),denH(0.01,5));
bode(sys1,{1,10});
grid;
```

Try with $\zeta = 0.1$ and $\omega_0 = 5$ rad/s:

```
sys2 = tf(numH(0.1,5),denH(0.1,5));
bode(sys2,{1,10});
grid;
```

Try with $\zeta = 0.5$ and $\omega_0 = 5$ rad/s:

```
sys3 = tf(numH(0.5,5),denH(0.5,5));
bode(sys3,{1,10});
grid;
```

Try with $\zeta = 1$ and $\omega_0 = 5$ rad/s:

```
sys4 = tf(numH(1,5),denH(1,5));
bode(sys4,{1,10});
grid;
```

Try with $\zeta = 2$ and $\omega_0 = 5$ rad/s:

```
sys5 = tf(numH(2,5),denH(2,5));
bode(sys5,{1,10});
grid;
```

Graph all five diagrams together for comparison.

```
bode(sys1,{1,10});
hold on;
bode(sys2,{1,10});
bode(sys3,{1,10});
bode(sys4,{1,10});
bode(sys5,{1,10});
hold off;
legend('\zeta=0.01','\zeta=0.1','\zeta=0.5','\zeta=1','\zeta=2',...
   'Location','SouthWest')
grid;
```

7.58.

a. Define $H(s)$, $H_1(s)$, and $H_1^{-1}(s)$.

```
H = zpk([-1,2],[-3,-4],1)
H1 = zpk([-1,-2],[-3,-4],1)
H1inv = zpk([-3,-4],[-1,-2],1)
```

b. Graph $|H(s)|$ and $\angle H(s)$.

```
omg = [-5:0.01:5];
Homg = freqresp(H,omg);
subplot(1,2,1);
plot(omg,abs(Homg(1,:)));
grid;
subplot(1,2,2);
plot(omg,angle(Homg(1,:)));
grid;
```

Graph $|H_1(s)|$ and $\angle H_1(s)$.

```
H1omg = freqresp(H1,omg);
subplot(1,2,1);
plot(omg,abs(H1omg(1,:)));
grid;
subplot(1,2,2);
plot(omg,angle(H1omg(1,:)));
grid;
```

Graph $\left|H_1^{-1}(s)\right|$ and $\angle H_1^{-1}(s)$.

```
H1invomg = freqresp(H1inv,omg);
subplot(1,2,1);
plot(omg,abs(H1invomg(1,:)));
grid;
subplot(1,2,2);
plot(omg,angle(H1invomg(1,:)));
grid;
```

c. Graph unit step response of $H(s)$.

```
clf;
step(H,[0:0.01:5])
grid;
```

Graph unit step response of $H(s)\,H_1^{-1}(s)$.

```
clf;
step(H*H1inv,[0:0.01:5])
grid;
```

7.59.

a. Recall that the differential equation is

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+8\,y(t)=8\,x(t)$$

Using the initial conditions

$$y(0) = 2 \qquad \text{and} \qquad \left.\frac{dy(t)}{dt}\right|_{t=0} = 4$$

with unilateral Laplace transform leads to

$$\left[s^2\,Y(s) - 2s - 4\right] + 2\left[s\,Y(s) - 2\right] + 8\,Y(s) = 8\,X(s)$$

Solve for the output transform $Y(s)$.

```
>> syms s t Ys
>> xt = heaviside(t);         % x(t)=u(t)
>> Xs = laplace(xt);          % Laplace transform of x(t)
>> Y2 = s*s*Ys-2*s-4;         % Laplace transform of d2y/dt2
>> Y1 = s*Ys-2;               % Laplace transform of dy/dt
>> Ys = solve(Y2+2*Y1+8*Ys-8*Xs,Ys)    % Solve for Y(s)
```

Find $y(t)$ through inverse Laplace transform.

```
>> yt = ilaplace(Ys)          % Inverse Laplace transform of Y(s)
```

Graph the output signal.

```
>> ezplot(yt,[0,5]); grid;
>> axis([0,5,-1,3]);
```

b. Solve for the output transform $Y(s)$.

```
>> syms s t Ys
>> xt = exp(-2*t)*heaviside(t);  % x(t)=exp(-2*t)*u(t)
>> Xs = laplace(xt);          % Laplace transform of x(t)
>> Y2 = s*s*Ys-2*s-4;         % Laplace transform of d2y/dt2
>> Y1 = s*Ys-2;               % Laplace transform of dy/dt
>> Ys = solve(Y2+2*Y1+8*Ys-8*Xs,Ys)    % Solve for Y(s)
```

Find $y(t)$ through inverse Laplace transform.

```
>> yt = ilaplace(Ys)          % Inverse Laplace transform of Y(s)
```

Graph the output signal.

```
>> ezplot(yt,[0,5]); grid;
>> axis([0,5,-1,3]);
```

7.60.

a. Recall that the differential equation is

$$\frac{d^2y(t)}{dt^2} + 3\,\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt}$$

Using the initial conditions

$$y(0) = 2 \qquad \text{and} \qquad \left.\frac{dy(t)}{dt}\right|_{t=0} = -6$$

with unilateral Laplace transform leads to

$$\left[s^2\,Y(s) - 2s + 6\right] + 3\left[s\,Y(s) - 2\right] + Y(s) = s\,X(s)$$

Solve for the output transform $Y(s)$.

```
>> syms s t Ys
>> xt = heaviside(t);          % x(t)=u(t)
>> Xs = laplace(xt);           % Laplace transform of x(t)
>> Y2 = s*s*Ys-2*s+6;          % Laplace transform of d2y/dt2
>> Y1 = s*Ys-2;                % Laplace transform of dy/dt
>> Ys = solve(Y2+3*Y1+Ys-s*Xs,Ys)   % Solve for Y(s)
```

Find $y(t)$ through inverse Laplace transform.

```
>> yt = ilaplace(Ys)           % Inverse Laplace transform of Y(s)
```

Graph the output signal.

```
>> ezplot(yt,[0,5]); grid;
>> axis([0,5,-1,3]);
```

b. Solve for the output transform $Y(s)$.

```
>> syms s t Ys
>> xt = exp(-2*t)*heaviside(t);  % x(t)=exp(-2*t)*u(t)
>> Xs = laplace(xt);           % Laplace transform of x(t)
>> Y2 = s*s*Ys-2*s+6;          % Laplace transform of d2y/dt2
>> Y1 = s*Ys-2;                % Laplace transform of dy/dt
>> Ys = solve(Y2+3*Y1+Ys-s*Xs,Ys)   % Solve for Y(s)
```

Find $y(t)$ through inverse Laplace transform.

```
>> yt = ilaplace(Ys)           % Inverse Laplace transform of Y(s)
```

Graph the output signal.

```
>> ezplot(yt,[0,5]); grid;
>> axis([0,5,-1,3]);
```

Chapter 8

z-Transform for Discrete-Time Signals and Systems

8.1.

a. Applying the z-transform definition:

$$X(z) = 1 + z^{-1} + z^{-2} = \frac{z^2 + z + 1}{z^2}$$

Zeros: $z_1 = -0.5 + j\,0.866\,,\quad z_2 = -0.5 - j\,0.866$

Poles: $p_1 = p_2 = 0$

The transform converges at every point except the origin of the z-plane.

ROC: $|z| > 0$

b. Applying the z-transform definition:

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} = \frac{z^4 + z^3 + z^2 + z + 1}{z^4}$$

Zeros: $z_1 = 0.309 + j\,0.9511\,,\quad z_2 = 0.309 - j\,0.9511\,,\quad z_3 = -0.809 + j\,0.5878\,,\quad z_4 = -0.809 - j\,0.5878$

Poles: $p_1 = p_2 = p_3 = p_4 = 0$

The transform converges at every point except the origin of the z-plane.

ROC: $|z| > 0$

c. Applying the z-transform definition:

$$X(z) = z^2 + z + 1 + z^{-1} + z^{-2} = \frac{z^4 + z^3 + z^2 + z + 1}{z^2}$$

Zeros: $z_1 = 0.309 + j\,0.9511\,,\quad z_2 = 0.309 - j\,0.9511\,,\quad z_3 = -0.809 + j\,0.5878\,,\quad z_4 = -0.809 - j\,0.5878$

Poles: $p_1 = p_2 = 0$

In addition to the two poles at the origin of the z-plane, the transform also has two poles at infinity (due to the z^2 term). Therefore the ROC must exclude both the origin of the z-plane and infinity.

ROC: $0 < |z| < \infty$

d. Applying the z-transform definition:

$$X(z) = z^4 + z^3 + z^2 + z + 1$$

Zeros: $z_1 = 0.309 + j\,0.9511\,,\quad z_2 = 0.309 - j\,0.9511\,,\quad z_3 = -0.809 + j\,0.5878\,,\quad z_4 = -0.809 - j\,0.5878$

The transform $X(z)$ has no finite poles. There are four poles at infinity. Therefore $X(z)$ converges at all points except where $|z| \rightarrow \infty$.

ROC: $|z| < \infty$

8.2.

a. No, since the ROC does not include the unit circle.

b. Yes, since the ROC includes the unit circle.

$$X(\Omega) = X(z)\Big|_{z=e^{j\Omega}} = \frac{e^{j\Omega}\left(e^{j\Omega}+2\right)}{\left(e^{j\Omega}+1/2\right)\left(e^{j\Omega}+3/2\right)}$$

c. Yes, since the ROC includes the unit circle.

$$X(\Omega) = X(z)\Big|_{z=e^{j\Omega}} = \frac{e^{j2\Omega}}{e^{j2\Omega}+5\,e^{j\Omega}+6}$$

d. Yes, since the ROC includes the unit circle.

$$X(\Omega) = X(z)\Big|_{z=e^{j\Omega}} = \frac{\left(e^{j\Omega}+1\right)\left(e^{j\Omega}-1\right)}{\left(e^{j\Omega}+2\right)\left(e^{j\Omega}-3\right)\left(e^{j\Omega}-4\right)}$$

8.3.

a. ROC: $|z| > 0.75$

b. Two complex poles are at $0.5 \mp j\,0.5 = \frac{1}{\sqrt{2}}\,e^{\mp j\pi/4}$

ROC: $\frac{1}{\sqrt{2}} < |z| < 1.25$

c. ROC: $|z| > 0.5$

d. ROC: $\frac{1}{3} < |z| < 1.25$

8.4.

a. Using z-transform definition

$$X(z) = \sum_{n=0}^{9} n\,z^{-n}$$

Let $A(z)$ be defined as

$$A(z) = \sum_{n=0}^{9} z^{-n}$$

Using the closed form formula for the finite-length geometric series

$$A(z) = \frac{1-z^{-10}}{1-z^{-1}}\,, \qquad |z| > 0$$

Differentiating $A(z)$ with respect to z yields

$$\frac{d}{dz}\left[A(z)\right] = -\sum_{n=0}^{9} n\, z^{-n-1}$$

and therefore

$$-z\frac{d}{dz}\left[A(z)\right] = \sum_{n=0}^{9} n\, z^{-n} = X(z)$$

The derivative of the closed form expression for $A(z)$ is

$$\frac{d}{dz}\left[A(z)\right] = \frac{-z^{-2}+10\,z^{-11}-9\,z^{-12}}{\left(1-z^{-1}\right)^2}\,, \qquad |z|>0$$

and $X(z)$ is found as

$$X(z) = -z\frac{d}{dz}\left[A(z)\right] = \frac{z^{-1}-10\,z^{-10}+9\,z^{-11}}{\left(1-z^{-1}\right)^2}\,, \qquad |z|>0$$

b. The transform $X(z)$ can be written as

$$X(z) = X_1(z) + X_2(z)$$

with

$$X_1(z) = \sum_{n=0}^{9} n\, z^{-n}$$

and

$$X_2(z) = \sum_{n=10}^{\infty} 10\, z^{-n}$$

From part (a) we have

$$X_1(z) = \frac{z^{-1}-10\,z^{-10}+9\,z^{-11}}{\left(1-z^{-1}\right)^2}\,, \qquad |z|>0$$

$X_2(z)$ is found by using the variable change $m = n-10$:

$$X_2(z) = 10\sum_{m=0}^{\infty} z^{-(m+10)} = \frac{10\,z^{-10}}{1-z^{-1}}\,, \qquad |z|>1$$

and

$$\begin{aligned} X(z) &= \frac{z^{-1}-10\,z^{-10}+9\,z^{-11}}{\left(1-z^{-1}\right)^2} + \frac{10\,z^{-10}}{1-z^{-1}} \\ &= \frac{z^{-1}-z^{-11}}{\left(1-z^{-1}\right)^2}\,, \qquad |z|>1 \end{aligned}$$

c. The transform $X(z)$ can be written as

$$X(z) = X_1(z) + X_2(z)$$

with

$$X_1(z) = \sum_{n=0}^{9} n\, z^{-n}$$

and

$$X_2(z) = \sum_{n=10}^{19} (-n+20)\, z^{-n}$$

From part (a) we have

$$X_1(z) = \frac{z^{-1} - 10\,z^{-10} + 9\,z^{-11}}{\left(1-z^{-1}\right)^2}\,, \qquad |z| > 0$$

$X_2(z)$ is found by using the variable change $m = n - 10$:

$$\begin{aligned}
X_2(z) &= \sum_{m=0}^{9} (-m+10)\, z^{-(m+10)} \\
&= -z^{-10} \sum_{m=0}^{9} m\, z^{-m} + 10\,z^{-10} \sum_{m=0}^{9} z^{-m} \\
&= -z^{-10} \left[\frac{z^{-1} - 10\,z^{-10} + 9\,z^{-11}}{\left(1-z^{-1}\right)^2} \right] + 10\,z^{-10} \left[\frac{1-z^{-10}}{1-z^{-1}} \right] \\
&= \frac{10\,z^{-10} - 11\,z^{-11} + z^{-21}}{\left(1-z^{-1}\right)^2}\,, \qquad |z| > 0
\end{aligned}$$

The transform $X(z)$ is

$$X(z) = X_1(z) + X_2(z) = \frac{z^{-1} - 2\,z^{-11} + z^{-21}}{\left(1-z^{-1}\right)^2}\,, \qquad |z| > 0$$

8.5. Let $X(z)$ be written as

$$X(z) = X_1(z) + X_2(z)$$

with

$$X_1(z) = \sum_{\substack{n=0 \\ n\ \text{even}}}^{\infty} z^{-n}$$

and

$$X_2(z) = \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} (2/3)^n\, z^{-n}$$

Using $n = 2m$ for the indices of the summation for $X_1(z)$ yields

$$X_1(z) = \sum_{m=0}^{\infty} z^{-2m} = \frac{1}{1-z^{-2}}\,, \qquad \left|z^{-2}\right| < 1 \quad \Rightarrow \quad |z| > 1$$

Similarly, using $n=2m+1$ for the indices of the summation for $X_2(z)$ yields

$$\begin{aligned}
X_2(z) &= \sum_{m=0}^{\infty} (2/3)^{2m+1}\, z^{-(2m+1)} \\
&= \left(\frac{2}{3}\, z^{-1}\right) \sum_{m=0}^{\infty} \left(\frac{4}{9}\, z^{-2}\right)^m \\
&= \frac{\frac{2}{3}\, z^{-1}}{1-\frac{4}{9}\, z^{-2}} = \frac{6\, z^{-1}}{9-4\, z^{-2}}\,, \qquad \left|\frac{4}{9}\, z^{-2}\right| < 1 \quad \Rightarrow \quad |z| > \frac{2}{3}
\end{aligned}$$

The transform $X(z)$ is

$$X(z) = \frac{1}{1-z^{-2}} + \frac{6\, z^{-1}}{9-4\, z^{-2}} = \frac{9+6\, z^{-1}-4\, z^{-2}-6\, z^{-3}}{9-13\, z^{-2}+4\, z^{-4}}\,, \qquad |z| > 1$$

or, using non-negative powers of z

$$X(z) = \frac{9\, z^4+6\, z^3-4\, z^2-6\, z}{9\, z^4-13\, z^2+4}\,, \qquad |z| > 1$$

8.6.

a. Using Euler's formula

$$g[n] = \frac{1}{2}\,(0.9)^n \left[e^{j\,0.3n} + e^{-j\,0.3n}\right] u[n]$$

The transform is

$$\begin{aligned}
G(z) &= \frac{1}{2} \sum_{n=0}^{\infty} \left(0.9\, e^{j\,0.3}\, z^{-1}\right)^n \frac{1}{2} \sum_{n=0}^{\infty} \left(0.9\, e^{-j\,0.3}\, z^{-1}\right)^n \\
&= \left(\frac{1}{2}\right) \frac{1}{1-0.9\, e^{j\,0.3}\, z^{-1}} + \left(\frac{1}{2}\right) \frac{1}{1-0.9\, e^{-j\,0.3}\, z^{-1}} \\
&= \frac{1-0.9\,\cos(0.3)\; z^{-1}}{1-1.8\,\cos(0.3)\; z^{-1}+0.81\, z^{-2}} \\
&= \frac{z\left[z-0.9\,\cos(0.3)\right]}{z^2-1.8\,\cos(0.3)\; z+0.81} \\
&= \frac{z\left(z-0.8598\right)}{z^2-1.7196\, z+0.81}\,, \qquad |z| > 0.9
\end{aligned}$$

b. Since $x[n] = g[2n]$, the corresponding transform is

$$\begin{aligned}
X(z) &= \frac{1}{2}\, G\left(\sqrt{z}\right) + \frac{1}{2}\, G\left(-\sqrt{z}\right) \\
&= \left(\frac{1}{2}\right) \frac{\sqrt{z}\left(\sqrt{z}-0.8598\right)}{z-1.7196\,\sqrt{z}+0.81} + \left(\frac{1}{2}\right) \frac{\sqrt{z}\left(\sqrt{z}+0.8598\right)}{z+1.7196\,\sqrt{z}+0.81} \\
&= \frac{z\,(z-0.6685)}{z^2-1.3370\, z+0.6561}\,, \qquad |z| > 0.81
\end{aligned}$$

c. The signal $x[n]$ is

$$
\begin{aligned}
x[n] &= g[2n] = (0.9)^{2n} \cos\left((0.3)\,2n\right)\,u[2n] \\
&= (0.81)^n \cos(0.6n)\,u[n]
\end{aligned}
$$

and the corresponding transform is

$$
\begin{aligned}
X(z) &= \frac{z\left[z-0.81\cos(0.6)\right]}{z^2-1.62\cos(0.6)\,z+0.6561} \\
&= \frac{z\,(z-0.6685)}{z^2-1.3370\,z+0.6561}\,, \qquad |z|>0.81
\end{aligned}
$$

8.7.

a. Using z-transform definition

$$X(z) = z^2\,, \qquad |z|<\infty$$

b. Using z-transform definition

$$X(z) = z^{-3}\,, \qquad |z|>0$$

c. Using z-transform definition with the closed form formula for the infinite-length geometric series

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}\,, \qquad |z|>\frac{1}{2}$$

d. From part (c)

$$\mathcal{Z}\left\{(1/2)^n\,u[n]\right\} = \frac{z}{z-\frac{1}{2}}\,, \qquad |z|>\frac{1}{2}$$

$$\mathcal{Z}\left\{(1/3)^n\,u[n]\right\} = \frac{z}{z-\frac{1}{3}}\,, \qquad |z|>\frac{1}{3}$$

Using linearity of the z-transform

$$
\begin{aligned}
X(z) &= \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}} \\
&= \frac{z\left(2z-\frac{5}{6}\right)}{z^2-\frac{5}{6}z+\frac{1}{6}}\,, \qquad |z|>\frac{1}{2}
\end{aligned}
$$

e. From part (c)

$$\mathcal{Z}\left\{(1/2)^n\,u[n]\right\} = \frac{z}{z-\frac{1}{2}}\,, \qquad |z|>\frac{1}{2}$$

Using the time shifting property of the z-transform

$$X(z) = \mathcal{Z}\left\{(1/2)^{n-1}\,u[n-1]\right\} = \left(z^{-1}\right)\frac{z}{z-\frac{1}{2}} = \frac{1}{z-\frac{1}{2}}\,, \qquad |z|>\frac{1}{2}$$

f. From part (c)

$$\mathcal{Z}\left\{(1/2)^n\, u[n]\right\} = \frac{z}{z-\frac{1}{2}}\,, \qquad |z| > \frac{1}{2}$$

Using linearity of the z-transform

$$\mathcal{Z}\left\{(1/2)^{n-1}\, u[n]\right\} = (1/2)^{-1}\, \mathcal{Z}\left\{(1/2)^n\, u[n]\right\} = \frac{2\,z}{z-\frac{1}{2}}\,, \qquad |z| > \frac{1}{2}$$

g. From part (c)

$$\mathcal{Z}\left\{(1/2)^n\, u[n]\right\} = \frac{z}{z-\frac{1}{2}}\,, \qquad |z| > \frac{1}{2}$$

Using linearity of the z-transform

$$\mathcal{Z}\left\{(1/2)^{n+1}\, u[n]\right\} = (1/2)\, \mathcal{Z}\left\{(1/2)^n\, u[n]\right\} = \frac{\frac{1}{2}\,z}{z-\frac{1}{2}}\,, \qquad |z| > \frac{1}{2}$$

h. From part (d)

$$\mathcal{Z}\left\{(1/3)^n\, u[n]\right\} = \frac{z}{z-\frac{1}{3}}\,, \qquad |z| > \frac{1}{3}$$

Applying the time reversal property of the z-transform

$$\mathcal{Z}\left\{(1/3)^{-n}\, u[-n]\right\} = \mathcal{Z}\left\{(3)^n\, u[-n]\right\} = \frac{z^{-1}}{z^{-1}-\frac{1}{3}} = \frac{3}{3-z}\,, \qquad |z| < 3$$

Replacing n with $n+1$ and using time shifting property of the z-transform

$$\mathcal{Z}\left\{(3)^{n+1}\, u[-n-1]\right\} = (z)\,\frac{3}{3-z} = \frac{3\,z}{3-z}\,, \qquad |z| < 3$$

and using linearity

$$X(z) = \mathcal{Z}\left\{(3)^n\, u[-n-1]\right\} = \left(\frac{1}{3}\right)\frac{3\,z}{3-z} = \frac{z}{3-z}\,, \qquad |z| < 3$$

i. From part (h)

$$\mathcal{Z}\left\{(3)^n\, u[-n]\right\} = \frac{3}{3-z}\,, \qquad |z| < 3$$

Replacing n with $n-1$ and using time shifting property of the z-transform

$$\mathcal{Z}\left\{(3)^{n-1}\, u[-n+1]\right\} = \left(z^{-1}\right)\frac{3}{3-z} = \frac{3\,z^{-1}}{3-z} = \frac{3}{z\,(3-z)}\,, \qquad 0 < |z| < 3$$

Note that the ROC for the new transform must also exclude the origin of the z-plane due to the pole introduced at $z=0$. Another way to explain the exclusion of the origin from the ROC is to realize that the rightmost sample of $x[n]$ with nonzero amplitude is now at $n=1$ due to the right shift. Using linearity we obtain

$$X(z) = \mathcal{Z}\left\{(3)^n\, u[-n+1]\right\} = \left(\frac{1}{3}\right)\frac{3}{z\,(3-z)} = \frac{1}{z\,(3-z)}\,, \qquad 0 < |z| < 3$$

8.8.

a.

$$x[n] = \begin{cases} (1/2)^n\,, & n \ge 0 \\ (1/2)^n\,, & n < 0 \end{cases}$$

Let

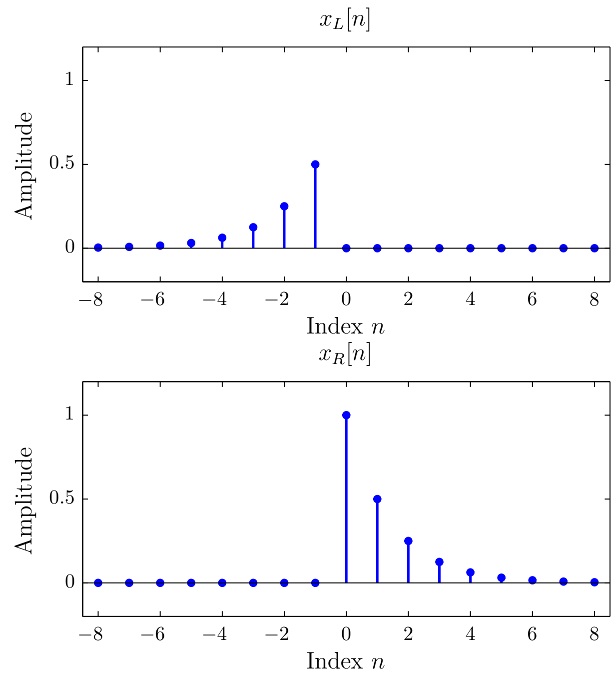
$$x_L[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1] = (2)^n\, u[-n-1]$$

$$x_R[n] = \left(\frac{1}{2}\right)^{n} u[n]$$

so that

$$x[n] = x_L[n] + x_R[n]$$

The two signal components are shown on the right.



b.

$$X_L(z) = \mathcal{Z}\left\{(2)^n\, u[-n-1]\right\} = -\frac{z}{z-2}\,, \qquad |z| < 2$$

$$X_R(z) = \mathcal{Z}\left\{\left(\frac{1}{2}\right)^n u[n]\right\} = \frac{z}{z-\frac{1}{2}}\,, \qquad |z| > \frac{1}{2}$$

c. Using linearity of the z-transform

$$\begin{aligned} X(z) &= X_L(z) + X_R(z) \\ &= -\frac{z}{z-2} + \frac{z}{z-\frac{1}{2}} \\ &= \frac{-\frac{3}{2}\,z}{z^2 - \frac{5}{2}\,z + 1}\,, \qquad \frac{1}{2} < |z| < 2 \end{aligned}$$

8.9.

a. We know that

$$\mathcal{Z}\left\{u[n]\right\} = \frac{z}{z-1}\,, \qquad |z| > 1$$

Using the time reversal property

$$X(z) = \mathcal{Z}\left\{u[-n]\right\} = \frac{z^1}{z^{-1}-1} = \frac{1}{1-z}\,, \qquad |z| < 1$$

b. Applying the time reversal property to the transform pair

$$\mathcal{Z}\left\{u[n-1]\right\} = \frac{1}{z-1}\,, \qquad |z| > 1$$

yields the transform pair

$$X(z) = \mathcal{Z}\left\{u[-n-1]\right\} = \frac{1}{z^{-1}-1} = \frac{z}{1-z}\,, \qquad |z| < 1$$

c. From part (a)

$$\mathcal{Z}\left\{u[-n]\right\} = \frac{1}{1-z}\,, \qquad |z| < 1$$

In addition we have

$$\mathcal{Z}\left\{u[n-5]\right\} = \frac{\left(z^{-5}\right)z}{z-1} = \frac{z^{-4}}{z-1}\,, \qquad |z| > 1$$

Applying the time reversal property

$$\mathcal{Z}\left\{u[-n-5]\right\} = \frac{z^{4}}{z^{-1}-1} = \frac{z^{5}}{1-z}\,, \qquad |z| < 1$$

and, applying the linearity property

$$X(z) = \frac{1}{1-z} + \frac{z^5}{1-z} = \frac{1+z^5}{1-z}\,, \qquad |z| < 1$$

d. The z transform of the unit-ramp function was found in Example 8-21 to be

$$\mathcal{Z}\left\{n\,u[n]\right\} = \frac{z}{(z-1)^2}\,, \qquad |z| > 1$$

Applying the time reversal property leads to

$$\mathcal{Z}\left\{-n\,u[-n]\right\} = \frac{z^{-1}}{\left(z^{-1}-1\right)^2} = \frac{z}{(1-z)^2} = \frac{z}{(z-1)^2}\,, \qquad |z| < 1$$

and, through the use of the linearity property, we get

$$X(z) = \mathcal{Z}\left\{n\,u[-n]\right\} = \frac{-z}{(z-1)^2}\,, \qquad |z| < 1$$

e. The transform of a cosine signal was found in Example 8-15 to be

$$\mathcal{Z}\left\{\cos(\Omega_0 n)\;u[n]\right\} = \frac{z\left[z-\cos(\Omega_0)\right]}{z^2-2\cos(\Omega_0)\;z+1}\,, \qquad |z| > 1$$

Applying the time reversal property yields

$$\begin{aligned}\mathcal{Z}\left\{\cos(-\Omega_0 n)\;u[-n]\right\} &= \frac{z^{-1}\left[z^{-1}-\cos(\Omega_0)\right]}{z^{-2}-2\cos(\Omega_0)\;z^{-1}+1}\\ &= \frac{1-\cos(\Omega_0)\;z}{1-2\cos(\Omega_0)\;z+z^2}\,, \qquad |z| < 1\end{aligned}$$

Since the cosine function is even, that is, $\cos(-\Omega_0 n) = \cos(\Omega_0 n)$, we get

$$X(z) = \mathcal{Z}\left\{\cos(\Omega_0 n)\; u[-n]\right\} = \frac{1-\cos(\Omega_0)\; z}{1-2\cos(\Omega_0)\; z+z^2}\,, \qquad |z|<1$$

8.10.

a. Starting with

$$\mathcal{Z}\left\{u[n]\right\} = \frac{z}{z-1}\,, \qquad |z|>1$$

the following relationships can be obtained through the use of the differentiation property of the z-transform:

$$\mathcal{Z}\left\{n\,u[n]\right\} = -z\,\frac{d}{dz}\left[\frac{z}{z-1}\right] = \frac{z}{(z-1)^2}\,, \qquad |z|>1$$

$$\mathcal{Z}\left\{n^2\,u[n]\right\} = -z\,\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = \frac{z\,(z+1)}{(z-1)^3}\,, \qquad |z|>1$$

Using the relationships found, $X(z)$ is obtained as

$$X(z) = \frac{z\,(z+1)}{(z-1)^3} + 3\,\frac{z}{(z-1)^2} + 5\,\frac{z}{z-1} = \frac{5\,z^3-6\,z^2+3\,z}{(z-1)^3}\,, \qquad |z|>1$$

b. Starting with

$$\mathcal{Z}\left\{u[-n-1]\right\} = \frac{-z}{z-1}\,, \qquad |z|<1$$

the following relationships can be obtained through the use of the differentiation property of the z-transform:

$$\mathcal{Z}\left\{n\,u[-n-1]\right\} = -z\,\frac{d}{dz}\left[\frac{-z}{z-1}\right] = \frac{-z}{(z-1)^2}\,, \qquad |z|<1$$

$$\mathcal{Z}\left\{n^2\,u[-n-1]\right\} = -z\,\frac{d}{dz}\left[\frac{-z}{(z-1)^2}\right] = \frac{-z\,(z+1)}{(z-1)^3}\,, \qquad |z|<1$$

Using the relationships found, $X(z)$ is obtained as

$$X(z) = \frac{-z\,(z+1)}{(z-1)^3} - 5\,\frac{-z}{z-1} = \frac{5\,z^3-11\,z^2+4\,z}{(z-1)^3} \qquad |z|<1$$

c. The transform of a cosine signal was found in Example 8-15 to be

$$\mathcal{Z}\left\{\cos(\Omega_0 n)\; u[n]\right\} = \frac{z\,[z-\cos(\Omega_0)]}{z^2-2\cos(\Omega_0)\; z+1}\,, \qquad |z|>1$$

Using the differentiation property of the z-transform

$$X(z) = \mathcal{Z}\left\{n\cos(\Omega_0 n)\; u[n]\right\} = -z\,\frac{d}{dz}\left[\frac{z\,[z-\cos(\Omega_0)]}{z^2-2\cos(\Omega_0)\; z+1}\right]\,, \qquad |z|>1$$

It can be shown that

$$\frac{d}{dz}\left[\frac{z\,[z-\cos(\Omega_0)]}{z^2-2\cos(\Omega_0)\; z+1}\right] = \frac{-\cos(\Omega_0)\; z^2+2\,z-\cos(\Omega_0)}{\left(z^2-2\cos(\Omega_0)\; z+1\right)^2}$$

and the transform $X(z)$ is

$$X(z) = \frac{z\left[\cos(\Omega_0)\, z^2 - 2\, z + \cos(\Omega_0)\right]}{\left(z^2 - 2\,\cos(\Omega_0)\, z + 1\right)^2}\,, \qquad |z| > 1$$

d. The transform of a sine signal was found in Example 8-16 to be

$$\mathcal{Z}\left\{\sin(\Omega_0 n)\; u[n]\right\} = \frac{\sin(\Omega_0)\, z}{z^2 - 2\,\cos(\Omega_0)\, z + 1}\,, \qquad |z| > 1$$

Using the differentiation property of the z-transform along with the linearity property

$$\begin{aligned} X(z) &= \mathcal{Z}\left\{n\sin(\Omega_0 n)\; u[n]\right\} + \mathcal{Z}\left\{\sin(\Omega_0 n)\; u[n]\right\} \\ &= -z\frac{d}{dz}\left[\frac{\sin(\Omega_0)\, z}{z^2 - 2\,\cos(\Omega_0)\, z + 1}\right] + \frac{\sin(\Omega_0)\, z}{z^2 - 2\,\cos(\Omega_0)\, z + 1}\,, \qquad |z| > 1 \end{aligned}$$

It can be shown that

$$\frac{d}{dz}\left[\frac{\sin(\Omega_0)\, z}{z^2 - 2\,\cos(\Omega_0)\, z + 1}\right] = \frac{-\sin(\Omega_0)\left(z^2 - 1\right)}{\left(z^2 - 2\,\cos(\Omega_0)\, z + 1\right)^2}$$

and the transform $X(z)$ is

$$\begin{aligned} X(z) &= \frac{\sin(\Omega_0)\, z\left(z^2 - 1\right)}{\left(z^2 - 2\,\cos(\Omega_0)\, z + 1\right)^2} + \frac{\sin(\Omega_0)\, z}{z^2 - 2\,\cos(\Omega_0)\, z + 1} \\ &= \frac{2\sin(\Omega_0)\, z^2\, [z - \cos(\Omega_0)]}{\left(z^2 - 2\,\cos(\Omega_0)\, z + 1\right)^2}\,, \qquad |z| > 1 \end{aligned}$$

8.11.

a.

$$x[0] = \lim_{z\to\infty}\left[X(z)\right] = \lim_{z\to\infty}\left[\frac{z^2}{z^2}\right] = 1$$

b.

$$x[0] = \lim_{z\to\infty}\left[X(z)\right] = \lim_{z\to\infty}\left[\frac{z^2}{z^3}\right] = 0$$

c. The first step is to write $X(z)$ using non-negative powers of z:

$$X(z) = \frac{z^2 + z - 1}{z^3 + 0.7\, z^2 + 1.2\, z - 1.5}$$

Applying the initial value property

$$x[0] = \lim_{z\to\infty}\left[X(z)\right] = \lim_{z\to\infty}\left[\frac{z^2}{z^3}\right] = 0$$

8.12.

a. Using the correlation property, the transform of the autocorrelation function is

$$R_{xx}(z) = X(z)\, X\left(z^{-1}\right)$$

The transform of $x[n]$ is

$$X(z) = 1 + z^{-1} + z^{-2}$$

Therefore

$$R_{xx}(z) = \left(1 + z^{-1} + z^{-2}\right)\left(1 + z + z^{2}\right) = z + 2 + z^{-1}$$

and

$$r_{xx}[m] = \delta[m+1] + 2\,\delta[m] + \delta[m-1]$$

b. The transform of $x[n]$ is

$$X(z) = \frac{1}{1 - z^{-1}}\left(1 - z^{-N}\right), \qquad |z| > 0$$

Replacing z with z^{-1} leads to

$$X\left(z^{-1}\right) = \frac{1}{1 - z}\,(1 - z), \qquad |z| < \infty$$

Therefore

$$R_{xx}(z) = X(z)\, X\left(z^{-1}\right) = \left(\frac{1 - z^{-N}}{1 - z^{-1}}\right)\left(\frac{1 - z^{N}}{1 - z}\right)$$
$$= \frac{z^{N+1} - 2\,z + z^{-N+1}}{(z-1)^2}, \qquad 0 < |z| < \infty$$

$R_{xx}(z)$ can be written as

$$R_{xx}(z) = \frac{z}{(z-1)^2}\left[z^{N} - 2 + z^{-N}\right]$$

which leads to the autocorrelation function

$$r_{xx}[m] = (m+N)\; u[m+N] - 2m\, u[m] + (m-N)\; u[m-N]$$

8.13.

The transforms of the two signals are

$$X(z) = 1 + z^{-1} + z^{-2}$$

and

$$Y(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$$

The transform of the cross correlation function is

$$R_{XY}(z) = X(z)\; Y\left(z^{-1}\right) = \left(1 + z^{-1} + z^{-2}\right)\left(1 + z + z^{2} + z^{3} + z^{4}\right)$$
$$= z^{4} + 2\,z^{3} + 3\,z^{2} + 3\,z + 3 + 2\,z^{-1} + z^{-2}$$

and the cross correlation function is found as

$$r_{XY}[m] = \{\,1,\ 2,\ 3,\ 3,\ \underset{\substack{\uparrow \\ m=0}}{3},2,\ 1\,\}$$

8.14.

a. The convolution of $x[m]$ and $y[m]$ is

$$x[m] * y[m] = \sum_{n=-\infty}^{\infty} x[n]\, y[m-n]$$

Replacing $y[m]$ with $y[-m]$ gives

$$x[m] * y[-m] = \sum_{n=-\infty}^{\infty} x[n]\, y[n-m] = r_{xy}[m]$$

b. Using correlation property of the z-transform

$$R_{xy}(z) = X(z)\; Y\left(z^{-1}\right)$$

Since

$$\mathcal{Z}^{-1}\left\{X(z)\right\} = x[m] \qquad \text{and} \qquad \mathcal{Z}^{-1}\left\{Y\left(z^{-1}\right)\right\} = y[-m]$$

it follows that

$$r_{xy}[m] = \mathcal{Z}^{-1}\left\{X(z)\; Y\left(z^{-1}\right)\right\} = x[m] * y[-m]$$

8.15.

a. The transform of $X(z)$ is

$$X(z) = \frac{z}{z-a}\,, \qquad |z| > |a|$$

Using the summation property

$$W(z) = \frac{z}{z-1}\, X(z) = \frac{z^2}{(z-1)\,(z-a)}\,, \qquad |z| > \max\left(1, |a|\right)$$

b. Partial fraction form of $W(z)$ is

$$W(z) = \frac{k_1\, z}{z-1} + \frac{k_2\, z}{z-a}$$

with the residues

$$k_1 = \frac{1}{1-a}\,, \qquad \text{and} \qquad k_2 = \frac{a}{a-1}$$

Inverse transform is

$$\begin{aligned} w[n] &= \frac{1}{1-a}\, u[n] + \frac{a}{a-1}\, a^n\, u[n] \\ &= \frac{1}{1-a}\left(1 - a^{n+1}\right) u[n] \end{aligned}$$

c. Using the closed form formula for finite-length geometric series

$$w[n] = \sum_{k=-\infty}^{n} a^k\, u[k] = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

for $n \geq 0$.

8.16. Let

$$x[n] = n\,u[n]$$

so that

$$w[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n} k\,u[k] = \sum_{k=0}^{n} k$$

The transform of $x[n]$ is

$$X(z) = \frac{z}{(z-1)^2}$$

Using the differentiation property of the z-transform

$$W(z) = \frac{z}{z-1}\,X(z) = \frac{z^2}{(z-1)^3}$$

Following transform relationships were derived earlier (see Example 8-22)

$$n\,u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{(z-1)^2} = \frac{z^2-z}{(z-1)^3}$$

$$n^2\,u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^2+z}{(z-1)^3}$$

Combining these transform pairs using linearity we get

$$\left(n^2+n\right)\,u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{2\,z^2}{(z-1)^3}$$

Therefore

$$w[n] = \frac{n\,(n+1)}{2}\,u[n]$$

8.17.

a.

$$\frac{X(z)}{z} = \frac{1}{(z+1)\,(z+2)} = \frac{k_1}{(z+1)} + \frac{k_2}{(z+2)}$$

Next step is to determine the residues:

$$k_1 = \left.\frac{1}{(z+2)}\right|_{z=-1} = 1$$

$$k_2 = \left.\frac{1}{(z+1)}\right|_{z=-2} = -1$$

Partial fraction expansion for $X(z)$ is

$$X(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

Based on the ROC specified, $x[n]$ is an anti-causal signal. Therefore

$$x[n] = -(-1)^n\,u[-n-1] + (-2)^n\,u[-n-1]$$

b.

$$\frac{X(z)}{z}=\frac{z+1}{z\,(z+1/2)\,(z+2/3)}=\frac{k_1}{z}+\frac{k_2}{(z+1/2)}+\frac{k_3}{(z+2/3)}$$

The residues are

$$k_1=\left.\frac{z+1}{(z+1/2)\,(z+2/3)}\right|_{z=0}=3$$

$$k_2=\left.\frac{z+1}{z\,(z+2/3)}\right|_{z=-1/2}=-6$$

and

$$k_3=\left.\frac{z+1}{z\,(z+1/2)}\right|_{z=-2/3}=3$$

Partial fraction expansion for $X(z)$ is

$$X(z)=3-\frac{6\,z}{z+1/2}+\frac{3\,z}{z+2/3}$$

The specified ROC indicates a causal signal, therefore

$$x[n]=3\,\delta[n]-6\,(-1/2)^n\,u[n]+3\,(-2/3)^n\,u[n]$$

c.

$$\frac{X(z)}{z}=\frac{z+1}{(z-0.4)\,(z+0.7)}=\frac{k_1}{(z-0.4)}+\frac{k_2}{(z+0.7)}$$

The residues are

$$k_1=\left.\frac{z+1}{(z-0.4)\,(z+0.7)}\right|_{z=0.4}=\frac{14}{11}$$

and

$$k_2=\left.\frac{z+1}{(z-0.4)\,(z+0.7)}\right|_{z=-0.7}=-\frac{3}{11}$$

Partial fraction expansion for $X(z)$ is

$$X(z)=\left(\frac{14}{11}\right)\frac{z}{(z-0.4)}-\left(\frac{3}{11}\right)\frac{z}{(z+0.7)}$$

The specified ROC indicates a causal signal, therefore

$$x[n]=\left(\frac{14}{11}\right)(0.4)^n\,u[n]-\left(\frac{3}{11}\right)(-0.7)^n\,u[n]$$

d.

$$\frac{X(z)}{z}=\frac{z+1}{(z+3/4)\,(z-1/2)\,(z-3/2)}=\frac{k_1}{(z+3/4)}+\frac{k_2}{(z-1/2)}+\frac{k_3}{(z-3/2)}$$

The residues are

$$k_1=\left.\frac{z+1}{(z-1/2)\,(z-3/2)}\right|_{z=-3/4}=\frac{4}{45}$$

$$k_2=\left.\frac{z+1}{(z+3/4)\,(z-3/2)}\right|_{z=1/2}=-\frac{6}{5}$$

and

$$k_3 = \frac{z+1}{(z+3/4)\,(z-1/2)} = \Big|_{z=3/2} = \frac{10}{9}$$

Partial fraction expansion for $X(z)$ is

$$X(z) = \frac{\frac{4}{45}\,z}{(z+3/4)} - \frac{\frac{6}{5}\,z}{(z-1/2)} + \frac{\frac{10}{9}\,z}{(z-3/2)}$$

Based on the ROC specified, the terms with poles at $z = 1/2$ and $z = -3/4$ correspond to causal signal components whereas the term with pole at $z = 3/2$ corresponds to an anti-causal signal component:

$$X(z) = \underbrace{\frac{\frac{4}{45}\,z}{(z+3/4)} - \frac{\frac{6}{5}\,z}{(z-1/2)}}_{\text{causal}} + \underbrace{\frac{\frac{10}{9}\,z}{(z-3/2)}}_{\text{anti-causal}}$$

The signal $x[n]$ is

$$x[n] = \left(\frac{4}{45}\right)(-3/4)^n\,u[n] - \left(\frac{6}{5}\right)(1/2)^n\,u[n] - \left(\frac{10}{9}\right)(3/2)^n\,u[-n-1]$$

e. Partial fraction expansion for $X(z)$ is as found in part (d). However, since the ROC is different than that of part (d), the terms with poles at $z = -3/4$ and $z = 3/2$ correspond to anti-causal signal components whereas the term with pole at $z = 1/2$ corresponds to a causal signal component:

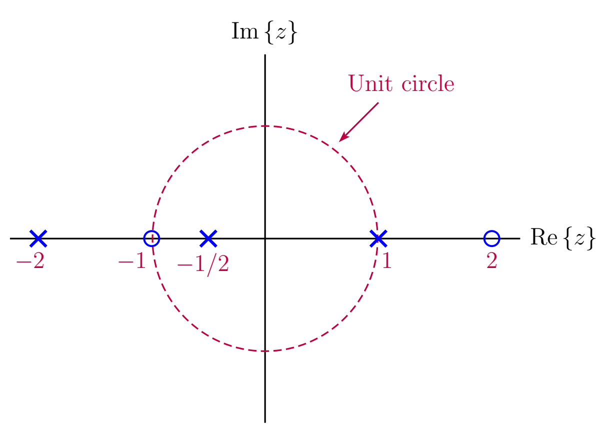
$$X(z) = \underbrace{\frac{\frac{4}{45}\,z}{(z+3/4)}}_{\text{anti-causal}} - \underbrace{\frac{\frac{6}{5}\,z}{(z-1/2)}}_{\text{causal}} + \underbrace{\frac{\frac{10}{9}\,z}{(z-3/2)}}_{\text{anti-causal}}$$

The signal $x[n]$ is

$$x[n] = -\left(\frac{4}{45}\right)(-3/4)^n\,u[-n-1] - \left(\frac{6}{5}\right)(1/2)^n\,u[n] - \left(\frac{10}{9}\right)(3/2)^n\,u[-n-1]$$

8.18.

a.



b.

$$\frac{X(z)}{z}=\frac{(z+1)\,(z-2)}{z\,(z+1/2)\,(z-1)\,(z+2)}=\frac{k_1}{z}+\frac{k_2}{(z+1/2)}+\frac{k_3}{(z-1)}+\frac{k_4}{(z+2)}$$

Next step is to determine the residues:

$$k_1=\left.\frac{(z+1)\,(z-2)}{(z+1/2)\,(z-1)\,(z+2)}\right|_{z=0}=2$$

$$k_2=\left.\frac{(z+1)\,(z-2)}{z\,(z-1)\,(z+2)}\right|_{z=-1/2}=-\frac{10}{9}$$

$$k_3=\left.\frac{(z+1)\,(z-2)}{z\,(z+1/2)\,(z+2)}\right|_{z=1}=-\frac{4}{9}$$

$$k_4=\left.\frac{(z+1)\,(z-2)}{z\,(z+1/2)\,(z-1)}\right|_{z=-2}=-\frac{4}{9}$$

Partial fraction expansion for $X(z)$ is

$$X(z)=2-\frac{\frac{10}{9}\,z}{(z+1/2)}-\frac{\frac{4}{9}\,z}{(z-1)}-\frac{\frac{4}{9}\,z}{(z+2)}$$

c. There are four possible choices for the ROC:

Case 1: $|z|<1/2$

$$x[n]=2\,\delta[n]+\left(\frac{10}{9}\right)(-1/2)^n\,u[-n-1]+\left(\frac{4}{9}\right)u[-n-1]+\left(\frac{4}{9}\right)(-2)^n\,u[-n-1]$$

Case 2: $1/2<|z|<1$

$$x[n]=2\,\delta[n]-\left(\frac{10}{9}\right)(-1/2)^n\,u[n]+\left(\frac{4}{9}\right)u[-n-1]+\left(\frac{4}{9}\right)(-2)^n\,u[-n-1]$$

Case 3: $1<|z|<2$

$$x[n]=2\,\delta[n]-\left(\frac{10}{9}\right)(-1/2)^n\,u[n]-\left(\frac{4}{9}\right)u[n]+\left(\frac{4}{9}\right)(-2)^n\,u[-n-1]$$

Case 4: $|z|>2$

$$x[n]=2\,\delta[n]-\left(\frac{10}{9}\right)(-1/2)^n\,u[n]-\left(\frac{4}{9}\right)u[n]-\left(\frac{4}{9}\right)(-2)^n\,u[n]$$

8.19.

a. Factored form of $X(z)$ is

$$X(z)=\frac{z\,(z+3)}{\big(z-0.7-j\,0.6\big)\,\big(z-0.7+j\,0.6\big)}$$

and

$$\frac{X(z)}{z}=\frac{(z+3)}{\big(z-0.7-j\,0.6\big)\,\big(z-0.7+j\,0.6\big)}=\frac{k_1}{z-0.7-j\,0.6}+\frac{k_2}{z-0.7+j\,0.6}$$

Residues are found as

$$k_1=\left.\frac{(z+3)}{\big(z-0.7+j\,0.6\big)}\right|_{z=0.7+j\,0.6}=0.5-j\,3.0833$$

and

$$k_2 = k_1^* = 0.5 + j\,3.0833$$

Partial fraction expansion of $X(z)$ is

$$X(z) = \frac{\left(0.5 - j\,3.0833\right)\,z}{z - 0.7 - j\,0.6} + \frac{\left(0.5 + j\,3.0833\right)\,z}{z - 0.7 + j\,0.6}$$

and the inverse transform is

$$x[n] = \left(0.5 - j\,3.0833\right)\,\left(0.7 + j\,0.6\right)^n\,u[n] + \left(0.5 + j\,3.0833\right)\,\left(0.7 - j\,0.6\right)^n\,u[n]$$

Since

$$0.5 - j\,3.0833 = 3.1236\,e^{-j\,1.41} \qquad \text{and} \qquad 0.7 + j\,0.6 = 0.922\,e^{j\,0.7086}$$

the inverse transform can be written as

$$\begin{aligned} x[n] &= \left(3.1236\,e^{-j\,1.41}\right)\,\left(0.922\,e^{j\,0.7086}\right)^n\,u[n] + \left(3.1236\,e^{j\,1.41}\right)\,\left(0.922\,e^{-j\,0.7086}\right)^n\,u[n] \\ &= 6.2472\,(0.922)^n\,\cos\left(0.7086n - 1.41\right)\,u[n] \end{aligned}$$

b. Factored form of $X(z)$ is

$$X(z) = \frac{z^2}{\left(z - 0.8 - j\,0.6\right)\,\left(z - 0.8 + j\,0.6\right)}$$

and

$$\frac{X(z)}{z} = \frac{z}{\left(z - 0.8 - j\,0.6\right)\,\left(z - 0.8 + j\,0.6\right)} = \frac{k_1}{z - 0.8 - j\,0.6} + \frac{k_2}{z - 0.8 + j\,0.6}$$

Residues are found as

$$k_1 = \left.\frac{z}{\left(z - 0.8 + j\,0.6\right)}\right|_{z=0.8+j\,0.6} = \frac{1}{2} - j\,\frac{2}{3}$$

and

$$k_2 = k_1^* = \frac{1}{2} + j\,\frac{2}{3}$$

Partial fraction expansion of $X(z)$ is

$$X(z) = \frac{\left(\frac{1}{2} - j\,\frac{2}{3}\right)\,z}{z - 0.8 - j\,0.6} + \frac{\left(\frac{1}{2} + j\,\frac{2}{3}\right)\,z}{z - 0.8 + j\,0.6}$$

and the inverse transform is

$$x[n] = \left(\frac{1}{2} - j\,\frac{2}{3}\right)\,\left(0.8 + j\,0.6\right)^n\,u[n] + \left(\frac{1}{2} + j\,\frac{2}{3}\right)\,\left(0.8 - j\,0.6\right)^n\,u[n]$$

Since

$$\frac{1}{2} - j\,\frac{2}{3} = \frac{5}{6}\,e^{-j\,0.9273} \qquad \text{and} \qquad 0.8 + j\,0.6 = e^{j\,0.6436}$$

the inverse transform can be written as

$$\begin{aligned} x[n] &= \left(\frac{5}{6}\,e^{-j\,0.9273}\right)\,e^{j\,0.7086n}\,u[n] + \left(\frac{5}{6}\,e^{j\,0.9273}\right)\,e^{-j\,0.7086n}\,u[n] \\ &= \frac{5}{3}\,\cos\left(0.9273n - 0.7086\right)\,u[n] \end{aligned}$$

c. Factored form of $X(z)$ is

$$X(z) = \frac{z\,(z+3)}{\left(z-\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\left(z-\frac{\sqrt{3}}{2}+j\,\frac{1}{2}\right)}$$

and

$$\frac{X(z)}{z} = \frac{z+3}{\left(z-\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\left(z-\frac{\sqrt{3}}{2}+j\,\frac{1}{2}\right)} = \frac{k_1}{z-\frac{\sqrt{3}}{2}-\frac{1}{2}} + \frac{k_2}{z-\frac{\sqrt{3}}{2}+j\,\frac{1}{2}}$$

Residues are found as

$$k_1 = \left.\frac{z+3}{\left(z-\frac{\sqrt{3}}{2}+j\,\frac{1}{2}\right)}\right|_{z=\frac{\sqrt{3}}{2}+j\,\frac{1}{2}} = 0.5-j\,3.866$$

and

$$k_2 = k_1^* = 0.5+j\,3.866$$

Partial fraction expansion of $X(z)$ is

$$X(z) = \frac{\left(0.5-j\,3.866\right)\,z}{z-\frac{\sqrt{3}}{2}-j\,\frac{1}{2}} + \frac{\left(0.5+j\,3.866\right)\,z}{z-\frac{\sqrt{3}}{2}+j\,\frac{1}{2}}$$

and the inverse transform is

$$x[n] = \left(0.5-j\,3.866\right)\left(\frac{\sqrt{3}}{2}+j\,\frac{1}{2}\right)^n u[n] + \left(0.5+j\,3.866\right)\left(\frac{\sqrt{3}}{2}-j\,\frac{1}{2}\right)^n u[n]$$

Since

$$0.5-j\,3.866 = 3.8982\,e^{-j\,1.4422} \qquad \text{and} \qquad \frac{\sqrt{3}}{2}+j\,\frac{1}{2} = e^{j\,0.5236}$$

the inverse transform can be written as

$$\begin{aligned} x[n] &= \left(3.8982\,e^{-j\,1.4422}\right)\,e^{j\,0.5236n}\,u[n] + \left(3.8982\,e^{j\,1.4422}\right)\,e^{-j\,0.5236n}\,u[n] \\ &= 7.7964\,\cos\left(0.5236n-1.4422\right)\,u[n] \end{aligned}$$

8.20.

a. Factored form of $X(z)$ is

$$X(z) = \frac{(z+1)\,(z+2)}{(z-1)^2}$$

and

$$\frac{X(z)}{z} = \frac{(z+1)\,(z+2)}{z\,(z-1)^2} = \frac{k_1}{z} + \frac{k_{2,1}}{z-1} + \frac{k_{2,2}}{(z-1)^2}$$

Residues are found as

$$k_1 = \left.\frac{(z+1)\,(z+2)}{(z-1)^2}\right|_{z=0} = 2$$

$$k_{2,2} = \left.\frac{(z+1)\,(z+2)}{z}\right|_{z=1} = 6$$

and

$$k_{2,1}=\frac{d}{dz}\left[\frac{(z+1)\,(z+2)}{z}\right]\bigg|_{z=1}=\left.\frac{z^2-2}{z^2}\right|_{z=1}=-1$$

Partial fraction expansion of $X(z)$ is

$$X(z)=2-\frac{z}{z-1}+\frac{6\,z}{(z-1)^2}$$

b. Dividing $X(z)$ by z yields

$$\frac{X(z)}{z}=\frac{z+1}{(z+0.9)^2\,(z-1)}=\frac{k_{1,1}}{z+0.9}+\frac{k_{1,2}}{(z+0.9)^2}+\frac{k_2}{z-1}$$

Residues are found as

$$k_{1,2}=\left.\frac{z+1}{z-1}\right|_{z=-0.9}=-0.0526$$

$$k_{1,1}=\frac{d}{dz}\left[\frac{z+1}{z-1}\right]\bigg|_{z=-0.9}=\left.\frac{-2}{(z-1)^2}\right|_{z=-0.9}=-0.5540$$

and

$$k_2=\left.\frac{z+1}{(z+0.9)^2}\right|_{z=1}=0.5540$$

Partial fraction expansion of $X(z)$ is

$$X(z)=-\frac{0.5540\,z}{z+0.9}-\frac{0.0526\,z}{(z+0.9)^2}+\frac{0.5540\,z}{z-1}$$

c. Dividing $X(z)$ by z yields

$$\frac{X(z)}{z}=\frac{z^2+4\,z-7}{z\,(z+0.9)^2\,(z-1.2)^2}=\frac{k_1}{z}+\frac{k_{2,1}}{z+0.9}+\frac{k_{2,2}}{(z+0.9)^2}+\frac{k_{3,1}}{z-1.2}+\frac{k_{3,2}}{(z-1.2)^2}$$

Residues are found as

$$k_1=\left.\frac{z^2+4\,z-7}{(z+0.9)^2\,(z-1.2)^2}\right|_{z=0}=-6.0014$$

$$k_{2,2}=\left.\frac{z^2+4\,z-7}{z\,(z-1.2)^2}\right|_{z=-0.9}=2.4666$$

$$k_{3,2}=\left.\frac{z^2+4\,z-7}{z\,(z+0.9)^2}\right|_{z=1.2}=-0.1436$$

$$k_{2,1}=\frac{d}{dz}\left[\frac{z^2+4\,z-7}{z\,(z-1.2)^2}\right]\bigg|_{z=-0.9}=\left.\frac{-z^4-8\,z^3+32.04\,z^2-33.6\,z+10.08}{z^2\,(z-1.2)^4}\right|_{z=-0.9}=4.5355$$

$$k_{3,1}=\frac{d}{dz}\left[\frac{z^2+4\,z-7}{z\,(z+0.9)^2}\right]\bigg|_{z=1.2}=\left.\frac{-z^4-8\,z^3+14.61\,z^2+25.2\,z+5.67}{z^2\,(z+0.9)^4}\right|_{z=1.2}=1.4658$$

Partial fraction expansion of $X(z)$ is

$$X(z) = -6.0014 + \frac{4.5355\,z}{z+0.9} + \frac{2.4666\,z}{(z+0.9)^2} + \frac{1.4658\,z}{z-1.2} - \frac{0.1436\,z}{(z-1.2)^2}$$

8.21.

a. Taking the z-transform of the difference equation

$$Y(z) = (1+c)\,z^{-1}\,Y(z) - X(z)$$

and

$$H(z) = \frac{-z}{z-(1+c)}$$

b.

$$X(z) = -A + B\,z^{-1}\left(\frac{z}{z-1}\right) = -A + \frac{B}{z-1} = \frac{-A\,z + A + B}{z-1}$$

c.

$$Y(z) = H(z)\,X(z) = \frac{A\,z\left(z - \dfrac{A+B}{A}\right)}{(z-1-c)\,(z-1)}$$

d. Partial fraction expansion for $Y(z)$ is found through

$$\frac{Y(z)}{z} = \frac{A\left(z - \dfrac{A+B}{A}\right)}{(z-1-c)\,(z-1)} = \frac{k_1}{z-1-c} + \frac{k_2}{z-1}$$

The residues are

$$k_1 = A - \frac{B}{c}, \qquad \text{and} \qquad k_2 = \frac{B}{c}$$

The solution is

$$y[n] = \left(A - \frac{B}{c}\right)(1+c)^n\,u[n] + \frac{B}{c}\,u[n]$$

Setting $y[N] = 0$ we obtain

$$B = \frac{A\,c\,(1+c)^N}{(1+c)^N - 1}$$

8.22.

a.

$$X(z) = \frac{z^2 + 3\,z}{z^2 - 1.4\,z + 0.85}$$

Long division is carried out as follows:

$$
\begin{array}{r|lllll}
 & \multicolumn{5}{l}{1+4.4z^{-1}+5.31z^{-2}+3.694z^{-3}+0.6581z^{-4}} \\
\cline{2-6}
z^2 \; -1.4z \; +0.85 & z^2 & +3z & & & \\
 & z^2 & -1.4z & +0.85 & & \\
\cline{2-4}
 & & 4.4z & -0.85 & & \\
 & & 4.4z & -6.16 & +3.74z^{-1} & \\
\cline{3-5}
 & & & 5.31 & -3.74z^{-1} & \\
 & & & 5.31 & -7.434z^{-1} & +4.5135z^{-2} \\
\cline{4-6}
 & & & & 3.694z^{-1} & -4.5135z^{-2} \\
 & & & & 3.694z^{-1} & -5.1716z^{-2} \; +3.1399z^{-3} \\
\cline{5-6}
 & & & & & 0.6581z^{-2} \; -3.1399z^{-3}
\end{array}
$$

The inverse transform $x[n]$ is

$$x[n]=\{\underset{\substack{\uparrow \\ n=0}}{1},\,4.4,\,5.31,\,3.694,\,0.6581,\,-2.2186;\,-3.6654,\,\ldots\}$$

b.

$$X(z)=\frac{z^2}{z^2-1.6z+1}$$

Long division is carried out as follows:

$$
\begin{array}{r|lllll}
 & \multicolumn{5}{l}{1+1.6z^{-1}+1.56z^{-2}+0.896z^{-3}-0.1264z^{-4}} \\
\cline{2-6}
z^2 \; -1.6z \; +1 & z^2 & & & & \\
 & z^2 & -1.6z & +1 & & \\
\cline{2-4}
 & & 1.6z & -1 & & \\
 & & 1.6z & -2.56 & +1.6z^{-1} & \\
\cline{3-5}
 & & & 1.56 & -1.6z^{-1} & \\
 & & & 1.56 & -2.496z^{-1} & +1.56z^{-2} \\
\cline{4-6}
 & & & & 0.896z^{-1} & -1.56z^{-2} \\
 & & & & 0.896z^{-1} & -1.4336z^{-2} \; +0.896z^{-3} \\
\cline{5-6}
 & & & & & -0.1264z^{-2} \; -0.896z^{-3}
\end{array}
$$

The inverse transform $x[n]$ is

$$x[n]=\{\underset{\substack{\uparrow \\ n=0}}{1},\,1.6,\,1.56,\,0.896,\,-0.1264,\,-1.0982;\,-1.6308,\,\ldots\}$$

c.

$$X(z)=\frac{z^2+3z}{z^2-1.7321z+1}$$

Long division is carried out as follows:

$$
\begin{array}{r|llllll}
 & \multicolumn{6}{l}{1+4.7321z^{-1}+7.1962z^{-2}+7.7324z^{-3}+6.1971z^{-4}} \\
\hline
z^2 \; -1.7321z \; +1 & z^2 & +3z & & & & \\
 & z^2 & -1.7321z & +1 & & & \\
\cline{2-4}
 & & 4.7321z & -1 & & & \\
 & & 4.7321z & -8.1962 & +4.7321z^{-1} & & \\
\cline{3-5}
 & & & 7.1962 & -4.7321z^{-1} & & \\
 & & & 7.1962 & -12.4645z^{-1} & +7.1962z^{-2} & \\
\cline{4-6}
 & & & & 7.7324z^{-1} & -7.1962z^{-2} & \\
 & & & & 7.7324z^{-1} & -13.3933z^{-2} & +7.7324z^{-3} \\
\cline{5-7}
 & & & & & 6.1971z^{-2} & -7.7324z^{-3}
\end{array}
$$

The inverse transform $x[n]$ is

$$x[n]=\{\underset{\substack{\uparrow\\ n=0}}{1},\,4.7321,\,7.1962,\,7.7324,\,6.1971,\,3,\,-1,\,\ldots\}$$

8.23. There are four possible choices for the ROC:

Case 1: $|z|<1/2$

$$X(z)=\frac{z^2-z-2}{z^3+1.5z^2-1.5z-1}=\frac{-2-z+z^2}{-1-1.5z+1.5z^2+z^3}$$

Long division is carried out as follows:

$$
\begin{array}{r|lllllll}
 & \multicolumn{7}{l}{2-2z-5z^2-8.5z^3+18.25z^4} \\
\hline
-1\;-1.5z\;+1.5z^2+z^3 & -2 & -z & +z^2 & & & & \\
 & -2 & -3z & +3z^2 & +2z^3 & & & \\
\cline{2-5}
 & & 2z & -2z^2 & -2z^3 & & & \\
 & & 2z & +3z^2 & -3z^3 & -2z^4 & & \\
\cline{3-6}
 & & & -5z^2 & +z^3 & +2z^4 & & \\
 & & & -5z^2 & -7.5z^3 & +7.5z^4 & 5z^5 & \\
\cline{4-7}
 & & & & 8.5z^3 & -5.5z^4 & -5z^5 & \\
 & & & & 8.5z^3 & +12.75z^4 & -12.75z^5 & -8.5z^6 \\
\cline{5-8}
 & & & & & -18.25z^4 & +7.75z^5 & +8.5z^6
\end{array}
$$

The inverse transform $x[n]$ is

$$x[n]=\{\ldots,\,-35.125,\,18.25,\,-8.5,\,5,\,-2,\,\underset{\substack{\uparrow\\ n=0}}{2},0,\,0,\,\ldots\}$$

Case 2: $1/2<|z|<1$

$$X(z)=2-\frac{\frac{10}{9}z}{z+\frac{1}{2}}-\frac{\frac{4}{9}z}{z-1}-\frac{\frac{4}{9}z}{z+2}=X_L(z)+X_R(z)$$

$$X_R(z) = 2 - \frac{\frac{10}{9}z}{z+\frac{1}{2}} = \frac{0.8889\,z+1}{z+0.5}$$

$$X_L(z) = -\frac{\frac{4}{9}z}{z-1} - \frac{\frac{4}{9}z}{z+2} = \frac{-0.4444\,z - 0.8889\,z^2}{-2+z+z^2}$$

For $X_R(z)$, long division is carried out as follows:

$$\begin{array}{r|llll}
 & 0.8889+0.5556z^{-1}-0.2778z^{-2}+0.1389z^{-3}-0.0694z^{-4} \\ \hline
z\ +0.5 & 0.8889z \quad +1 \\
 & 0.8889z \quad +0.4444 \\ \cline{2-2}
 & \quad 0.5556 \\
 & \quad 0.5556 \quad +0.2778z^{-1} \\ \cline{2-2}
 & \quad\quad -0.2778z^{-1} \\
 & \quad\quad -0.2778z^{-1} \quad -0.1389z^{-2} \\ \cline{2-2}
 & \quad\quad\quad 0.1389z^{-2} \\
 & \quad\quad\quad 0.1389z^{-2} \quad +0.0694z^{-3} \\ \cline{2-2}
 & \quad\quad\quad\quad -0.0694z^{-3}
\end{array}$$

For $X_L(z)$, long division is carried out as follows:

$$\begin{array}{r|l}
 & 0.2222z+0.5556z^2+0.3889z^3+0.4722z^4+0.4306z^5 \\ \hline
-2\ +z\ +z^2 & -0.4444z \quad -0.8889z^2 \\
 & -0.4444z \quad +0.2222z^2 \quad +0.2222z^3 \\ \cline{2-2}
 & \quad -1.3333z^2 \quad -0.2222z^3 \\
 & \quad -1.3333z^2 \quad +0.5556z^3 \quad +0.5556z^4 \\ \cline{2-2}
 & \quad\quad -0.7778z^3 \quad -0.5556z^4 \\
 & \quad\quad -0.7778z^3 \quad +0.3889z^4 \quad +0.3889z^5 \\ \cline{2-2}
 & \quad\quad\quad -0.9445z^4 \quad -0.3889z^5 \\
 & \quad\quad\quad -0.9445z^4 \quad +0.4722z^5 \quad +0.4722z^6 \\ \cline{2-2}
 & \quad\quad\quad\quad -0.8611z^5 \quad -0.4722z^6
\end{array}$$

The inverse transform $x[n]$ is

$$x[n] = \{\ldots,\ 0.4306,\ 0.4722,\ 0.3889,\ 0.5556,\ 0.2222,\ \underset{\substack{\uparrow \\ n=0}}{0.8889},\ 0.5556,\ -0.2778,\ 0.1389,\ -0.0694,\ \ldots\}$$

Case 3: $\quad 1 < |z| < 2$

$$X(z) = 2 - \frac{\frac{10}{9}z}{z+\frac{1}{2}} - \frac{\frac{4}{9}z}{z-1} - \frac{\frac{4}{9}z}{z+2} = X_L(z) + X_R(z)$$

$$X_R(z) = 2 - \frac{\frac{10}{9}z}{z+\frac{1}{2}} - \frac{\frac{4}{9}z}{z-1} = \frac{0.4444\,z^2 - 0.1111\,z - 1}{z^2 - 0.5\,z - 0.5}$$

$$X_L(z) = -\frac{\frac{4}{9}z}{z+2} = \frac{-0.4444\,z}{2+z}$$

For $X_R(z)$, long division is carried out as follows:

$$\begin{array}{r|llllll}
 & 0.4444 + 0.1111z^{-1} - 0.7222z^{-2} - 0.3056z^{-3} - 0.5139z^{-4} \\
\hline
z^2 \; -0.5z \; -0.5 & 0.4444z^2 & -0.1111z & & & & \\
 & 0.4444z^2 & -0.2222z & -0.2222 & & & \\
\cline{2-4}
 & & 0.1111z & -0.7778 & & & \\
 & & 0.1111z & -0.0556 & -0.0556z^{-1} & & \\
\cline{3-5}
 & & & -0.7222 & +0.0556z^{-1} & & \\
 & & & -0.7222 & +0.3611z^{-1} & +0.3611z^{-2} & \\
\cline{4-6}
 & & & & -0.3056z^{-1} & -0.3611z^{-2} & \\
 & & & & -0.3056z^{-1} & +0.1528z^{-2} & +0.1528z^{-3} \\
\cline{5-7}
 & & & & & -0.5139z^{-2} & -0.1528z^{-3}
\end{array}$$

For $X_L(z)$, long division is carried out as follows:

$$\begin{array}{r|lllll}
 & -0.2222z + 0.1111z^2 - 0.0556z^3 + 0.0278z^4 \\
\hline
2 \; +z & -0.4444z & & & & \\
 & -0.4444z & -0.2222z^2 & & & \\
\cline{2-3}
 & & 0.2222z^2 & & & \\
 & & 0.2222z^2 & +0.1111z^3 & & \\
\cline{3-4}
 & & & -0.1111z^3 & & \\
 & & & -0.1111z^3 & -0.0556z^4 & \\
\cline{4-5}
 & & & & 0.0556z^4 & \\
 & & & & 0.0556z^4 & +0.0278z^5 \\
\cline{5-6}
 & & & & & -0.0278z^5
\end{array}$$

The inverse transform $x[n]$ is

$$x[n] = \{\ldots,\ 0.0278,\ -0.0556,\ 0.1111,\ -0.2222,\ \underset{\underset{n=0}{\uparrow}}{0.4444},\ 0.1111,\ -0.7222,\ -0.3056,\ -0.5139\ \ldots\}$$

Case 4: $\quad |z| > 2$

$$X(z) = \frac{z^2 - z - 2}{z^3 + 1.5\,z^2 - 1.5\,z - 1}$$

Long division is carried out as follows:

$$
\begin{array}{r|llllll}
 & z^{-1}-2.5z^{-2}+3.25z^{-3}-7.625z^{-4}+13.8125z^{-5} & & & & & \\
\hline
z^3 \; +1.5z^2 \; -1.5z-1 & z^2 \quad -z \quad -2 & & & & & \\
 & z^2 \; +1.5z \quad -1.5 & -z^{-1} & & & & \\
\cline{2-3}
 & \quad -2.5z \quad -0.5 & +z^{-1} & & & & \\
 & \quad -2.5z \; -3.75 & +3.75z^{-1} & +2.5z^{-2} & & & \\
\cline{2-4}
 & \qquad\qquad 3.25 & -2.75z^{-1} & -2.5z^{-2} & & & \\
 & \qquad\qquad 3.25 & +4.875z^{-1} & -4.875z^{-2} & -3.25z^{-3} & & \\
\cline{3-5}
 & & -7.625z^{-1} & +2.375z^{-2} & +3.25z^{-3} & & \\
 & & -7.625z^{-1} & -11.4375z^{-2} & +11.4375z^{-3} & +7.625z^{-4} & \\
\cline{3-6}
 & & & 13.8125z^{-2} & -8.1875z^{-3} & -7.625z^{-4} &
\end{array}
$$

The inverse transform $x[n]$ is

$$
x[n] = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 1, -2.5, 3.25, -7.625, 13.8125, -28.9063, \ldots \}
$$

8.24.

a.

$$
X(z) = \frac{z}{z^2+3z+2}
$$

Long division:

$$
\begin{array}{r|lllll}
 & z^{-1}-3z^{-2}+7z^{-3}-15z^{-4}+31z^{-5} & & & & \\
\hline
z^2 \; +3z \; +2 & z & & & & \\
 & z \; +3 & +2z^{-1} & & & \\
\cline{2-3}
 & \quad -3 & -2z^{-1} & & & \\
 & \quad -3 & -9z^{-1} & -6z^{-2} & & \\
\cline{2-4}
 & & 7z^{-1} & +6z^{-2} & & \\
 & & 7z^{-1} & +21z^{-2} & +14z^{-3} & \\
\cline{3-5}
 & & & 15z^{-2} & -14z^{-3} & \\
 & & & 15z^{-2} & -45z^{-3} & -30z^{-4} \\
\cline{4-6}
 & & & & 31z^{-3} & +30z^{-4}
\end{array}
$$

The inverse transform $x[n]$ is

$$
x[n] = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 1, -3, 7, -15, 31, -63, \ldots \}
$$

b.

$$
X(z) = \frac{z+1}{z^2+1.1667\,z+0.3333}
$$

Long division:

$$
\begin{array}{r|l}
 & z^{-1} - 0.1667z^{-2} - 0.1388z^{-3} + 0.2175z^{-4} - 0.2075z^{-5} \\
\cline{2-2}
z^2 \;+1.1667z \;+0.3333 & z \qquad +1 \\
 & z \;+1.1667 \;+0.3333z^{-1} \\
\cline{2-2}
 & \quad -0.1667 \;-0.3333z^{-1} \\
 & \quad -0.1667 \;-0.1945z^{-1} \;-0.0556z^{-2} \\
\cline{2-2}
 & \qquad\qquad -0.1388z^{-1} \;+0.0556z^{-2} \\
 & \qquad\qquad -0.1388z^{-1} \;-0.1619z^{-2} \;-0.0463z^{-3} \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad 0.2175z^{-2} \;+0.0463z^{-3} \\
 & \qquad\qquad\qquad\qquad 0.2175z^{-2} \;+0.2538z^{-3} \;+0.0725z^{-4} \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad\qquad\qquad 0.2075z^{-3} \;-0.0725z^{-4}
\end{array}
$$

The inverse transform $x[n]$ is

$$x[n] = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 1, -0.1667, -0.1388, 0.2175, -0.2075, 0.1696, \ldots \}$$

c.

$$X(z) = \frac{z^2 + z}{z^2 + 0.3z - 0.28}$$

Long division:

$$
\begin{array}{r|l}
 & 1 + 0.7z^{-1} + 0.07z^{-2} + 0.175z^{-3} - 0.0329z^{-4} \\
\cline{2-2}
z^2 \;+0.3z \;-0.28 & z^2 \qquad +z \\
 & z^2 \;+0.3z \;-0.28 \\
\cline{2-2}
 & \quad 0.7z \;+0.28 \\
 & \quad 0.7z \;+0.21 \;-0.196z^{-1} \\
\cline{2-2}
 & \qquad\qquad 0.07 \;+0.196z^{-1} \\
 & \qquad\qquad 0.07 \;+0.021z^{-1} \;-0.0196z^{-2} \\
\cline{2-2}
 & \qquad\qquad\qquad 0.175z^{-1} \;+0.0196z^{-2} \\
 & \qquad\qquad\qquad 0.175z^{-1} \;+0.0525z^{-2} \;-0.049z^{-3} \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad\qquad -0.0329z^{-2} \;+0.049z^{-3}
\end{array}
$$

The inverse transform $x[n]$ is

$$x[n] = \{ \underset{\substack{\uparrow \\ n=0}}{1}, 0.7, 0.07, 0.175, -0.0329, 0.0589, -0.0269, \ldots \}$$

8.25.

The partial fraction for $X(z)$ is

$$X(z) = \frac{-\frac{6}{5}\,z}{z-\frac{1}{2}} + \frac{\frac{4}{45}\,z}{z+\frac{3}{4}} + \frac{\frac{10}{9}\,z}{z-\frac{3}{2}}$$

a. For the specified ROC, the poles at $z = 1/2$ and $z = -3/4$ correspond to causal terms, and the pole at $z = 3/2$ corresponds to an anti-causal term. The causal part of the transform is

$$X_R(z) = \frac{-\frac{6}{5}\,z}{z-\frac{1}{2}} + \frac{\frac{4}{45}\,z}{z+\frac{3}{4}} = \frac{-1.1111\,z^2 - 0.9444\,z}{z^2 + 0.25\,z - 0.375}$$

Long division:

$$\begin{array}{r|llllll}
 & -1.1111 - 0.6667z^{-1} - 0.25z^{-2} - 0.1875z^{-3} - 0.0329z^{-4} & & & & & \\
\hline
z^2 \ +0.25z \ -0.375 & -0.1111z^2 & -0.9444z & & & & \\
 & -0.1111z^2 & -0.2778z & +0.4167 & & & \\
\cline{2-4}
 & & -0.6667z & +0.4167 & & & \\
 & & -0.6667z & +0.1667 & +0.25z^{-1} & & \\
\cline{3-5}
 & & & -0.25 & -0.25z^{-1} & & \\
 & & & -0.25 & -0.0625z^{-1} & 0.0938z^{-2} & \\
\cline{4-6}
 & & & & -0.1875z^{-1} & -0.0938z^{-2} & \\
 & & & & -0.1875z^{-1} & -0.0469z^{-2} & +0.0703z^{-3} \\
\cline{5-7}
 & & & & & -0.0329z^{-2} & +0.049z^{-3}
\end{array}$$

The anti-causal part of the transform is

$$X_L(z) = \frac{\frac{10}{9}\,z}{z-\frac{3}{2}} = \frac{1.1111\,z}{-1.5+z}$$

Long division:

$$\begin{array}{r|lllll}
 & -0.7407z - 0.4938z^2 - 0.3292z^3 - 0.2195z^4 & & & & \\
\hline
-1.5 \ +z & 0.1111z & & & & \\
 & 0.1111z & -0.7407z^2 & & & \\
\cline{2-3}
 & & 0.7407z^2 & & & \\
 & & 0.7407z^2 & -0.4938z^3 & & \\
\cline{3-4}
 & & & 0.4938z^3 & & \\
 & & & 0.4938z^3 & -0.3292z^4 & \\
\cline{4-5}
 & & & & 0.3292z^4 & \\
 & & & & 0.3292z^4 & -0.2195z^5 \\
\cline{5-6}
 & & & & & 0.2195z^5
\end{array}$$

The inverse transform $x[n]$ is

$$x[n] = \{\ldots,\ -0.1463,\ -0.2195,\ -0.3292,\ -0.4938,\ -0.7407,\ \underset{\substack{\uparrow \\ n=0}}{-1.1111},\ -0.6667,\ -0.25,\ -0.1875,\ -0.0469,\ -0.0586,\ \ldots\}$$

b. For the specified ROC, the pole at $z = 1/2$ corresponds to a causal term. The poles at and $z = -3/4$ and $z = 3/2$ correspond to anti-causal terms. The causal part of the transform is

$$X_R(z) = \frac{-\frac{6}{5}\,z}{z-\frac{1}{2}} = \frac{-1.2\,z}{-0.5+z}$$

Long division:

$$\begin{array}{r|llllll}
 & -1.2 - 0.6z^{-1} - 0.3z^{-2} - 0.15z^{-3} - 0.075z^{-4} \\ \hline
z\ -0.5 & -1.2z \\
 & -1.2z & +0.6 \\ \cline{2-3}
 & & -0.6 \\
 & & -0.6 & +0.3z^{-1} \\ \cline{3-4}
 & & & -0.3z^{-1} \\
 & & & -0.3z^{-1} & +0.15z^{-2} \\ \cline{4-5}
 & & & & -0.15z^{-2} \\
 & & & & -0.15z^{-2} & +0.075z^{-3} \\ \cline{5-6}
 & & & & & -0.075z^{-3}
\end{array}$$

The anti-causal part of the transform is

$$X_L(z) = \frac{\frac{4}{45}\,z}{z+\frac{3}{4}} + \frac{\frac{10}{9}\,z}{z-\frac{3}{2}} = \frac{1.2\,z^2 + 0.7\,z}{z^2 - 0.75\,z - 1.125}$$

Long division:

$$\begin{array}{r|llllll}
 & -0.6222z - 0.6519z^2 - 0.1185z^3 - 0.5004z^4 + 0.2283z^5 \\ \hline
-1.125\ -0.75z\ +z^2 & 0.7z & +1.2z^2 \\
 & 0.7z & +0.4667z^2 & -0.6222z^3 \\ \cline{2-4}
 & & 0.7333z^2 & +0.6222z^3 \\
 & & 0.7333z^2 & +0.4889z^3 & -0.6519z^4 \\ \cline{3-5}
 & & & 0.1333z^3 & +0.6519z^4 \\
 & & & 0.1333z^3 & +0.0889z^4 & -0.1185z^5 \\ \cline{4-6}
 & & & & 0.5630z^4 & +0.1185z^5 \\
 & & & & 0.5630z^4 & +0.3753z^5 & -0.5004z^6 \\ \cline{5-7}
 & & & & & -0.2568z^5 & +0.5004z^6
\end{array}$$

The inverse transform $x[n]$ is

$$x[n] = \{\ldots,\ 0.2283,\ -0.5004,\ -0.1185,\ -0.6519,\ -0.6222,\ \underset{\substack{\uparrow \\ n=0}}{-1.2},\ -0.6,\ -0.3,\ -0.15,\ -0.075,\ -0.0375,\ \ldots\}$$

8.26.

a. Take the z-transform of both sides of the difference equation:

$$Y(z) = 0.9\,z^{-1}\,Y(z) + X(z) + z^{-1}\,X(z)$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-0.9\,z^{-1}} = \frac{z-1}{z-0.9}$$

and its partial fraction form is

$$H(z) = \frac{10}{9} - \frac{\frac{1}{9}\,z}{z-0.9}$$

The corresponding impulse response is

$$h[n] = \frac{10}{9}\,\delta[n] - \frac{1}{9}\,(0.9)^n\,u[n]$$

b. Take the z-transform of both sides of the difference equation:

$$Y(z) = 1.7\,z^{-1}\,Y(z) - 0.72\,z^{-2}\,Y(z) + X(z) - 2\,z^{-1}\,X(z)$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2\,z^{-1}}{1-1.7\,z^{-1}+0.72\,z^{-2}} = \frac{z\,(z-2)}{(z-0.9)\,(z-0.8)}$$

$$\frac{H(z)}{z} = \frac{z-2}{(z-0.9)\,(z-0.8)} = \frac{k_1}{z-0.9} + \frac{k_2}{z-0.8}$$

The residues are

$$k_1 = \left.\frac{z-2}{z-0.8}\right|_{z=0.9} = -11$$

and

$$k_2 = \left.\frac{z-2}{z-0.9}\right|_{z=0.8} = 12$$

and the partial fraction form of the system function is

$$H(z) = -\frac{11\,z}{z-0.9} + \frac{12\,z}{z-0.8}$$

and the impulse response is found as

$$h[n] = -11\,(0.9)^n\,u[n] + 12\,(0.8)^n\,u[n]$$

c. Take the z-transform of both sides of the difference equation:

$$Y(z) = 1.7\,z^{-1}\,Y(z) - 0.72\,z^{-2}\,Y(z) + X(z) + z^{-1}\,X(z) + z^{-2}\,X(z)$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}+z^{-2}}{1-1.7\,z^{-1}+0.72\,z^{-2}} = \frac{z^2+z+1}{(z-0.9)\,(z-0.8)}$$

Division of $H(z)$ by z yields

$$\frac{H(z)}{z} = \frac{z^2+z+1}{z\,(z-0.9)\,(z-0.8)} = \frac{k_1}{z} + \frac{k_2}{z-0.9} + \frac{k_3}{z-0.8}$$

The residues are

$$k_1 = \left. \frac{z^2+z+1}{(z-0.9)\,(z-0.8)} \right|_{z=0} = 1.3889$$

$$k_2 = \left. \frac{z^2+z+1}{z\,(z-0.8)} \right|_{z=0.9} = 30.111$$

and

$$k_3 = \left. \frac{z^2+z+1}{z\,(z-0.9)} \right|_{z=0.8} = -30.5$$

Partial fraction form of the system function is

$$H(z) = 1.3889 + \frac{30.111\,z}{z-0.9} - \frac{30.5\,z}{z-0.8}$$

The impulse response is found as

$$h[n] = 1.3889\,\delta[n] + 30.1111\;(0.9)^n\;u[n] - 30.5\;(0.8)^n\;u[n]$$

d. Take the z-transform of both sides of the difference equation:

$$Y(z) = z^{-1}\,Y(z) - 0.11\,z^{-2}\,Y(z) - 0.07\,z^{-3}\,Y(z) + z^{-1}\,X(z)$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} + 0.11\,z^{-2} + 0.07\,z^{-3}} = \frac{z^2}{z^3 - z^2 + 0.11\,z + 0.07}$$

In factored form the system function is

$$H(z) = \frac{z^2}{(z-0.7)\,(z-0.5)\,(z+0.2)}$$

Dividing $H(z)$ by z we get

$$\frac{H(z)}{z} = \frac{z}{(z-0.7)\,(z-0.5)\,(z+0.2)}$$

The residues are

$$k_1 = \left. \frac{z}{(z-0.5)\,(z+0.2)} \right|_{z=0.7} = 2.7222$$

$$k_2 = \left. \frac{z}{(z-0.7)\,(z+0.2)} \right|_{z=0.5} = -1.7857$$

and

$$k_3 = \left. \frac{z}{(z-0.7)\,(z-0.5)} \right|_{z=-0.2} = 0.0635$$

Partial fraction form of the system function is

$$H(z) = \frac{2.7222\,z}{z-0.7} - \frac{1.7857\,z}{z-0.5} + \frac{0.0635\,z}{z+0.2}$$

The impulse response is

$$h[n] = 2.7222\,(0.7)^n\,u[n] - 1.7857\,(0.5)^n\,u[n] + 0.0635\,(-0.2)^n\,u[n]$$

8.27.

a. Begin by writing $H(z)$ using negative powers of z:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 + 5\,z^{-1} + 6\,z^{-2}}$$

The relationship between $X(z)$ and $Y(z)$ is

$$Y(z)\,\left[1 + 5\,z^{-1} + 6\,z^{-2}\right] = X(z)\,\left[z^{-1} + z^{-2}\right]$$

or, equivalently

$$Y(z) + 5\,z^{-1}\,Y(z) + 6\,z^{-2}\,Y(z) = z^{-1}\,X(z) + z^{-2}\,X(z)$$

Taking inverse transforms of both sides leads to

$$y[n] + 5\,y[n-1] + 6\,y[n-2] = x[n-1] + x[n-2]$$

b. Multiply out numerator and denominator factors, and write the result using negative powers of z to obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2\,z^{-1} + z^{-2}}{1 + \dfrac{11}{30}\,z^{-1} - \dfrac{3}{5}\,z^{-2} - \dfrac{4}{15}\,z^{-3}}$$

Using the same steps as in part (a)

$$Y(z)\,\left[1 + \frac{11}{30}\,z^{-1} - \frac{3}{5}\,z^{-2} - \frac{4}{15}\,z^{-3}\right] = X(z)\,\left[1 - 2\,z^{-1} + z^{-2}\right]$$

$$Y(z) + \frac{11}{30}\,z^{-1}\,Y(z) - \frac{3}{5}\,z^{-2}\,Y(z) - \frac{4}{15}\,z^{-3}\,Y(z) = X(z) - 2\,z^{-1}\,X(z) + z^{-2}\,X(z)$$

and

$$y[n] + \frac{11}{30}\,y[n-1] - \frac{3}{5}\,y[n-2] - \frac{4}{15}\,y[n-3] = x[n] - 2\,x[n-1] + x[n-2]$$

c. The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-3}}{1 + 1.2\,z^{-1} - 1.8\,z^{-3}}$$

which leads to

$$Y(z)\,\left[1 + 1.2\,z^{-1} - 1.8\,z^{-3}\right] = X(z)\,\left[z^{-1} + z^{-3}\right]$$

and to the corresponding difference equation

$$y[n] + 1.2\,y[n-1] - 1.8\,y[n-3] = x[n-1] + x[n-3]$$

8.28.

a. Take the z-transform of both sides of the difference equation:

$$Y(z) = -0.1\,z^{-1}\,Y(z) + 0.56\,z^{-2}\,Y(z) + X(z) - 2\,z^{-1}\,X(z)$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2\,z^{-1}}{1 + 0.1\,z^{-1} - 0.56\,z^{-2}} = \frac{z\,(z-2)}{(z+0.8)\,(z-0.7)}$$

b. Division of $H(z)$ by z yields

$$\frac{H(z)}{z} = \frac{(z-2)}{(z+0.8)\,(z-0.7)} = \frac{k_1}{z+0.8} + \frac{k_2}{z-0.7}$$

The residues are found as

$$k_1 = \left.\frac{z-2}{z-0.7}\right|_{z=-0.8} = 1.8667$$

and

$$k_2 = \left.\frac{z-2}{z+0.8}\right|_{z=0.7} = -0.8667$$

Partial fraction form of the system function is

$$H(z) = \frac{1.8667\,z}{z+0.8} - \frac{0.8667\,z}{z-0.7}\,, \qquad \text{ROC:} \quad |z| > 0.8$$

and the impulse response is found as

$$h[n] = 1.8667\,(-0.8)^n\,u[n] - 0.8667\,(0.7)^n\,u[n]$$

c. If $x[n] = u[n]$ then

$$X(z) = \frac{z}{z-1}\,, \qquad \text{ROC:} \quad |z| > 1$$

and the transform of the output signal is

$$Y(z) = H(z)\,X(z) = \frac{z^2\,(z-2)}{(z+0.8)\,(z-0.7)\,(z-1)} \qquad \text{ROC:} \quad |z| > 1$$

Partial fraction form of $Y(z)$ is

$$Y(z) = \frac{0.8296\,z}{z+0.8} + \frac{2.0222\,z}{z-0.7} - \frac{1.8519\,z}{z-1}$$

which leads to the output signal

$$y[n] = \left[0.8296\,(-0.8)^n + 2.0222\,(0.7)^n - 1.8519\right]\,u[n]$$

d. If $x[n] = u[-n]$ then

$$X(z) = \frac{-1}{z-1}\,, \qquad \text{ROC:} \quad |z| < 1$$

and the transform of the output signal is

$$Y(z) = H(z)\,X(z) = \frac{-z\,(z-2)}{(z+0.8)\,(z-0.7)\,(z-1)} \qquad \text{ROC:}\quad 0.8 < |z| < 1$$

Partial fraction form of $Y(z)$ is

$$Y(z) = \frac{1.0370\,z}{z+0.8} - \frac{2.8889\,z}{z-0.7} + \frac{1.8519\,z}{z-1}$$

which leads to the output signal

$$y[n] = 1.0370\,(-0.8)^n\,u[n] - 2.8889\,(0.7)^n\,u[n] - 1.8519\,u[-n-1]$$

8.29.

a. Take the z-transform of both sides of the difference equation:

$$Y(z) = -\frac{5}{6}\,z\,Y(z) - \frac{1}{6}\,z^2\,Y(z) + \frac{1}{6}\,z\,X(z) + \frac{1}{6}\,z^2\,X(z)$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z\,(z+1)}{(z+2)\,(z+3)}\,, \qquad \text{ROC:}\quad |z| < 2$$

b. Division of $H(z)$ by z yields

$$\frac{H(z)}{z} = \frac{(z+1)}{(z+2)\,(z+3)} = \frac{k_1}{z+2} + \frac{k_2}{z+3}$$

The residues are found as

$$k_1 = \left.\frac{z+1}{z+3}\right|_{z=-2} = -1$$

and

$$k_2 = \left.\frac{z+1}{z+2}\right|_{z=-3} = 2$$

Partial fraction form of the system function is

$$H(z) = \frac{-z}{z+2} + \frac{2\,z}{z+3}\,, \qquad \text{ROC:}\quad |z| < 2$$

and the impulse response is found as

$$h[n] = (-2)^n\,u[-n-1] - 2\,(-3)^n\,u[-n-1]$$

c. If $x[n] = u[n]$ then

$$X(z) = \frac{z}{z-1}\,, \qquad \text{ROC:}\quad |z| > 1$$

and the transform of the output signal is

$$Y(z) = H(z)\,X(z) = \frac{z^2\,(z+1)}{(z-1)\,(z+2)\,(z+3)} \qquad \text{ROC:}\quad 1 < |z| < 2$$

Partial fraction form of $Y(z)$ is

$$Y(z) = \frac{\frac{1}{6}\,z}{z-1} + \frac{-\frac{2}{3}\,z}{z+2} + \frac{\frac{3}{2}\,z}{z+3}$$

which leads to the output signal

$$y[n] = \frac{1}{6}\,u[n] + \frac{2}{3}\,(-2)^n\,u[-n-1] - \frac{3}{2}\,(-3)^n\,u[-n-1]$$

d. If $x[n] = u[-n]$ then

$$X(z) = \frac{-1}{z-1}\,, \qquad \text{ROC:}\quad |z| < 1$$

and the transform of the output signal is

$$Y(z) = H(z)\,X(z) = \frac{-z\,(z+1)}{(z-1)\,(z+2)\,(z+3)} \qquad \text{ROC:}\quad |z| < 1$$

Partial fraction form of $Y(z)$ is

$$Y(z) = \frac{-\frac{1}{6}\,z}{z-1} + \frac{-\frac{1}{3}\,z}{z+2} + \frac{\frac{1}{2}\,z}{z+3}$$

which leads to the output signal

$$y[n] = \frac{1}{6}\,u[-n-1] + \frac{1}{3}\,(-2)^n\,u[-n-1] - \frac{1}{2}\,(-3)^n\,u[-n-1]$$

8.30.

a. The transform of the causal sinusoidal signal is

$$X(z) = \frac{\sin(0.01)\,z}{z^2 - 2\cos(0.01)\,z + 1} = \frac{\sin(0.01)\,z}{\left(z - e^{j\,0.01}\right)\left(z - e^{-j\,0.01}\right)}$$

and the transform of the output signal is

$$\begin{aligned} Y(z) = H(z)\,X(z) =& \frac{0.04\sin(0.01)\,z^2}{(z-0.96)\left(z - e^{j\,0.01}\right)\left(z - e^{-j\,0.01}\right)} \\ =& \frac{0.2264\,z}{z-0.96} + \frac{0.4856\,e^{-j\,1.8061}\,z}{z - e^{j\,0.01}} + \frac{0.4856\,e^{j\,1.8061}\,z}{z - e^{-j\,0.01}} \end{aligned}$$

The output signal is

$$\begin{aligned} y[n] =& 0.2264\,(0.96)^n\,u[n] + 0.4856\,e^{-j\,1.8061}\,e^{j\,0.01n} + 0.4856\,e^{j\,1.8061}\,e^{-j\,0.01n} \\ =& 0.2264\,(0.96)^n\,u[n] + 0.9713\cos(0.01n - 1.8061)\,u[n] \end{aligned}$$

b. The system function evaluated at $\omega_0 = 0.01$ rad/s is

$$H(0.01) = H(z)\Big|_{z=e^{j\,0.01}} = 0.9445 - j\,0.2264 = 0.9713\,e^{-j\,0.2353}$$

and the steady-state response of the system is

$$
\begin{aligned}
y_{ss}[n] =& 0.9713 \sin(0.01n - 0.2353) \\
=& 0.9713 \cos(0.01n - 0.2353 - \pi/2) \\
=& 0.9713 \cos(0.01n - 1.8061)
\end{aligned}
$$

c. The difference between the two responses is the transient term

$$y_t[n] = 0.2264\ (0.96)^n\ u[n]$$

It can be shown that $y_t[140] \approx 0.0007$ which is less than 0.1 percent of the amplitude of the steady-state response.

8.31.

a. Starting with

$$
\begin{aligned}
\mathcal{Z}\left\{\cos(0.2\pi n)\ u[n]\right\} =& \frac{z\,[z-\cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)\ z + 1} \\
=& \frac{z\,(z-0.8090)}{z^2 - 1.6180\,z + 1}
\end{aligned}
$$

and using the multiplication by an exponential property of the z-transform, the system function is obtained as

$$
\begin{aligned}
\mathcal{Z}\left\{(0.8)^n \cos(0.2\pi n)\ u[n]\right\} =& \frac{\left(\frac{z}{0.8}\right)\left[\left(\frac{z}{0.8}\right) - 0.8090\right]}{\left(\frac{z}{0.8}\right)^2 - 1.6180\left(\frac{z}{0.8}\right) + 1} \\
=& \frac{z\,(z-0.6472)}{z^2 - 1.2944\,z + 0.64}
\end{aligned}
$$

b.

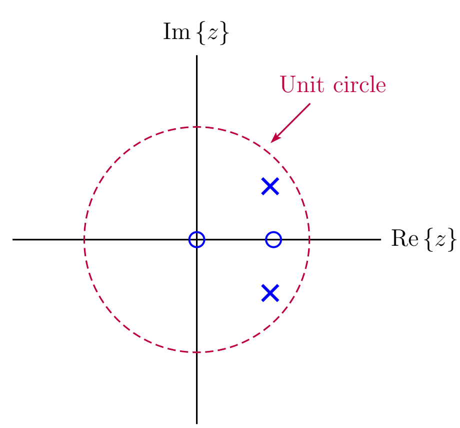
The zeros of the system function are at

$$z_1 = 0\ ,\ z_2 = 0.6472$$

and its poles are at

$$p_{1,2} = 0.6472 \pm 0.4702$$

Pole zero plot is shown. Poles are inside the unit circle. Since $h[n]$ is causal, it is also stable.



c. The difference equation for the system is

$$y[n]-1.2944\,y[n-1]+0.64\,y[n-2]=x[n]-0.5178\,x[n-1]$$

d. The z-transform of the output signal is

$$\begin{aligned}
Y(z) &= H(z)\,X(z)\\
&= \frac{z^2\,(z-0.6472)}{z^3-2.2944\,z^2+1.9344\,z-0.64}\\
&= \frac{1.0209\,z}{z-1}+\frac{\left(-0.0104-j\,0.6804\right)\,z}{z-0.6472-j\,0.4702}+\frac{\left(-0.0104+j\,0.6804\right)\,z}{z-0.6472+j\,0.4702}\\
&= \frac{1.0209\,z}{z-1}+\frac{0.6804\,e^{-j\,1.5861}\,z}{z-0.8\,e^{j\,0.6283}}+\frac{0.6804\,e^{j\,1.5861}\,z}{z-0.8\,e^{-j\,0.6283}}
\end{aligned}$$

The output signal is

$$\begin{aligned}
y[n] &= 1.0209\,u[n]+0.6804\,e^{-j\,1.5861}\left(0.8\,e^{j\,0.6283}\right)^n u[n]+0.6804\,e^{j\,1.5861}\left(0.8\,e^{j\,0.6283}\right)^n u[n]\\
&= 1.0209\,u[n]+1.3608\,(0.8)^n\cos(0.6283n-1.5861)
\end{aligned}$$

8.32.

a.

$$X(z)=\frac{z}{z-\frac{1}{2}}\,,\qquad \text{ROC:}\quad |z|>\frac{1}{2}$$

$$\begin{aligned}
Y(z) &= \frac{3\,z}{z-\frac{1}{2}}+\frac{2\,z}{z-\frac{3}{4}}\\
&= \frac{5\,z\left(z-\frac{13}{20}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{4}\right)}\,,\qquad \text{ROC:}\quad |z|>\frac{3}{4}
\end{aligned}$$

The system function is

$$H(z)=\frac{Y(z)}{X(z)}=\frac{5\left(z-\frac{13}{20}\right)}{z-\frac{3}{4}}\,,\qquad \text{ROC:}\quad |z|>\frac{3}{4}$$

The system is both causal and stable.

b.

$$X(z)=\frac{z}{z-\frac{1}{2}}\,,\qquad \text{ROC:}\quad |z|>\frac{1}{2}$$

The output signal may be written as

$$y[n]=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n+2}u[n+2]+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n+1}u[n+1]$$

and leads to the transform

$$Y(z) = \left(\frac{1}{2}\right) z^2 \left(\frac{z}{z-\frac{1}{2}}\right) + \left(\frac{1}{2}\right) z \left(\frac{z}{z-\frac{1}{2}}\right)$$

$$= \frac{\left(\frac{1}{2}\right) z^2 (z+1)}{z-\frac{1}{2}}\,, \qquad \text{ROC:} \quad \frac{1}{2} < |z| < \infty$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{1}{2}\right) z (z+1)\,, \qquad \text{ROC:} \quad |z| < \infty$$

and the impulse response is

$$h[n] = \{ 0.5\,, \underset{\substack{\uparrow \\ n=-1}}{0.5} \}$$

The system is stable but not causal.

c.

$$X(z) = \frac{z}{z-1}\,, \qquad \text{ROC:} \quad |z| > 1$$

$$Y(z) = z^{-1} \left[\frac{z}{(z-1)^2}\right] = \frac{1}{(z-1)^2}\,, \qquad \text{ROC:} \quad |z| > 1$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z(z-1)}\,, \qquad \text{ROC:} \quad |z| > 1$$

The system is causal but not stable.

d.

$$X(z) = 1.25 - \frac{0.25\,z}{z-0.8} = \frac{z-1}{z-0.8}\,, \qquad \text{ROC:} \quad |z| > 0.8$$

$$Y(z) = \frac{z}{z-0.8}\,, \qquad \text{ROC:} \quad |z| > 0.8$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-1}\,, \qquad \text{ROC:} \quad |z| > 1$$

The system is causal but not stable.

8.33.

a.

$$Y(z) = H_1(z)\, E(z)$$

$$E(z) = X(z) + H_2(z)\, Y(z)$$

Combining the two relationships we get

$$Y(z) = H_1(z)\, [X(z) + H_2(z)\, Y(z)]$$

Therefore

$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 - H_1(z)\, H_2(z)}$$

b.

$$h_1[n] = u[n] \quad \Rightarrow \quad H_1(z) = \frac{z}{z-1}$$

$$h_2[n] = K\,\delta[n-1] \quad \Rightarrow \quad H_2(z) = K\,z^{-1} = \frac{K}{z}$$

$$H(z) = \frac{H_1(z)}{1 - H_1(z)\,H_2(z)} = \frac{\dfrac{z}{z-1}}{1 - \left(\dfrac{z}{z-1}\right)\left(\dfrac{K}{z}\right)} = \frac{z}{z-1-K}$$

c. For a causal system, the ROC of the system function found in part (b) is

$$\text{ROC:} \quad |z| > |1+K|$$

For the system to be stable, the ROC must include the unit circle of the z-plane. This requires

$$|1+K| < 1 \quad \Rightarrow \quad -1 < 1+K < 1 \quad \Rightarrow \quad -2 < K < 0$$

d. With $K = 3/2$ the system function is

$$H(z) = \frac{z}{z - \frac{5}{2}}$$

The transform of the unit step response is

$$Y(z) = \left(\frac{z}{z-\frac{5}{2}}\right)\left(\frac{z}{z-1}\right) = \frac{z^2}{\left(z-\frac{5}{2}\right)(z-1)}$$

Partial fraction form of $Y(z)$ is

$$Y(z) = \frac{1.6667\,z}{z-\frac{5}{2}} - \frac{0.6667\,z}{z-1}$$

and the unit-step response is

$$y[n] = \left[-0.6667 + 1.6667\left(\frac{5}{2}\right)^n\right] u[n]$$

8.34.

Using the z-transform definition we have

$$H(z_0) = \sum_{n=-\infty}^{\infty} h[n]\,z_0^{-n}$$

and

$$H(z_0^*) = \sum_{n=-\infty}^{\infty} h[n]\,(z_0^*)^{-n}$$

Conjugating both sides of the transform relationship for $H(z_0)$ yields

$$[H(z_0)]^* = \left[\sum_{n=-\infty}^{\infty} h[n]\,z_0^{-n}\right]^* = \sum_{n=-\infty}^{\infty} (h[n])^*\,(z_0^*)^{-n}$$

Since $h[n]$ is specified to be real, $(h[n])^* = h[n]$, and therefore

$$\left[H(z_0)\right]^* = \sum_{n=-\infty}^{\infty} h[n]\,\left(z_0^*\right)^{-n} = H\left(z_0^*\right)$$

which implies that

$$\left|H(z_0)\right| = \left|H\left(z_0^*\right)\right| = H_0 \qquad \text{and} \qquad \measuredangle\, H\left(z_0^*\right) = -\measuredangle\, H(z_0) = -\Theta_0$$

8.35.

a. The input signal can be expressed as $x[n] = z_0^n$ with

$$z_0 = 0.8\, e^{j\,0.4\pi}$$

The steady-state response of the system for a complex exponential signal is

$$y[n] = H(z_0)\, x[n] = H(z_0)\, z_0^n$$

The system function can be evaluated at $z = z_0$ to yield

$$H(z_0) = H\left(0.8\, e^{j\,0.4\pi}\right) = \frac{\left(0.8\, e^{j\,0.4\pi}\right)^2 + 3\,\left(0.8\, e^{j\,0.4\pi}\right)}{\left(0.8\, e^{j\,0.4\pi}\right)^2 - 1.4\,\left(0.8\, e^{j\,0.4\pi}\right) + 0.85} = 3.8717\, e^{j\,3.0777}$$

Therefore

$$y[n] = 3.8717\,(0.8)^n\, e^{j\,(0.4\pi n + 3.0777)}$$

b. Let $z_1 = (0.9)^n\, e^{0.3\pi}$. The input signal can be expressed as

$$x[n] = \frac{1}{2}\, z_1^n + \frac{1}{2}\,\left(z_1^*\right)^n$$

The system function can be evaluated at $z = z_1$ and at $z = z_1^*$ to yield

$$H(z_1) = H\left(0.9\, e^{j\,0.3\pi}\right) = \frac{\left(0.9\, e^{j\,0.3\pi}\right)^2 + 3\,\left(0.9\, e^{j\,0.3\pi}\right)}{\left(0.9\, e^{j\,0.3\pi}\right)^2 - 1.4\,\left(0.9\, e^{j\,0.3\pi}\right) + 0.85} = 11.3347\, e^{-j\,3.0515}$$

and

$$H\left(z_1^*\right) = H\left(0.9\, e^{-j\,0.3\pi}\right) = H^*\left(0.9\, e^{j\,0.3\pi}\right) = 11.3347\, e^{j\,3.0515}$$

In the last step we recognized that $H(z)$ has only real coefficients and therefore $H\left(z_1^*\right) = H^*(z_1)$ (see Problem 8.34). The output signal is

$$\begin{aligned} y[n] &= \left(\frac{1}{2}\right) 11.3347\, e^{-j\,3.0515} \left(0.9\, e^{j\,0.3\pi}\right)^n + \left(\frac{1}{2}\right) 11.3347\, e^{j\,3.0515} \left(0.9\, e^{-j\,0.3\pi}\right)^n \\ &= 11.3347\,(0.9)^n \cos(0.3\pi n - 3.0515) \end{aligned}$$

8.36.

a.

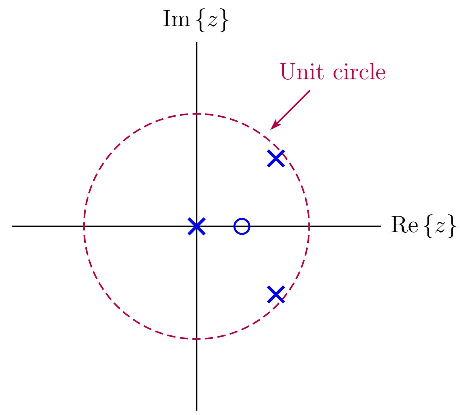
The zero of the system function is at

$$z_1 = 0.4$$

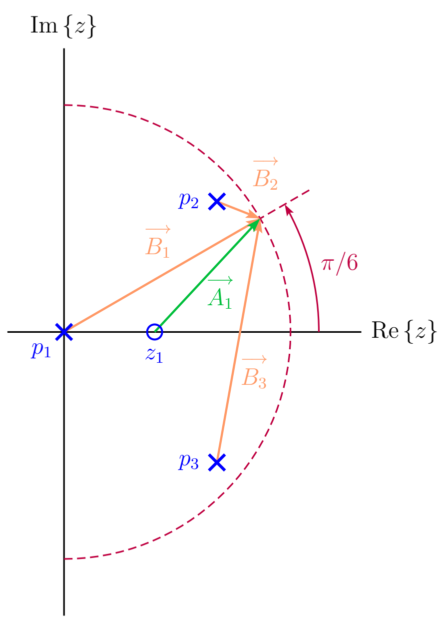
and its poles are at

$$p_1 = 0\,, \quad p_{2,3} = 0.7 \pm 0.6$$

Pole zero plot is shown.



b.



$$\begin{aligned}
\left|A_1\right| &= 0.6835\,, & \measuredangle A_1 &= 0.8206\text{ rad}\\
\left|B_1\right| &= 1\,, & \measuredangle B_1 &= 0.5236\text{ rad}\\
\left|B_2\right| &= 0.1938\,, & \measuredangle B_2 &= -0.5421\text{ rad}\\
\left|B_3\right| &= 1.1125\,, & \measuredangle B_3 &= 1.4210\text{ rad}
\end{aligned}$$

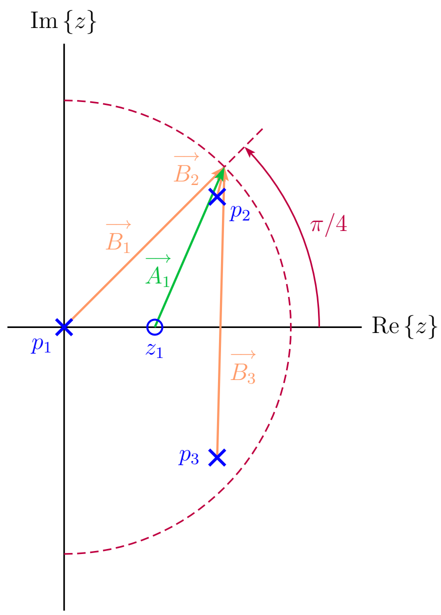
The magnitude and the phase of the system function are

$$\left|H(\pi/6)\right| = \frac{0.6835}{(1)\;(0.1938)\;(1.1125)} = 3.1701$$

and

$$\measuredangle H(\pi/6) = 0.8206-0.5236+0.5421-1.4210 = -0.5819\text{ rad}$$

c.



$$
\begin{aligned}
\left|A_1\right| &= 0.7709\,, & \measuredangle A_1 &= 1.1611 \text{ rad}\\
\left|B_1\right| &= 1\,, & \measuredangle B_1 &= 0.7854 \text{ rad}\\
\left|B_2\right| &= 0.1073\,, & \measuredangle B_2 &= 1.5045 \text{ rad}\\
\left|B_3\right| &= 1.3071\,, & \measuredangle B_3 &= 1.5654 \text{ rad}
\end{aligned}
$$

The magnitude and the phase of the system function are

$$\left|H(\pi/4)\right| = \frac{0.7709}{(1)\ (0.1073)\ (1.3071)} = 5.4944$$

and

$$\measuredangle H(\pi/4) = 1.1611-0.7854-1.5045-1.5654 = -2.6942 \text{ rad}$$

8.37.

a. The system has zeros at $z = \pm 2$ and a pole at $z = -1/2$. The system function has a second-order numerator and a first-order denominator, and therefore does not converge at $|z| \to \infty$. The ROC may be one of the following:

$$
\begin{aligned}
\left|z\right| < \frac{1}{2} \quad &: \quad \text{Neither stable nor causal}\\
\frac{1}{2} < \left|z\right| < \infty \quad &: \quad \text{Stable, but not causal}
\end{aligned}
$$

In the first case the ROC includes neither the unit circle nor infinity. In the second case it includes the unit circle but excludes infinity. Consequently, this cannot be the system function of a system that is both stable and causal.

b. The system has two zeros at $z = -1$ and poles at $z = -1/3$ and $z = 1/2$. The ROC may be one of the following:

$$
\begin{aligned}
\left|z\right| < \frac{1}{3} \quad &: \quad \text{Neither stable nor causal}\\
\frac{1}{3} < \left|z\right| < \frac{1}{2} \quad &: \quad \text{Neither stable nor causal}\\
\left|z\right| > \frac{1}{2} \quad &: \quad \text{Stable and causal}
\end{aligned}
$$

This system function could correspond to a system that is both stable and causal.

c. The system has zeros at $z = 0$ and $z = -1$. Its poles are at $z = -1/2$ and $z = -3/2$. The ROC may be one of the following:

$$\begin{aligned} |z| < \frac{1}{2} \quad &: \quad \text{Neither stable nor causal} \\ \frac{1}{2} < |z| < \frac{3}{2} \quad &: \quad \text{Stable, but not causal} \\ |z| > \frac{3}{2} \quad &: \quad \text{Causal, but not stable} \end{aligned}$$

This cannot be the system function of a system that is both stable and causal.

8.38.

a. Taking the z-transform of the difference equation leads to

$$Y(z) = 1.5\,z^{-1}\,Y(z) - 0.54\,z^{-2}\,Y(z) + X(z) + 3\,z^{-1}\,X(z)$$

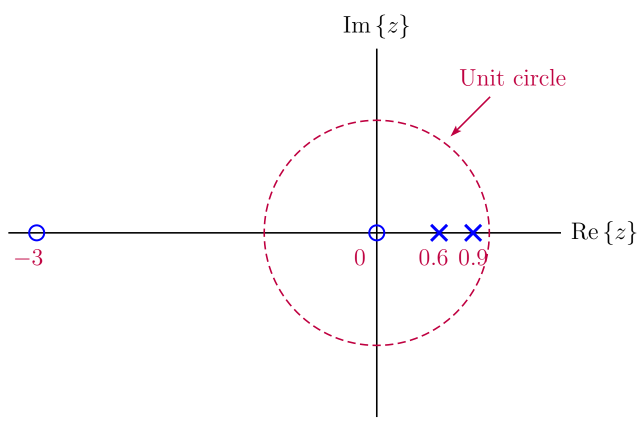
from which the system function can be obtained as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 3\,z^{-1}}{1 - 1.5\,z^{-1} + 0.54\,z^{-2}}$$

The factored form of the system function is

$$H(z) = \frac{z\,(z+3)}{(z-0.6)\,(z-0.9)}$$

and the pole-zero diagram is shown below.



Since the system is specified to be causal, the ROC of the system function is

$$|z| > 0.9$$

The ROC includes the unit circle, therefore the system is stable.

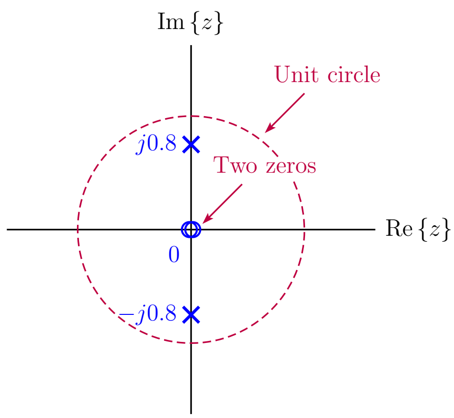
b. Taking the z-transform of the difference equation leads to

$$Y(z) = -0.64\,z^{-2}\,Y(z) + 2\,X(z)$$

from which the system function can be obtained as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 + 0.64\, z^{-2}} = \frac{2\, z^2}{\left(z - j\,0.8\right)\left(z + j\,0.8\right)}$$

The pole-zero diagram is shown below.



Since the system is specified to be causal, the ROC of the system function is

$$|z| > 0.8$$

The ROC includes the unit circle, therefore the system is stable.

c. Taking the z-transform of the difference equation leads to

$$Y(z) = 0.25\, z^{-1}\, Y(z) - 0.125\, z^{-2}\, Y(z) - 0.5\, z^{-3}\, Y(z) + X(z)$$

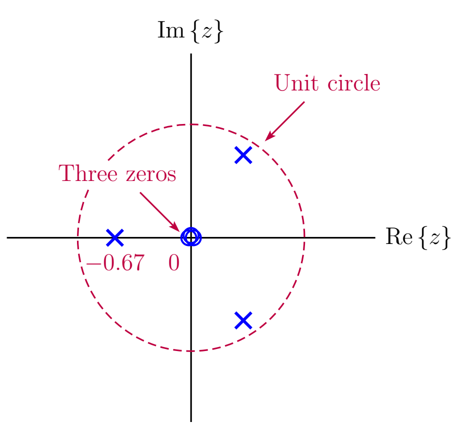
from which the system function can be obtained as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.25\, z^{-1} + 0.125\, z^{-2} + 0.5\, z^{-3}}$$

or, using non-negative powers of z

$$H(z) = \frac{z^3}{z^3 - 0.25\, z^2 + 0.125\, z + 0.5}$$

There are three zeros at $z = 0$. Poles are at $z = -0.6718$ and $z = 0.4609 \pm 0.7293$. The pole-zero diagram is shown below.



Since the system is specified to be causal, the ROC of the system function is

$$|z| > 0.8627$$

The ROC includes the unit circle, therefore the system is stable.

d. Taking the z-transform of the difference equation leads to

$$Y(z) = 0.25\,z^{-1}\,Y(z) - 0.5\,z^{-2}\,Y(z) - 0.75\,z^{-3}\,Y(z) + X(z) + z^{-1}\,X(z)$$

from which the system function can be obtained as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.25\,z^{-1}+0.5\,z^{-2}+0.75\,z^{-3}}$$

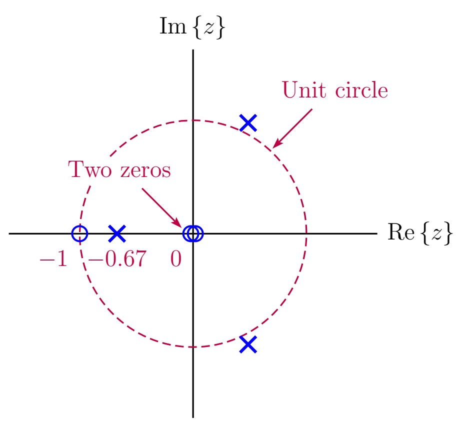
or, using non-negative powers of z

$$H(z) = \frac{z^2\,(z+1)}{z^3-0.25\,z^2+0.5\,z+0.75}$$

The factored form of the system function is

$$H(z) = \frac{z^2\,(z+1)}{(z+0.6709)\,\big(z-0.4605-j0.9518\big)\,\big(z-0.4605+j0.9518\big)}$$

and the pole-zero diagram is shown below.



Since the system is specified to be causal, the ROC of the system function is

$$|z| > 1.0573$$

The ROC does not include the unit circle, therefore the system is not stable.

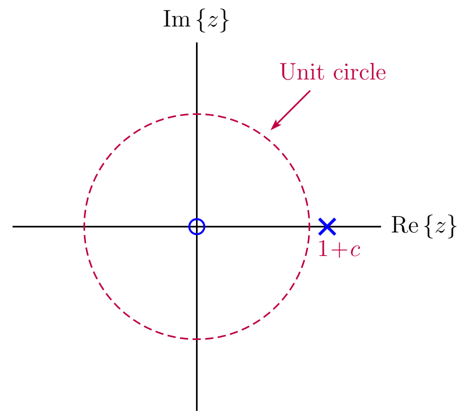
8.39.

a.

The system function is

$$H(z) = \frac{-z}{z-(1+c)}$$

It has a zero at $z_1 = 0$ and a pole at $p_1 = 1+c$. The ROC of the system function is $|z| > 1+c$ which excludes the unit circle for any positive interest rate c. Therefore, the system is unstable for any positive interest rate c.



b.

Consider the input signal

$$x[n] = -A\,\delta[n]$$

which corresponds to borrowing the amount A initially and never making a payment. The transform of the input signal is $X(z) = -A$, and the transform of the output signal is

$$Y(z) = H(z)\,X(z) = \frac{A\,z}{z-(1+c)}$$

which corresponds to the output signal

$$y[n] = A\,(1+c)^n\,u[n]$$

Alternatively, let

$$x[n] = -A\,\delta[n] + D\,u[n-1]$$

The transform of this input signal is

$$X(z) = \frac{-A\left(z - \dfrac{A+D}{A}\right)}{z-1}$$

The output transform is

$$Y(z) = H(z)\,X(z) = \frac{A\,z\left(z - \dfrac{A+D}{A}\right)}{(z-1)\,(z-1-c)}$$

which can be expressed in partial fraction form as

$$Y(z) = \frac{(D/c)\,z}{z-1} + \frac{(A-D/c)\,z}{z-1-c}$$

The output signal is

$$\left(\frac{D}{c}\right)u[n] + \left(A - \frac{D}{c}\right)(1+c)^n\,u[n]$$

If $A - D/c \neq 0$, the output signal would grow unbounded even though the input signal is bounded.

c. For the second input signal in part (b), let

$$A - \frac{D}{c} = 0 \qquad \Rightarrow \qquad D = A\,c$$

This results in the output signal $x[n] = A\,u[n]$. From a practical perspective this corresponds to paying exactly the amount of interest each month, and yields a constant monthly balance. The result found is intuitively satisfying.

8.40.

Frequency response of the allpass filter is

$$H(\Omega) = H(z)\Big|_{z=e^{j\Omega}} = \frac{e^{j\Omega} - r\,e^{j\Omega_0}}{e^{j\Omega} - (1/r)\,e^{j\Omega_0}}$$

Multiply both the numerator and the denominator with the complex conjugate of the denominator:

$$\begin{aligned}
H(z)\Big|_{z=e^{j\Omega}} &= \left(\frac{e^{j\Omega} - r\,e^{j\Omega_0}}{e^{j\Omega} - (1/r)\,e^{j\Omega_0}}\right)\left(\frac{e^{-j\Omega} - (1/r)\,e^{-j\Omega_0}}{e^{-j\Omega} - (1/r)\,e^{-j\Omega_0}}\right)\\
&= \frac{2 - (1/r)\cos(\Omega-\Omega_0) - r\cos(\Omega-\Omega_0) - j\,(1/r)\sin(\Omega-\Omega_0) + j\,r\,\sin(\Omega-\Omega_0)}{1 + 1/r^2 - (2/r)\cos(\Omega-\Omega_0)}\\
&= \frac{2r - \left(r^2+1\right)\cos(\Omega-\Omega_0)}{\left(r^2+1\right)/r - 2\cos(\Omega-\Omega_0)} + j\,\frac{\left(r^2-1\right)\sin(\Omega-\Omega_0)}{\left(r^2+1\right)/r - 2\cos(\Omega-\Omega_0)}
\end{aligned}$$

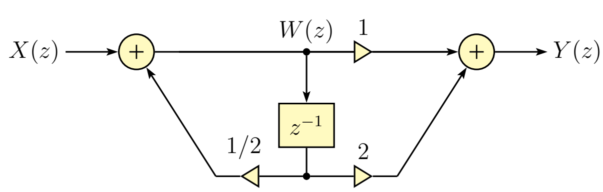
The phase characteristic is

$$\measuredangle H(\Omega) = \tan^{-1}\left[\frac{\text{Im}\{H(\Omega)\}}{\text{Re}\{H(\Omega)\}}\right] = \tan^{-1}\left[\frac{\left(r^2-1\right)\sin(\Omega-\Omega_0)}{2r - \left(r^2+1\right)\cos(\Omega-\Omega_0)}\right]$$

8.41.

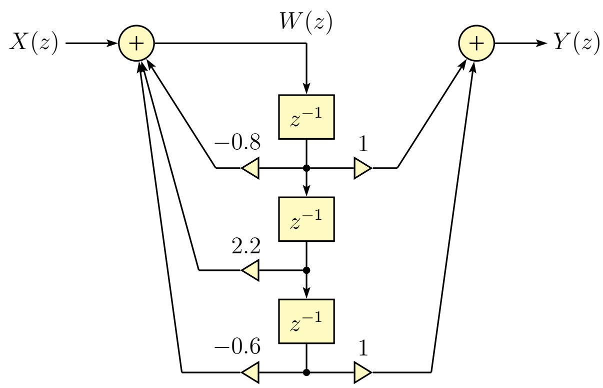
a.

$$H(z) = \frac{1 + 2\,z^{-1}}{1 - (1/2)\,z^{-1}}$$



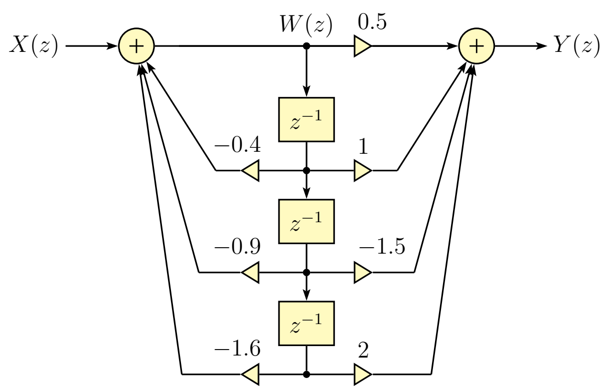
b.

$$H(z)=\frac{z^{-1}+z^{-3}}{1+0.8\,z^{-1}-2.2\,z^{-2}+0.6\,z^{-3}}$$



c.

$$\begin{aligned}H(z)&=\frac{1+2\,z^{-1}-3\,z^{-2}+4}{2+0.8\,z^{-1}+1.8\,z^{-2}+3.2\,z^{-3}}\\&=\frac{0.5+z^{-1}-1.5\,z^{-2}+2}{1+0.4\,z^{-1}+0.9\,z^{-2}+1.6\,z^{-3}}\end{aligned}$$



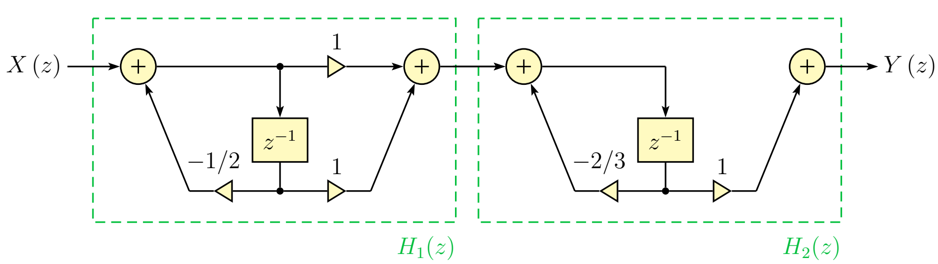
8.42.

a.

$$H(z)=H_1(z)\;H_2(z)$$

$$H_1(z)=\frac{z+1}{z+1/2}=\frac{1+z^{-1}}{1+(1/2)\;z^{-1}}$$

$$H_2(z)=\frac{1}{z+2/3}=\frac{z^{-1}}{1+(2/3)\;z^{-1}}$$

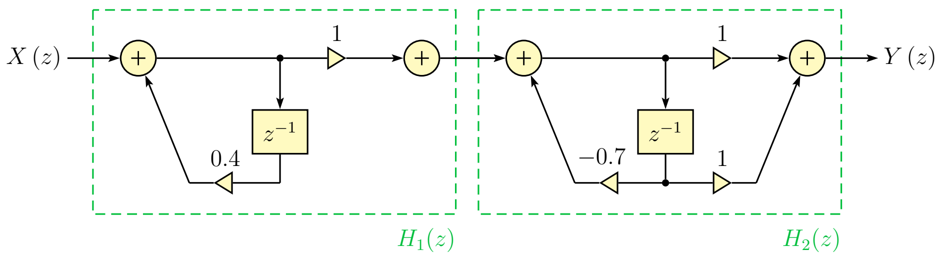


b.

$$H(z) = H_1(z)\, H_2(z)$$

$$H_1(z) = \frac{z}{z - 0.4} = \frac{1}{1 - 0.4\, z^{-1}}$$

$$H_2(z) = \frac{z + 1}{z + 0.7} = \frac{1 + z^{-1}}{1 + 0.7\, z^{-1}}$$

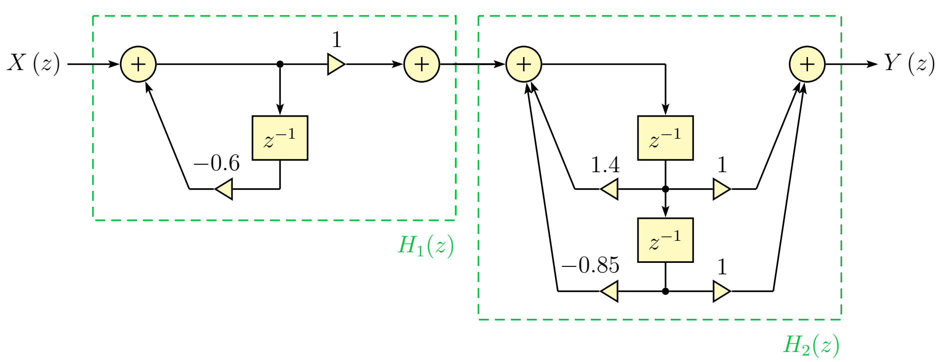


c.

$$H(z) = H_1(z)\, H_2(z)$$

$$H_1(z) = \frac{z}{z + 0.6} = \frac{1}{1 + 0.6\, z^{-1}}$$

$$H_2(z) = \frac{z + 1}{z^2 - 1.4, z + 0.85} = \frac{z^{-1} + z^{-2}}{1 - 1.4, z^{-1} + 0.85\, z^{-2}}$$

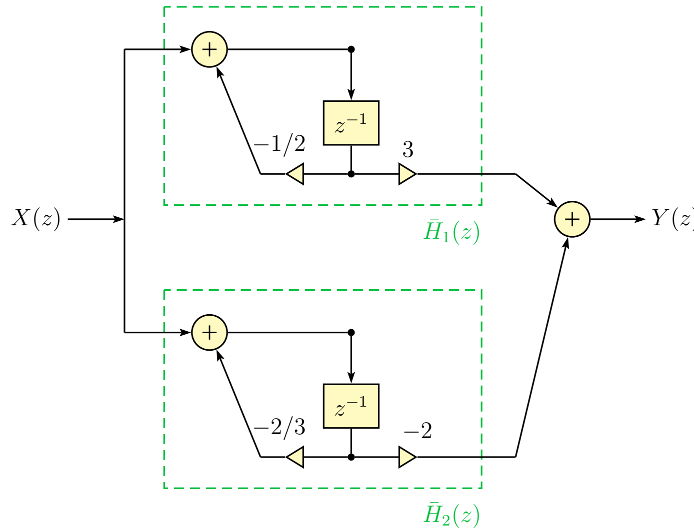


8.43.

a.

The system function can be written as

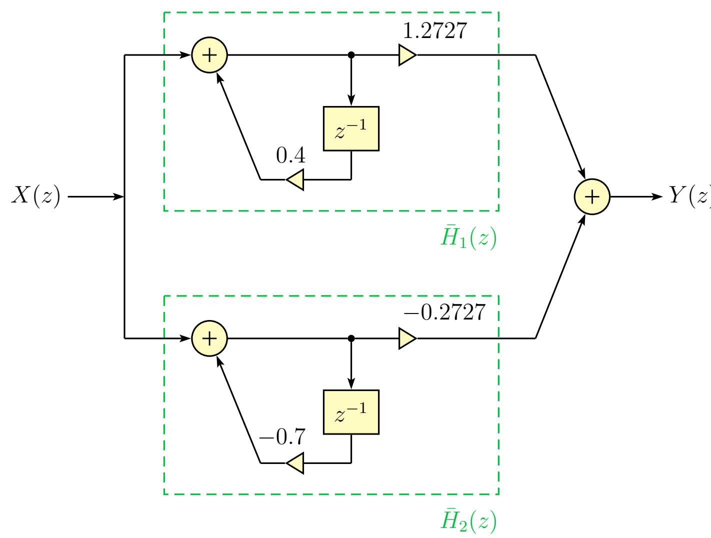
$$
\begin{aligned}
H(z) &= \bar{H}_1(z) + \bar{H}_2(z) \\
&= \frac{3}{z+1/2} + \frac{-2}{z+2/3} \\
&= \frac{3\,z^{-1}}{1+(1/2)\,z^{-1}} + \frac{-2\,z^{-1}}{1+(2/3)\,z^{-1}}
\end{aligned}
$$



b.

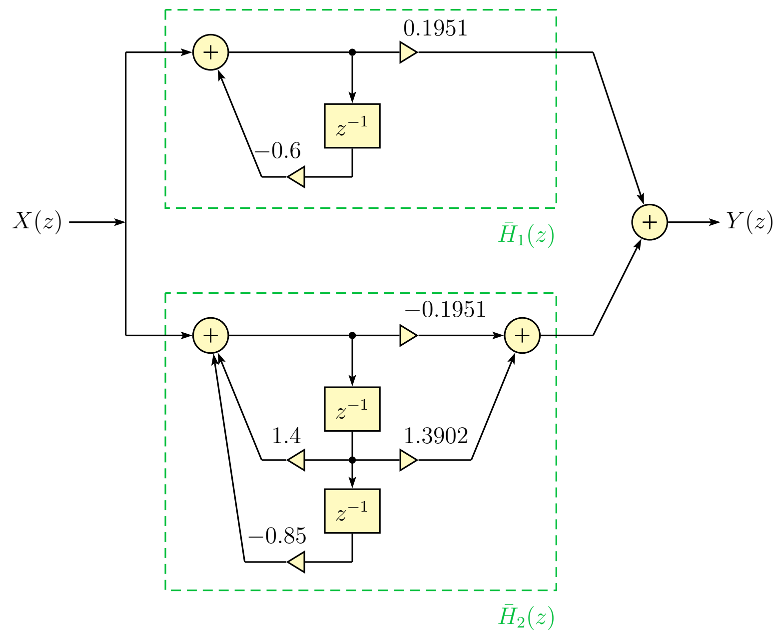
The system function can be written as

$$
\begin{aligned}
H(z) &= \bar{H}_1(z) + \bar{H}_2(z) \\
&= \frac{1.2727\,z}{z-0.4} + \frac{-0.2727\,z}{z+0.7} \\
&= \frac{1.2727}{1-0.4\,z^{-1}} + \frac{-0.2727}{1+0.7\,z^{-1}}
\end{aligned}
$$



c. Write $H(z)$ as

$$
\begin{aligned}
H(z) &= \bar{H}_1(z) + \bar{H}_2(z) \\
&= \frac{0.1951\,z}{z+0.6} + \frac{\left(-0.0976 - j\,1.0447\right) z}{z-0.7-j\,0.6} + \frac{\left(-0.0976 + j\,1.0447\right) z}{z-0.7+j\,0.6} \\
&= \frac{0.1951\,z}{z+0.6} + \frac{-0.1951\,z^2 + 1.3902\,z}{z^2 - 1.4\,z + 0.85} \\
&= \frac{0.1951}{1+0.6\,z^{-1}} + \frac{-0.1951 + 1.3902\,z^{-1}}{1 - 1.4\,z^{-1} + 0.85\,z^{-2}}
\end{aligned}
$$



8.44.

a. Using the definition of the unilateral z-transform

$$\mathcal{Z}_u\left\{x[n+1]\right\} = \sum_{n=0}^{\infty} x[n+1]\, z^{-n}$$

Let a new variable m be defined as $m = n+1$:

$$\mathcal{Z}_u\left\{x[n+1]\right\} = \sum_{m=1}^{\infty} x[m]\, z^{-m+1}$$

It is possible to start the summation at $m = 0$ by adding and subtracting the missing term:

$$\begin{aligned}\mathcal{Z}_u\left\{x[n+1]\right\} &= \sum_{m=0}^{\infty} x[m]\, z^{-m+1} - z\,x[0]\\ &= z \sum_{m=0}^{\infty} x[m]\, z^{-m} - z\,x[0]\\ &= z\,X_u(z) - z\,x[0]\end{aligned}$$

b.

$$\mathcal{Z}_u\left\{x[n+2]\right\} = \sum_{n=0}^{\infty} x[n+2]\, z^{-n}$$

Let a new variable m be defined as $m = n+2$:

$$\mathcal{Z}_u\left\{x[n+2]\right\} = \sum_{m=2}^{\infty} x[m]\, z^{-m+2}$$

It is possible to start the summation at $m = 0$ by adding and subtracting the two missing terms:

$$\begin{aligned}
\mathcal{Z}_u\left\{x[n+2]\right\} =& \sum_{m=0}^{\infty} x[m]\, z^{-m+2} - z^2\, x[0] - z\, x[1] \\
=& z^2 \sum_{m=0}^{\infty} x[m]\, z^{-m} - z^2\, x[0] - z\, x[1] \\
=& z^2\, X_u(z) - z^2\, x[0] - z\, x[1]
\end{aligned}$$

c.

$$\mathcal{Z}_u\left\{x[n+k]\right\} = \sum_{n=0}^{\infty} x[n+k]\, z^{-n}$$

Let a new variable m be defined as $m = n+k$:

$$\mathcal{Z}_u\left\{x[n+k]\right\} = \sum_{m=k}^{\infty} x[m]\, z^{-m+k}$$

Since

$$\begin{aligned}
\sum_{m=0}^{\infty} x[m]\, z^{-m+k} =& \sum_{m=0}^{k-1} x[m]\, z^{-m+k} + \sum_{m=k}^{\infty} x[m]\, z^{-m+k} \\
=& \sum_{m=0}^{k-1} x[m]\, z^{k-m} + z^k\, X_u(z)
\end{aligned}$$

$$\mathcal{Z}_u\left\{x[n+k]\right\} = \sum_{m=k}^{\infty} x[m]\, z^{-m+k} = z^k\, X_u(z) - \sum_{n=0}^{k-1} x[n]\, z^{k-n}$$

8.45.

a.

$$Y_u(z) - 1.4\left(z^{-1} Y_u(z) + y[-1]\right) + 0.85\left(z^{-2} Y_u(z) + z^{-1}\, y[-1] + y[-2]\right) = 0$$

Using initial values we obtain

$$Y_u(z) = \frac{1.05 - 4.25\, z^{-1}}{1 - 1.4\, z^{-1} + 0.85\, z^{-2}} = \frac{z\,(1.05\, z - 4.25)}{z^2 - 1.4\, z + 0.85}$$

The partial fraction form of $Y_u(z)$ is

$$Y_u(z) = \frac{\left(0.5250 + j\,2.9292\right)\, z}{z - 0.7 - j\,0.6} + \frac{\left(0.5250 - j\,2.9292\right)\, z}{z - 0.7 + j\,0.6}$$

$$\begin{aligned}
y[n] =& \left(0.5250 + j\,2.9292\right)\left(0.9220\, e^{j0.7086}\right)^n u[n] + \left(0.5250 - j\,2.9292\right)\left(0.9220\, e^{-j0.7086}\right)^n u[n] \\
=& 1.05\,(0.9220)^n \cos(0.7086 n)\; u[n] + 5.8583\,(0.9220)^n \sin(0.7086 n)\; u[n]
\end{aligned}$$

b.

$$Y_u(z) - 1.6\left(z^{-1} Y_u(z) + y[-1]\right) + 0.64\left(z^{-2} Y_u(z) + z^{-1}\, y[-1] + y[-2]\right) = 0$$

Using initial values we obtain

$$Y_u(z) = \frac{5.12 - 1.28\,z^{-1}}{1 - 1.6\,z^{-1} + 0.64\,z^{-2}} = \frac{z\,(5.12\,z - 1.28)}{z^2 - 1.6\,z + 0.64} = \frac{z\,(5.12\,z - 1.28)}{(z - 0.8)^2}$$

The partial fraction form of $Y_u(z)$ is

$$Y_u(z) = \frac{2.8160\,z}{(z - 0.8)^2} + \frac{5.12\,z}{z - 0.8}$$

$$y[n] = \left(\frac{2.8160}{0.8}\right) n\,(0.8)^n\,u[n] + 5.12\,(0.8)^n\,u[n]$$

8.46.

a. The transform of the input signal is

$$X_u(z) = \frac{z\,(z - \cos(0.2\pi))}{z^2 - 2\cos(0.2\pi)\,z + 1} = \frac{z\,(z - 0.8090)}{z^2 - 1.6180\,z + 1}$$

Using the unilateral z-transform we have

$$Y_u(z) - 2\left[z^{-1}\,Y_u(z) + 5\right] = X_u(z)$$

Solving for $Y_u(z)$ yields

$$Y_u(z) = \frac{11\,z\left(z^2 - 1.5445\,z + 0.9091\right)}{(z - 2)\left(z^2 - 1.6180\,z + 1\right)}$$

The partial fraction form of $Y_u(z)$ is

$$Y_u(z) = \frac{11.3503\,z}{z - 2} + \frac{0.3765\,e^{-j2.0548}}{z - e^{j0.6283}} + \frac{0.3765\,e^{j2.0548}}{z - e^{-j0.6283}}$$

and the output signal is

$$\begin{aligned} y[n] &= 11.3503\,(2)^n\,u[n] + 0.3765\,e^{-j2.0548}\,e^{j0.6283\,n} + 0.3765\,e^{j2.0548}\,e^{-j0.6283\,n} \\ &= 11.3503\,(2)^n\,u[n] + 0.7530\cos(0.6283\,n - 2.0548)\,u[n] \end{aligned}$$

b. The transform of the input signal is

$$X_u(z) = \frac{z}{z - 1}$$

Using the unilateral z-transform we have

$$Y_u(z) + 0.6\left[z^{-1}\,Y_u(z) - 3\right] = X_u(z) + z^{-1}\,X_u(z)$$

Solving for $Y_u(z)$ yields

$$Y_u(z) = \frac{z + 1}{z + 0.6}\,X_u(z) + \frac{1.8\,z}{z + 0.6} = \frac{2.8\,z\,(z - 0.2857)}{(z - 1)\,(z + 0.6)}$$

The partial fraction form of $Y_u(z)$ is

$$Y_u(z) = \frac{1.25\,z}{z-1} + \frac{1.55\,z}{z+0.6}$$

and the output signal is

$$y[n] = \left[1.25 + 1.55\,(-0.6)^n\right]\,u[n]$$

c. The transform of the input signal is

$$X_u(z) = \frac{z}{z-1}$$

Using the unilateral z-transform we have

$$Y_u(z) - 0.2\left[z^{-1}\,Y_u(z) - 1\right] - 0.48\left[z^{-2}\,Y_u(z) + 2 - z^{-1}\right] = X_u(z)$$

Solving for $Y_u(z)$ yields

$$\begin{aligned} Y_u(z) &= \frac{z^2}{z^2-0.2\,z-0.48}\,X_u(z) + \frac{0.76\,z^2-0.48\,z}{z^2-0.2\,z-0.48} \\ &= \frac{z\left(1.76\,z^2-1.24\,z+0.48\right)}{(z-1)\left(z^2-0.2\,z-0.48\right)} \end{aligned}$$

The partial fraction form of $Y_u(z)$ is

$$Y_u(z) = \frac{3.125\,z}{z-1} - \frac{2.1943\,z}{z-0.8} + \frac{0.8293\,z}{z+0.6}$$

and the output signal is

$$y[n] = \left[3.125 - 2.1943\,(0.8)^n + 0.8293\,(-0.6)^n\right]\,u[n]$$

8.47.

a.

$$y[n] = y[n-1] + y[n-2] \qquad \Rightarrow \qquad y[n] - y[n-1] - y[n-2] = 0\,, \quad y[1] = 1\,, \quad y[-2] = 0$$

b. Using unilateral z-transform yields

$$Y_u(z) - \left(z^{-1}\,Y_u(z) - y[-1]\right) - \left(z^{-2}\,Y_u(z) - z^{-1}\,y[-1] - y[-2]\right) = 0$$

and

$$Y_u(z) = \frac{1+z^{-1}}{1-z^{-1}-z^{-2}} = \frac{z\,(z+1)}{z^2-z-1}$$

The transform can be put into partial fraction form as follows:

$$Y_u(z) = \frac{1.17802\,z}{z-1.618034} - \frac{0.17082\,z}{z+0.618034}$$

The sequence can be found as the inverse transform.

$$y[n] = 1.17082\,(1.618034)^n - 0.17082\,(-0.618034)^n$$

c. Using the result found in part (b), $y[n+1]$ is

$$y[n+1] = 1.17082\,(1.618034)^{n+1} - 0.17082\,(-0.618034)^{n+1}$$

and the ratio of two consecutive numbers in the sequence is

$$\frac{y[n+1]}{y[n]} = \frac{1.17082\,(1.618034)^{n+1} - 0.17082\,(-0.618034)^{n+1}}{1.17082\,(1.618034)^{n} - 0.17082\,(-0.618034)^{n}}$$

For large n, the terms $(-0.618034)^{n+1}$ and $(-0.618034)^{n}$ approach zero. Therefore we have

$$\varphi = \lim_{n\to\infty} \frac{y[n+1]}{y[n]} = \lim_{n\to\infty} \left[\frac{1.17082\,(1.618034)^{n+1}}{1.17082\,(1.618034)^{n}}\right] = 1.618034$$

8.48.

a. Begin with

$$\mathcal{Z}\left\{u[n] - u[n-8]\right\} = \frac{z^8 - 1}{z^7\,(z-1)}$$

Using the multiplication by an exponential signal property of the z-transform

$$X(z) = \mathcal{Z}\left\{(0.8)^n\,(u[n] - u[n-8])\right\} = \frac{\left(\dfrac{z}{0.8}\right)^8 - 1}{\left(\dfrac{z}{0.8}\right)^7 \left(\dfrac{z}{0.8} - 1\right)}$$

The zeros of the transform $X(z)$ are found by solving

$$\left(\frac{z}{0.8}\right)^8 = 1\,e^{j2\pi k}\,, \qquad k = 0, \ldots, 7$$

The zeros are at

$$z_k = 1\,e^{j2\pi k/8};, \qquad k = 1, \ldots, 7$$

The zero for $k = 1$ is canceled by the denominator factor $(z-1)$ so that there is neither a zero nor a pole at $z = 1$. The transform has seven poles at $z = 0$.

b.

```
% Anonymous function for the transform
X = @(z) ((z/0.8).^8-1)./(((z/0.8).^7).*((z/0.8)-1));
% Create a grid of z values
[zr,zi] = meshgrid([-1.5:0.05:1.5],[-1.5:0.05:1.5]);
z = (zr+eps)+j*(zi+eps);  % Avoid division by 0
% Evaluate the magnitude |X(z)|
Xmag = abs(X(z));
% Clip the peak
Xmag = Xmag.*(Xmag<=10)+10.*(Xmag>10);
```

c.

```
clf;
shading interp;
colormap copper;
ml = mesh(zr,zi,Xmag);
xlabel('Real(z)');
ylabel('Imag(z)');
zlabel('|X(z)|');
```

d.

```
r = 1;
omg = [0:0.005:1]*2*pi+eps;
tr = r*exp(j*omg);  % Circular trajectory
shading interp;
colormap copper;
ml = mesh(zr,zi,Xmag);
hold on;
p2 = plot3(...
    real(tr),imag(tr),zeros(size(tr)),...  % Draw unit circle
    real(tr),imag(tr),abs(X(tr)),...       % Draw |X(z)| evaluated on the unit circle
    [-1.5,1.5],[0,0],[0,0],...
    [0,0],[-1.5,1.5],[0,0]);
set(ml,'FaceAlpha',0.4,'EdgeAlpha',0.6);
set(p2(1),'Color',[0.8,0,0],'Linewidth',1.5);
set(p2(2),'Color',[0,0,1],'Linewidth',1.5);
set(p2(3),'Color',[0,0,0]);
set(p2(4),'Color',[0,0,0]);
hold off;
xlabel('Real(z)');
ylabel('Imag(z)');
zlabel('|X(z)|');
```

8.49.

a. In the script in part (b), modify line 2 as follows:

```
2 X = @(z) ((z/0.6).^8-1)./(((z/0.6).^7).*((z/0.6)-1));
```

b. In the script in part (b), modify line 2 as follows:

```
2 X = @(z) ((z/0.4).^8-1)./(((z/0.4).^7).*((z/0.4)-1));
```

8.50.

a. A system object can be obtained as follows:

```
>> sf = zpk([0],[0.96],0.04,1)

Zero/pole/gain:
```

```
  0.04 z
 --------
 (z-0.96)

 Sampling time: 1
```

b. The output signal is computed and graphed with the following statements:

```
>> n = [0:49];
>> x = sin(0.01*n);
>> y = lsim(sf,x,n);
>> stem(n,y)
```

8.51.

a. The script below computes and displays the impulse response by iterating through the difference equation:

```
% Set initial conditions to zero
ynm1 = 0;  % y[-1]
ynm2 = 0;  % y[-2]
xn = 1;    % x[0] = 1 since x[n] is a unit impulse
xnm1 = 0;  % x[-1] = 0 since x[n] is a unit impulse
out = [];  % Empty vector to start
for n=0:10,
  yn = 1.2944*ynm1-0.64*ynm2+xn-0.6472*xnm1;
  out = [out,yn];
  ynm2 = ynm1;
  ynm1 = yn;
  xnm1 = xn;
  xn = 0;  % x[n] = 0 for n > 0
end;
% Display output signal
n = [0:10];
[n',out']
```

For comparison, the specified impulse response can be computed from its analytical expression and displayed for $n=0,\ldots,10$ using the following code:

```
% Analytical result for the impulse response
y = (0.8).^n.*cos(0.2*pi*n);
[n',y']
```

b. The script below computes and displays the unit-step response by iterating through the difference equation:

```
% Set initial conditions to zero
ynm1 = 0;  % y[-1]
ynm2 = 0;  % y[-2]
```

```
xn = 1;     % x[0] = 1 since x[n] is a unit step
xnm1 = 0;   % x[-1] = 0 since x[n] is a unit step
out = [];   % Empty vector to start
for n=0:10,
  yn = 1.2944*ynm1-0.64*ynm2+xn-0.6472*xnm1;
  out = [out,yn];
  ynm2 = ynm1;
  ynm1 = yn;
  xnm1 = xn;
end;
% Display output signal
n = [0:10];
[n',out']
```

For comparison, the unit-step response can be computed from the analytical result found in Problem 8.31 and displayed for $n = 0, \ldots, 10$ using the following code:

```
% Compute and display the analytical solution
y = 1.0209+1.3608*(0.8).^n.*cos(0.6283*n-1.5861);
[n',y']
```

c. The script below creates a system object and uses it to compute and display the impulse response:

```
sys = tf([1,-0.6472,0],[1,-1.2944,0.64],-1)
n = [0:10];
h = impulse(sys,n);
% Display the impulse response
[n',h]
```

For the unit-step response, use the following:

```
y = step(sys,n);
% Display the unit-step response
[n',y]
```

8.52.

a. The code listed below can be used for computing and graphing the frequency response $H(\Omega)$:

```
% Anonymous function for H(z)
H = @(z) (z.*z+3*z)./(z.*z-1.4*z+0.85);
% Evaluate H(z) at z=exp(j*Omega)
Omg = [-1:0.001:1]*pi;
sf = H(exp(j*Omg));
% Graph the magnitude
sfmag = abs(sf);
plot(Omg,sfmag);
title('Magnitude of the system function');
xlabel('\Omega (rad)');
```

```
ylabel('Magnitude');
grid;
% Graph the phase
sfmag = angle(sf);
plot(Omg, sfmag);
title('Phase of the system function');
xlabel('\Omega (rad)');
ylabel('Phase (rad)');
grid;
```

The code below graphs the magnitude and the phase of $H(z)$ evaluated for $z = 0.8\,e^{j\Omega}$. Critical points $|H(z_0)|$ and $\measuredangle\, H(z_0)$ are marked on magnitude and phase graphs.

```
% Evaluate H(z) at z=0.8*exp(j*Omega)
sf1 = H(0.8*exp(j*Omg));
% Graph the magnitude and the phase
sf1mag = abs(sf1);
sf1phs = angle(sf1);
z0 = 0.8*exp(j*0.4*pi);
clf;
subplot(2,1,1);
plot(Omg, sf1mag,0.4*pi,abs(H(z0)),'r*');
title('H(z) evaluated at z=0.8*exp(j*Omega)');
ylabel('Magnitude');
grid;
subplot(2,1,2);
plot(Omg, sf1phs,0.4*pi,angle(H(z0)),'r*');
xlabel('\Omega (rad)');
ylabel('Phase (rad)');
grid;
```

The code below graphs the magnitude and the phase of $H(z)$ evaluated for $z = 0.9\,e^{j\Omega}$. Critical points $|H(z_1)|$, $\left|H\left(z_1^*\right)\right|$, $\measuredangle\, H(z_1)$ $\measuredangle\, H\left(z_1^*\right)$ are marked on magnitude and phase graphs.

```
% Evaluate H(z) at z=0.9*exp(j*Omega)
sf1 = H(0.9*exp(j*Omg));
% Graph the magnitude and the phase
sf1mag = abs(sf1);
sf1phs = angle(sf1);
z1 = 0.9*exp(j*0.3*pi);
clf;
subplot(2,1,1);
plot(Omg, sf1mag,[-0.3*pi,0.3*pi],[abs(H(conj(z1))),abs(H(z1))],'r*');
title('H(z) evaluated at z=0.9*exp(j*Omega)');
ylabel('Magnitude');
grid;
subplot(2,1,2);
plot(Omg, sf1phs,[-0.3*pi,0.3*pi],[angle(H(conj(z1))),angle(H(z1))],'r*');
xlabel('\Omega (rad)');
ylabel('Phase (rad)');
grid;
```

b. The steady state response for the input signal $x[n]=(0.8)^n\, e^{j0.4\pi n}$ is computed and graphed as follows:

```
% Steady-state response found in part (a) of Problem 8.35:
n = [0:19];
y = H(z0)*(0.8).^n.*exp(j*0.4*pi*n);
clf;
subplot(2,1,1);
stem(n,real(y));
title('y[n]');
ylabel('Real part');
grid;
subplot(2,1,2);
stem(n,imag(y));
xlabel('Sample index n');
ylabel('Imag. part');
grid;
```

The steady state response for the input signal $x[n]=(0.9)^n \cos(0.3\pi n)$ is computed and graphed as follows:

```
% Steady-state response found in part (b) of Problem 8.35:
n = [0:19];
y = 0.5*H(z1)*(0.9*exp(j*0.3*pi)).^n + 0.5*H(conj(z1))*(0.9*exp(-j*0.3*pi)).^n;
clf;
stem(n,real(y));  % Take real part as a precaution
title('y[n]');
xlabel('Sample index n');
grid;
```

8.53.

```
% Anonymous function for allpass H(z) with specified r and theta
H = @(r,theta,z) (z-r*exp(j*theta))./(z-1/r*exp(j*theta));
% Create vector z=exp(j*Omega)
Omg = [-1:0.002:1]*pi;
z = exp(j*Omg);
% Evaluate system function for r=0.2,0.4,0.6,0.8
H1 = H(0.2,0,z);
H2 = H(0.4,0,z);
H3 = H(0.6,0,z);
H4 = H(0.8,0,z);
plot(Omg,angle(H1),Omg,angle(H2),Omg,angle(H3),Omg,angle(H4));
legend('r=0.2','r=0.4','r=0.6','r=0.8');
title('Phase of first-order allpass system');
xlabel('\Omega (rad)');
ylabel('Phase (rad)');
grid;
```

8.54.

a.

```
% Anonymous function for allpass H(z) with specified r and theta
H = @(r,theta,z) (z-r*exp(j*theta))./(z-1/r*exp(j*theta));
% Create vector z=exp(j*Omega)
Omg = [-1:0.002:1]*pi;
z = exp(j*Omg);
% Evaluate system function
H1 = H(0.4,pi/6,z).*H(0.4,-pi/6,z);
H2 = H(0.6,pi/6,z).*H(0.6,-pi/6,z);
H3 = H(0.8,pi/6,z).*H(0.8,-pi/6,z);
plot(Omg,angle(H1),Omg,angle(H2),Omg,angle(H3));
legend('r=0.4','r=0.6','r=0.8');
title('Phase of second-order allpass system, \theta=\pi/6');
xlabel('\Omega (rad)');
ylabel('Phase (rad)');
grid;
```

b.

```
% Evaluate system function
H1 = H(0.8,pi/6,z).*H(0.8,-pi/6,z);
H2 = H(0.8,pi/4,z).*H(0.8,-pi/4,z);
H3 = H(0.8,pi/3,z).*H(0.8,-pi/3,z);
plot(Omg,angle(H1),Omg,angle(H2),Omg,angle(H3));
legend('\theta=\pi/6','\theta=pi/4','\theta=pi/3');
title('Phase of second-order allpass system, r=0.8');
xlabel('\Omega (rad)');
ylabel('Phase (rad)');
grid;
```

8.55.

a. The first difference equation can be solved iteratively with the following:

```
ynm1 = 5;  % Placeholder for y[n-1], initially set to y[-1]
ynm2 = 7;  % Placeholder for y[n-2], initially set to y[-2]
y = [];    % Empty array (output stream)
for n=0:10,
  yn = 1.4*ynm1-0.85*ynm2;  % y[n] = 1.4y[n-1]-0.85y[n-2]
  y = [y,yn];               % Append to output stream
  ynm2 = ynm1;              % Update y[n-2] for next index
  ynm1 = yn;                % Update y[n-1] for next index
end;
[[0:10]',y']                % Tabulate solution
```

For the second difference equation, use

```
ynm1 = 2;    % Placeholder for y[n-1], initially set to y[-1]
ynm2 = -3;   % Placeholder for y[n-2], initially set to y[-2]
y = [];      % Empty array (output stream)
for n=0:10,
  yn = 1.6*ynm1-0.64*ynm2;  % y[n] = 1.6y[n-1]-0.64y[n-2]
  y = [y,yn];               % Append to output stream
  ynm2 = ynm1;              % Update y[n-2] for next index
  ynm1 = yn;                % Update y[n-1] for next index
end;
[[0:10]',y']                % Tabulate solution
```

b. The first difference equation can be solved symbolically and then evaluated numerically with the following:

```
syms z n Yz
Y1 = 5+z^(-1)*Yz;              % z-transform of y[n-1]
Y2 = 7+5*z^(-1)+z^(-2)*Yz;     % z-transform of y[n-2]
Yz = solve(Yz-1.4*Y1+0.85*Y2,Yz)    % Solve for Y(z)
yn = iztrans(Yz)                    % Inverse z-transform of Y(z)
y = eval(subs(yn,'n',[0:10]))       % Evaluate y[n] for n=0,..,10
```

For the second difference equation, use

```
syms z n Yz
Y1 = 2+z^(-1)*Yz;              % z-transform of y[n-1]
Y2 = -3+2*z^(-1)+z^(-2)*Yz;    % z-transform of y[n-2]
Yz = solve(Yz-1.6*Y1+0.64*Y2,Yz)    % Solve for Y(z)
yn = iztrans(Yz)                    % Inverse z-transform of Y(z)
y = eval(subs(yn,'n',[0:10]))       % Evaluate y[n] for n=0,..,10
```

Chapter 9

State-Space Analysis of Systems

9.1.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 2 & 2 & -1 \\ -3 & 0 & 2 \\ 1 & 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{r}(t)$$

and

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{r}(t)$$

9.2.

a.

$$\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{F}\,r(t) \qquad \Rightarrow \qquad \mathbf{x}(t) = \mathbf{z}(t) + \mathbf{F}\,r(t)$$

Differentiating both sides yields

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{z}}(t) + \mathbf{F}\frac{r(t)}{dt}$$

Substituting this result into the state equation we obtain

$$\dot{\mathbf{z}}(t) + \mathbf{F}\frac{r(t)}{dt} = \mathbf{A}\left[\mathbf{z}(t) + \mathbf{F}\,r(t)\right] + \mathbf{B}\,r(t) + \mathbf{E}\frac{dr(t)}{dt}$$

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \left(\mathbf{B} + \mathbf{A}\mathbf{F}\right) r(t) + \left(\mathbf{E} - \mathbf{F}\right)\frac{dr(t)}{dt}$$

Choose $\mathbf{F} = \mathbf{E}$ to eliminate the term $dr(t)/dt$ so that

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \left(\mathbf{B} + \mathbf{A}\mathbf{E}\right) r(t)$$

b.

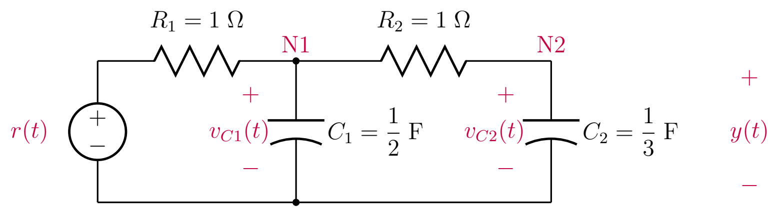
$$\mathbf{B} + \mathbf{A}\mathbf{E} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

The state-space model is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 3 \\ -2 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -5 \end{bmatrix} r(t)$$

9.3.



Begin by writing KCL at node "N1":

$$\frac{v_{C1}(t) - r(t)}{R_1} + \frac{v_{C1}(t) - v_{C2}(t)}{R_2} + C_1 \frac{dv_{C1}(t)}{dt} = 0$$

Rearranging terms we obtain

$$\frac{dv_{C1}(t)}{dt} = -\frac{R_1 + R_2}{R_1 R_2 C_1} v_{C1}(t) + \frac{1}{R_2 C_1} v_{C2}(t) + \frac{1}{R_2 C_1} r(t)$$

Similarly, writing the KCL at node "N2" yields

$$\frac{v_{C2}(t) - v_{C1}(t)}{R_2} + C_2 \frac{dv_{C2}(t)}{dt} = 0$$

which can be rearranged to become

$$\frac{dv_{C2}(t)}{dt} = \frac{1}{R_2 C_2} v_{C1}(t) - \frac{1}{R_2 C_2} v_{C2}(t)$$

Defining the two state variables as $x_1(t) = v_{C1}(t)$ and $x_2(t) = v_{C2}(t)$ and substituting numerical values of circuit components we get

$$\frac{dx_1(t)}{dt} = -4\, x_1(t) + 2\, x_2(t) + 2\, r(t)$$
$$\frac{dx_2(t)}{dt} = 3\, x_1(t) - 3\, x_2(t)$$

In matrix form the state equation is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} r(t)$$

Since $y(t) = x_2(t) = v_{C2}(t)$ the output equation is

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t)$$

9.4.

The inductor and the capacitor have the same voltage, therefore

$$L \frac{di_L(t)}{dt} = v_C(t) \quad \Rightarrow \quad \frac{di_L(t)}{dt} = \frac{1}{L} v_C(t)$$

Next step is to write the KCL to obtain

$$\frac{r(t) - v_C(t)}{R} = i_L(t) + C\,\frac{d v_C(t)}{dt}$$

which can be rearranged to yield

$$\frac{d v_C(t)}{dt} = -\frac{1}{C}\,i_L(t) - \frac{1}{RC}\,v_C(t) + \frac{1}{RC}\,r(t)$$

Defining the two state variables as $x_1(t) = i_L(t)$ and $x_2(t) = v_C(t)$ and substituting numerical values of circuit components we get

$$\begin{aligned}
\frac{d x_1(t)}{dt} &= 1.2\,x_2(t) \\
\frac{d x_2(t)}{dt} &= -5\,x_1(t) - 5\,x_2(t) + 5\,r(t)
\end{aligned}$$

In matrix form the state equation is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1.2 \\ -5 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 5 \end{bmatrix} r(t)$$

Since $y(t) = x_2(t) = v_C(t)$ the output equation is

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t)$$

9.5.

a. State variables can be defined as

$$\begin{aligned}
x_1(t) &= y(t) \\
x_2(t) &= \frac{d y(t)}{dt} \quad \Longrightarrow \quad \frac{d x_1(t)}{dt} = x_2(t)
\end{aligned}$$

Recognizing that

$$\frac{d^2 y(t)}{dt^2} = \frac{d x_2(t)}{dt}$$

the differential equation can be written as

$$\frac{d x_2(t)}{dt} = -3\,x_2(t) - 2\,x_1(t) + 2\,r(t)$$

In matrix form, the state-space model is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

b. State variables can be defined as

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt} \quad \Longrightarrow \quad \frac{dx_1(t)}{dt} = x_2(t)$$

Recognizing that

$$\frac{d^2y(t)}{dt^2} = \frac{dx_2(t)}{dt}$$

the differential equation can be written as

$$\frac{dx_2(t)}{dt} = -4\,x_2(t) - 3\,x_1(t) + 3\,r(t)$$

In matrix form, the state-space model is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

c. State variables can be defined as

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt} \quad \Longrightarrow \quad \frac{dx_1(t)}{dt} = x_2(t)$$

Recognizing that

$$\frac{d^2y(t)}{dt^2} = \frac{dx_2(t)}{dt}$$

the differential equation can be written as

$$\frac{dx_2(t)}{dt} = -x_1(t) + r(t)$$

In matrix form, the state-space model is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

d. State variables can be defined as

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt} \quad \Longrightarrow \quad \frac{dx_1(t)}{dt} = x_2(t)$$

$$x_3(t) = \frac{d^2y(t)}{dt^2} \quad \Longrightarrow \quad \frac{dx_2(t)}{dt} = x_3(t)$$

Recognizing that

$$\frac{d^3 y(t)}{dt^3} = \frac{dx_3(t)}{dt}$$

the differential equation can be written as

$$\frac{dx_3(t)}{dt} = -6x_3(t) - 11x_2(t) - 6x_1(t) + r(t)$$

In matrix form, the state-space model is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t)$$

e. Begin by writing the differential equation as

$$\frac{d^3 y(t)}{dt^3} - 2\frac{dr(t)}{dt} = -6\frac{d^2 y(t)}{dt^2} - 11\frac{dy(t)}{dt} - 6y(t) + r(t)$$

Let

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt} \qquad \Longrightarrow \qquad \frac{dx_1(t)}{dt} = x_2(t)$$

$$x_3(t) = \frac{d^2 y(t)}{dt^2} - 2r(t) \qquad \Longrightarrow \qquad \frac{dx_2(t)}{dt} = x_3(t) + 2r(t)$$

It follows that

$$\frac{dx_3(t)}{dt} = -6\left[x_3(t) + 2r(t)\right] - 11x_2(t) - 6x_1(t) + r(t)$$

In matrix form, the state-space model is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 2 \\ -11 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t)$$

9.6.

a.

$$G(s) = \frac{Y(s)}{R(s)} = \frac{X_1(s)}{R(s)} + \frac{X_2(s)}{R(s)} = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

Using residue formulas

$$k_1 = \left.\frac{2}{s+2}\right|_{s=-1} = 2$$

and

$$k_2 = \left.\frac{2}{s+1}\right|_{s=-2} = -2$$

Therefore

$$\frac{X_1(s)}{R(s)} = \frac{2}{s+1} \quad \Rightarrow \quad \frac{dx_1(t)}{dt} = -x_1(t) + 2\,r(t)$$

and

$$\frac{X_2(s)}{R(s)} = \frac{-2}{s+2} \quad \Rightarrow \quad \frac{dx_2(t)}{dt} = -2\,x_2(t) - 2\,r(t)$$

In matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 \\ -2 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t)$$

b.

$$G(s) = \frac{Y(s)}{R(s)} = \frac{X_1(s)}{R(s)} + \frac{X_2(s)}{R(s)} = \frac{k_1}{s+2} + \frac{k_2}{s+4}$$

Using residue formulas

$$k_1 = \left.\frac{10\,s+1}{s+4}\right|_{s=-2} = -9.5$$

and

$$k_2 = \left.\frac{10\,s+1}{s+2}\right|_{s=-4} = -19.5$$

Therefore

$$\frac{X_1(s)}{R(s)} = \frac{-9.5}{s+2} \quad \Rightarrow \quad \frac{dx_1(t)}{dt} = -2\,x_1(t) - 9.5\,r(t)$$

and

$$\frac{X_2(s)}{R(s)} = \frac{-19.5}{s+4} \quad \Rightarrow \quad \frac{dx_2(t)}{dt} = -4\,x_2(t) - 19.5\,r(t)$$

In matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -9.5 \\ -19.5 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t)$$

c.

$$G(s) = \frac{Y(s)}{R(s)} = \frac{X_1(s)}{R(s)} + \frac{X_2(s)}{R(s)} + \frac{X_3(s)}{R(s)} = \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{s+3}$$

Using residue formulas

$$k_1 = \left.\frac{(s-1)\,(s+4)}{(s+2)\,(s+3)}\right|_{s=-1} = -3$$

$$k_2 = \left.\frac{(s-1)\,(s+4)}{(s+1)\,(s+3)}\right|_{s=-2} = 6$$

and

$$k_3 = \left.\frac{(s-1)\,(s+4)}{(s+1)\,(s+2)}\right|_{s=-3} = -2$$

Therefore

$$\frac{X_1(s)}{R(s)} = \frac{-3}{s+1} \quad \Rightarrow \quad \frac{dx_1(t)}{dt} = -x_1(t) - 3\,r(t)$$

$$\frac{X_2(s)}{R(s)} = \frac{6}{s+2} \quad \Rightarrow \quad \frac{dx_2(t)}{dt} = -2\,x_2(t) + 6\,r(t)$$

and

$$\frac{X_3(s)}{R(s)} = \frac{-2}{s+3} \quad \Rightarrow \quad \frac{dx_3(t)}{dt} = -3\,x_3(t) - 2\,r(t)$$

In matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathbf{x}(t)$$

9.7.

a. Taking the Laplace transform of both sides of the differential equation yields

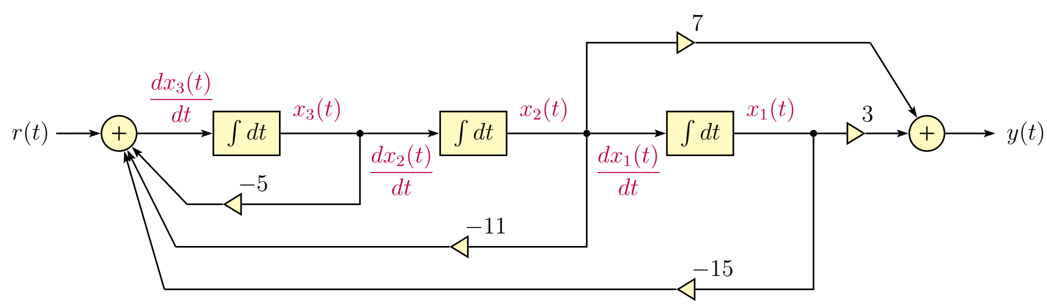
$$\left(s^3 + 5\,s^2 + 11\,s + 15\right)\, Y(s) = (3 + 7\,s)\, X(s)$$

which leads to the system function

$$G(s) = \frac{Y(s)}{R(s)} = \frac{3 + 7\,s}{s^3 + 5\,s^2 + 11\,s + 15}$$

Note: We assume zero initial conditions in taking the Laplace transform of the differential equation, consistent with the facts that the system is CTLTI.

b. A simulation diagram can be constructed for $G(s)$ as shown.



$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = x_3(t)$$

$$\frac{dx_3(t)}{dt} = -15\,x_1(t) - 11\,x_2(t) - 5\,x_3(t) + r(t)$$

$$y(t) = 3\,x_1(t) + 7\,x_2(t)$$

In matrix form we have

$$\dot{\mathbf{x}}(\mathbf{t}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -11 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{r}(t)$$

and

$$\mathbf{y}(t) = \begin{bmatrix} 3 & 7 & 0 \end{bmatrix} \mathbf{x}(t)$$

9.8.

a. In order to find the transformation matrix $\mathbf{P}$ we need to determine the eigenvalues of the state matrix $\mathbf{A}$. Eigenvalues are the solutions of

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = 0$$

$$\lambda\mathbf{I} - \mathbf{A} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda+4 & -3 \\ 1 & \lambda \end{bmatrix}$$

$$\begin{aligned} \left|\lambda\mathbf{I} - \mathbf{A}\right| =& \lambda\,(\lambda+4) - (-3)\,(1) \\ =& \lambda^2 + 4\lambda + 3 = 0 \end{aligned}$$

Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -3$. Let the corresponding eigenvectors be

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ b \end{bmatrix} \qquad \text{and} \qquad \mathbf{v_2} = \begin{bmatrix} a \\ 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{v_1} = \lambda_1\mathbf{v_1} \qquad \Rightarrow \qquad \begin{bmatrix} -4 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix} = - \begin{bmatrix} 1 \\ b \end{bmatrix}$$

$$-4 + 3b = -1 \qquad \Rightarrow \qquad b = 1$$

$$\mathbf{A}\mathbf{v_2} = \lambda_2\mathbf{v_2} \qquad \Rightarrow \qquad \begin{bmatrix} -4 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} = -3 \begin{bmatrix} a \\ 1 \end{bmatrix}$$

$$-4a + 3 = -3a \qquad \Rightarrow \qquad a = 3$$

The eigenvectors of the state matrix $\mathbf{A}$ are

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and the transformation matrix is

$$\mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

b. The similarity transformation $\mathbf{z}(t) = \mathbf{P}\mathbf{x}(t)$ transforms the state-space model from

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\,r(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + d\,r(t)$$

to

$$\dot{\mathbf{z}}(t) = \tilde{\mathbf{A}}\mathbf{z}(t) + \tilde{\mathbf{B}}\,r(t)$$

$$\mathbf{y}(t) = \tilde{\mathbf{C}}\mathbf{z}(t) + \tilde{d}\,r(t)$$

The coefficient matrices are

$$\tilde{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -0.5 & 1.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\tilde{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} -0.5 & 1.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{P} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 10 \end{bmatrix}$$

$$\tilde{d} = d = 2$$

The equivalent state-space model is

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} r(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 4 & 10 \end{bmatrix} \mathbf{z}(t) + (2)\,r(t)$$

9.9.

Let $\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t)$.

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -6 & -5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}$$

$$\tilde{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{P} = \begin{bmatrix} -5 & -7 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -7 & -5 \end{bmatrix}$$

$$\tilde{d} = d = 1$$

9.10.

a. Equations in open form are

$$\frac{dx_1(t)}{dt} = -2\,x_1(t) - 2\,x_2(t) + r(t) \tag{P.9.10.1}$$

$$\frac{dx_2(t)}{dt} = x_1(t) - 5\,x_2(t) \tag{P.9.10.2}$$

$$y(t) = 5\,x_2(t) \tag{P.9.10.3}$$

Differentiating Eqn. (P.9.10.3) and using it in conjunction with Eqn. (P.9.10.2) yields

$$\begin{aligned}\frac{dy(t)}{dt} &= 5\,\frac{dx_2}{dt}\\ &= 5\,x_1(t) - 25\,x_2(t)\end{aligned} \tag{P.9.10.4}$$

Differentiating Eqn. (P.9.10.4) one more time we obtain

$$\begin{aligned}\frac{d^2y(t)}{dt^2} &= 5\,\frac{dx_1(t)}{dt} - 25\,\frac{dx_2(t)}{dt}\\ &= 5\left[-2\,x_1(t) - 2\,x_2(t) + r(t)\right] - 25\left[x_1(t) - 5\,x_2(t)\right]\\ &= -35\,x_1(t) + 115\,x_2(t) + 5\,r(t)\end{aligned} \tag{P.9.10.5}$$

b. Solving for $x_2(t)$ from Eqn. (P.9.10.3) we get

$$x_2(t) = \left(\frac{1}{5}\right) y(t)$$

Substituting this result into Eqn. (P.9.10.4) leads to

$$x_1(t) = \left(\frac{1}{5}\right) \frac{dy(t)}{dt} + y(t)$$

These two results can be used in Eqn. (P.9.10.5) to produce the differential equation

$$\frac{d^2y(t)}{dt^2} = -7\,\frac{dy(t)}{dt} - 12\,y(t) + 5\,r(t)$$

or in scaled form

$$\frac{1}{5}\,\frac{d^2y(t)}{dt^2} = -\frac{7}{5}\,\frac{dy(t)}{dt} - \frac{12}{5}\,y(t) + r(t)$$

c. New state variables can be defined as

$$\begin{aligned}z_1(t) &= \left(\frac{1}{5}\right) y(t)\\ z_2(t) &= \left(\frac{1}{5}\right) \frac{dy(t)}{dt} \quad\Longrightarrow\quad z_2(t) = \frac{dz_1(t)}{dt}\end{aligned}$$

Recognizing that

$$\left(\frac{1}{5}\right) \frac{d^2y(t)}{dt^2} = \frac{dz_2(t)}{dt}$$

the differential equation can be written as

$$\frac{dz_2(t)}{dt} = -7\,z_2(t) - 12\,z_1(t) + r(t)$$

In matrix form, the state-space model is

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 5 & 0 \end{bmatrix} \mathbf{x}(t)$$

9.11.

The original state-space model is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 0 & 5 \end{bmatrix} \mathbf{x}(t)$$

and the state-space model in phase-variable canonical form is

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 5 & 0 \end{bmatrix} \mathbf{z}(t)$$

Let the new state vector $\mathbf{z}$ be defined through the transformation

$$\mathbf{z}(t) = \mathbf{P}\mathbf{x}(t)$$

where $\mathbf{P}$ is a transformation matrix in the form

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

The first state variable $z_1(t)$ can be written as

$$z_1(t) = p_{11}\, x_1(t) + p_{12}\, x_2(t)$$

Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be the row vectors constructed using the rows of matrix $\mathbf{P}$, that is,

$$\mathbf{P_1} = \begin{bmatrix} p_{11} & p_{12} \end{bmatrix} \qquad \text{and} \qquad \mathbf{P_2} = \begin{bmatrix} p_{21} & p_{22} \end{bmatrix}$$

so that

$$z_1(t) = \mathbf{P_1}\mathbf{x}(t) \tag{P.9.11.1}$$

and

$$z_2(t) = \mathbf{P_2}\mathbf{x}(t) \tag{P.9.11.2}$$

Differentiating Eqn. (P.9.11.1) we get

$$\begin{aligned} \frac{dz_1(t)}{dt} &= \mathbf{P_1}\dot{\mathbf{x}}(t) \\ &= \mathbf{P_1}\left[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\, r(t)\right] \\ &= \mathbf{P_1}\mathbf{A}\mathbf{x}(t) + \mathbf{P_1}\mathbf{B}\, r(t) \end{aligned}$$

In the phase-variable canonical form $dz_1(t)/dt$ should not have a $r(t)$ term. Therefore we require that $\mathbf{P_1}\mathbf{B} = \mathbf{0}$, and

$$\frac{dz_1(t)}{dt} = z_2(t) = \mathbf{P_1}\mathbf{A}\mathbf{x}(t) \tag{P.9.11.3}$$

Similarly, differentiating Eqn. (P.9.11.2) yields

$$\begin{aligned}
\frac{dz_2(t)}{dt} =& \mathbf{P_2}\,\dot{\mathbf{x}}(t) \\
=& \mathbf{P_2}\left[\mathbf{A}\mathbf{x}(t)+\mathbf{B}\,r(t)\right] \\
=& \mathbf{P_2}\,\mathbf{A}\mathbf{x}(t)+\mathbf{P_2}\,\mathbf{B}\,r(t)
\end{aligned} \tag{P.9.11.4}$$

which leads to the requirement $\mathbf{P_2}\,\mathbf{B}=1$. It should be noted that $dz_2(t)/dt$ can also be obtained by differentiating Eqn. (P.9.11.3) which results in

$$\begin{aligned}
\frac{dz_2(t)}{dt} =& \mathbf{P_1}\,\mathbf{A}\dot{\mathbf{x}}(t) \\
=& \mathbf{P_1}\,\mathbf{A}\left[\mathbf{A}\mathbf{x}(t)+\mathbf{B}\,r(t)\right] \\
=& \mathbf{P_1}\,\mathbf{A}^2\mathbf{x}(t)+\mathbf{P_1}\,\mathbf{A}\mathbf{B}\,r(t)
\end{aligned} \tag{P.9.11.5}$$

Comparing Eqns. (P.9.11.4) and (P.9.11.5) we conclude that $\mathbf{P_2}=\mathbf{P_1}\,\mathbf{A}$ and $\mathbf{P_1}\,\mathbf{A}\mathbf{B}=0$. Therefore, the transformation matrix $\mathbf{P}$ is in the form

$$\mathbf{P}=\begin{bmatrix} \mathbf{P_1} \\ \mathbf{P_1}\,\mathbf{A} \end{bmatrix}$$

Furthermore, we have the condition

$$\mathbf{P_1}\begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} \end{bmatrix}=\begin{bmatrix} 0 & 1 \end{bmatrix}$$

which leads to

$$\mathbf{P_1}=\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} \end{bmatrix}^{-1}$$

Using the original state-space model, we have

$$\mathbf{B}=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A}\mathbf{B}=\begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The vector $\mathbf{P_1}$ is found as

$$\mathbf{P_1}=\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 1 \end{bmatrix}$$

and the vector $\mathbf{P_2}$ is

$$\mathbf{P_2}=\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}=\begin{bmatrix} 1 & -5 \end{bmatrix}$$

The transformation matrix is

$$\mathbf{P}=\begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}$$

It can easily be verified that the matrix $\mathbf{P}$ converts the original state-space model to phase-variable canonical form:

$$\tilde{\mathbf{A}}=\mathbf{P}\mathbf{A}\mathbf{P}^{-1}=\begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \qquad \tilde{\mathbf{B}}=\mathbf{P}\mathbf{B}=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \tilde{\mathbf{C}}=\mathbf{C}\mathbf{P}^{-1}=\begin{bmatrix} 5 & 0 \end{bmatrix}$$

9.12.

a. State equations in open form are

$$\frac{dx_1(t)}{dt} = -2\,x_1(t) - 2\,x_2(t) + R(s)$$

$$\frac{dx_2(t)}{dt} = x_1(t) - 5\,x_2(t)$$

and the output equation is

$$y(t) = 5\,x_2(t)$$

Taking the Laplace transform of each state equation yields

$$s\,X_1(s) = -2\,X_1(s) - 2\,X_2(s) + R(s) \qquad \text{(P.9.12.1)}$$

$$s\,X_2(s) = X_1(s) - 5\,X_2(s) \qquad \text{(P.9.12.2)}$$

The Laplace transform of the output equation is

$$Y(s) = 5\,X_2(s) \qquad \text{(P.9.12.3)}$$

b. Solving for $X_1(s)$ from Eqn. (P.9.12.2) we obtain

$$X_1(s) = (s+5)\,X_2(s)$$

Substituting this result into Eqn. (P.9.12.1) leads to

$$s\,(s+5)\,X_2(s) = -2\,(s+5)\,X_2 - 2\,X_2 + R(s)$$

and therefore

$$X_2(s) = \frac{R(s)}{s^2+7\,s+12}$$

Substituting this result into the s-domain output equation given by Eqn. (P.9.12.3) yields the system function

$$G(s) = \frac{Y(s)}{R(s)} = \frac{5}{s^2+7\,s+12}$$

c. From the system function we get

$$\left(s^2+7\,s+12\right)\,Y(s) = 5\,R(s)$$

and the corresponding differential equation is

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12\,y(t) = 5\,r(t)$$

9.13.

Choosing the state variables as

$$x_1(t) = y_1(t) \qquad \text{and} \qquad x_2(t) = y_2(t) - r_1(t)$$

the state equations become

$$\frac{dx_1(t)}{dt} = -2\,x_1(t) - \left[x_2(t) + r_1(t)\right] + r_1(t) + r_2(t)$$

$$= -2\,x_1(t) - x_2(t) + r_2(t)$$

and

$$\frac{dx_2(t)}{dt} = x_1(t) - 3\,x_2(t) - 3\,r_1(t)$$

Output equations are

$$y_1(t) = x_1(t)$$

$$y_2(t) = x_2(t) + r_1(t)$$

In matrix form we have

$$\dot{\mathbf{x}}(\mathbf{t}) = \begin{bmatrix} -2 & -1 \\ 1 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} \mathbf{r}(t)$$

and

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{r}(t)$$

9.14.

a. In order to find the transformation matrix **P** we need to determine the eigenvalues of the state matrix **A**. Eigenvalues are the solutions of

$$\left|\lambda\,\mathbf{I} - \mathbf{A}\right| = 0$$

$$\lambda\,\mathbf{I} - \mathbf{A} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -3 & -6 \end{bmatrix} = \begin{bmatrix} \lambda+1 & -2 \\ 3 & \lambda+6 \end{bmatrix}$$

$$\left|\lambda\,\mathbf{I} - \mathbf{A}\right| = (\lambda+1)\,(\lambda+6) - (3)\,(-2)$$

$$= \lambda^2 + 7\,\lambda + 12 = 0$$

Eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -4$. Let the corresponding eigenvectors be

$$\mathbf{v_1} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \qquad \text{and} \qquad \mathbf{v_2} = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$$

$$\mathbf{A}\mathbf{v_1} = \lambda_1\,\mathbf{v_1} \qquad \Rightarrow \qquad \begin{bmatrix} -1 & 2 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = -3 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

$$-v_{11} + 2\,v_{12} = -3\,v_{11} \qquad \Rightarrow \qquad v_{11} = -v_{12}$$

$$\mathbf{A}\mathbf{v_2} = \lambda_2\,\mathbf{v_2} \qquad \Rightarrow \qquad \begin{bmatrix} -1 & 2 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = -4 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$$

$$-v_{21} + 2\,v_{12} = -4\,v_{21} \qquad \Rightarrow \qquad v_{21} = -(2/3)\;v_{22}$$

Arbitrarily choosing $v_{12} = 1$ and $v_{22} = 3$ the other elements of vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ can be determined from these relationships, and lead to eigenvectors

$$\mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

and the transformation matrix is

$$\mathbf{P} = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$$

The similarity transformation $\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t)$ transforms the state-space model from

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{r}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{r}(t)$$

to

$$\dot{\mathbf{z}}(t) = \tilde{\mathbf{A}}\mathbf{z}(t) + \tilde{\mathbf{B}}\mathbf{r}(t)$$

$$\mathbf{y}(t) = \tilde{\mathbf{C}}\mathbf{z}(t) + \tilde{\mathbf{D}}\mathbf{r}(t)$$

The coefficient matrices are

$$\tilde{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\tilde{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{P} = \begin{bmatrix} -1 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -7 \\ -2 & -4 \end{bmatrix}$$

$$\tilde{\mathbf{D}} = \mathbf{D} = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix}$$

The equivalent state-space model is

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \mathbf{r}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} -2 & -7 \\ -2 & -4 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} \mathbf{r}(t)$$

b. Using the state equation

$$\frac{dz_1(t)}{dt} = -3\,z_1(t) - 3\,r_1(t) - 4\,r_2(t) \quad \Rightarrow \quad s\,Z_1(s) = -3\,Z_1(s) - 3\,R_1(s) - 4\,R_2(s)$$

$$Z_1(s) = -\frac{3}{s+3}R_1(s) - \frac{4}{s+3}R_2(s)$$

$$\frac{dz_2(t)}{dt} = -4\,z_2(t) + r_1(t) + r_2(t) \quad \Rightarrow \quad s\,Z_2(s) = -4\,Z_2(s) + R_1(s) + R_2(s)$$

$$Z_2(s) = \frac{1}{s+4}R_1(s) + \frac{1}{s+4}R_2(s)$$

Using the output equation

$$y_1(t) = -2\,z_1(t) - 7\,z_2(t) + r_2(t)$$

$$\begin{aligned} Y_1(s) &= -2\,Z_1(s) - 7\,Z_2(s) + R_2(s) \\ &= -2\left[-\frac{3}{s+3}\,R_1(s) - \frac{4}{s+3}\,R_2(s)\right] - 7\left[\frac{1}{s+4}\,R_1(s) + \frac{1}{s+4}\,R_2(s)\right] + R_2(s) \\ &= \left[\frac{s^2+6\,s+15}{s^2+7\,s+12}\right] R_1(s) + \left[\frac{s+11}{s^2+7\,s+12}\right] R_2(s) \end{aligned}$$

$$y_2(t) = -2\,z_1(t) - 4\,z_2(t) + 4\,r_1(t) - r_2(t)$$

$$\begin{aligned} Y_2(s) &= -2\,Z_1(s) - 4\,Z_2(s) + 4\,R_1(s) - R_2(s) \\ &= -2\left[-\frac{3}{s+3}\,R_1(s) - \frac{4}{s+3}\,R_2(s)\right] - 4\left[\frac{1}{s+4}\,R_1(s) + \frac{1}{s+4}\,R_2(s)\right] + 4\,R_1(s) - R_2(s) \\ &= \left[\frac{4\,s^2+30\,s+60}{s^2+7\,s+12}\right] R_1(s) + \left[\frac{-s^2-3\,s+8}{s^2+7\,s+12}\right] R_2(s) \end{aligned}$$

c.

$$\mathbf{Y}(s) = \begin{bmatrix} \dfrac{s^2+6\,s+15}{s^2+7\,s+12} & \dfrac{s+11}{s^2+7\,s+12} \\ \dfrac{4\,s^2+30\,s+60}{s^2+7\,s+12} & \dfrac{-s^2-3\,s+8}{s^2+7\,s+12} \end{bmatrix} \mathbf{R}(s)$$

9.15.

The state variables were found as

$$\begin{aligned} x_1(t) &= \left(\frac{7}{6} + \frac{5}{2}\,e^{-t} - \frac{11}{2}\,e^{-2t} + \frac{11}{6}\,e^{-3t}\right) u(t) \\ x_2(t) &= \left(\frac{7}{2} + 5\,e^{-t} - \frac{11}{2}\,e^{-2t}\right) u(t) \\ x_3(t) &= \left(1 + e^{-t}\right) u(t) \end{aligned}$$

Evaluating each state variable at $t = 0$ we get

$$\begin{aligned} x_1(0) &= \frac{7}{6} + \frac{5}{2} - \frac{11}{2} + \frac{11}{6} = 0 \\ x_2(0) &= \frac{7}{2} + 5 - \frac{11}{2} = 3 \\ x_3(0) &= 1 + 1 = 2 \end{aligned}$$

matching the desired initial state vector. Derivatives of state variables are

$$\frac{dx_1(t)}{dt} = \left(-\frac{5}{2}\,e^{-t} + 11\,e^{-2t} - \frac{11}{2}\,e^{-3t}\right)\,u(t)$$

$$\frac{dx_2(t)}{dt} = \left(-5\,e^{-t} + 11\,e^{-2t}\right)\,u(t)$$

$$\frac{dx_3(t)}{dt} = -\,e^{-t}\,u(t)$$

It can be shown that

$$\frac{dx_1(t)}{dt} = -3\,x_1(t) + x_2(t)$$

$$\frac{dx_2(t)}{dt} = -2\,x_1(t) + 5\,x_2(t) + 2\,u(t)$$

$$\frac{dx_3(t)}{dt} = -\,x_3(t) + u(t)$$

9.16.

The state equations in open form are

$$\frac{dx_1(t)}{dt} = -3\,x_1(t) + 2\,x_2(t)$$

$$\frac{dx_2(t)}{dt} = -\,x_2(t) + r(t)$$

The second equation can be solved with $r(t) = u(t)$ and $x_2(0) = 3$ to yield

$$\begin{aligned} x_2(t) =& e^{-t}\,x_2(0) + \int_0^t e^{-(t-\tau)}\,u(\tau)\,d\tau \\ =& 3\,e^{-t} + e^{-t}\int_0^t e^{\tau}\,d\tau \\ =& \left(1 + 2\,e^{-t}\right)\,u(t) \end{aligned}$$

Now the first state equation can be written as

$$\begin{aligned} \frac{dx_1(t)}{dt} =& -3\,x_1(t) + 2\left(1 + 2\,e^{-t}\right)\,u(t) \\ =& -3\,x_1(t) + \left(2 + 4\,e^{-t}\right)\,u(t) \end{aligned}$$

It can be solved with the initial condition $x_1(0) = -2$ to yield

$$\begin{aligned} x_1(t) =& e^{-3t}\,x_1(0) + \int_0^t e^{-3(t-\tau)}\left(2 + 4\,e^{-\tau}\right)\,u(\tau)\,d\tau \\ =& -2\,e^{-3t} + e^{-3t}\int_0^t \left(2\,e^{3\tau} + 4\,e^{2\tau}\right)\,d\tau \\ =& \left(\frac{2}{3} + 2\,e^{-t} - \frac{14}{3}\,e^{-3t}\right)\,u(t) \end{aligned}$$

The output signal is

$$\begin{aligned} y(t) &= x_1(t) + x_2(t) + r(t) \\ &= \left(\frac{8}{3} + 4e^{-t} - \frac{14}{3}e^{-3t}\right) u(t) \end{aligned}$$

9.17.

The eigenvalues of the state matrix $\mathbf{A}$ are found as follows:

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = 0$$

$$\lambda\mathbf{I} - \mathbf{A} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda+3 & -2 \\ 0 & \lambda+1 \end{bmatrix}$$

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = (\lambda+3)\,(\lambda+1) = 0$$

Eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. Let the corresponding eigenvectors be

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ a \end{bmatrix} \qquad \text{and} \qquad \mathbf{v_2} = \begin{bmatrix} 1 \\ b \end{bmatrix}$$

$$\mathbf{A}\mathbf{v_1} = \lambda_1\mathbf{v_1} \quad \Rightarrow \quad \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = -3 \begin{bmatrix} 1 \\ a \end{bmatrix} \quad \Rightarrow \quad a = 0$$

$$\mathbf{A}\mathbf{v_2} = \lambda_2\mathbf{v_2} \quad \Rightarrow \quad \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix} = - \begin{bmatrix} 1 \\ b \end{bmatrix} \quad \Rightarrow \quad b = 1$$

Thus the transformation matrix is

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The similarity transformation $\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t)$ transforms the state-space model from

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{r}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + d\,r(t)$$

to

$$\dot{\mathbf{z}}(t) = \tilde{\mathbf{A}}\mathbf{z}(t) + \tilde{\mathbf{B}}\mathbf{r}(t)$$

$$\mathbf{y}(t) = \tilde{\mathbf{C}}\mathbf{z}(t) + \tilde{d}\mathbf{r}(t)$$

The coefficient matrices are

$$\tilde{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\tilde{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{P} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$\tilde{d} = d = 1$$

The equivalent state-space model is

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} --1 \\ 1 \end{bmatrix} r(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{z}(t) + r(t)$$

The initial conditions must also be translated to the new state-space model.

$$\mathbf{z}(0) = \mathbf{P}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

The two state equations are now decoupled, and can be solved independently of each other. With $r(t) = u(t)$ the first state equation is

$$\frac{dz_1(t)}{dt} = -3\,z_1(t) - u(t)\,, \qquad z_1(0) = -5$$

It can be solved for $z_1(t)$ as

$$\begin{aligned} z_1(t) =& e^{-3t}\,z_1(0) - \int_0^t e^{-3(t-\tau)}\,u(\tau)\,d\tau \\ =& -5\,e^{-3t} - e^{-3t}\int_0^t e^{3\tau}\,d\tau \\ =& \left(-\frac{1}{3} - \frac{14}{3}\,e^{-3t}\right) u(t) \end{aligned}$$

The second state equation is

$$\frac{dz_2(t)}{dt} = -z_2(t) + u(t)\,, \qquad z_2(0) = 3$$

and can be solved to yield

$$\begin{aligned} z_2(t) =& e^{-t}\,z_2(0) + \int_0^t e^{-(t-\tau)}\,u(\tau)\,d\tau \\ =& 3\,e^{-t} + e^{-t}\int_0^t e^{\tau}\,d\tau \\ =& \left(1 + 2\,e^{-t}\right) u(t) \end{aligned}$$

The output signal is

$$\begin{aligned} y(t) =& z_1(t) + 2\,z_2(t) + u(t) \\ =& \left(\frac{8}{3} + 4\,e^{-t} - \frac{14}{3}\,e^{-3t}\right) u(t) \end{aligned}$$

9.18.

a.

$$s\,\mathbf{I}-\mathbf{A}=s\left[\begin{array}{cc}1&0\\0&1\end{array}\right]-\left[\begin{array}{cc}-3&2\\0&-1\end{array}\right]=\left[\begin{array}{cc}s+3&-2\\0&s+1\end{array}\right]$$

Resolvent matrix:

$$\Phi(s)=\left[s\,\mathbf{I}-\mathbf{A}\right]^{-1}=\frac{1}{(s+1)\,(s+3)}\left[\begin{array}{cc}s+1&2\\0&s+3\end{array}\right]$$

State transition matrix:

$$\phi(t)=\mathscr{L}^{-1}\left\{\Phi(s)\right\}$$

The elements of the state transition matrix are:

$$\phi_{11}(t)=\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}=e^{-3t}\,u(t)$$

$$\phi_{12}(t)=\mathscr{L}^{-1}\left\{\frac{2}{(s+1)\,(s+3)}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{s+1}-\frac{1}{s+3}\right\}=\left(e^{-t}-e^{-3t}\right)\,u(t)$$

$$\phi_{21}(t)=\mathscr{L}^{-1}\left\{0\right\}=0$$

$$\phi_{11}(t)=\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}=e^{-t}\,u(t)$$

Thus, the state transition matrix is

$$\phi(t)=\left[\begin{array}{cc}e^{-3t}\,u(t)&\left(e^{-t}-e^{-3t}\right)\,u(t)\\0&e^{-t}\,u(t)\end{array}\right]$$

b.

$$\mathbf{x}(2)=\phi(2)\,\mathbf{x}(0)=\left[\begin{array}{cc}e^{-6}&\left(e^{-2}-e^{-6}\right)\\0&e^{-2}\end{array}\right]\left[\begin{array}{c}-2\\3\end{array}\right]=\left[\begin{array}{c}0.3936\\0.4060\end{array}\right]$$

c.

$$\mathbf{x}(5)=\phi(5)\,\mathbf{x}(0)=\left[\begin{array}{cc}e^{-15}&\left(e^{-5}-e^{-15}\right)\\0&e^{-5}\end{array}\right]\left[\begin{array}{c}-2\\3\end{array}\right]=\left[\begin{array}{c}0.0202\\0.0202\end{array}\right]$$

9.19.

a. Let the vector $\mathbf{X}(s)$ be defined as

$$\mathbf{X}(s)=\left[\begin{array}{c}X_1(s)\\X_2(s)\end{array}\right]$$

Taking unilateral Laplace transform of both sides of the state equation yields

$$s\mathbf{X}(s)-\mathbf{x}(0)=\mathbf{A}\mathbf{X}(s)+\mathbf{B}\,R(s) \qquad \text{(P.9.19.1)}$$

b. Rearranging the terms in Eqn. (P.9.19.1) we get

$$\left[s\,\mathbf{I}-\mathbf{A}\right]\mathbf{X}(s)=\mathbf{x}(0)+\mathbf{B}\,R(s)$$

Multiplying both sides of this result by $[s\,\mathbf{I}-\mathbf{A}]^{-1}$ leads to

$$\mathbf{X}(s)=\left[s\,\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{x}(0)+\left[s\,\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}\,R(s)$$

Recall that

$$\mathscr{L}^{-1}\left\{\left[s\,\mathbf{I}-\mathbf{A}\right]^{-1}\right\}=\phi(t)=e^{\mathbf{A}\,t}$$

Therefore

$$\mathscr{L}^{-1}\left\{\left[s\,\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{x}(0)\right\}=e^{\mathbf{A}\,t}\,\mathbf{x}(0) \tag{P.9.19.2}$$

and

$$\begin{aligned}\mathscr{L}^{-1}\left\{\left[s\,\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}\,R(s)\right\}=&\mathscr{L}^{-1}\left\{\left[s\,\mathbf{I}-\mathbf{A}\right]^{-1}\right\}*\mathscr{L}^{-1}\left\{\mathbf{B}\,R(s)\right\}\\ =&e^{\mathbf{A}\,t}*B\,r(t)\\ =&\int_0^t e^{\mathbf{A}(t-\tau)}\,B\,r(\tau)\,d\tau\end{aligned} \tag{P.9.19.3}$$

Combining Eqns. (P.9.19.2) and (P.9.19.3) we get

$$\mathbf{x}(t)=e^{\mathbf{A}\,t}\,\mathbf{x}(0)+\int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,r(\tau)\,d\tau$$

c.

$$\begin{aligned}Y(s)=&\mathbf{C}\mathbf{X}(s)+d\,R(s)\\ =&\mathbf{C}\left[s\,\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{x}(0)+\mathbf{C}\left[s\,\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}\,R(s)\end{aligned}$$

d.

$$y(t)=\mathbf{C}\,e^{\mathbf{A}\,t}\,\mathbf{x}(0)+\int_0^t \mathbf{C}\,e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,r(\tau)\,d\tau$$

9.20.

The state variables were found as

$$x_1(t)=\left(\frac{5}{12}+\frac{28}{3}\,e^{-3t}-\frac{27}{4}\,e^{-4t}\right)u(t)$$

$$x_2(t)=\left(\frac{1}{12}+\frac{14}{3}\,e^{-3t}-\frac{27}{4}\,e^{-4t}\right)u(t)$$

Evaluating each state variable at $t=0$ we get

$$x_1(0)=\frac{5}{12}+\frac{28}{3}-\frac{27}{4}=3$$

$$x_2(0)=\frac{1}{12}+\frac{14}{3}-\frac{27}{4}=-2$$

matching the desired initial state vector. Derivatives of state variables are

$$\frac{dx_1(t)}{dt} = \left(-28\,e^{-3t} + 27\,e^{-4t}\right)\,u(t)$$
$$\frac{dx_2(t)}{dt} = \left(-14\,e^{-3t} + 27\,e^{-4t}\right)\,u(t)$$

It can be shown that

$$\frac{dx_1(t)}{dt} = -2\,x_1(t) - 2\,x_2(t) + u(t)$$
$$\frac{dx_2(t)}{dt} = x_1(t) - 5\,x_2(t)$$

9.21.

$$G(s) = \frac{Y(s)}{R(s)} = \mathbf{C}\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B} + d$$

$$s\mathbf{I}-\mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 26 & 0 \end{bmatrix} = \begin{bmatrix} s+2 & 1 \\ -26 & s \end{bmatrix}$$

$$\left[s\mathbf{I}-\mathbf{A}\right]^{-1} = \frac{1}{s^2+2s+26}\begin{bmatrix} s & -1 \\ 26 & s+2 \end{bmatrix}$$

The system function is

$$G(s) = \frac{1}{s^2+2s+26}\begin{bmatrix} -2 & 0 \end{bmatrix}\begin{bmatrix} s & -1 \\ 26 & s+2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1$$
$$= \frac{s^2+26}{s^2+2s+26}$$

9.22.

a. Expressing $y[n]$ as a function of the other terms leads to

$$y[n] = 0.9\,y[n-1] + r[n]$$

Define the state variable $x_1[n]$ as

$$x_1[n] = y[n-1]$$

so that the difference equation can be written as

$$x_1[n+1] = 0.9\,x[n] + r[n]$$

In matrix form, the state-space model is

$$\mathbf{x}[n+1] = \begin{bmatrix} 0.9 \end{bmatrix}\mathbf{x}[n] + \begin{bmatrix} 1 \end{bmatrix}r[n]$$

$$y[n] = \begin{bmatrix} 0.9 \end{bmatrix}\mathbf{x}[n] + r[n]$$

b. Expressing $y[n]$ as a function of the other terms leads to

$$y[n] = 1.7\,y[n-1] - 0.72\,y[n-2] + 3\,r[n]$$

State variables can be defined as

$$\begin{aligned} x_1[n] &= y[n-2] \\ x_2[n] &= y[n-1] \quad \Longrightarrow \quad x_1[n+1] = x_2[n] \end{aligned}$$

Recognizing that

$$y[n] = x_2[n+1]$$

the difference equation can be written as

$$x_2[n+1] = 1.7\,x_2[n] - 0.72\,x_1[n] + 3\,r[n]$$

In matrix form, the state-space model is

$$\mathbf{x}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 0 \\ 3 \end{bmatrix} r[n]$$

$$y[n] = \begin{bmatrix} -0.72 & 1.7 \end{bmatrix} \mathbf{x}[n] + 3\,r[n]$$

c. Expressing $y[n]$ as a function of the other terms leads to

$$y[n] = y[n-1] - 0.11\,y[n-2] - 0.07\,y[n-3] + r[n]$$

State variables can be defined as

$$\begin{aligned} x_1[n] &= y[n-3] \\ x_2[n] &= y[n-2] \quad \Longrightarrow \quad x_1[n+1] = x_2[n] \\ x_3[n] &= y[n-1] \quad \Longrightarrow \quad x_2[n+1] = x_3[n] \end{aligned}$$

Recognizing that

$$y[n] = x_3[n+1]$$

the difference equation can be written as

$$x_3[n+1] = x_3[n] - 0.11\,x_2[n] - 0.07\,x_1[n] + r[n]$$

In matrix form, the state-space model is

$$\mathbf{x}[n+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.07 & -0.11 & 1 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r[n]$$

$$y[n] = \begin{bmatrix} -0.07 & -0.11 & 1 \end{bmatrix} \mathbf{x}[n] + r[n]$$

9.23.

a. Partial fraction expansion of the system function is

$$H(z) = \frac{Y(z)}{R(z)} = \frac{X_1(z)}{R(z)} + \frac{X_2(z)}{R(z)} = \frac{k_1}{z+1/2} + \frac{k_2}{z+1/3}$$

The residues are

$$k_1 = \left.\frac{z+1}{z+2/3}\right|_{z=-1/2} = 3$$

and

$$k_2 = \left.\frac{z+1}{z+1/2}\right|_{z=-2/3} -2$$

Therefore

$$\frac{X_1(z)}{R(z)} = \frac{3}{z+1/2} \qquad \Rightarrow \qquad x_1[n+1] = -\frac{1}{2}\,x_1[n] + 3\,r[n]$$

and

$$\frac{X_2(z)}{R(z)} = -\frac{2}{z+2/3} \qquad \Rightarrow \qquad x_2[n+1] = -\frac{2}{3}\,x_2[n] - 2\,r[n]$$

In matrix form

$$\mathbf{x}[n+1] = \begin{bmatrix} -1/2 & 0 \\ 0 & -2/3 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 3 \\ -2 \end{bmatrix} r[n]$$

$$y[n] = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}[n]$$

b. In this case the numerator order is the same as the denominator order and we will therefore use a slightly different form of partial fraction expansion compared to part (a):

$$H(z) = \frac{Y(z)}{R(z)} = \frac{X_1(z)}{R(z)} + \frac{X_2(z)}{R(z)} = \frac{k_1\,z}{z+1/2} + \frac{k_2\,z}{z+1/3}$$

The residues are

$$k_1 = \left.\frac{z+1}{z+0.7}\right|_{z=0.4} = 1.2727$$

and

$$k_2 = \left.\frac{z+1}{z-0.4}\right|_{z=-0.7} = -0.2727$$

Therefore

$$\frac{W_1(z)}{R(z)} = \frac{1.2727\,z}{z-0.4} \qquad \Rightarrow \qquad w_1[n+1] = 0.4\,w_1[n] + 1.2727\,r[n+1]$$

and

$$\frac{W_2(z)}{R(z)} = -\frac{0.2727\,z}{z+2/3} \qquad \Rightarrow \qquad w_2[n+1] = -0.7\,w_2[n] - 0.2727\,r[n+1]$$

The terms $w_1[n]$ and $w_2[n]$ can not be selected as the state variables of the system since we do not want the $r[n+1]$ terms to appear in state equations. Let us substitute $n \to n-1$ and write the two equations as

$$w_1[n] = 0.4\,w_1[n-1] + 1.2727\,r[n]$$

$$w_2[n] = -0.7\,w_2[n-1] - 0.2727\,r[n]$$

and define $x_1[n]$ and $x_2[n]$ as follows:

$$x_1[n+1] = w_1[n] \qquad x_2[n+1] = w_2[n]$$

It follows that

$$\begin{aligned} x_1[n+1] &= 0.4\,x_1[n] + 1.2727\,r[n] \\ x_2[n+1] &= -0.7\,x_2[n] - 0.2727\,r[n] \end{aligned}$$

The output signal is

$$\begin{aligned} y[n] &= w_1[n] + w_2[n] \\ &= x_1[n+1] + x_2[n+1] \\ &= 0.4\,x_1[n] + 1.2727\,r[n] - 0.7\,x_2[n] - 0.2727\,r[n] \\ &= 0.4\,x_1[n] - 0.7\,x_2[n] + r[n] \end{aligned}$$

In matrix form

$$\mathbf{x}[n+1] = \begin{bmatrix} 0.4 & 0 \\ 0 & -0.7 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 1.2727 \\ -0.2727 \end{bmatrix} r[n]$$

$$y[n] = \begin{bmatrix} 0.4 & -0.7 \end{bmatrix} \mathbf{x}[n] + r[n]$$

c. Partial fraction expansion of the system function is

$$H(z) = \frac{Y(z)}{R(z)} = \frac{X_1(z)}{R(z)} + \frac{X_2(z)}{R(z)} + \frac{X_3(z)}{R(z)} = \frac{k_1}{z+3/4} + \frac{k_2}{z-1/2} + \frac{k_3}{z-3/2}$$

The residues are

$$k_1 = \left.\frac{z\,(z+1)}{(z-1/2)\,(z-3/2)}\right|_{z=-3/4} = -1/15$$

$$k_2 = \left.\frac{z\,(z+1)}{(z+3/4)\,(z-3/2)}\right|_{z=1/2} = -3/5$$

and

$$k_3 = \left.\frac{z\,(z+1)}{(z+3/4)\,(z-1/2)}\right|_{z=3/2} = 5/3$$

Therefore

$$\frac{X_1(z)}{R(z)} = \frac{-1/15}{z+3/4} \quad \Rightarrow \quad x_1[n+1] = -\frac{3}{4}\,x_1[n] - \frac{1}{15}\,r[n]$$

$$\frac{X_2(z)}{R(z)} = \frac{-3/5}{z-1/2} \quad \Rightarrow \quad x_2[n+1] = \frac{1}{2}\,x_2[n] - \frac{3}{5}\,r[n]$$

$$\frac{X_3(z)}{R(z)} = \frac{5/3}{z-3/2} \quad \Rightarrow \quad x_3[n+1] = \frac{3}{2}\,x_3[n] + \frac{5}{3}\,r[n]$$

In matrix form

$$\mathbf{x}[n+1] = \begin{bmatrix} -3/4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} -1/15 \\ -3/5 \\ 1/3 \end{bmatrix} r[n]$$

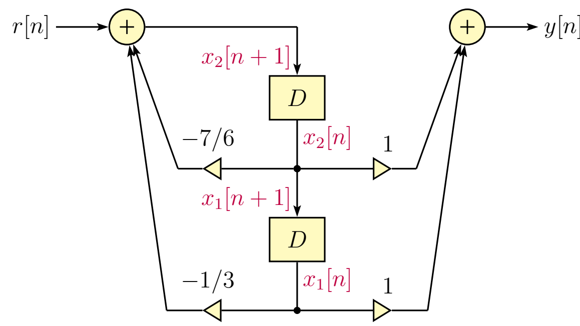
$$y[n] = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathbf{x}[n]$$

9.24.

a.

The system function is

$$H(z) = \frac{z+1}{z^2 + (7/6)\,z + 1/3}$$



$$x_1[n+1] = x_2[n]$$

$$x_2[n+1] = -\frac{1}{3}\,x_1[n] - \frac{7}{6}\,x_2[n] + r[n]$$

$$y[n] = x_1[n] + x_2[n]$$

In matrix form

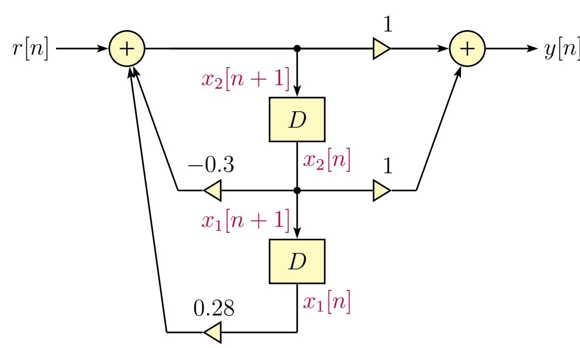
$$\mathbf{x}[n+1] = \begin{bmatrix} 0 & 1 \\ -1/3 & -7/6 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r[n]$$

$$y[n] = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}[n]$$

b.

The system function is

$$H(z) = \frac{z^2 + z}{z^2 + 0.3\,z - 0.28}$$



$$x_1[n+1] = x_2[n]$$

$$x_2[n+1] = 0.28\,x_1[n] - 0.3\,x_2[n] + r[n]$$

$$\begin{aligned} y[n] &= x_2[n+1] + x_2[n] \\ &= 0.28\,x_1[n] - 0.3\,x_2[n] + r[n] + x_2[n] \\ &= 0.28\,x_1[n] + 0.7\,x_2[n] + r[n] \end{aligned}$$

In matrix form

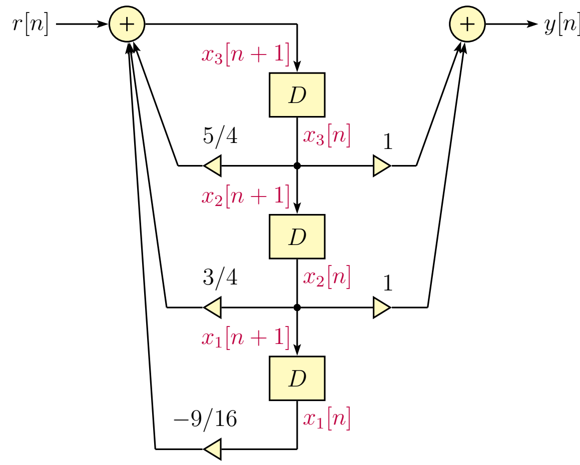
$$\mathbf{x}[n+1] = \begin{bmatrix} 0 & 1 \\ 0.28 & -0.3 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r[n]$$

$$y[n] = \begin{bmatrix} 0.28 & 0.7 \end{bmatrix} \mathbf{x}[n] + r[n]$$

c.

The system function is

$$H(z) = \frac{z^2+z}{z^3-(5/4)\,z^2-(3/4)\,z+9/16}$$



$$\begin{aligned}
x_1[n+1] &= x_2[n] \\
x_2[n+1] &= x_3[n] \\
x_3[n+1] &= -\frac{9}{16}\,x_1[n]+\frac{3}{4}\,x_2[n]+\frac{5}{4}\,x_3[n]+r[n] \\
y[n] &= x_2[n]+x_3[n]
\end{aligned}$$

In matrix form

$$\mathbf{x}[n+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9/16 & 3/4 & 5/4 \end{bmatrix}\mathbf{x}[n] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r[n]$$

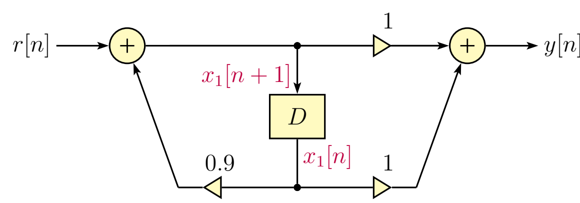
$$y[n] = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\mathbf{x}[n]$$

9.25.

a.

The system function is

$$H(z) = \frac{1+z^{-1}}{1-0.9\,z^{-1}} = \frac{z+1}{z-0.9}$$



$$\begin{aligned}
x_1[n+1] &= 0.9\,x_1[n]+r[n] \\
y[n] &= x_1[n]+x_1[n+1] \\
&= x_1[n]+0.9\,x_1[n]+r[n] \\
&= 1.9\,x_1[n]+r[n]
\end{aligned}$$

In matrix form

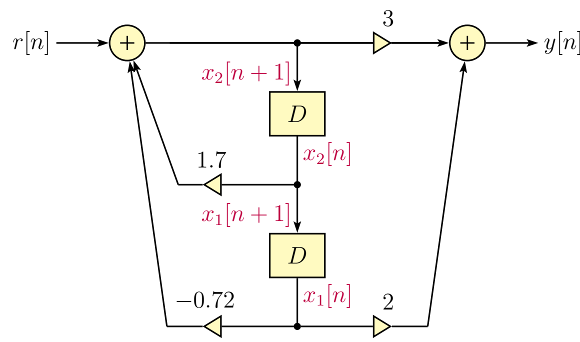
$$\mathbf{x}[n+1] = \begin{bmatrix} 0.9 \end{bmatrix}\mathbf{x}[n] + \begin{bmatrix} 1 \end{bmatrix} r[n]$$

$$y[n] = \begin{bmatrix} 1.9 \end{bmatrix}\mathbf{x}[n] + r[n]$$

b.

The system function is

$$H(z) = \frac{3+2z^{-2}}{1-1.7z^{-1}+0.72z^{-2}} = \frac{3z^2+2}{z^2-1.7z+0.72}$$



$$\begin{aligned}
x_1[n+1] &= x_2[n] \\
x_2[n+1] &= -0.72\,x_1[n] + 1.7\,x_2[n] + r[n] \\
y[n] &= 2\,x_1[n] + 3\,x_2[n+1] \\
&= 2\,x_1[n] + 3\left(-0.72\,x_1[n] + 1.7\,x_2[n] + r[n]\right) \\
&= -0.16\,x_1[n] + 5.1\,x_2[n] + 3\,r[n]
\end{aligned}$$

In matrix form

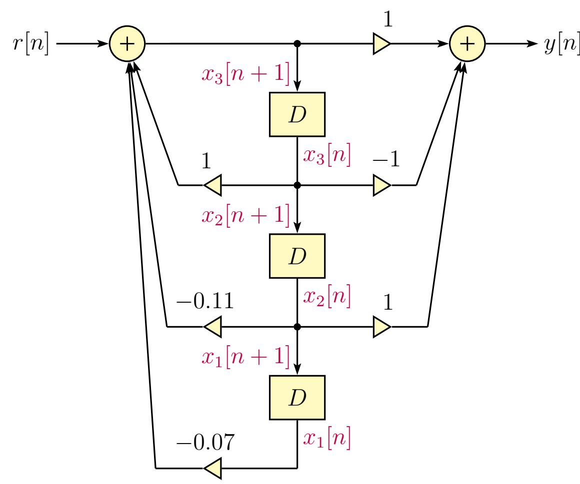
$$\mathbf{x}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r[n]$$

$$y[n] = \begin{bmatrix} -0.16 & 5.1 \end{bmatrix} \mathbf{x}[n] + 3\,r[n]$$

c.

The system function is

$$\begin{aligned}
H(z) &= \frac{1-z^{-1}+z^{-2}}{1-z^{-1}+0.11z^{-2}+0.07z^{-3}} \\
&= \frac{z^3-z^2+z}{z^3-z^2+0.11z+0.07}
\end{aligned}$$



$$\begin{aligned}
x_1[n+1] &= x_2[n] \\
x_2[n+1] &= x_3[n] \\
x_3[n+1] &= -0.07\,x_1[n] - 0.11\,x_2[n] + x_3[n] + r[n] \\
y[n] &= x_2[n] - x_3[n] + x_3[n+1] \\
&= -0.07\,x_1[n] + 0.89\,x_2[n] + r[n]
\end{aligned}$$

In matrix form

$$\mathbf{x}[n+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.07 & -0.11 & 1 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r[n]$$

$$y[n] = \begin{bmatrix} -0.07 & 0.89 & 0 \end{bmatrix} \mathbf{x}[n] + r[n]$$

9.26.

$$\mathbf{x}[1] = \mathbf{A}\mathbf{x}[0] + \mathbf{B}\,r[0] = \begin{bmatrix} 0 & -0.5 \\ 0.25 & 0.75 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} (1) = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$$

$$y[0] = \mathbf{C}\mathbf{x}[0] = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 6$$

$$\mathbf{x}[2] = \mathbf{A}\mathbf{x}[1] + \mathbf{B}\,r[1] = \begin{bmatrix} 0 & -0.5 \\ 0.25 & 0.75 \end{bmatrix} \begin{bmatrix} 2 \\ 1.5 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} (1) = \begin{bmatrix} 1.25 \\ 2.625 \end{bmatrix}$$

$$y[1] = \mathbf{C}\mathbf{x}[1] = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1.5 \end{bmatrix} = 7.5$$

$$\mathbf{x}[3] = \mathbf{A}\mathbf{x}[2] + \mathbf{B}\,r[2] = \begin{bmatrix} 0 & -0.5 \\ 0.25 & 0.75 \end{bmatrix} \begin{bmatrix} 1.25 \\ 2.625 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} (1) = \begin{bmatrix} 0.6875 \\ 3.2813 \end{bmatrix}$$

$$y[2] = \mathbf{C}\mathbf{x}[2] = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1.25 \\ 2.625 \end{bmatrix} = 6.375$$

$$\mathbf{x}[4] = \mathbf{A}\mathbf{x}[3] + \mathbf{B}\,r[3] = \begin{bmatrix} 0 & -0.5 \\ 0.25 & 0.75 \end{bmatrix} \begin{bmatrix} 0.6875 \\ 3.2813 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} (1) = \begin{bmatrix} 0.3594 \\ 3.6328 \end{bmatrix}$$

$$y[3] = \mathbf{C}\mathbf{x}[3] = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 0.6875 \\ 3.2813 \end{bmatrix} = 5.3438$$

9.27.

a. Resolvent matrix is found as

$$\Phi(z) = z\left[z\mathbf{I} - \mathbf{A}\right]^{-1}$$

The first step is to find the matrix $\left[z\mathbf{I} - \mathbf{A}\right]$.

$$z\mathbf{I} - \mathbf{A} = z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -0.1 & -0.7 \\ -0.8 & 0 \end{bmatrix} = \begin{bmatrix} z+0.1 & 0.7 \\ 0.8 & z \end{bmatrix}$$

$$\left[z\mathbf{I} - \mathbf{A}\right]^{-1} = \frac{1}{(z-0.7)\,(z+0.8)} \begin{bmatrix} z & -0.7 \\ -0.8 & z+0.1 \end{bmatrix}$$

The resolvent matrix is

$$\Phi(z) = z\left[z\mathbf{I} - \mathbf{A}\right]^{-1} = \frac{1}{(z-0.7)\,(z+0.8)} \begin{bmatrix} z^2 & -0.7\,z \\ -0.8\,z & z\,(z+0.1) \end{bmatrix}$$

b. The state transition matrix is

$$\phi[n] = \left[\Phi(z)\right]^{-1} = \begin{bmatrix} \phi_{11}[n] & \phi_{12}[n] \\ \phi_{21}[n] & \phi_{22}[n] \end{bmatrix}$$

$$\Phi_{11}(z) = \frac{z^2}{(z-0.7)\,(z+0.8)} = \frac{(7/15)\,z}{z-0.7} + \frac{(8/15)\,z}{z+0.8}$$

$$\phi_{11}[n] = \left(\frac{7}{15}\right)(0.7)^n\, u[n] + \left(\frac{8}{15}\right)(-0.8)^n\, u[n]$$

$$\Phi_{12}(z) = \frac{-0.7\,z}{(z-0.7)\,(z+0.8)} = -\frac{(7/15)\,z}{z-0.7} + \frac{(7/15)\,z}{z+0.8}$$

$$\phi_{12}[n] = -\left(\frac{7}{15}\right)(0.7)^n\, u[n] + \left(\frac{7}{15}\right)(-0.8)^n\, u[n]$$

$$\Phi_{21}(z) = \frac{-0.8\,z}{(z-0.7)\,(z+0.8)} = -\frac{(8/15)\,z}{z-0.7} + \frac{(8/15)\,z}{z+0.8}$$

$$\phi_{21}[n] = -\left(\frac{8}{15}\right)(0.7)^n\, u[n] + \left(\frac{8}{15}\right)(-0.8)^n\, u[n]$$

$$\Phi_{22}(z) = \frac{z\,(z+1)}{(z-0.7)\,(z+0.8)} = \frac{(8/15)\,z}{z-0.7} + \frac{(7/15)\,z}{z+0.8}$$

$$\phi_{22}[n] = \left(\frac{8}{15}\right)(0.7)^n\, u[n] + \left(\frac{7}{15}\right)(-0.8)^n\, u[n]$$

c. Using Eqn. (9.177) we have

$$\mathbf{X}(z) = \Phi(z)\,\mathbf{x}(0) + z^{-1}\,\Phi(z)\,\mathbf{B}\,R(z)$$

The resolvent matrix was found in part (a).

$$\Phi(z)\,\mathbf{x}(0) = \frac{1}{(z-0.7)\,(z+0.8)}\begin{bmatrix} z^2 & -0.7\,z \\ -0.8\,z & z\,(z+0.1)\end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{(z-0.7)\,(z+0.8)}\begin{bmatrix} 2\,z^2 \\ -1.6\,z \end{bmatrix}$$

The second term in the expression for $\mathbf{X}(z)$ is

$$z^{-1}\,\Phi(z)\,\mathbf{B}\,R(z) = z^{-1}\,\frac{1}{(z-0.7)\,(z+0.8)}\begin{bmatrix} z^2 & -0.7\,z \\ -0.8\,z & z\,(z+0.1)\end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\left(\frac{z}{z-1}\right)$$

$$= \frac{1}{(z-1)\,(z-0.7)\,(z+0.8)}\begin{bmatrix} 3\,z^2 - 0.7\,z \\ z^2 - 2.3\,z \end{bmatrix}$$

Thus $\mathbf{X}(z)$ is found as

$$\mathbf{X}(z) = \frac{1}{(z-1)\,(z-0.7)\,(z+0.8)}\begin{bmatrix} 2\,z^3 + z^2 - 0.7\,z \\ -0.6\,z^2 - 0.7\,z \end{bmatrix}$$

and the output transform is

$$Y(z) = \frac{z\left(4\,z^2 + 2.6\,z - 0.7\right)}{(z-1)\,(z-0.7)\,(z+0.8)}$$

The output signal can be found using partial fraction expansion:

$$y[n] = 10.9259 - 0.0815\,(-0.8)^n - 6.8444\,(0.7)^n\ , \qquad n \ge 0$$

9.28.

The system function is found as

$$H(z) = \mathbf{C}\left[z\,\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B}$$

The first step is to find the matrix $\left[z\,\mathbf{I} - \mathbf{A}\right]$ and its inverse:

$$z\,\mathbf{I} - \mathbf{A} = z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -0.1 & -0.7 \\ -0.8 & 0 \end{bmatrix} = \begin{bmatrix} z+0.1 & 0.7 \\ 0.8 & z \end{bmatrix}$$

$$\left[z\,\mathbf{I} - \mathbf{A}\right]^{-1} = \frac{1}{(z-0.7)\,(z+0.8)}\begin{bmatrix} z & -0.7 \\ -0.8 & z+0.1 \end{bmatrix}$$

$$H(z) = \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} \dfrac{z}{(z-0.7)\,(z+0.8)} & \dfrac{-0.7}{(z-0.7)\,(z+0.8)} \\ \dfrac{-0.8}{(z-0.7)\,(z+0.8)} & \dfrac{z+0.1}{(z-0.7)\,(z+0.8)} \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \frac{5\,z+0.9}{(z-0.7)\,(z+0.8)}$$

9.29.

The system function must first be written using powers of z^{-1}:

$$H(z) = \frac{0 + z^{-1} + 0\,z^{-2} - 7\,z^{-3} + 6\,z^{-4}}{1 - 0.2\,z^{-1} - 0.93\,z^{-2} + 0.198\,z^{-3} - 0.1296\,z^{-4}}$$

It is important to ensure that numerator and denominator polynomials have the same length for the use of the function **tf2ss** (..) . We also account for missing terms using zero-valued coefficients. The state-space model for the system is found with the statements

```
>>  num = [0,1,0,-7,6];
>>  den = [1,-0.2,-0.93,0.198,0.1296];
>>  [A,B,C,d] = tf2ss(num,den)

A =

    0.2000    0.9300   -0.1980   -0.1296
    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0

B =

     1
     0
     0
     0

C =
```

```
    1     0    -7     6

d =

    0
```

The transformation matrix **P** to convert this state-space model to an alternative one with a diagonal state matrix is found using the following:

```
>> [P,lambda] = eig(A)

P =

   0.4211    0.3367    0.1743   -0.0258
  -0.4678    0.4209    0.2904    0.0859
   0.5198    0.5262    0.4841   -0.2862
  -0.5776    0.6577    0.8068    0.9540

lambda =

  -0.9000         0         0         0
        0    0.8000         0         0
        0         0    0.6000         0
        0         0         0   -0.3000
```

Coefficient matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ for the alternative state-space model are determined using the following:

```
>> A_tilde = inv(P)*A*P

A_tilde =

  -0.9000   -0.0000   -0.0000   -0.0000
   0.0000    0.8000   -0.0000   -0.0000
  -0.0000    0.0000    0.6000   -0.0000
   0.0000   -0.0000   -0.0000   -0.3000

>> B_tilde = inv(P)*B

B_tilde =

   1.1316
   4.0654
  -4.5906
   1.7647

>> C_tilde = C*P

C_tilde =

  -6.6833    0.5998    1.6265    7.7014
```

9.30.

Code for the script is listed below:

```
% Set up the state space model
A = [0,-0.5;0.25,0.75];
B = [2;1];
C = [3,1];
xn = [2;0];  % "xn" represents the current state vector x[n]
% Solve iteratively
n = [0:99];
out = [];     % Placeholder for the output signal for n=0,...,99
rn = 1;       % Unit-step input; always 1 for n>=0
for nn=0:99,
  xnp1 = A*xn+B*rn;  % "xnp1" represents vector x[n+1]
  yn = C*xn;
  out = [out,yn];    % Append y[n] to the output stream
  xn = xnp1;         % Get ready for next iteration
end;
% Graph the output signal
stem(n,out);
```

9.31.

Set up the state space model:

```
A = [-0.1,-0.7;-0.8,0];
B = [3;1];
C = [2,-1];
```

Compute the resolvent matrix and the state transition matrix.

```
z = sym('z');
tmp = z*eye(2)-A;
rsm = z*inv(tmp)      % Resolvent matrix
stm = iztrans(rsm)    % State transition matrix
```

Find $\mathbf{X}(z)$, the z-transform of the state vector.

```
Xz = rsm*[2;0]+1/(z-1)*rsm*B
xn = iztrans(Xz)
```

Find $Y(z)$, the output transform. Afterwards compute and graph the output signal $y[n]$.

```
Yz = C*Xz
yn = iztrans(Yz)
n = [0:60];
y = subs(yn,n);
stem(n,y);
axis([-0.5,60.5,0,12]);
xlabel('n');
ylabel('y[n]');
```

9.32.

Code for the script is listed below:

```
%% Set up the state-space model
A = [-0.1,-0.7;-0.8,0];
B = [3;1];
C = [2,-1];
% Find H(z) using symbolic math functions
z = sym('z');
tmp = z*eye(2)-A;
H = C*inv(tmp)*B;
simplify(H)
```

The answer produced by the script is

```
31/(3*(5*z + 4)) + 88/(3*(10*z - 7))
```

and is equivalent to

$$H(z) = \frac{31/3}{5z+4} + \frac{88/3}{10z-7} = \frac{5z+0.9}{(z-0.7)\,(z+0.8)}$$

9.33. Set up the continuous-time state space model and convert to a discrete-time model using Euler's method with $T = 0.02$ seconds:

```
A = [-2,-2;1,-5];
B = [1;0];
C = [0,5];
d = 0;
Ts = 0.02;
A_bar = eye(2)+A*Ts
B_bar = B*Ts
C_bar = C
d_bar = d
```

Solve the approximate discrete-time model iteratively:

```
xn = [3;-2];          % Initial value of state vector
n = [0:150];           % Vector of indices
yn = [];              % Empty vector to start
for nn=0:150,
  xnp1 = A_bar*xn+B_bar;  % 'xnp1' represents x[n+1]
  yn = [yn,C*xn];          % Append to vector 'yn'
  xn = xnp1;               % New becomes old for next iteration
end;
```

Graph actual and approximate solutions together:

```
t = [0:0.01:3];
y = 5/12+70/3*exp(-3*t)-135/4*exp(-4*t);    % From Example 9.13
plot(n*Ts,yn,'r.',t,y,'b-'); grid;
ht = title('y_{a}(t) and y[n]');
hx = xlabel('t (sec)');
```

9.34.

a. Set up the continuous-time state space model and convert to a discrete-time model with $T = 0.1$ seconds:

```
A = [0,1,0;0,0,1;-15,-11,-5]
B = [0;0;3]
C = [1,0,0]
d = 0
Ts = 0.1;
A_bar = expm(A*Ts)
B_bar = inv(A)*(A_bar-eye(3))*B
C_bar = C
d_bar = d
```

b. Solve the approximate discrete-time model iteratively:

```
xn = [0;0;0];          % Initial value of state vector
n = [0:60];            % Vector of indices
yn = [];               % Empty vector to start
for nn=0:60,
  xnp1 = A_bar*xn+B_bar;  % 'xnp1' represents x[n+1]
  yn = [yn,C*xn];         % Append to vector 'yn'
  xn = xnp1;              % New becomes old for next iteration
end;
```

The system function for the continuous-time system is

$$H(s) = \frac{3}{s^3 + 5s^2 + 11s + 15}$$

Create a system object "sys" and use it to compute the unit-step response of the system through the use of the function **lsim**(..) Afterwards graph the output signal along with that obtained through the iterative solution of the discretized state space model:

```
sys = tf([3],[1,5,11,15]);
t = [0:0.01:6];
xa = ones(size(t));
ya = lsim(sys,xa,t);
plot(n*Ts,yn,'r.',t,ya,'b-'); grid;
ht = title('Actual and approximate solutions');
hx = xlabel('t (sec)');
```

9.35.

a. Set up the continuous-time state space model and convert to a discrete-time model using Euler's method with $T = 0.1$ seconds:

```
A = [0,1,0;0,0,1;-15,-11,-5];
B = [0;7;-32];
C = [1,0,0];
d = 0;
Ts = 0.1;
A_bar = eye(3)+A*Ts
B_bar = B*Ts
C_bar = C
d_bar = d
```

b. Solve the approximate discrete-time model iteratively:

```
xn = [0;0;0];       % Initial value of state vector
n = [0:60];         % Vector of indices
yn = [];            % Empty vector to start
for nn=0:60,
  xnp1 = A_bar*xn+B_bar;  % 'xnp1' represents x[n+1]
  yn = [yn,C*xn];         % Append to vector 'yn'
  xn = xnp1;              % New becomes old for next iteration
end;
```

Create a system object "sys" and use it to compute the unit-step response of the system through the use of the function **lsim**(..) Afterwards graph the output signal along with that obtained through the iterative solution of the discretized state space model:

```
sys = tf([7,3],[1,5,11,15]);
t = [0:0.01:6];
xa = ones(size(t));
ya = lsim(sys,xa,t);
plot(n*Ts,yn,'r.',t,ya,'b-'); grid;
ht = title('Actual and approximate solutions');
hx = xlabel('t (sec)');
```

Chapter 10

Analysis and Design of Filters

10.1.

a.

$$\omega_c = \frac{1}{RC} = \frac{1}{1000 \times 10^{-6}} = 1000 \ \text{rad/s}$$

The peak magnitude occurs at $\omega = 0$ and its value is $|H(0)| = 1$. For at most 1 percent deviation from the peak magnitude

$$\omega_0 = 1000 \sqrt{\left(\frac{100}{100-1}\right)^2 - 1} = 142.5 \ \text{rad/s}$$

Therefore we require $|\omega| \le 142.5$ rad/s.

b. The slope of the phase characteristic at $\omega = 0$ is

$$\left.\frac{d\Theta}{d\omega}\right|_{\omega=0} = -\frac{1}{\omega_c} = -\frac{1}{1000}$$

If the phase characteristic were perfectly linear, the phase at $\omega_0 = 142.5$ rad/s would be

$$\hat{\Theta}(142.5) = -\frac{142.5}{1000} = -0.1425 \ \text{radians}$$

The actual phase at $\omega_0 = 142.5$ rad/s is

$$\Theta(142.5) = -\tan^{-1}\left(\frac{142.5}{1000}\right) == -0.1415 \ \text{radians}$$

and the percent deviation from linear phase is

$$100\left(\frac{-0.1415 + 0.1425}{-0.1425}\right) = -0.67 \ \text{percent}$$

c. If the phase characteristic were linear, the time delay for the frequency ω_0 would be

$$\hat{t}_d(\omega_0) = -\frac{\hat{\Theta}(\omega_0)}{\omega_0} = \frac{0.1425}{142.5} = 0.001 \ \text{sec}$$

The actual time delay for the frequency ω_0 is

$$t_d(\omega_0) = -\frac{\Theta(\omega_0)}{\omega_0} = \frac{0.1415}{142.5} = 0.000993 \ \text{sec}$$

d. For at most 2 percent deviation from the peak magnitude

$$\omega_0 = 1000\sqrt{\left(\frac{100}{100-2}\right)^2 - 1} = 203.1 \text{ rad/s}$$

Therefore we require $|\omega| \le 203.1$ rad/s. The slope of the phase characteristic at $\omega = 0$ is

$$\left.\frac{d\Theta}{d\omega}\right|_{\omega=0} = -\frac{1}{\omega_c} = -\frac{1}{1000}$$

If the phase characteristic were perfectly linear, the phase at $\omega_0 = 203.1$ rad/s would be

$$\hat{\Theta}(203.1) = -\frac{203.1}{1000} = -0.2031 \text{ radians}$$

The actual phase at $\omega_0 = 203.1$ rad/s is

$$\Theta(203.1) = -\tan^{-1}\left(\frac{203.1}{1000}\right) == -0.2003 \text{ radians}$$

and the percent deviation from linear phase is

$$100\left(\frac{-0.2003 + 0.2031}{-0.2031}\right) = -1.34 \text{ percent}$$

If the phase characteristic were linear, the time delay for the frequency ω_0 would be

$$\hat{t}_d(\omega_0) = -\frac{\hat{\Theta}(\omega_0)}{\omega_0} = \frac{0.2031}{203.1} = 0.001 \text{ sec}$$

The actual time delay for the frequency ω_0 is

$$t_d(\omega_0) = -\frac{\Theta(\omega_0)}{\omega_0} = \frac{0.2003}{203.1} = 0.000987 \text{ sec}$$

10.2.

a.

$$H(\omega) = \left.\frac{1}{s^2 + \sqrt{2}\,s + 1}\right|_{s=j\omega} = \frac{1}{(1-\omega^2) + j\sqrt{2}\omega}$$

$$|H(\omega)| = \frac{1}{\sqrt{(1-\omega^2)^2 + 2\omega^2}} = \frac{1}{\sqrt{1+\omega^4}}$$

For the magnitude response to stay within p percent of the of the peak we need

$$\frac{1}{\sqrt{1+\omega^4}} \le \frac{100-p}{p} \qquad \Rightarrow \qquad \omega_0 = \left\{\left(\frac{100}{100-p}\right)^2 - 1\right\}^{1/4}$$

For $p = 1$

$$\omega_0 = \left\{\left(\frac{100}{99}\right)^2 - 1\right\}^{1/4} = 0.3775 \text{ rad/s}$$

b.

$$\Theta(\omega) = \measuredangle H(\omega) = -\tan^{-1}\left(\frac{\sqrt{2}\,\omega}{1-\omega^2}\right)$$

It can be shown that

$$\frac{d\Theta(\omega)}{d\omega} = \frac{-\sqrt{2}\left(1+\omega^2\right)}{1+\omega^4}$$

At $\omega = 0$ we get

$$\left.\frac{d\Theta(\omega)}{d\omega}\right|_{\omega=0} = -\sqrt{2}$$

If the phase characteristic were perfectly linear, the phase at $\omega_0 = 0.3775$ rad/s would be

$$\hat{\Theta}(0.3775) = -\sqrt{2}\,(0.3775) = -0.5338 \text{ radians}$$

The actual phase at $\omega_0 = 0.3775$ rad/s is

$$\Theta(0.3775) = -\tan^{-1}\left(\frac{\sqrt{2}\,(0.3775)}{1-0.3775^2}\right) = -0.5568 \text{ radians}$$

and the percent deviation from linear phase is

$$100\left(\frac{-0.5568+0.5338}{-0.5338}\right) = 4.31 \text{ percent}$$

c. If the phase characteristic were linear, the time delay for the frequency ω_0 would be

$$\hat{t}_d(\omega_0) = -\frac{\hat{\Theta}(\omega_0)}{\omega_0} = \frac{0.5338}{0.3775} = 1.4142 \text{ sec}$$

The actual time delay for the frequency ω_0 is

$$t_d(\omega_0) = -\frac{\Theta(\omega_0)}{\omega_0} = \frac{0.5568}{0.3775} = 1.4751 \text{ sec}$$

d. For at most 2 percent deviation from the peak magnitude

$$\omega_0 = \left\{\left(\frac{100}{98}\right)^2 - 1\right\}^{1/4} = 0.4506 \text{ rad/s}$$

Therefore we require $|\omega| \le 0.4506$ rad/s. The slope of the phase characteristic at $\omega = 0$ is

$$\left.\frac{d\Theta(\omega)}{d\omega}\right|_{\omega=0} = -\sqrt{2}$$

If the phase characteristic were perfectly linear, the phase at $\omega_0 = 0.4506$ rad/s would be

$$\hat{\Theta}(0.4506) = -\sqrt{2}\,(0.4506) = -0.6373 \text{ radians}$$

The actual phase at $\omega_0 = 0.4506$ rad/s is

$$\Theta(0.4506) = -\tan^{-1}\left(\frac{\sqrt{2}\,(0.4506)}{1-0.4506^2}\right) = -0.6745 \text{ radians}$$

and the percent deviation from linear phase is

$$100\left(\frac{-0.6745+0.6373}{-0.6373}\right)=5.85\ \text{percent}$$

If the phase characteristic were linear, the time delay for the frequency ω_0 would be

$$\hat{t}_d(\omega_0)=-\frac{\hat{\Theta}(\omega_0)}{\omega_0}=\frac{0.6373}{0.4506}=1.4142\ \text{sec}$$

The actual time delay for the frequency ω_0 is

$$t_d(\omega_0)=-\frac{\Theta(\omega_0)}{\omega_0}=\frac{0.6745}{0.4506}=1.4969\ \text{sec}$$

10.3.

a. **a.** Using Eqn. (10.29)

$$H_{LP}(\omega)=\Pi\left(\frac{\omega}{400\pi}\right)e^{-j0.1\omega}$$

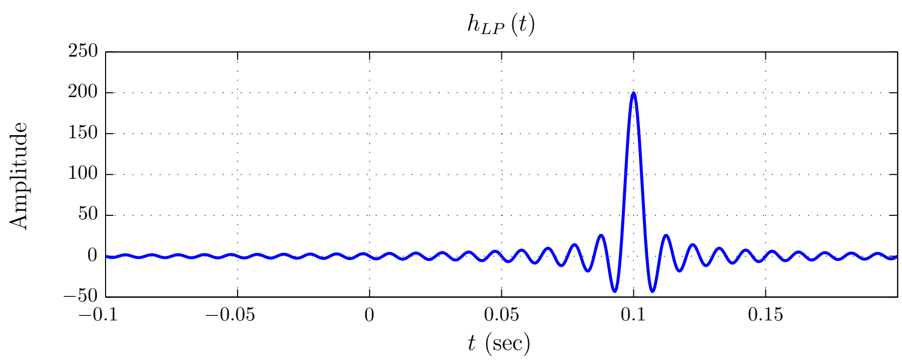
or in terms of f

$$H_{LP}(f)=\Pi\left(\frac{f}{200}\right)e^{-j0.2\pi\omega}$$

b. Using Eqn. (10.31)

$$h_{LP}(t)=\frac{200\pi}{\pi}\ \text{sinc}\left(\frac{200\pi}{\pi}\,(t-0.1)\right)=200\ \text{sinc}\left(200\,(t-0.1)\right)$$

c.



10.4.

a. Using Eqn. (10.33) with $\omega_b=450\pi$ rad/s and $2\omega_0=500\,pi$ rad/s and incorporating the time delay of $t_d=0.3$ seconds we obtain

$$H_{BP}(\omega)=\left[\Pi\left(\frac{\omega-450\pi}{500\pi}\right)+\Pi\left(\frac{\omega+450\pi}{500\pi}\right)\right]e^{-j0.3\omega}$$

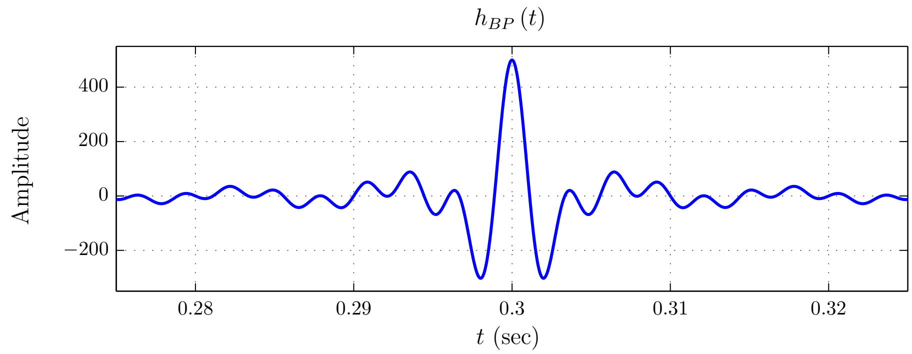
or in terms of f

$$H_{BP}(f) = \left[\Pi\left(\frac{f-225}{250}\right) + \Pi\left(\frac{f+225}{250}\right)\right] e^{-j0.6\pi f}$$

b. Using Eqn. (10.37) with $f_b = 225$ Hz and $2f_0 = 250$ Hz we obtain

$$h_{BP}(t) = 500\,\text{sinc}\big(250\,(t-0.3)\big)\,\cos\big(450\pi\,(t-0.3)\big)$$

c.



10.5.

Using parameters $\omega_c = 2$ rad/s and $N = 4$, the Butterworth squared magnitude function is

$$|H(\omega)|^2 = \frac{1}{1+\left(\dfrac{\omega}{2}\right)^8}$$

Using Eqn. (10.49)

$$H(s)\,H(-s) = |H(\omega)|^2\Big|_{\omega^2=-s^2} = \frac{1}{1+\left(\dfrac{-s^2}{4}\right)^4} = \frac{256}{256+s^8}$$

The poles of $H(s)\,H(-s)$ are the solutions of

$$s^8 = -256$$

which can be written in the equivalent form

$$s^8 = 256\,e^{j\pi}\,e^{j2\pi k} = 256\,e^{j(2k+1)\pi}$$

The poles of $H(s)\,H(-s)$ are at

$$p_k = 4\,e^{j(2k+1)\pi/8}\,, \qquad k = 0,\ldots,7$$

The poles in the left half s-plane are associated with $H(s)$.

$$H(s) = \frac{K}{(s-p_2)\,(s-p_3)\,(s-p_4)\,(s-p_5)}$$

Poles p_2 and p_5 form a conjugate pair:

$$p_2 = e^{j5\pi/8}$$
$$p_5 = e^{j11\pi/8} = e^{-j5\pi/8}$$

$$\begin{aligned}(s-p_2)\,(s-p_5) &= \left(s-2\,e^{j5\pi/8}\right)\left(s-2\,e^{-j5\pi/8}\right)\\ &= s^2+1.5307\,s+4\end{aligned}$$

Similarly, poles p_3 and p_4 form a conjugate pair:

$$p_3 = e^{j7\pi/8}$$
$$p_4 = e^{j9\pi/8} = e^{-j7\pi/8}$$

$$\begin{aligned}(s-p_3)\,(s-p_4) &= \left(s-2\,e^{j7\pi/8}\right)\left(s-2\,e^{-j7\pi/8}\right)\\ &= s^2+3.6955\,s+4\end{aligned}$$

The system function is

$$\begin{aligned}H(s) &= \frac{K}{\left(s^2+1.5307\,s+4\right)\left(s^2+3.6955\,s+4\right)}\\ &= \frac{16}{s^4+5.2263\,s^3+13.6569\,s^2+20.9050\,s+16.0000}\end{aligned}$$

where the gain factor K is adjusted to achieve $|H(0)|=1$.

10.6.

Use Eqn. (10.65) to determine N:

$$N \ge \frac{\log_{10}\sqrt{\left(10^3-1\right)/\left(10^{0.1}-1\right)}}{\log_{10}(4.5/3)} = 10.18$$

Filter order must be chosen as $N=11$. If the excess tolerance that results from rounding up N to 11 is to be used for improving the stopband response, we need to obtain ω_c from Eqn. (10.62):

$$\left(\frac{3}{\omega_c}\right)^{22} = 10^{1/10}-1 \qquad \Rightarrow \qquad \omega_c = 3.19 \text{ rad/s}$$

Alternatively, the excess tolerance can be used for improving the response in the passband by solving for ω_c from Eqn. (10.63):

$$\left(\frac{4.5}{\omega_c}\right)^{22} = 10^{30/10}-1 \qquad \Rightarrow \qquad \omega_c = 3.2875 \text{ rad/s}$$

10.7.

In order to evenly distribute the excess tolerance, we need a passband ripple of $\left(R_p - \Delta\right)$ dB and a stopband attenuation of $(A_s + \Delta)$ dB where Δ is a positive quantity. Using Eqns. (10.62) and (10.63) we get

$$\left(\frac{\omega 1}{\omega_c}\right)^{2N} = 10^{\left(R_p-\Delta\right)/10} - 1 \tag{P.10.7.1}$$

and

$$\left(\frac{\omega 2}{\omega_c}\right)^{2N} = 10^{\left(A_s+\Delta\right)/10} - 1 \tag{P.10.7.2}$$

Combining the two requirements yields

$$\left(\frac{\omega_1}{\omega_2}\right)^{2N} = \frac{10^{R_p/10}\,10^{-\Delta/10} - 1}{10^{A_s/10}\,10^{\Delta/10} - 1}$$

Let

$$K = \left(\frac{\omega_1}{\omega_2}\right)^{2N}, \qquad A_1 = 10^{R_p/10}, \qquad A_2 = 10^{A_s/10} \qquad \text{and} \qquad D = 10^{\Delta/10}$$

The condition for equal distribution of the excess tolerance between the passband and the stopband is

$$K = \frac{A_1/D - 1}{A_2\,D - 1}$$

or, equivalently

$$A_2 K\,D^2 + (1-K)\,D - A_1 = 0$$

Solving for D we obtain

$$D = \frac{(K-1) \pm \sqrt{(1-K)^2 + 4A_1 A_2 K}}{2A_2 K}$$

Use the positive solution since $\Delta = 10\log_{10}(D)$. Once Δ is found, solve for ω_c from either Eqn. (P.10.7.1) or Eqn. (P.10.7.2).

10.8.

Parameter α_k is found as

$$\alpha_k = \frac{(2k+1)\,\pi}{8}, \qquad k = 0,\ldots,7$$

and the parameter β_k is

$$\beta_k = \frac{\sinh^{-1}(1/0.3)}{4} = 0.4797$$

Poles of $H(s)\,H(-s)$ are found through the use of Eqn. (10.86) as

$$p_k = 2\left[\cos\left(\frac{(2k+1)\,\pi}{8}\right)\cosh(0.4797) - j\,\sin\left(\frac{(2k+1)\,\pi}{8}\right)\sinh(0.4797)\right], \qquad k = 0,\ldots,7$$

Locations of the poles of $H(s)\ H(-s)$ are as follows:

$$\begin{aligned}
p_0 =& 0.3814 + j\,2.0645 \\
p_1 =& 0.9208 + j\,0.8551 \\
p_2 =& 0.9208 - j\,0.8551 \\
p_3 =& 0.3814 - j\,2.0645 \\
p_4 =& -0.3814 - j\,2.0645 \\
p_5 =& -0.9208 - j\,0.8551 \\
p_6 =& -0.9208 + j\,0.8551 \\
p_7 =& -0.3814 + j\,2.0645
\end{aligned}$$

Poles p_4, p_5, p_6 and p_7 are in the left half s-plane, and are therefore associated with $H(s)$. System function is constructed as

$$\begin{aligned}
H(s) =& \frac{A}{\left(s+0.3814+j\,2.0645\right)\left(s+0.9208+j\,0.8551\right)\left(s+0.9208-j\,0.8551\right)\left(s+0.3814-j\,2.0645\right)} \\
=& \frac{A}{s^4+2.6044\,s^3+7.3915\,s^2+9.3217\,s+6.9602}
\end{aligned}$$

The gain factor A is adjusted to achieve $|H(\omega)|_{\max} = 1$. With $N = 4$, this requires that (see Fig. 10.21)

$$\left|H(0)\right| = \frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{1+(0.3)^2}} = 0.9578$$

and therefore

$$\frac{A}{6.9602} = 0.9578 \qquad \Rightarrow \qquad A = 6.6667$$

The system function is

$$H(s) = \frac{6.6667}{s^4+2.6044\,s^3+7.3915\,s^2+9.3217\,s+6.9602}$$

10.9.

Parameter α_k is found as

$$\alpha_k = \frac{(2k+1)\,\pi}{8}\,, \qquad k = 0,\ldots,7$$

and the parameter β_k is

$$\beta_k = \frac{\sinh^{-1}(1/0.3)}{4} = 0.4797$$

Poles of $H(s)\ H(-s)$ are found through the use of Eqn. (10.109) as

$$p_k = \frac{j2}{\cos\left(\dfrac{(2k+1)\,\pi}{8}\right)\cosh(0.4797) - j\,\sin\left(\dfrac{(2k+1)\,\pi}{8}\right)\sinh(0.4797)}\,, \qquad k = 0,\ldots,7$$

Locations of the poles of $H(s)\, H(-s)$ are as follows:

$$\begin{aligned}
p_0 &= -0.3461 - j\,1.8736 \\
p_1 &= -2.3324 - j\,2.1661 \\
p_2 &= -2.3324 + j\,2.1661 \\
p_3 &= -0.3461 + j\,1.8736 \\
p_4 &= 0.3461 + j\,1.8736 \\
p_5 &= 2.3324 + j\,2.1661 \\
p_6 &= 2.3324 - j\,2.1661 \\
p_7 &= 0.3461 - j\,1.8736
\end{aligned}$$

Poles p_0, p_1, p_2 and p_3 are in the left half s-plane, and are therefore associated with $H(s)$. Zeros of the system function $H(s)$ are found using Eqns. (10.111) and (10.112):

$$z_k = \frac{\pm j\omega_2}{\cos\left(\dfrac{(2k-1)\,\pi}{2N}\right)}\;, \qquad k = 1,2$$

Locations of the zeros of $H(s)$ are as follows:

$$\begin{aligned}
z_1 &= 0 \pm j\,2.1648 \\
z_2 &= 0 \pm j\,5.2263
\end{aligned}$$

System function is constructed as

$$\begin{aligned}
H(s) &= \frac{A\left(s+j\,2.1648\right)\left(s-j\,2.1648\right)\left(s+j\,5.2263\right)\left(s-j\,5.2263\right)}{\left(s+0.3461+j\,1.8736\right)\left(s+2.3324+j\,2.1661\right)\left(s+2.3324-j\,2.1661\right)\left(s+0.3461-j\,1.8736\right)} \\
&= \frac{0.2873\,s^4+9.1951\,s^2+36.7805}{s^4+5.3571\,s^3+16.9916\,s^2+23.9481\,s+36.7805}
\end{aligned}$$

The peak of the system function occurs at $\omega = 0$. The gain factor A is adjusted to achieve $|H(0)| = 1$.

10.10.

a. The minimum filter order is found using Eqn. (10.65):

$$N \ge \frac{\log_{10}\sqrt{\left(10^{20/10}-1\right)/\left(10^{1/10}-1\right)}}{\log_{10}(3.5/2)} = 5.3129$$

Filter order must be chosen as $N = 6$. If the excess tolerance that results from rounding up N to 6 is to be used for improving the passband response, we need to obtain ω_c from Eqn. (10.63):

$$\left(\frac{3.5}{\omega_c}\right)^{12} = 10^{20/10}-1 \qquad \Rightarrow \qquad \omega_c = 2.3865 \ \text{rad/s}$$

The poles of $H(s)\, H(s)$ are at

$$p_k = 2.3865\, e^{j(2k+1)\pi/12}\;, \qquad k = 0,\ldots,11$$

Using the poles in the left half s-plane, $H(s)$ is constructed as

$$H(s) = \frac{184.75}{s^6 + 9.2208\, s^5 + 42.5116\, s^4 + 124.256\, s^3 + 242.124\, s^2 + 299.109\, s + 184.75}$$

b. Minimum filter order is found using Eqn. (10.94) and Eqn. (10.101):

$$F = \sqrt{\frac{10^{20/10} - 1}{10^{1/10} - 1}} = 19.5538$$

$$N \geq \frac{\cosh^{-1}(19.5538)}{\cosh^{-1}(3.5/2)} = 3.1633$$

which must be rounded up to $N = 4$. The parameter ε is found using Eqn. (10.89):

$$\varepsilon = \sqrt{10^{1/10} - 1} = 0.5088$$

The poles of the product $H(s)\, H(-s)$ can now be found using Eqns. (10.82), (10.84) and (10.85). Parameter α_k is found as

$$\alpha_k = \frac{(2k+1)\,\pi}{8}\,, \qquad k = 0, \ldots, 7$$

and the parameter β_k is

$$\beta_k = \frac{\sinh^{-1}(1/0.5088)}{4} = 0.3570$$

Poles of $H(s)\, H(-s)$ are found through the use of Eqn. (10.86) as

$$p_k = 2\left[\cos\left(\frac{(2k+1)\,\pi}{8}\right)\cosh(0.3570) - j\,\sin\left(\frac{(2k+1)\,\pi}{8}\right)\sinh(0.3570)\right]\,, \qquad k = 0, \ldots, 7$$

Locations of the poles of $H(s)\, H(-s)$ are as follows:

$$\begin{aligned}
p_0 &= 0.2791 - j\,1.9668 \\
p_1 &= 0.6737 - j\,0.8147 \\
p_2 &= 0.6737 + j\,0.8147 \\
p_3 &= 0.2791 + j\,1.9668 \\
p_4 &= -0.2791 + j\,1.9668 \\
p_5 &= -0.6737 + j\,0.8147 \\
p_6 &= -0.6737 - j\,0.8147 \\
p_7 &= -0.2791 - j\,1.9668
\end{aligned}$$

Poles p_4, p_5, p_6 and p_7 are in the left half s-plane, and are therefore associated with $H(s)$. System function is constructed as

$$\begin{aligned}
H(s) &= \frac{A}{\left(s + 0.2791 - j\,1.9668\right)\left(s + 0.6737 - j\,0.8147\right)\left(s + 0.6737 + j\,0.8147\right)\left(s + 0.2791 + j\,1.9668\right)} \\
&= \frac{A}{s^4 + 1.9056\, s^3 + 5.8157\, s^2 + 5.9410\, s + 4.4100}
\end{aligned}$$

The gain factor A is adjusted to achieve $|H(\omega)|_{\max}=1$. With $N=4$, this requires that (see Fig. 10.21)

$$\left|H(0)\right| = \frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{1+(0.5088)^2}} = 0.8913$$

and therefore

$$\frac{A}{4.4100} = 0.8913 \qquad \Rightarrow \qquad A = 3.9305$$

The system function is

$$H(s) = \frac{3.9305}{s^4 + 1.9056\,s^3 + 5.8157\,s^2 + 5.9410\,s + 4.4100}$$

c. Minimum filter order is found using Eqn. (10.117) and Eqn. (10.118):

$$F = \sqrt{\frac{10^{20/10}-1}{10^{1/10}-1}} = 19.5538$$

$$N \ge \frac{\cosh^{-1}(19.5538)}{\cosh^{-1}(3.5/2)} = 3.1633$$

which must be rounded up to $N=4$. The parameter ε is found using Eqn. (10.114):

$$\varepsilon = \frac{1}{\sqrt{10^{20/10}-1}} = 0.1005$$

Parameter α_k is found as

$$\alpha_k = \frac{(2k+1)\,\pi}{8}\,, \qquad k = 0,\ldots,7$$

and the parameter β_k is

$$\beta_k = \frac{\sinh^{-1}(1/0.1005)}{4} = 0.7483$$

Poles of $H(s)\;H(-s)$ are found through the use of Eqn. (10.109) as

$$p_k = \frac{j2}{\cos\left(\dfrac{(2k+1)\,\pi}{8}\right)\cosh(0.7483) - j\,\sin\left(\dfrac{(2k+1)\,\pi}{8}\right)\sinh(0.0.7483)}\,, \qquad k = 0,\ldots,7$$

Locations of the poles of $H(s)\;H(-s)$ are as follows:

$$\begin{aligned}
p_0 &= -0.7198 + j\,2.7402\\
p_1 &= -3.2378 + j\,2.1149\\
p_2 &= -3.2378 - j\,2.1149\\
p_3 &= -0.7198 - j\,2.7402\\
p_4 &= 0.7198 - j\,2.7402\\
p_5 &= 3.2378 - j\,2.1149\\
p_6 &= 3.2378 + j\,2.1149\\
p_7 &= 0.7198 + j\,2.7402
\end{aligned}$$

Poles p_0, p_1, p_2 and p_3 are in the left half s-plane, and are therefore associated with $H(s)$. Zeros of the system function $H(s)$ are found using Eqns. (10.111) and (10.112):

$$z_k = \frac{\pm j\omega_2}{\cos\left(\dfrac{(2k-1)\,\pi}{2N}\right)}\,, \qquad k=1,2$$

Locations of the zeros of $H(s)$ are as follows:

$$z_1 = 0 \pm j\,3.7884$$
$$z_2 = 0 \pm j\,9.1459$$

System function is constructed as

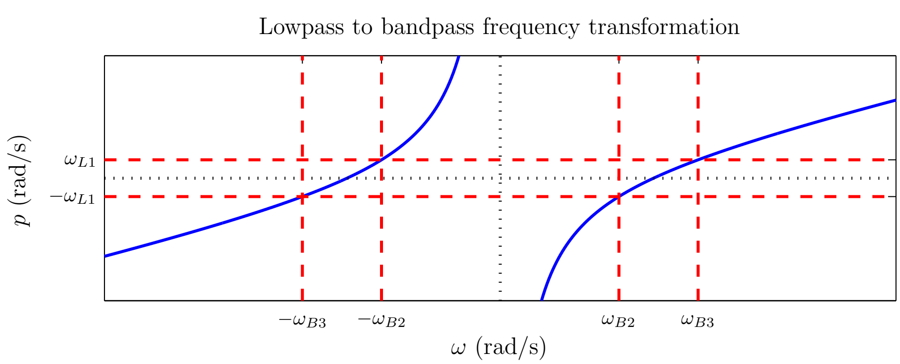
$$H(s) = \frac{A\,(s+j3.7884)\,(s-j3.7884)\,(s+j9.1459)\,(s-j9.1459)}{(s+0.7198-j2.7402)\,(s+3.2378-j2.1149)\,(s+3.2378+j2.1149)\,(s+0.7198+j2.7402)}$$
$$= \frac{0.1\,s^4+9.8\,s^2+120.05}{s^4+7.9152\,s^3+32.3049\,s^2+73.5079\,s+120.05}$$

The peak of the system function occurs at $\omega=0$. The gain factor A is adjusted to achieve $|H(0)|=1$.

10.11.

Let the system functions for lowpass and bandpass filters be $H_{LP}(s)$ and $H_{BP}(\lambda)$ respectively. Furthermore, let ω and p be the radian frequency variables in s and λ domains so that $s=j\,\omega$ and $\lambda = j\,p$. The lowpass to bandpass frequency transformation is

$$s = \frac{\lambda^2+\omega_0^2}{B\,\lambda} \quad\Rightarrow\quad j\,\omega = \frac{-p^2+\omega_0^2}{j\,B\,p} \quad\Rightarrow\quad -\omega = \frac{-p^2+\omega_0^2}{B\,p}$$



The goal is to map the band of frequencies $-\omega_{L1} < \omega < \omega_{L1}$ for the lowpass filter to the band of frequencies $\omega_{B2} < p < \omega_{B3}$ for the bandpass filter. At the two critical frequencies $\omega = \pm\omega_{L1}$ we have

$$-\,\omega_{L1} = \frac{-\omega_{B3}^2+\omega_0^2}{B\,\omega_{B3}} \qquad \text{and} \qquad \omega_{L1} = \frac{-\omega_{B2}^2+\omega_0^2}{B\,\omega_{B2}}$$

which can be solved together to yield

$$\frac{\omega_{B3}^2-\omega_0^2}{B\,\omega_{B3}}=\frac{-\omega_{B2}^2+\omega_0^2}{B\,\omega_{B2}} \qquad \text{and} \qquad \omega_0=\sqrt{\omega_{B2}\,\omega_{B3}}$$

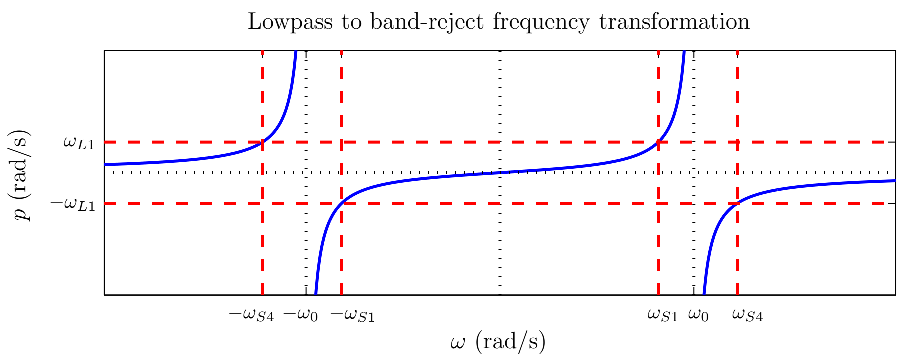
and

$$B=\frac{\omega_{B3}-\omega_{B2}}{\omega_{L1}}$$

10.12.

Let the system functions for lowpass and band-reject filters be $H_{LP}(s)$ and $H_{BR}(\lambda)$ respectively. Furthermore, let ω and p be the radian frequency variables in s and λ domains so that $s=j\,\omega$ and $\lambda=j\,p$. The lowpass to bandpass frequency transformation is

$$s=\frac{B\,\lambda}{\lambda^2+\omega_0^2} \qquad \Rightarrow \qquad j\,\omega=\frac{j\,B\,p}{-p^2+\omega_0^2} \qquad \Rightarrow \qquad \omega=\frac{-B\,p}{p^2-\omega_0^2}$$



The goal is to map the band of frequencies $-\omega_{L1}<\omega<\omega_{L1}$ for the lowpass filter to the three bands of frequencies, namely

Band 1: $\quad -\infty<p<\omega_{S4}$

Band 2: $\quad \omega_{S1}<p<\omega_{S1}$

Band 3: $\quad \omega_{S4}<p<\infty$

for the band-reject filter. At the two critical frequencies $\omega=\pm\omega_{L1}$ we have

$$-\omega_{L1}=\frac{-B\,\omega_{S4}}{\omega_{S4}^2-\omega_0^2} \qquad \text{and} \qquad \omega_{L1}=\frac{-B\,\omega_{S1}}{\omega_{S1}^2-\omega_0^2}$$

which can be solved together to yield

$$\frac{B\,\omega_{S4}}{\omega_{S4}^2-\omega_0^2}=\frac{-B\,\omega_{S1}}{\omega_{S1}^2-\omega_0^2} \qquad \text{and} \qquad \omega_0=\sqrt{\omega_{S1}\,\omega_{S4}}$$

and

$$B=(\omega_{S4}-\omega_{S1})\,\omega_{L1}$$

10.13.

a.

$$\omega_0 = \omega_{L1}\,\omega_{H2} = (2)\,(5) = 10$$

The transformation is

$$s = \frac{10}{\lambda}$$

b. Let $s = j\omega$ and $\lambda = jp$ so that

$$j\omega = \frac{10}{jp} \qquad \Rightarrow \qquad \omega = -\frac{10}{p}$$

If $\omega_1 = 6$ rad/s, the corresponding frequency of the highpass filter is

$$p_1 = -\frac{10}{6} = -1.6667 \text{ rad/s}$$

c.

$$H(\lambda) = \frac{2}{\left(\dfrac{10}{\lambda}\right) + 2} = \frac{\lambda}{\lambda + 5}$$

At $\lambda = j5$

$$\left|H(\lambda)\right|\Big|_{\lambda=j5} = \left|\frac{j5}{j5+5}\right| = \frac{1}{\sqrt{2}}$$

Similarly, for $G(s)$ at $s = j2$ we have

$$\left|G(s)\right|\Big|_{s=j2} = \left|\frac{j2}{j2+2}\right| = \frac{1}{\sqrt{2}}$$

10.14.

a. Let the passband edge frequency for the lowpass filter be $\omega_{L1} = 1$ rad/s. For the lowpass to highpass transformation we have

$$\omega_0 = \omega_{L1}\,\omega_{H2} = (1)\,(5) = 5$$

Therefore, the transformation is

$$s = \frac{5}{\lambda}$$

The frequency ω_{L1} for the lowpass filter corresponds to the frequency $-\omega_{H2}$ for the highpass filter. Similarly, the frequency ω_{L2} for the lowpass filter should correspond to the frequency $-\omega_{H1}$ for the highpass filter, therefore

$$\omega_{L2} = \frac{5}{\omega_{H1}} = 2.5 \text{ rad/s}$$

The specifications for the lowpass prototype are

$$\omega_{L1} = 2 \text{ rad/s} \qquad \omega_{L2} = 2.5 \text{ rad/s} \qquad R_p = 1 \text{ dB} \qquad A_s = 30 \text{ dB}$$

b. The minimum filter order is found using Eqn. (10.65):

$$N \geq \frac{\log_{10} \sqrt{\left(10^{30/10}-1\right) / \left(10^{1/10}-1\right)}}{\log_{10}(2.5/1)} = 4.5062$$

Filter order must be chosen as $N = 5$. If the excess tolerance that results from rounding up N to 5 is to be used for improving the passband response, we need to obtain ω_c from Eqn. (10.63):

$$\left(\frac{2.5}{\omega_c}\right)^{10} = 10^{30/10} - 1 \qquad \Rightarrow \qquad \omega_c = 1.2531 \ \ \text{rad/s}$$

c. The poles of $G(s)\,G(s)$ are at

$$p_k = 1.2531\, e^{jk\pi/5} \ , \qquad k = 0, \ldots, 9$$

Using the poles in the left half s-plane, $G(s)$ is constructed as

$$G(s) = \frac{3.0897}{s^5 + 4.0551\, s^4 + 8.2219\, s^3 + 10.3028\, s^2 + 7.9791\, s + 3.0897}$$

d.

$$H(\lambda) = G(s)\Big|_{s=5/\lambda} = \frac{3.0897}{\left(\frac{5}{\lambda}\right)^5 + 4.0551\left(\frac{5}{\lambda}\right)^4 + 8.2219\left(\frac{5}{\lambda}\right)^3 + 10.3028\left(\frac{5}{\lambda}\right)^2 + 7.9791\left(\frac{5}{\lambda}\right) + 3.0897}$$

$$= \frac{\lambda^5}{\lambda^5 + 12.91\,\lambda^4 + 83.36\,\lambda^3 + 332.63\,\lambda^2 + 820.28\,\lambda + 1011.4}$$

e.

$$20\log_{10}\left|H(2)\right| = -30 \ \ \text{dB}$$

$$20\log_{10}\left|H(5)\right| = -0.43 \ \ \text{dB}$$

10.15.

a. The partial fraction form of $G(s)$ is

$$G(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

and the corresponding impulse response is

$$g(t) = 2\,e^{-t}\,u(t) - 2\,e^{-2t}\,u(t)$$

b.

$$h[n] = 0.5\,g(0.5\,n) = e^{-0.5\,n}\,u[n] - e^{-n}\,u[n]$$

c.

$$H(z) = \frac{z}{z - e^{-0.5}} - \frac{z}{z - e^{-1}} = \frac{0.2387\,z}{z^2 - 0.9744\,z + 0.2231}$$

10.16.

$$s = \frac{2}{T}\,\frac{1 - z^{-1}}{1 + z^{-1}} = \frac{4\,(z-1)}{z+1}$$

The system function is

$$\begin{aligned}
H(z) &= \frac{2}{\left(\dfrac{4\,(z-1)}{z+1} + 1\right)\left(\dfrac{4\,(z-1)}{z+1} + 2\right)} \\
&= \frac{2\,(z+1)^2}{(5\,z-3)\,(6\,z-2)} \\
&= \frac{2\,z^2 + 4\,z + 2}{30\,z^2 - 28\,z + 6}
\end{aligned}$$

10.17.

a. The system function is

$$H(\Omega) = \sum_{n=0}^{N-1} h[n]\,e^{-j\Omega n}$$

Since N is even, the summation can be split into two halves:

$$H(\Omega) = \sum_{n=0}^{N/2-1} h[n]\,e^{-j\Omega n} + \sum_{n=N/2}^{N-1} h[n]\,e^{-j\Omega n}$$

Using the variable change $n = N - 1 - m$, the second summation can be written as

$$\sum_{n=N/2}^{N-1} h[n]\,e^{-j\Omega n} = \sum_{m=N/2-1}^{0} h[N-1-m]\,e^{-j\Omega\,(N-1-m)}$$

Recognizing that $h[N-1-m] = h[m]$ and that the summation limits can be swapped, we have

$$\sum_{n=N/2}^{N-1} h[n]\,e^{-j\Omega n} = e^{-j\,(N-1)} \sum_{m=0}^{N/2-1} h[m]\,e^{j\Omega\,m}$$

and $H(\Omega)$ is

$$\begin{aligned}
H(\Omega) &= \sum_{n=0}^{N/2-1} h[n]\,e^{-j\Omega n} + e^{-j\,(N-1)} \sum_{n=0}^{N/2-1} h[n]\,e^{j\Omega n} \\
&= \sum_{n=0}^{N/2-1} \left\{ h[n]\,e^{-j\Omega n} + e^{-j\,(N-1)}\,h[n]\,e^{j\Omega n} \right\}
\end{aligned}$$

Factoring out $e^{-j(N-1)/2}$ the result can be written as

$$\begin{aligned}
H(\Omega) =& e^{-j(N-1)/2} \sum_{n=0}^{N/2-1} \left\{ h[n]\, e^{-j\Omega n}\, e^{j\Omega(N-1)/2} + h[n]\, e^{j\Omega n}\, e^{-j\Omega(N-1)/2} \right\} \\
=& e^{-j(N-1)/2} \underbrace{\sum_{n=0}^{N/2-1} 2\, h[n] \cos\left[\Omega\left(n-\frac{N-1}{2}\right)\right]}_{\text{Purely real}} \\
=& e^{-j(N-1)/2} A(\Omega)
\end{aligned}$$

b. The system function is

$$H(\Omega) = \sum_{n=0}^{N-1} h[n]\, e^{-j\Omega n}$$

Since N is even, the summation can be split into two halves:

$$H(\Omega) = \sum_{n=0}^{N/2-1} h[n]\, e^{-j\Omega n} + \sum_{n=N/2}^{N-1} h[n]\, e^{-j\Omega n}$$

Using the variable change $n = N-1-m$, the second summation can be written as

$$\sum_{n=N/2}^{N-1} h[n]\, e^{-j\Omega n} = \sum_{m=N/2-1}^{0} h[N-1-m]\, e^{-j\Omega(N-1-m)}$$

Recognizing that $h[N-1-m] = -h[m]$ and that the summation limits can be swapped, we have

$$\sum_{n=N/2}^{N-1} h[n]\, e^{-j\Omega n} = -e^{-j(N-1)} \sum_{m=0}^{N/2-1} h[m]\, e^{j\Omega m}$$

and $H(\Omega)$ is

$$\begin{aligned}
H(\Omega) =& \sum_{n=0}^{N/2-1} h[n]\, e^{-j\Omega n} - e^{-j(N-1)} \sum_{n=0}^{N/2-1} h[n]\, e^{j\Omega n} \\
=& \sum_{n=0}^{N/2-1} \left\{ h[n]\, e^{-j\Omega n} - e^{-j(N-1)}\, h[n]\, e^{j\Omega n} \right\}
\end{aligned}$$

Factoring out $e^{-j(N-1)/2}$ the result can be written as

$$\begin{aligned}
H(\Omega) =& e^{-j(N-1)/2} \sum_{n=0}^{N/2-1} \left\{ h[n]\, e^{-j\Omega n}\, e^{j\Omega(N-1)/2} - h[n]\, e^{j\Omega n}\, e^{-j\Omega(N-1)/2} \right\} \\
=& e^{-j(N-1)/2} \sum_{n=0}^{N/2-1} -j\, 2\, h[n] \sin\left[\Omega\left(n-\frac{N-1}{2}\right)\right] \\
=& e^{-j(N-1)/2}\, e^{j\pi/2} \underbrace{\sum_{n=0}^{N/2-1} -2\, h[n] \sin\left[\Omega\left(n-\frac{N-1}{2}\right)\right]}_{\text{Purely real}} \\
=& e^{-j(N-1)/2}\, e^{j\pi/2}\, B(\Omega)
\end{aligned}$$

10.18.

a. The system function is

$$H(\Omega)=\sum_{n=0}^{N-1} h[n]\,e^{-j\Omega n}$$

Since N is odd, the summation must be split as follows:

$$H(\Omega)=\sum_{n=0}^{(N-3)/2} h[n]\,e^{-j\Omega n}+h[(N-1)/2]\,e^{-j\omega(N-1)/2}+\sum_{n=(N+1)/2}^{N-1} h[n]\,e^{-j\Omega n}$$

Using the variable change $n=N-1-m$, the second summation can be written as

$$\sum_{n=(N+1)/2}^{N-1} h[n]\,e^{-j\Omega n}=\sum_{m=(N-3)/2}^{0} h[N-1-m]\,e^{-j\Omega(N-1-m)}$$

Recognizing that $h[N-1-m]=h[m]$ and that the summation limits can be swapped, we have

$$\sum_{n=(N+1)/2}^{N-1} h[n]\,e^{-j\Omega n}=e^{-j(N-1)}\sum_{m=0}^{(N-3)/2} h[m]\,e^{j\Omega m}$$

and $H(\Omega)$ is

$$\begin{aligned}H(\Omega)&=\sum_{n=0}^{(N-3)/2} h[n]\,e^{-j\Omega n}+h[(N-1)/2]\,e^{-j\omega(N-1)/2}+e^{-j(N-1)}\sum_{n=0}^{(N-3)/2} h[n]\,e^{j\Omega n}\\&=\sum_{n=0}^{(N-3)/2}\left\{h[n]\,e^{-j\Omega n}+e^{-j(N-1)}\,h[n]\,e^{j\Omega n}\right\}+h[(N-1)/2]\,e^{-j\omega(N-1)/2}\end{aligned}$$

Factoring out $e^{-j(N-1)/2}$ the result can be written as

$$\begin{aligned}H(\Omega)=&e^{-j(N-1)/2}\left[\sum_{n=0}^{(N-3)/2}\left\{h[n]\,e^{-j\Omega n}\,e^{j\Omega(N-1)/2}+h[n]\,e^{j\Omega n}\,e^{-j\Omega(N-1)/2}\right\}+h[(N-1)/2]\right]\\=&e^{-j(N-1)/2}\underbrace{\left[\sum_{n=0}^{(N-3)/2} 2\,h[n]\,\cos\left[\Omega\left(n-\frac{N-1}{2}\right)\right]+h[(N-1)/2]\right]}_{\text{Purely real}}\\=&e^{-j(N-1)/2}\,A(\Omega)\end{aligned}$$

b. The system function is

$$H(\Omega)=\sum_{n=0}^{N-1} h[n]\,e^{-j\Omega n}$$

It should be noted that, for the odd symmetry to work in this case, the center sample must be zero, that is, $h[(N-1)/2]=0$ as no other value would be equal to its own negative. The summation can be split into two halves as follows:

$$H(\Omega)=\sum_{n=0}^{(N-3)/2} h[n]\,e^{-j\Omega n}+\sum_{n=(N+1)/2}^{N-1} h[n]\,e^{-j\Omega n}$$

Using the variable change $n = N-1-m$, the second summation can be written as

$$\sum_{n=(N+1)/2}^{N-1} h[n]\, e^{-j\Omega n} = \sum_{m=(N-3)/2}^{0} h[N-1-m]\, e^{-j\Omega(N-1-m)}$$

Recognizing that $h[N-1-m] = -h[m]$ and that the summation limits can be swapped, we have

$$\sum_{n=(N+1)/2}^{N-1} h[n]\, e^{-j\Omega n} = -e^{-j(N-1)} \sum_{m=0}^{(N-3)/2} h[m]\, e^{j\Omega m}$$

and $H(\Omega)$ is

$$\begin{aligned} H(\Omega) &= \sum_{n=0}^{(N-3)/2} h[n]\, e^{-j\Omega n} - e^{-j(N-1)} \sum_{n=0}^{(N-3)/2} h[n]\, e^{j\Omega n} \\ &= \sum_{n=0}^{(N-3)/2} \left\{ h[n]\, e^{-j\Omega n} - e^{-j(N-1)}\, h[n]\, e^{j\Omega n} \right\} \end{aligned}$$

Factoring out $e^{-j(N-1)/2}$ the result can be written as

$$\begin{aligned} H(\Omega) =& e^{-j(N-1)/2} \sum_{n=0}^{(N-3)/2} \left\{ h[n]\, e^{-j\Omega n}\, e^{j\Omega(N-1)/2} - h[n]\, e^{j\Omega n}\, e^{-j\Omega(N-1)/2} \right\} \\ =& e^{-j(N-1)/2} \sum_{n=0}^{(N-3)/2} -j2\, h[n]\, \sin\left[\Omega\left(n - \frac{N-1}{2}\right)\right] \\ =& e^{-j(N-1)/2}\, e^{j\pi/2} \underbrace{\sum_{n=0}^{(N-3)/2} -2\, h[n]\, \sin\left[\Omega\left(n - \frac{N-1}{2}\right)\right]}_{\text{Purely real}} \\ =& e^{-j(N-1)/2}\, e^{j\pi/2}\, B(\Omega) \end{aligned}$$

10.19.

Spectrum of an ideal bandpass filter can be expressed as the difference of the spectra of two lowpass filters:

$$\begin{aligned} H_{BP}(\Omega) =& \frac{\Omega_2}{\pi}\, \text{sinc}\left(\frac{\Omega_2 n}{\pi}\right) - \frac{\Omega_2}{\pi}\, \text{sinc}\left(\frac{\Omega_2 n}{\pi}\right) \\ =& 0.7\, \text{sinc}(0.7\, n) - 0.3\, \text{sinc}(0.3\, n) \end{aligned}$$

Since $N = 2M+1 = 19$ we have $M = 9$. Hamming window with $M = 9$ is

$$w[n] = 0.54 - 0.46\, \cos\left(\frac{\pi\,(n+M)}{M}\right)$$

Coefficients are listed in the table on the left. The impulse response of the designed filter is

$$h[n] = h_T[n-9]\, w[n-9]$$

and is listed in the table on the right.

n	$h_T[n]$	$w[n]$	$h_T[n]\,w[n]$
−9	0.0000	0.0800	0.0000
−8	0.0757	0.1077	0.0082
−7	0.0000	0.1876	0.0000
−6	−0.0624	0.3100	−0.0193
−5	0.0000	0.4601	0.0000
−4	−0.0935	0.6199	−0.0580
−3	0.0000	0.7700	0.0000
−2	0.3027	0.8924	0.2702
−1	0.0000	0.9723	0.0000
0	−0.4000	1.0000	−0.4000
1	0.0000	0.9723	0.0000
2	0.3027	0.8924	0.2702
3	0.0000	0.7700	0.0000
4	−0.0935	0.6199	−0.0580
5	0.0000	0.4601	0.0000
6	−0.0624	0.3100	−0.0193
7	0.0000	0.1876	0.0000
8	0.0757	0.1077	0.0082
9	0.0000	0.0800	0.0000

n	$h[n]$
0	0.0000
1	0.0082
2	0.0000
3	−0.0193
4	0.0000
5	−0.0580
6	0.0000
7	0.2702
8	0.0000
9	−0.4000
10	0.0000
11	0.2702
12	0.0000
13	−0.0580
14	0.0000
15	−0.0193
16	0.0000
17	0.0082
18	0.0000

10.20.

Using the inverse DTFT with the desired spectrum $H_d(\Omega)$ yields

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\Omega)\, e^{j\Omega n}\, d\Omega \\ &= \int_{-\pi}^{\pi} (1)\, e^{-j\Omega(N-1)/2}\, e^{j\Omega n}\, d\Omega \\ &= \frac{\Omega_c}{\pi}\,\text{sinc}\left[\frac{\Omega_c}{\pi}\left(n-\frac{N-1}{2}\right)\right] \end{aligned}$$

If normalized frequency F_c is used, then

$$h_d[n] = 2F_c\,\text{sinc}\left[2F_c\left(n-\frac{N-1}{2}\right)\right]$$

Truncating the ideal impulse response $h_d[n]$ to keep only the samples for $n = 0, \ldots, N-1$ yields

$$h_T[n] = \begin{cases} h_d[n]\,, & n = 0, \ldots, N-1 \\ 0\,, & \text{otherwise} \end{cases}$$

Check for linear phase:

$$\begin{aligned} h_d[N-1-n] =& 2F_c \operatorname{sinc}\left[2F_c\left(N-1-n-\frac{N-1}{2}\right)\right] \\ =& 2F_c \operatorname{sinc}\left[2F_c\left(-n+\frac{N-1}{2}\right)\right] \\ =& h_d[n]\,, \qquad n=0,\ldots,N-1 \end{aligned}$$

10.21.

a. Triangular window:

$$w[n] = 1-\alpha\left|n+\beta\right|$$

The center of the window function needs to be at $n(N-1)/2$. Therefore

$$\beta = -\frac{N-1}{2}$$

We also need $w[0] = w[N-1] = 0$. Using the value at either end results in

$$\alpha = \frac{2}{N-1}$$

Using the values of α and β found, the triangular window function is

$$w[n] = 1-\left(\frac{2}{N-1}\right)\left|n-\frac{N-1}{2}\right|$$

b. Hanning window

$$w[n] = 0.5-0.5\cos\left(\alpha\, n+\beta\right)$$

The center of the window function needs to be at $n(N-1)/2$. Therefore

$$\cos\left(\alpha\,\frac{N-1}{2}+\beta\right) = -1 \qquad \Rightarrow \qquad \alpha\,\frac{N-1}{2}+\beta = \pi$$

We also need $w[0] = w[N-1] = 0$. Using the value at $n=0$ results in

$$\cos\left(\beta\right) = 1 \qquad \Rightarrow \qquad \beta = 0 \qquad \Rightarrow \qquad \alpha = \frac{2\pi}{N-1}$$

Using the values of α and β found, the Hamming window function is

$$w[n] = 0.5-0.5\cos\left(\frac{2\pi n}{N-1}\right)$$

c. Hamming window

$$w[n] = 0.54-0.46\cos\left(\alpha\, n+\beta\right)$$

The center of the window function needs to be at $n(N-1)/2$. Therefore

$$\cos\left(\alpha\,\frac{N-1}{2}+\beta\right) = -1 \qquad \Rightarrow \qquad \alpha\,\frac{N-1}{2}+\beta = \pi$$

We also need $w[0]$ and $w[N-1]$ to be minimum, that is, $w[0]=w[N-1]=0.08$. Using the value at $n=0$ results in

$$\cos\left(\beta\right)=1 \qquad \Rightarrow \qquad \beta=0 \qquad \Rightarrow \qquad \alpha=\frac{2\pi}{N-1}$$

Using the values of α and β found, the Hamming window function is

$$w[n]=0.54-0.46\,\cos\left(\frac{2\pi n}{N-1}\right)$$

d. Blackman window

$$w[n]=0.42-0.5\,\cos\left(\alpha\, n+\beta\right)+0.08\,\cos\left(\alpha\, n+\beta\right)$$

Parameters α and β are the same as in parts (b) and (c):

$$\alpha=\frac{2\pi}{N-1}\,, \qquad \beta=0$$

$$0.42-0.5\,\cos\left(\frac{2\pi n}{N-1}\right)+0.08\,\cos\left(\frac{4\pi n}{N-1}\right)$$

10.22. a.

```
p = 1;
% Determine the frequency omg0 for p percent magnitude variation
omg0 = 1000*sqrt((100/(100-p))^2-1);
% Define anonymous function for H(omg)
H = @(omg) 1./(1+j*omg/1000);
% Graph the magnitude in the range -omg0,+omg0
omg = [-omg0:0.1:omg0];
subplot(3,1,1);
plot(omg,abs(H(omg)));
axis([-omg0,omg0,0.99,1]);
title('|H(\omega)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
% Graph the phase in the range -omg0,+omg0
subplot(3,1,2);
plot(omg,angle(H(omg)));
axis([-omg0,omg0,-pi/18,pi/18]);
title('Phase of H(\omega)');
xlabel('\omega (rad/s)');
ylabel('Phase (rad)');
grid;
% Graph the time delay in the range -omg0,omg0
subplot(3,1,3);
omg = omg+eps; % Avoid division by 0
tdelay = -angle(H(omg))./omg;
plot(omg,tdelay);
axis([-omg0,omg0,0.0009,0.0011]);
```

```
29 title('Time delay for H(\omega)');
30 xlabel('\omega (rad/s)');
31 ylabel('Delay (sec)');
32 grid;
```

b. Modify line 1 as follows and repeat.

```
1 p = 2;
```

10.23.

a.

```
p = 1;
% Determine the frequency omg0 for 1 percent magnitude variation
tmp = (100/(100-p))^2-1;
omg0 = sqrt(sqrt(tmp));
% Define anonymous function for H(omg)
H = @(omg) 1./((1-omg.*omg)+j*sqrt(2)*omg);
% Graph the magnitude in the range -omg0,+omg0
omg = [-omg0:0.01:omg0];
subplot(3,1,1);
plot(omg,abs(H(omg)));
axis([-omg0,omg0,0.99,1]);
title('|H(\omega)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
% Graph the phase in the range -omg0,+omg0
subplot(3,1,2);
plot(omg,angle(H(omg)));
axis([-omg0,omg0,-0.75,0.75]);
title('Phase of H(\omega)');
xlabel('\omega (rad/s)');
ylabel('Phase (rad)');
grid;
% Graph the time delay in the range -omg0,omg0
subplot(3,1,3);
omg = omg+eps;  % Avoid division by 0
tdelay = -angle(H(omg))./omg;
plot(omg,tdelay);
axis([-omg0,omg0,1.2,1.5]);
title('Time delay for H(\omega)');
xlabel('\omega (rad/s)');
ylabel('Delay (sec)');
grid;
```

b. Modify line 1 as follows and repeat.

```
p = 2;
```

10.24.

a.

```
function h = ss_ilp(omg0,td,t)
  f0 = omg0/(2*pi);
  h = 2*f0*sinc(2*f0*(t-td));        % Eqn. (10.32)
end
```

b. Duplicate the result of Problem 10.3.

```
t = [-0.1:0.0001:0.2];
hLP = ss_ilp(200*pi,0.1,t);
plot(t,hLP);
grid;
```

10.25.

a.

```
function h = ss_ibp(omg1,omg2,td,t)
  h = ss_ilp(omg2,td,t)-ss_ilp(omg1,td,t);
end
```

b. Duplicate the result of Problem 10.4.

```
t = [0.275:0.00005:0.325];
hBP = ss_ibp(200*pi,700*pi,0.3,t);
plot(t,hBP);
axis([0.275,0.325,-350,550]);
xlabel('t (sec)');
ylabel('Amplitude');
title('h_{BP}(t)');
grid;
```

10.26.

The filter can be designed using the statement

```
[num,den] = butter(4,2,'s')
```

which results in numerator and denominator polynomials

```
num =
         0         0         0         0   16.0000

den =
    1.0000    5.2263   13.6569   20.9050   16.0000
```

The system function can be evaluated at a set of radian frequencies using the following code:

```
omg = [0:0.01:5];
H = freqs(num,den,omg);
```

Graph the magnitude response:

```
plot(omg,abs(H));
axis([0,5,0,1.2]);
xlabel('\omega (rad/s)');
ylabel('Magnitude');
title('|H(\omega)|');
grid;
```

Graph the phase response:

```
plot(omg,angle(H));
axis([0,5,-pi,pi]);
xlabel('\omega (rad/s)');
ylabel('Phase (rad)');
title('Phase of H(\omega)');
grid;
```

10.27.

The filter is designed using the following statements:

```
N = 11;
omgc = 3.2875;
[num,den] = butter(N,omgc,'s')
```

Evaluate the system function for $\omega = 0, \ldots, 6$ rad/s.

```
omg = [0:0.01:6];
H = freqs(num,den,omg);
```

Graph the dB magnitude of the system function.

```
plot(omg,20*log10(abs(H)));
axis([0,6,-60,10]);
xlabel('\omega (rad/s)');
ylabel('Magnitude (dB)');
title('|H(\omega)|_{dB}');
grid;
```

10.28.

a. The system function for the filter is found using the following statements:

```
omg1 = 2;
epsilon = 0.3;
% Maximum passband ripple
Rp = 10*log10(1+epsilon^2)
[num,den] = cheby1(4,Rp,omg1,'s')
```

b. Evaluate the system function in the interval $\omega = 0, \ldots, 5$ rad/s.

```
omg = [0:0.01:5];
H = freqs(num,den,omg);
```

Graph $|H(\omega)|$:

```
plot(omg,abs(H));
axis([0,5,0,1.2]);
title('|H(s)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
```

Graph $\measuredangle H(\omega)$:

```
plot(omg,angle(H));
axis([0,5,-pi,pi]);
title('Phase of H(s)');
xlabel('\omega (rad/s)');
ylabel('Phase (rad)');
grid;
```

c. Pole-zero plot is obtained through the following:

```
p = roots(den)
z = roots(num)
plot(real(p),imag(p),'rx',real(z),imag(z),'ro');
axis([-2,2,-3,3]);
xlabel('Real');
ylabel('Imag');
grid;
```

10.29.

a. The system function for the filter is found using the following statements:

```
omg2 = 2;
epsilon = 0.3;
% Minimum stopband attenuation
As = 10*log10((1+epsilon^2)/epsilon^2);
% Compute system function using MATLAB function cheby2()
[num,den] = cheby2(4,As,omg2,'s')
```

b. Evaluate the system function in the interval $\omega = 0, \ldots, 5$ rad/s.

```
omg = [0:0.01:5];
H = freqs(num,den,omg);
```

Graph $|H(\omega)|$:

```
plot(omg,abs(H));
axis([0,5,0,1.2]);
title('|H(s)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
```

Graph $\measuredangle H(\omega)$:

```
plot(omg,angle(H));
axis([0,5,-pi,pi]);
title('Phase of H(s)');
xlabel('\omega (rad/s)');
ylabel('Phase (rad)');
grid;
```

c. Pole-zero plot is obtained through the following:

```
p = roots(den)
z = roots(num)
plot(real(p),imag(p),'rx',real(z),imag(z),'ro');
axis([-3,3,-6,6]);
xlabel('Real');
ylabel('Imag');
grid;
```

10.30.

a.

```
omg1 = 2;
omg2 = 3.5;
Rp = 1;
As = 20;
% Butterworth design
[N1,omgc] = buttord(omg1,omg2,Rp,As,'s')
[num1,den1] = butter(N1,omgc,'s')
% Chebyshev type-I design
N2 = cheb1ord(omg1,omg2,Rp,As,'s')
[num2,den2] = cheby1(N2,Rp,omg1,'s')
% Chebyshev type-II design
N3 = cheb2ord(omg1,omg2,Rp,As,'s')
[num3,den3] = cheby2(N3,As,omg2,'s')
```

b. The following code evaluates the system function for each design.

```
omg = [0:0.01:6];
H1 = freqs(num1,den1,omg);
H2 = freqs(num2,den2,omg);
H3 = freqs(num3,den3,omg);
```

Graph $|H(s)|$ for the Butterworth design.

```
plot(omg,abs(H1));
axis([0,6,0,1.2]);
title('|H(s)| for the Butterworth design');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
```

Graph $|H(s)|$ for the Chebyshev type-I design.

```
plot(omg,abs(H2));
axis([0,6,0,1.2]);
title('|H(s)| for the Chebyshev type-I design');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
```

Graph $|H(s)|$ for the Chebyshev type-II design.

```
plot(omg,abs(H3));
axis([0,6,0,1.2]);
title('|H(s)| for the Chebyshev type-II design');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
```

c. Plot the dB magnitude of all three designs simultaneously.

```
plot(omg,20*log10(abs(H1)),omg,20*log10(abs(H2)),omg,20*log10(abs(H3)));
axis([0,6,-50,10]);
title('|H(s)|');
xlabel('\omega (rad/s)');
ylabel('dB Magnitude');
legend('Butterworth','Chebyshev type-I','Chebyshev type-II');
grid;
```

10.31.

Design the filter:

```
% Set up filter specifications
omg1 = 1;
omg2 = 2.5;
Rp = 1;
As = 30;
% Design the lowpass prototype
[N, omgc] = buttord(omg1,omg2,Rp,As,'s')
[numL,denL] = butter(N,omgc,'s')
% Lowpass to highpass transformation
[numH,denH] = lp2hp(num,den,5)
```

Evaluate and graph the dB magnitude:

```
omg = [1:0.01:10];
H = freqs(numH,denH,omg);
plot(omg,20*log10(abs(H)));
axis([0,10,-70,10]);
title('|H(\omega)|_{dB}');
xlabel('\omega (rad/s)');
ylabel('Magnitude (dB)');
grid;
```

10.32.

a. The discrete-time filter can be designed using the following statements:

```
num = [2];
den = [1,3,2];
[numz,denz] = impinvar(num,den,2)
```

MATLAB response is

```
numz =
          0      0.2387

denz =
     1.0000    -0.9744     0.2231
```

and corresponds to

$$H(z) = \frac{0 + 0.2387\,z^{-1}}{1 - 0.9744\,z^{-1} + 0.2231\,z^{-2}} = \frac{0.2387\,z}{z^2 - 0.9744\,z + 0.2231}$$

b.

```
omg = [0:0.01:2*pi];
G = freqs(num,den,omg);
plot(omg,abs(G));
axis([0,2*pi,0,1.2]);
title('|G(\omega)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
```

c.

```
Omg = [0:0.01:1]*pi;
H = freqz(numz,denz,Omg);
plot(Omg,abs(H),'r');
axis([0,pi,0,1.2]);
title('|H(\Omega)|');
xlabel('\Omega (rad)');
ylabel('Magnitude');
grid;
```

d. Use $\omega = \Omega / T_s = 2\,\Omega$. Graph $|G(\omega)|$ and $|Y_a(\omega)/X_a(\omega)|$ up to $f_{max} = f_s/2 = 1$ Hz, or equivalently, up to $\omega_{max} = \omega_s/2 = 2\pi$ rad/s.

```
plot(omg,abs(G),'b',2*Omg,abs(H),'r');
axis([0,2*pi,0,1.2]);
title('|G(\omega)|   and   |Y_{a}(\omega)/X_{a}(\omega)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
legend('|G(\omega)|','|Y_{a}(\omega)/X_{a}(\omega)|');
grid;
```

10.33.

a. The discrete-time filter can be designed using the following statements:

```
num = [2];
den = [1,3,2];
[numz,denz] = impinvar(num,den,4)
```

MATLAB response is

```
numz =
        0    0.0861

denz =
    1.0000   -1.3853    0.4724
```

and corresponds to

$$H(z) = \frac{0 + 0.0861\,z^{-1}}{1 - 1.3853\,z^{-1} + 0.4724\,z^{-2}} = \frac{0.0861\,z}{z^2 - 1.3853\,z + 0.4724}$$

b.

```
omg = [0:0.01:4*pi];
G = freqs(num,den,omg);
plot(omg,abs(G));
axis([0,4*pi,0,1.2]);
title('|G(\omega)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
```

c.

```
Omg = [0:0.01:1]*pi;
H = freqz(numz,denz,Omg);
plot(Omg,abs(H),'r');
axis([0,pi,0,1.2]);
title('|H(\Omega)|');
xlabel('\Omega (rad)');
ylabel('Magnitude');
grid;
```

d. Use $\omega = \Omega/T_s = 4\Omega$. Graph $|G(\omega)|$ and $|Y_a(\omega)/X_a(\omega)|$ up to $f_{max} = f_s/2 = 2$ Hz, or equivalently, up to $\omega_{max} = \omega_s/2 = 4\pi$ rad/s.

```
plot(omg,abs(G),'b',4*Omg,abs(H),'r');
axis([0,4*pi,0,1.2]);
title('|G(\omega)|  and  |Y_{a}(\omega)/X_{a}(\omega)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
legend('|G(\omega)|','|Y_{a}(\omega)/X_{a}(\omega)|');
grid;
```

This design has a lower degree of aliasing compared to the one in Problem 10.32 due to the use of a higher sampling rate. It should also be noted, however, that the bandwidth of the equivalent analog system is approximately halved in the process.

10.34.

a. The discrete-time filter can be designed using the following statements:

```
num = [2];
den = [1,3,2];
[numz, denz] = bilinear(num, den, 2)
```

MATLAB response is

```
numz =
    0.0667    0.1333    0.0667

denz =
    1.0000   -0.9333    0.2000
```

and corresponds to

$$H(z) = \frac{0.0667 + 0.1333\,z^{-1} + 0.0667}{1 - 0.9333\,z^{-1} + 0.2\,z^{-2}} = \frac{0.0667\,z^2 + 0.1333\,z + 0.0667}{z^2 - 0.9333\,z + 0.2}$$

b.

```
omg = [0:0.01:2*pi];
G = freqs(num, den, omg);
plot(omg, abs(G));
axis([0,2*pi,0,1.2]);
title('|G(\omega)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
grid;
```

c.

```
Omg = [0:0.01:1]*pi;
H = freqz(numz, denz, Omg);
plot(Omg, abs(H), 'r');
axis([0, pi, 0, 1.2]);
title('|H(\Omega)|');
xlabel('\Omega (rad)');
ylabel('Magnitude');
grid;
```

d. Use $\omega = \Omega / T_s = 2\Omega$. Graph $|G(\omega)|$ and $|Y_a(\omega) / X_a(\omega)|$ up to $f_{max} = f_s/2 = 1$ Hz, or equivalently, up to $\omega_{max} = \omega_s/2 = 2\pi$ rad/s.

```
plot(omg,abs(G),'b',2*Omg,abs(H),'r');
axis([0,2*pi,0,1.2]);
title('|G(\omega)|  and  |Y_{a}(\omega)/X_{a}(\omega)|');
xlabel('\omega (rad/s)');
ylabel('Magnitude');
legend('|G(\omega)|','|Y_{a}(\omega)/X_{a}(\omega)|');
grid;
```

10.35.

a.

```
h = [0.0000,0.0082,0.0000,-0.0193,0.0000,-0.0580,0.0000,0.2702, ...
     0.0000,-0.4000,0.0000,0.2702,0.0000,-0.0580,0.0000,-0.0193,...
     0.0000,0.0082,0.0000];
Omg = [-256:255]/256*pi;               % Compute DFT frequencies
Hmag = abs(fftshift(fft(h,512)));      % 512-point DFT
plot(Omg,Hmag);
axis([-pi,pi,0,1.2]);
title('|H(\Omega)|');
xlabel('\Omega (rad)');
ylabel('Magnitude');
grid;
```

b.

```
% Ideal lowpass filter impulse response
hLP = @(n,Omgc) Omgc/pi*sinc(Omgc*n/pi);          % Eqn. (10.172)
% Ideal bandpass filter impulse response
hBP = @(n,Omg1,Omg2) hLP(n,Omg2)-hdLP(n,Omg1);    % Bandpass
% Anonymous function for Hamming window
wHamm = @(n,M) 0.54-0.46*cos(pi*(n+M)/M);         % Eqn. (10.181)
% Truncated impulse response for bandpass filter
n = [-22:22]';                                    % N=45=2M+1 therefore M=22
hT = hBP(n,0.7*pi,0.3*pi);
% Let h1 be the impulse response of the bandpas filter designed using
% the Hamming window
h1 = hT.*wHamm(n,22);
% Compute and graph the magnitude characteristic
H1mag = abs(fftshift(fft(h1,512)));
plot(Omg,10*log10(Hmag),Omg,10*log10(H1mag));
axis([-pi,pi,-50,20]);
title('|H(\Omega)|_{dB}  and  |H_{1}(\Omega)|_{dB}');
xlabel('\Omega (rad)');
ylabel('Magnitude');
legend('|H(\Omega)|_{dB}','|H_{1}(\Omega)|_{dB}','Location','North');
grid;
```

c.

```
% Anonymous function for Blackman window
wBlk = @(n,M) 0.42-0.5*cos(pi*(n+M)/M)+...
       0.08*cos(2*pi*(n+M)/M);                    % Eqn. (10.183)
% Let h2 be the impulse response of the bandpas filter designed using
% the Blackman window
h2 = hT.*wBlk(n,22);
% Compute and graph the magnitude characteristic
H2mag = abs(fftshift(fft(h2,512)));
plot(Omg,10*log10(H1mag),Omg,10*log10(H2mag));
axis([-pi,pi,-50,20]);
title('|H_{1}(\Omega)|_{dB}   and   |H_{2}(\Omega)|_{dB}');
xlabel('\Omega (rad)');
ylabel('Magnitude');
legend('|H_{1}(\Omega)|_{dB}','|H_{2}(\Omega)|_{dB}','Location','North');
grid;
```

10.36.

a.

```
function hd = ss_fir1(Omgc,N)
  n = [0:N-1];
  tdelay = (N-1)/2;
  hd = Omgc/pi*sinc(Omgc*(n-tdelay)/pi);
end
```

b. Compute and graph the impulse response $h_d[n]$:

```
n = [0:35];
hd = ss_fir1(0.4*pi,36);
stem(n,hd);
title('Impulse response');
xlabel('n');
ylabel('h[n]');
```

Compute and graph the magnitude $|H_d()|$:

```
Omg = [-256:255]/256*pi;
Hmag = abs(fftshift(fft(hd,512)));
plot(Omg,Hmag);
axis([-pi,pi,0,1.2]);
title('|H(\Omega)|');
xlabel('\Omega (rad)');
ylabel('Magnitude');
grid;
```

Chapter 11

Amplitude Modulation

11.1.

a. $\mu = \dfrac{A_m}{A_c} = \dfrac{2}{A_c} = 0.7 \qquad \Rightarrow \qquad A_c = \dfrac{2}{0.7} = 2.86$

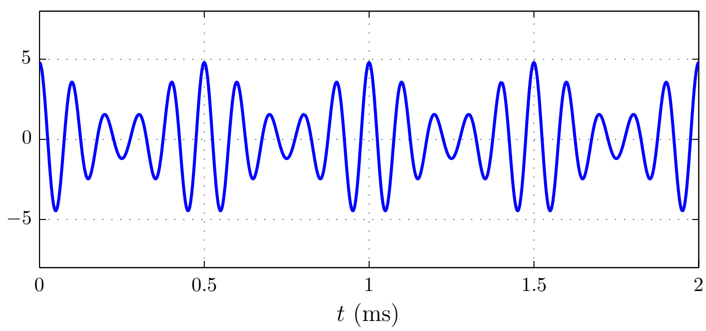
b. $\mu = \dfrac{A_m}{A_c} = \dfrac{8}{A_c} = 0.85 \qquad \Rightarrow \qquad A_c = \dfrac{3}{0.85} = 3.53$

c. $\mu = \dfrac{A_m}{A_c} = \dfrac{A_m}{2} = 0.9 \qquad \Rightarrow \qquad A_m = 1.8$

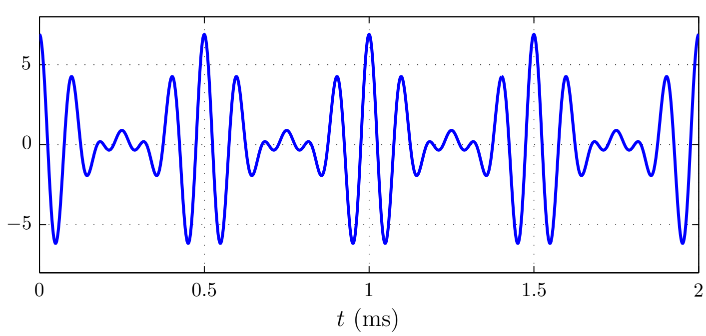
d. $\mu = \dfrac{A_m}{A_c} = \dfrac{2.5}{3} = 0.83$

11.2.

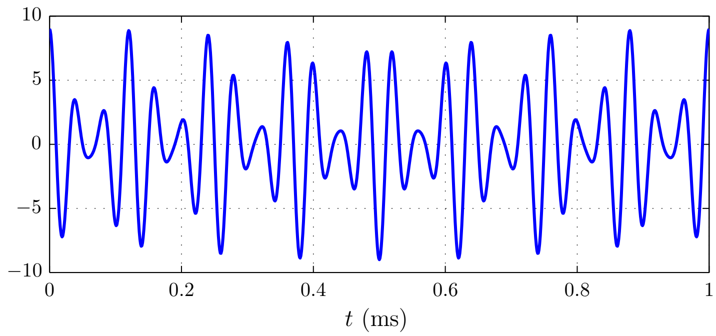
a.



b.



c.

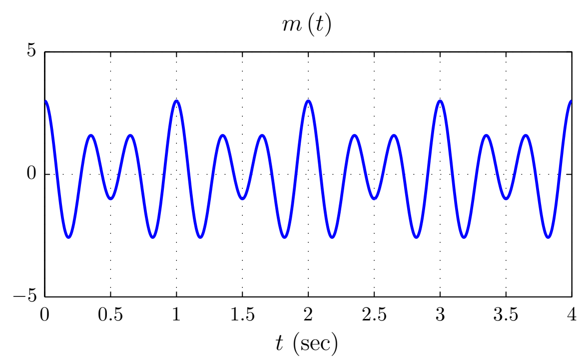


11.3.

a.

$$m_{\min} = -2.5746$$

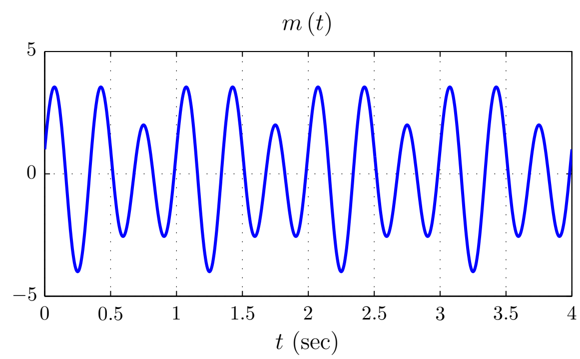
$$A_c = \frac{|m_{\min}|}{\mu} = \frac{2.5746}{0.7} = 3.68$$



b.

$$m_{\min} = -4$$

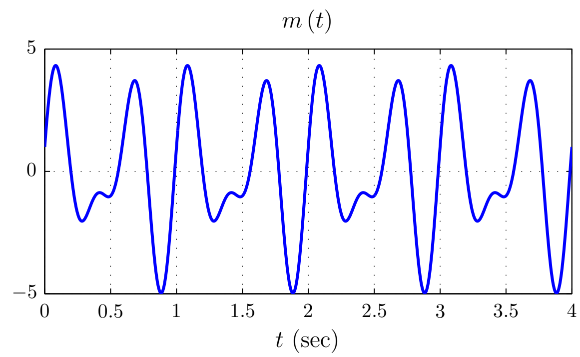
$$A_c = \frac{|m_{\min}|}{\mu} = \frac{4}{0.7} = 5.71$$



c.

$$m_{\text{min}} = -4.971$$

$$A_c = \frac{|m_{\text{min}}|}{\mu} = \frac{4.971}{0.7} = 7.10$$



11.4.

a. The signal $x_{AM}\,(t)$ is

$$\begin{aligned} x_{AM}\,(t) &= A_c\left[1+\mu\cos\left(2\pi f_m t\right)\right]\cos\left(2\pi f_c t\right) \\ &= 3\left[1+0.6\,\cos\left(4\pi t\right)\right]\cos\left(20\pi t\right) \\ &= 3\cos\left(20\pi t\right)+0.9\,\cos\left(16\pi t\right)+0.9\,\cos\left(24\pi t\right) \end{aligned}$$

The carrier signal can be written as

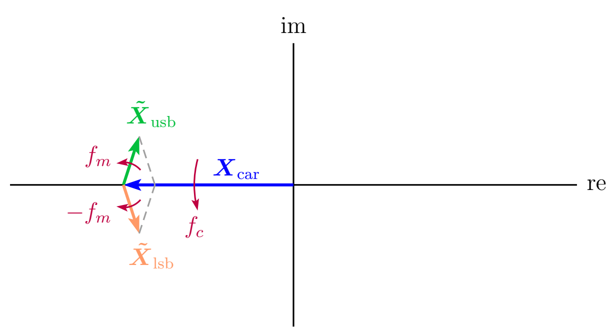
$$x_{\text{car}}\,(t) = \text{Re}\left\{\boldsymbol{X}_{\text{car}}\,e^{j2\pi f_c t}\right\} = \text{Re}\left\{3\,e^{j20\pi t}\right\}$$

Similarly, lower and upper sideband signals are

$$x_{\text{lsb}}\,(t) = \text{Re}\left\{\boldsymbol{X}_{\text{lsb}}\,e^{j2\pi(f_c-f_m)t}\right\} = \text{Re}\left\{0.9\,e^{j16\pi t}\right\} \qquad \text{and} \qquad x_{\text{usb}}\,(t) = \text{Re}\left\{\boldsymbol{X}_{\text{usb}}\,e^{j2\pi(f_c+f_m)t}\right\} = \text{Re}\left\{0.9\,e^{j24\pi t}\right\}$$

At the time instant $t = 0.35$ seconds we get

$$\begin{aligned} x_{\text{car}}\,(0.35) &= \text{Re}\left\{3\,e^{j21.9911}\right\} = \text{Re}\left\{-3+j\,0\right\} \\ x_{\text{lsb}}\,(0.35) &= \text{Re}\left\{0.9\,e^{j17.5929}\right\} = \text{Re}\left\{0.2781-j\,0.8560\right\} \\ x_{\text{usb}}\,(0.35) &= \text{Re}\left\{0.9\,e^{j26.3894}\right\} = \text{Re}\left\{0.2781+j\,0.8560\right\} \end{aligned}$$



b. The signal $x_{AM}(t)$ is

$$\begin{aligned} x_{AM}(t) &= A_c\left[1+\mu\cos\left(2\pi f_m t\right)\right]\cos\left(2\pi f_c t\right) \\ &= 3\left[1+1.3\cos\left(4\pi t\right)\right]\cos\left(20\pi t\right) \\ &= 3\cos\left(20\pi t\right)+1.95\cos\left(16\pi t\right)+1.95\cos\left(24\pi t\right) \end{aligned}$$

The carrier signal can be written as

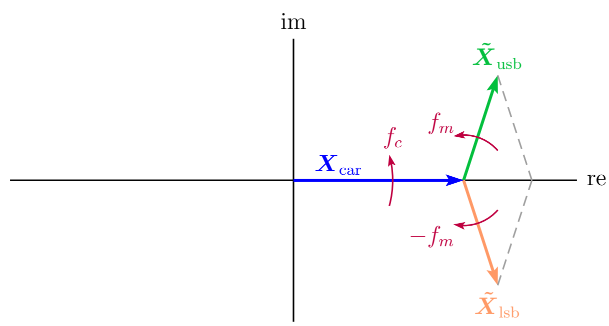
$$x_{\text{car}}(t) = \text{Re}\left\{\boldsymbol{X}_{\text{car}}\,e^{j2\pi f_c t}\right\} = \text{Re}\left\{3\,e^{j20\pi t}\right\}$$

Similarly, lower and upper sideband signals are

$$x_{\text{lsb}}(t) = \text{Re}\left\{\boldsymbol{X}_{\text{lsb}}\,e^{j2\pi(f_c-f_m)t}\right\} = \text{Re}\left\{1.95\,e^{j16\pi t}\right\} \qquad \text{and} \qquad x_{\text{usb}}(t) = \text{Re}\left\{\boldsymbol{X}_{\text{usb}}\,e^{j2\pi(f_c+f_m)t}\right\} = \text{Re}\left\{1.95\,e^{j24\pi t}\right\}$$

At the time instant $t = 0.6$ seconds we get

$$\begin{aligned} x_{\text{car}}(0.6) &= \text{Re}\left\{3\,e^{j37.6991}\right\} = \text{Re}\left\{3+j\,0\right\} \\ x_{\text{lsb}}(0.6) &= \text{Re}\left\{1.95\,e^{j30.1593}\right\} = \text{Re}\left\{0.6026-j\,1.8546\right\} \\ x_{\text{usb}}(0.35) &= \text{Re}\left\{1.95\,e^{j45.2389}\right\} = \text{Re}\left\{0.6026+j\,1.8546\right\} \end{aligned}$$



c. The modulation index is

$$\mu = \frac{A_m}{A_c} = \frac{4}{5} = 0.8$$

and the signal $x_{AM}(t)$ is

$$\begin{aligned} x_{AM}(t) &= A_c\left[1+\mu\cos\left(2\pi f_m t\right)\right]\cos\left(2\pi f_c t\right) \\ &= 5\left[1+0.8\cos\left(16\pi t\right)\right]\cos\left(50\pi t\right) \\ &= 5\cos\left(50\pi t\right)+2\cos\left(34\pi t\right)+2\cos\left(66\pi t\right) \end{aligned}$$

The carrier signal can be written as

$$x_{\text{car}}(t) = \text{Re}\left\{\boldsymbol{X}_{\text{car}}\,e^{j2\pi f_c t}\right\} = \text{Re}\left\{5\,e^{j50\pi t}\right\}$$

Similarly, lower and upper sideband signals are

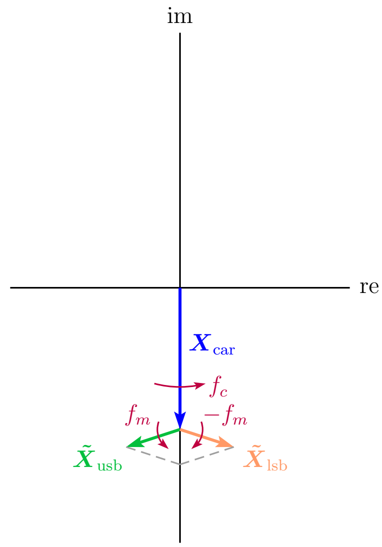
$$x_{\text{lsb}}(t) = \text{Re}\left\{\boldsymbol{X}_{\text{lsb}}\, e^{j2\pi(f_c-f_m)t}\right\} = \text{Re}\left\{2\, e^{j34\pi t}\right\} \qquad \text{and} \qquad x_{\text{usb}}(t) = \text{Re}\left\{\boldsymbol{X}_{\text{usb}}\, e^{j2\pi(f_c+f_m)t}\right\} = \text{Re}\left\{2\, e^{j66\pi t}\right\}$$

At the time instant $t = 150$ milliseconds we get

$$x_{\text{car}}(0.6) = \text{Re}\left\{5\, e^{j23.5619}\right\} = \text{Re}\left\{-j\,5\right\}$$

$$x_{\text{lsb}}(0.6) = \text{Re}\left\{2\, e^{j16.0221}\right\} = \text{Re}\left\{-1.9021 - j\,0.6180\right\}$$

$$x_{\text{usb}}(0.35) = \text{Re}\left\{2\, e^{j31.1018}\right\} = \text{Re}\left\{1.9021 - j\,0.6180\right\}$$



11.5.

The AM signal is in the form

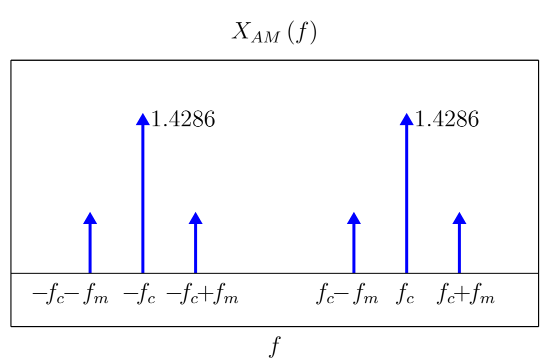
$$x_{AM}(t) = A_c \cos\left(2\pi f_c t\right) + \frac{\mu A_c}{2} \cos\left(2\pi\left[f_c - f_m\right] t\right) + \frac{\mu A_c}{2} \cos\left(2\pi\left[f_c + f_m\right] t\right)$$

and the frequency spectrum is in the form

$$X_{AM}(f) = \frac{A_c}{2}\,\delta\left(f - f_c\right) + \frac{A_c}{2}\,\delta\left(f + f_c\right) + \frac{\mu A_c}{4}\,\delta\left(f - f_c - f_m\right) + \frac{\mu A_c}{4}\,\delta\left(f - f_c + f_m\right)$$
$$+ \frac{\mu A_c}{4}\,\delta\left(f + f_c + f_m\right) + \frac{\mu A_c}{4}\,\delta\left(f + f_c - f_m\right)$$

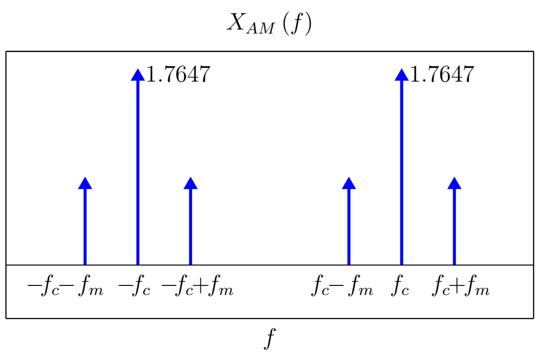
a.

$$X_{AM}(f) = 1.4286\,\delta\left(f - f_c\right) + 1.4286\,\delta\left(f + f_c\right) + 0.5\,\delta\left(f - f_c - f_m\right) + 0.5\,\delta\left(f - f_c + f_m\right)$$
$$+ 0.5\,\delta\left(f + f_c + f_m\right) + 0.5\,\delta\left(f + f_c - f_m\right)$$



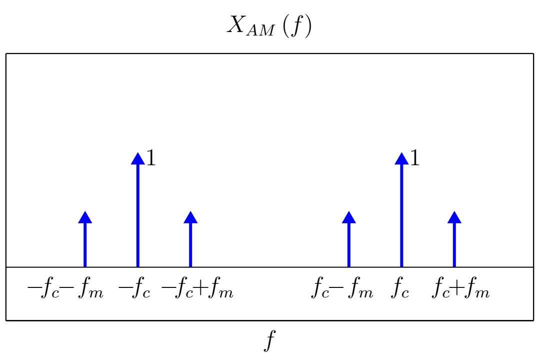
b.

$$X_{AM}(f) = 1.7647\,\delta(f - f_c) + 1.7647\,\delta(f + f_c) + 0.75\,\delta(f - f_c - f_m) + 0.75\,\delta(f - f_c + f_m)$$
$$+ 0.75\,\delta(f + f_c + f_m) + 0.75\,\delta(f + f_c - f_m)$$



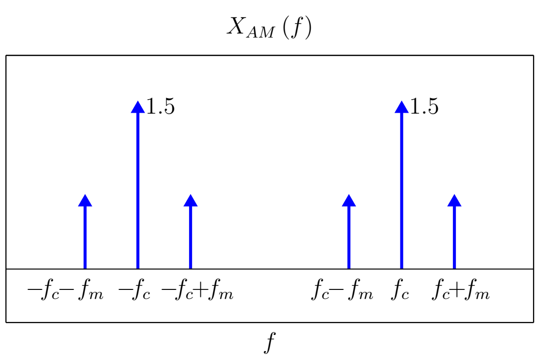
c.

$$X_{AM}(f) = \delta(f - f_c) + \delta(f + f_c) + 0.45\,\delta(f - f_c - f_m) + 0.45\,\delta(f - f_c + f_m)$$
$$+ 0.45\,\delta(f + f_c + f_m) + 0.45\,\delta(f + f_c - f_m)$$



d.

$$X_{AM}(f) = 1.5\,\delta(f - f_c) + 1.5\,\delta(f + f_c) + 0.625\,\delta(f - f_c - f_m) + 0.625\,\delta(f - f_c + f_m)$$
$$+ 0.625\,\delta(f + f_c + f_m) + 0.625\,\delta(f + f_c - f_m)$$

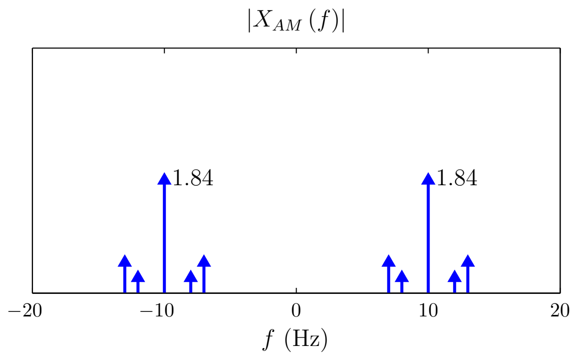


11.6.

a. Using $A_c = 3.68$ we obtain

$$x_{AM}(t) = 3.68\cos(20\pi t) + \cos(4\pi t)\cos(20\pi t) + 2\cos(6\pi t)\cos(20\pi t)$$
$$= 3.68\cos(20\pi t) + \frac{1}{2}\cos(16\pi t) + \frac{1}{2}\cos(24\pi t) + \cos(14\pi t) + \cos(26\pi t)$$

$$X_{AM}(f) = 1.84\,\delta(f-10) + 1.84\,\delta(f+10) + 0.25\,\delta(f-8) + 0.25\,\delta(f+8) + 0.25\,\delta(f-12) + 0.25\,\delta(f+12)$$
$$+\,0.5\,\delta(f-7) + 0.5\,\delta(f+7) + 0.5\,\delta(f-13) + 0.5\,\delta(f+13)$$



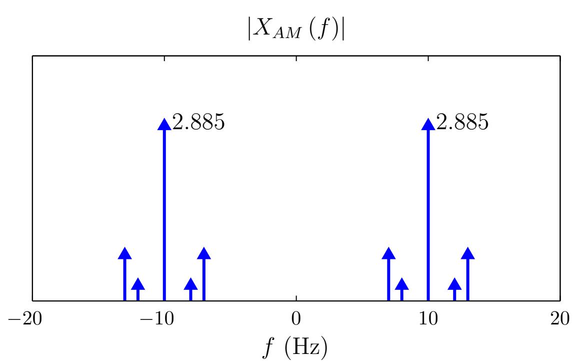
b. Let us write $m(t)$ as

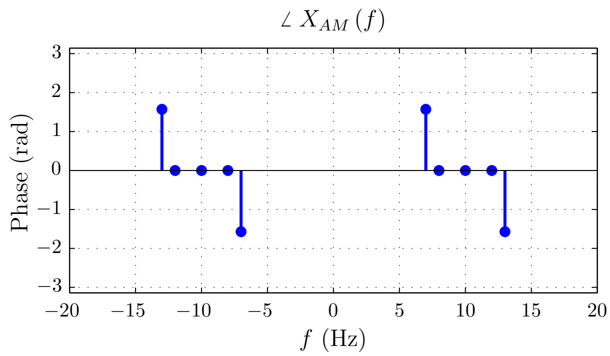
$$m(t) = \cos(4\pi t) + 3\cos(6\pi t - \pi/2)$$

Using $A_c = 5.71$ we obtain

$$x_{AM}(t) = 5.71\cos(20\pi t) + \cos(4\pi t)\cos(20\pi t) + 3\cos(6\pi t - \pi/2)\cos(20\pi t)$$
$$= 5.71\cos(20\pi t) + \frac{1}{2}\cos(16\pi t) + \frac{1}{2}\cos(24\pi t) + \frac{3}{2}\cos(14\pi t + \pi/2) + \frac{3}{2}\cos(26\pi t - \pi/2)$$

$$X_{AM}(f) = 2.855\,\delta(f-10) + 2.855\,\delta(f+10) + 0.25\,\delta(f-8) + 0.25\,\delta(f+8) + 0.25\,\delta(f-12) + 0.25\,\delta(f+12)$$
$$+\,j\,0.75\,\delta(f-7) - j\,0.75\,\delta(f+7) - j\,0.75\,\delta(f-13) + j\,0.75\,\delta(f+13)$$





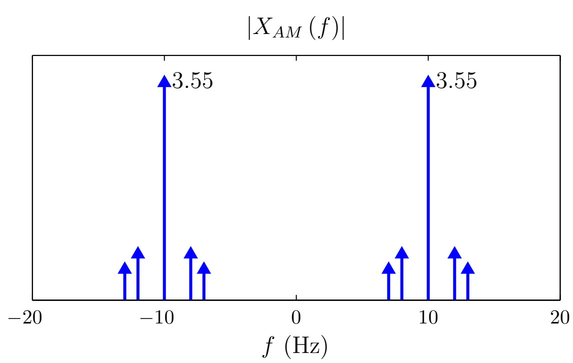
c. Let us write $m(t)$ as

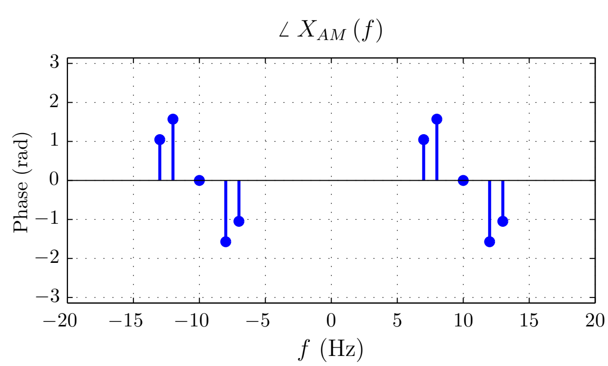
$$m(t) = 3\cos(4\pi t - \pi/2) + 2\cos(6\pi t - \pi/3)$$

Using $A_c = 7.10$ we obtain

$$\begin{aligned} x_{AM}(t) &= 7.10\cos(20\pi t) + 3\cos(4\pi t - \pi/2)\cos(20\pi t) + 2\cos(6\pi t - \pi/3)\cos(20\pi t) \\ &= 7.10\cos(20\pi t) + \frac{3}{2}\cos(16\pi t + \pi/2) + \frac{3}{2}\cos(24\pi t - \pi/2) + \cos(14\pi t + \pi/3) + \cos(26\pi t - \pi/3) \end{aligned}$$

$$\begin{aligned} X_{AM}(f) = {} & 3.55\,\delta(f-10) + 3.55\,\delta(f+10) + j\,0.75\,\delta(f-8) - j\,0.75\,\delta(f+8) - j\,0.75\,\delta(f-12) + j\,0.75\,\delta(f+12) \\ & + 0.5\,e^{j\pi/3}\,\delta(f-7) + 0.5\,e^{-j\pi/3}\,\delta(f+7) + 0.5\,e^{-j\pi/3}\,\delta(f-13) + 0.5\,e^{j\pi/3}\,\delta(f+13) \end{aligned}$$

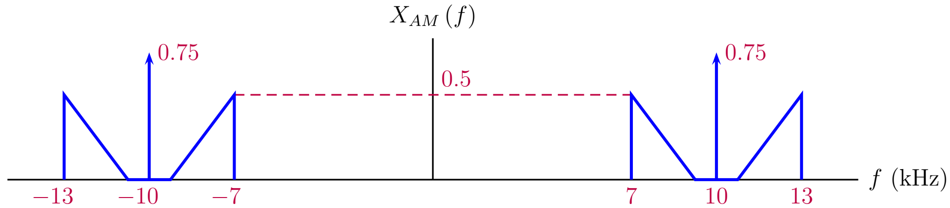




11.7.

$$x_{AM}(t) = A_c \cos\left(2\pi f_c t\right) + m(t)\cos\left(2\pi f_c t\right)$$

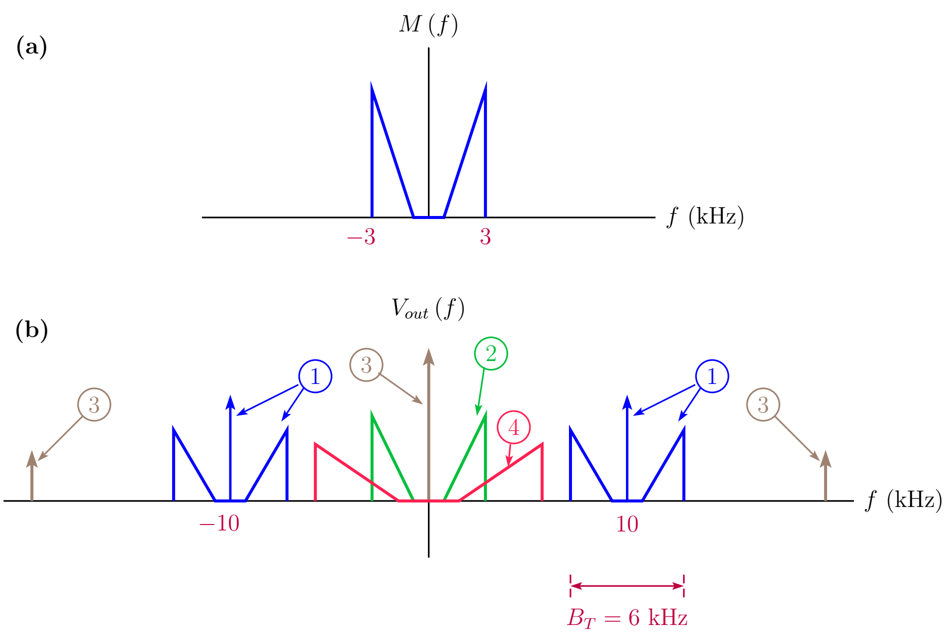
$$X_{AM}\left(f\right) = \frac{A_c}{2}\delta\left(f - f_c\right) + \frac{A_c}{2}\delta\left(f + f_c\right) + \frac{1}{2}M\left(f - f_c\right) + \frac{1}{2}M\left(f + f_c\right)$$



11.8.

a. The terms in the spectrum of the AM signal are listed in the table below:

Index	Term	Frequencies
1	$3\left[1+0.8\,m(t)\right]\cos(20\pi t)$	$7 \le f \le 13$ kHz
2	$2\,m(t)$	$-3 \le f \le 3$ kHz
3	$1.8\cos^2(20\pi t)$	$f = 0$, and $f = 20$ kHz
4	$0.8\,m^2(t)$	$-6 \le f \le 6$ kHz



b. To select the desirable terms (row 1 of the table), the passband of the filter should span the frequency range

$$7 \le f \le 13 \text{ kHz}$$

11.9.

a. The carrier completes 15 full cycles in a duration of 1 ms. Therefore the carrier frequency is $f_c = 15$ kHz.

b. Observing the envelope, we see that it completes one full cycle in about 0.84 ms. The message frequency is approximately $f_m = 1.2$ kHz.

c. For a single-tone modulated AM carrier the maximum value of the positive envelope is

$$\left|x_{AM}(t)\right|_{\max} \approx A_c + A_m$$

and the minimum value of the positive envelope is

$$\left|x_{AM}(t)\right|_{\min} \approx |A_c - A_m|$$

From the graph we observe

$$A_c + A_m \approx 7 \qquad \text{and} \qquad A_c - A_m \approx 7$$

yielding carrier and message amplitudes

$$A_c \approx 4 \qquad \text{and} \qquad A_m \approx 3$$

The modulation index is

$$\mu = \frac{A_m}{A_c} \approx 0.75$$

d. Approximate efficiency is

$$\eta \approx \frac{(0.75)^2}{2 + (0.75)^2} = 0.2195$$

11.10.

a.

$$A_c^2 = (3.68)^2 = 13.5424$$

$$\left\langle m^2(t) \right\rangle = \frac{1}{2} + \frac{4}{2} = 2.5$$

$$\eta = \frac{\left\langle m^2(t) \right\rangle}{A_c^2 + \left\langle m^2(t) \right\rangle} = \frac{2.5}{13.5424 + 2.5} = 0.1558$$

b.

$$A_c^2 = (5.71)^2 = 32.6041$$

$$\left\langle m^2(t) \right\rangle = \frac{1}{2} + \frac{9}{2} = 5$$

$$\eta = \frac{\langle m^2(t) \rangle}{A_c^2 + \langle m^2(t) \rangle} = \frac{5}{32.6041 + 5} = 0.1330$$

c.

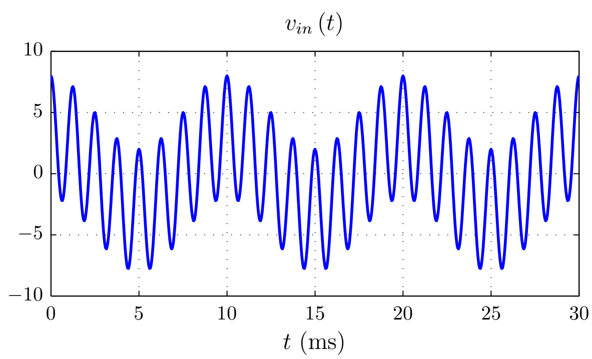
$$A_c^2 = (7.10)^2 = 50.41$$

$$\langle m^2(t) \rangle = \frac{9}{2} + \frac{4}{2} = 6.5$$

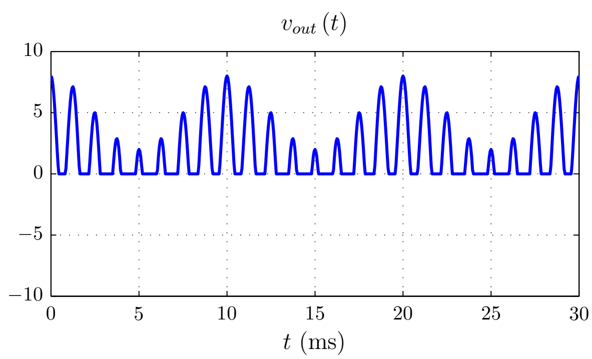
$$\eta = \frac{\langle m^2(t) \rangle}{A_c^2 + \langle m^2(t) \rangle} = \frac{6.5}{50.41 + 6.5} = 0.1142$$

11.11.

a.



b.



c. The message bandwidth is $W = 100$ Hz. The carrier frequency is $f_c = 800$ Hz. The significant frequency components of the modulated carrier are at 700, 800 and 900 Hz. A reasonable choice for a bandpass filter to isolate the AM signal may be one with a passband for

$$600 < f < 1000 \text{ Hz}$$

11.12.

The output of the first AM modulator:

$$w_1(t) = [A_c + m(t)] \cos(2\pi f_c t)$$

The output of the second AM modulator:

$$w_2(t) = [A_c - m(t)] \cos(2\pi f_c t)$$

Adding the two signals yields

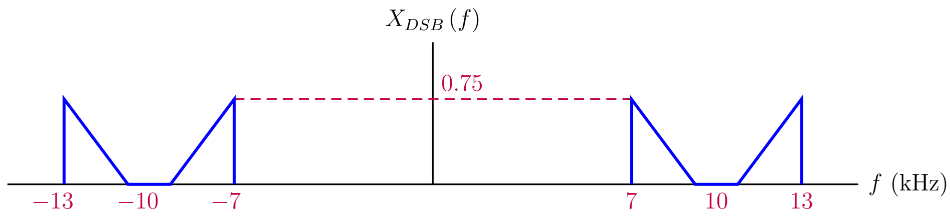
$$x_{DSB}(t) = w_1(t) + w_2(t) = 2\, A_c \cos(2\pi f_c t)$$

11.13.

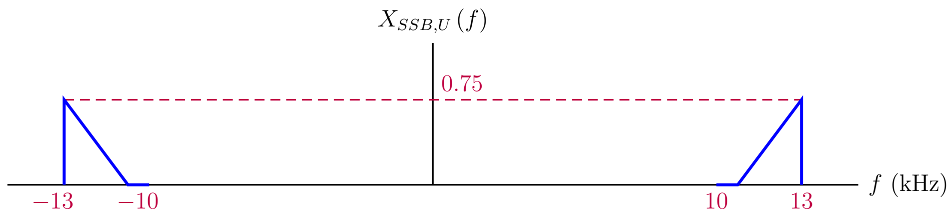
a.

$$x_{DSB}(t) = A_c\, m(t) \cos(2\pi f_c t)$$

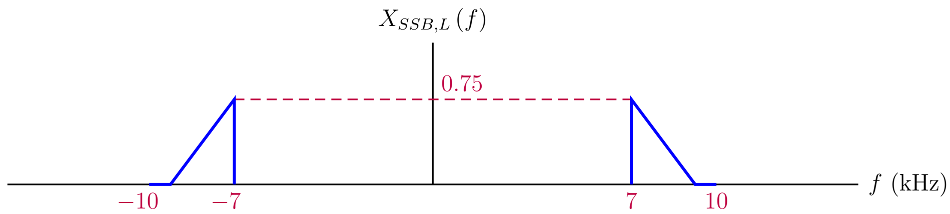
$$X_{DSB}(f) = \frac{A_c}{2} M(f - f_c) + \frac{A_c}{2} M(f + f_c)$$



b.



c.



11.14.

The modulation index is

$$\mu = \frac{3.2}{4} = 0.8$$

Using Eqn. (11.64) with $\mu = 0.8$ and $f_m = 100$ Hz, we obtain

$$\tau \le \frac{1}{200\pi}\left(\frac{\sqrt{1-(0.8)^2}}{0.8}\right) = 0.0012$$

Let $C = 1\ \mu$F. Required resistor value is $R = 1200\ \Omega$.

11.15.

The signal $r(t)$ is

$$r(t) = 2\left[A_c + m(t)\right]\cos\left(2\pi f_c t\right)\cos\left(2\pi f_c t + \theta\right)$$

Using the appropriate trigonometric identity

$$r(t) = A_c \cos\left(4\pi f_c t + \theta\right) + A_c \cos(\theta) + m(t)\cos\left(4\pi f_c t + \theta\right) + m(t)\cos(\theta)$$

The bandpass filter removes the first three terms, resulting in

$$\hat{m}(t) = m(t)\cos(\theta)$$

The effect of the phase error is multiplication of the message signal by the factor $\cos(\theta)$ which is a constant as long as the phase error θ is constant. For values of θ close to $\pi/2$ radians, the factor $\cos(\theta)$ is small, potentially degrading the performance of the demodulator in the presence of random noise.

11.16.

The signal $r(t)$ is

$$r(t) = 2\left[A_c + m(t)\right]\cos\left(2\pi f_c t\right)\cos\left(2\pi\left[f_c + \Delta f\right] t\right)$$

Using the appropriate trigonometric identity

$$r(t) = A_c \cos\left(2\pi\left[2f_c + \Delta f\right] t\right) + A_c \cos\left(2\pi \Delta f\, t\right) + m(t)\cos\left(2\pi\left[2f_c + \Delta f\right] t\right) + m(t)\cos\left(2\pi \Delta f\, t\right)$$

The bandpass filter removes the first three terms, resulting in

$$\hat{m}(t) = m(t)\cos\left(2\pi \Delta f\, t\right)$$

The effect of the frequency error is multiplication of the message signal by the factor $\cos\left(2\pi \Delta f\, t\right)$. The output signal $\hat{m}(t)$ is still a modulated term, with the carrier frequency Δf.

11.17.

Let us begin by defining an anonymous function for the AM signal:

```
xAM = @(t,Ac,mu,fc,fm) Ac*(1+mu*cos(2*pi*fm*t)).*cos(2*pi*fc*t);
```

Afterwards, each AM signal in question can be produced using this anonymous function.

a.

```
t = [0:2e-6:2e-3];
plot(1000*t,xAM(t,3,0.6,10000,2000));
axis([0,2,-8,8]);
xlabel('t (ms)');
```

b.

```
t = [0:2e-6:2e-3];
plot(1000*t,xAM(t,3,1.3,10000,2000));
axis([0,2,-8,8]);
xlabel('t (ms)');
```

c.

```
mu = 4/5;
t = [0:1e-6:1e-3];
plot(1000*t,xAM(t,3,mu,25000,8000));
axis([0,1,-10,10]);
xlabel('t (ms)');
```

11.18.

a.

```
t = [0:0.005:4];
Ac = 3.68;
m = cos(4*pi*t)+2*cos(6*pi*t);
x_am = (Ac+m).*cos(20*pi*t);
plot(t,x_am);
xlabel('t (sec)');
title('x_{AM}(t)');
grid;
```

b.

```
Ac = 5.71;
m = cos(4*pi*t)+3*sin(6*pi*t);
x_am = (Ac+m).*cos(20*pi*t);
plot(t,x_am);
xlabel('t (sec)');
title('x_{AM}(t)');
grid;
```

c.

```
Ac = 7.10;
m = 3*sin(4*pi*t)+2*cos(6*pi*t-pi/3);
x_am = (Ac+m).*cos(20*pi*t);
plot(t,x_am);
xlabel('t (sec)');
title('x_{AM}(t)');
grid;
```

11.19.

a.

Using $A_c = 3.68$ we obtain

$$x_{AM}(t) = 3.68\cos(20\pi t) + \cos(4\pi t)\cos(20\pi t) + 2\cos(6\pi t)\cos(20\pi t)$$

$$= 3.68\cos(20\pi t) + \frac{1}{2}\cos(16\pi t) + \frac{1}{2}\cos(24\pi t) + \cos(14\pi t) + \cos(26\pi t)$$

The frequency components in $x_{AM}(t)$ are

$$f_1 = 10 \text{ Hz}, \qquad f_2 = 8 \text{ Hz}, \qquad f_3 = 12 \text{ Hz}, \qquad f_4 = 7 \text{ Hz}, \qquad f_5 = 13 \text{ Hz}$$

Fundamental frequency f_0 is found through (see Section 1.3.4)

$$\frac{1}{f_0} = \frac{m_1}{10} = \frac{m_2}{8} = \frac{m_3}{12} = \frac{m_4}{7} = \frac{m_5}{13}$$

Using the integers

$$m_1 = 10, \qquad m_2 = 8, \qquad m_3 = 12, \qquad m_4 = 7, \qquad m_5 = 13$$

the fundamental frequency is found as $f_0 = 1$ Hz corresponding to a fundamental period of $T_0 = 1$ s. In order to generate 1024 samples in one period, samples need to be taken $1/1024$ seconds apart.

```
t = [0:1023]/1024;
x_am = 3.68*cos(20*pi*t)+cos(4*pi*t).*cos(20*pi*t)+2*cos(6*pi*t).*cos(20*pi*t);
k=[-20:20];
c = ss_efsapprox(x_am,k);
stem(k,c)
axis([-20.5,20.5,0,3]);
xlabel('k');
title('|c_{k}|');
```

b.

Writing $m(t)$ as

$$m(t) = \cos(4\pi t) + 3\cos(6\pi t - \pi/2)$$

and using $A_c = 5.71$ we obtain

$$x_{AM}(t) = 5.71\cos(20\pi t) + \cos(4\pi t)\cos(20\pi t) + 3\cos(6\pi t - \pi/2)\cos(20\pi t)$$

$$= 5.71\cos(20\pi t) + \frac{1}{2}\cos(16\pi t) + \frac{1}{2}\cos(24\pi t) + \frac{3}{2}\cos(14\pi t + \pi/2) + \frac{3}{2}\cos(26\pi t - \pi/2)$$

As in part (a) of the problem, the fundamental frequency and the fundamental period are $f_0 = 1$ Hz and $T_0 = 1$ s respectively.

```
t = [0:1023]/1024;
x_am = 5.71*cos(20*pi*t)+cos(4*pi*t).*cos(20*pi*t)+3*cos(6*pi*t-pi/2).*cos(20*pi*t);
k=[-20:20];
c = ss_efsapprox(x_am,k);
stem(k,abs(c));
axis([-20.5,20.5,0,3]);
xlabel('k');
title('|c_{k}|');
```

c. Writing $m(t)$ as

$$m(t) = 3\cos(4\pi t - \pi/2) + 2\cos(6\pi t - \pi/3)$$

and using $A_c = 7.10$ we obtain

$$x_{AM}(t) = 7.10\cos(20\pi t) + 3\cos(4\pi t - \pi/2)\cos(20\pi t) + 2\cos(6\pi t - \pi/3)\cos(20\pi t)$$

$$= 7.10\cos(20\pi t) + \frac{3}{2}\cos(16\pi t + \pi/2) + \frac{3}{2}\cos(24\pi t - \pi/2) + \frac{3}{2}\cos(14\pi t + \pi/3) + \frac{3}{2}\cos(26\pi t - \pi/3)$$

As in parts (a) and (b) of the problem, the fundamental frequency and the fundamental period are $f_0 = 1$ Hz and $T_0 = 1$ s respectively.

```
t = [0:1023]/1024;
x_am = 7.10*cos(20*pi*t)+3*cos(4*pi*t-pi/2).*cos(20*pi*t)+2*cos(6*pi*t-pi/3).*cos(20*pi*t);
k=[-20:20];
c = ss_efsapprox(x_am,k)
stem(k,abs(c));
axis([-20.5,20.5,0,4]);
xlabel('k');
title('|c_{k}|');
```

11.20. Compute and graph the signal $v_{in}(t)$.

```
t = [0:0.05:30]/1000;
Bc = 5;
fc = 800;
carrier = Bc*cos(2*pi*fc*t);
message = 3*cos(200*pi*t);
vin = carrier+message;
plot(1000*t,vin);
axis([0,30,-10,10]);
xlabel('t (ms)');
title('v_{in}(t)');
grid;
```

Compute and graph the signal $v_{out}(t)$.

```
vout = vin.*(vin>=0);
plot(1000*t,vout);
axis([0,30,-10,10]);
xlabel('t (ms)');
title('v_{out}(t)');
grid;
```

Compute and graph the signal $x_{AM}(t)$ using the function **ss_switchmod**(..)

```
x_am = ss_switchmod(message,Bc,fc,0.05e-3,600,1000);
plot(1000*t,x_am);
axis([0,30,-5,5]);
xlabel('t (ms)');
title('x_{AM}(t)');
grid;
```

11.21. Function **ss_switchmod2**(..).

```
function x_am = ss_switchmod2(msg,Bc,fc,Ts,f1,f2)
  nSamp = length(msg);            % Number of samples in "msg".
  t = [0:nSamp-1]*Ts;             % Vector of time instants.
  carrier = Bc*cos(2*pi*fc*t);
  % Compute input to the diode switch.
  v_in = carrier+msg;             % Eqn. (11.36)
  % Simulate the diode switch.
  v_out = (v_in-0.6).*(v_in>=0.6);
  % Design the bandpass filter.
  [numz,denz] = butter(5,[2*f1*Ts,2*f2*Ts],'bandpass');
  % Process switch output through bandpass filter.
  x_am = filter(numz,denz,v_out);
```

Compute and graph the signal $x_{AM}(t)$ using the functions **ss_switchmod**(..) and **ss_switchmod2**(..)

```
x_am1 = ss_switchmod(message,Bc,fc,0.05e-3,600,1000);
x_am2 = ss_switchmod2(message,Bc,fc,0.05e-3,600,1000);
plot(1000*t,x_am1,'b',1000*t,x_am2,'r');
axis([0,30,-5,5]);
xlabel('t (ms)');
grid;
```

11.22.

Script to simulate envelope detector using the value $\tau = 0.0012$ seconds found in Problem 11.14:

```
t = [0:20e-6:20e-3];
x_am = (4+3.2*cos(200*pi*t)).*cos(2000*pi*t);
Ts = 20e-6;
tau = 0.0012;
x_env = ss_envdet(x_am,Ts,tau);
plot(t,x_am,'b',t,x_env,'r');
grid;
```

Repeat using half the time constant value found:

```
x_env = ss_envdet(x_am,Ts,0.5*tau);
plot(t,x_am,'b',t,x_env,'r');
grid;
```

Repeat using twice the time constant value found:

```
x_env = ss_envdet(x_am,Ts,2*tau);
plot(t,x_am,'b',t,x_env,'r');
grid;
```

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