**Problem 1.7(H)**

Consider an orthonormal basis set $|b_i\rangle$ where $i \in 1, 2, \ldots, N$ and where an arbitrary ket is expressed as $|A\rangle = \sum_{i=1}^{N} \alpha_i |b_i\rangle$ where $\alpha_i \in \mathbb{C}$. Let $\hat{P}_1 = |b_1\rangle\langle b_1|$. Compute $\hat{P}_1|A\rangle$ and $\hat{P}_1\hat{P}_1|A\rangle$. Justify in words why $\hat{P}_1$ is called a projection operator.

Let $\{|b_i\rangle \mid i = 1, \ldots, N\}$ be an orthonormal basis of a complex vector space $\mathcal{H}$.

$$\mathcal{H} = \{ |A\rangle \mid |A\rangle = \sum_{i=1}^{N} \alpha_i |b_i\rangle, \alpha_i \in \mathbb{C} \}.$$

Let $\hat{P}_1 = |b_1\rangle\langle b_1|$ be a linear operator.

$\hat{P}_1$ is called a projection operator; it projects to the subspace generated by $|b_1\rangle$;

If $|A\rangle = \sum_{i=1}^{N} \alpha_i |b_i\rangle$, we want $\hat{P}_1|A\rangle = \alpha_1|b_1\rangle$.

Indeed
$$\hat{P}_1 = |b_1\rangle\langle b_1| \sum_{i=1}^{N} \alpha_i |b_i\rangle$$
$$= \sum_{i=1}^{N} \alpha_i \langle b_1 | b_i\rangle |b_1\rangle$$
$$= \alpha_1 \|\langle b_1 | b_1\rangle\|^2 |b_1\rangle$$

$$= \alpha_1 |b_1\rangle, \text{ since } \|\langle b_1 | b_1\rangle\| = 1.$$

We have used that $\langle b_i | b_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$

This justifies the terminology.

Also,
$$
\begin{aligned}
\hat{P}_1 \hat{P}_1 |A\rangle &= \hat{P}_1 \left( \hat{P}_1 |A\rangle \right) \\
&= |b_1\rangle\langle b_1| \left( \alpha_1 |b_1\rangle \right) \\
&= \alpha_1 |b_1\rangle \langle b_1|b_1\rangle \\
&= \alpha_1 |b_1\rangle \\
&= \hat{P}_1 |A\rangle
\end{aligned}
$$

By induction, one may then prove that $|\hat{P}_1\rangle^n |A\rangle = \hat{P}_1 |A\rangle$ for all $n \geq 1$.

**Problem 1.8(X)**

What is a quantum state? In your answer, explain how a quantum state is different from a classical state and how it is represented mathematically.

A Quantum state gives completely describes a mechanical system in quantum physics

They are vectors in complex vector spaces, of infinite dimension. In fact, they are mathematically represented as elements of a Hilbert Space; a complete inner product space, under the induced metric space by the inner product norm.

It supposedly describes everything of interest.

In classical mechanics, the system is described by vectors $\vec{x}, \vec{v}$, but they are over the real field and each of dimension 3.

**Problem 1.6(H)**

Consider the functions $\delta_\epsilon(x) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2+x^2}$ where $\epsilon$ is a parameter.

a) Show that $\int_{-\infty}^{\infty} dx \delta_\epsilon(x) = 1$

b) Verify that $\delta_{\epsilon/k}(x) = k\delta_\epsilon(kx)$ for a positive number $k$.

Such a sequence of functions is called a $\delta$-sequence and constitutes one possible formal

definition of a Dirac delta-function.

$$\int dx' \delta(x-x') f(x') \equiv \lim_{\epsilon\to 0} \int dx' \delta_\epsilon(x-x') f(x') = f(x)$$

c) Make numerical plots of $\delta_\epsilon(x)$ on the interval $x \in [-1,1]$ for three values of $\epsilon$, pick $\epsilon = \{0.01, 0.1, 1\}$.

d) Assume that you are given a table of numerical arguments $x$ and values $y \equiv \delta_1(x)$ of the function $\delta_1$. Explain how you can obtain a similar table of function arguments and values of the function $\delta_{0.1}$ using the numbers in the given table.

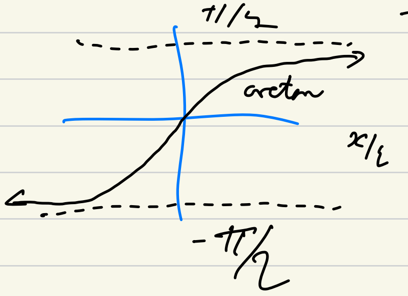
Let $\varepsilon \in \mathbb{R}$ be a parameter. Consider the function $\delta_\varepsilon(x) = \frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+x^2}$.

a)
$$\int dx\, \delta_\varepsilon(x) = \frac{1}{\pi}\int \frac{\varepsilon}{\varepsilon^2+x^2}dx$$

$$= \frac{1}{\pi}\int \frac{\varepsilon}{\varepsilon^2\left(1+\left(\frac{x}{\varepsilon}\right)^2\right)}dx$$

$$= \frac{1}{\pi}\int \frac{1}{1+\left(\frac{x}{\varepsilon}\right)^2}\frac{dx}{\varepsilon}$$

$$= \frac{1}{\pi}\arctan\left(\frac{x}{\varepsilon}\right)$$



Hence, $$\int_{-\infty}^{\infty} dx\, \delta_\varepsilon(x) = \frac{1}{\pi} \arctan\left(\frac{x}{\varepsilon}\right)\Bigg|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \underline{1}.$$

b) Let $k \in \mathbb{R}^+$. We shall prove that

$$\delta_{\varepsilon/k}(x) = k\,\delta_\varepsilon(kx).$$

$$\delta_{\varepsilon/k}(x) = \frac{1}{\pi} \frac{\varepsilon/k}{(\varepsilon/k)^2 + x^2} \cdot \frac{k^2}{k^2}$$

$$= \frac{1}{\pi} \frac{\varepsilon k}{\varepsilon^2 + x^2 k^2}$$

$$= \frac{1}{\pi} \frac{\varepsilon k}{\varepsilon^2 + (xk)^2}$$

$$= k\left(\frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + (xk)^2}\right)$$

$$= k\,\delta_\varepsilon(xk)$$

Let the Dirac delta function, $\delta(x)$, be such that

$$\int dx' \, \delta(x - x') f(x') \equiv \lim_{\varepsilon \to 0} \int dx' \, \delta_\varepsilon(x - x') f(x') = f(x),$$

for any $f$ and any $x$.

definition of a Dirac delta-function.

$$\int dx' \delta(x - x') f(x') \equiv \lim_{\epsilon \to 0} \int dx' \delta_\epsilon(x - x') f(x') = f(x)$$

c) Make numerical plots of $\delta_\epsilon(x)$ on the interval $x \in [-1, 1]$ for three values of $\epsilon$, pick $\epsilon = \{0.01, 0.1, 1\}$.
d) Assume that you are given a table of numerical arguments $x$ and values $y \equiv \delta_1(x)$ of the function $\delta_1$. Explain how you can obtain a similar table of function arguments and values of the function $\delta_{0.1}$ using the numbers in the given table.

Suppose $X \subset \mathbb{R}$ is a finite set whose evaluations by the function $\delta_1$ are known, for each $x \in X$.

This gives a table of function values:

| $x$ | $x_1$ | $x_2$ | $\dots$ | $x_N$ |
|---|---|---|---|---|
| $y$ | $y_1 = \delta_1(x_1)$ | | $\dots$ | $y_N = \delta_1(x_N)$ |

Now, $\delta_{0.1}(x) = \delta_{1/10}(x)$

$$= 10\,\delta_1(10x)$$

If the array of $x$- values is sufficiently "dense", we may assume that the evaluation of $\delta_1(10x)$ is already calculated in the previous table. Hence, we obtain new values.

If $x \neq 0$ $\quad \delta_1(x) = \frac{1}{\pi}\frac{1}{1+x^2}$,

but $\quad \delta_{1/10} = 10 \cdot \frac{1}{\pi}\frac{1}{1+10^2x^2}$

The denominator grows with the square, while the numerator grows linearly

, resulting in a limit towards zero.

At $x = 0$, on the other hand, this is not the case. The values grow linearly in $\varepsilon$.