Project 1 - Problem 6

5. september 2022

The general expression for $A\vec{v} = \vec{g}$ with $\vec{a}$, $\vec{b}$ and $\vec{c}$ being the sub-, main- and supdiagonal respectively is:

$$\begin{bmatrix} b_1 & c_2 & 0 & 0 \\ a_1 & b_2 & c_3 & 0 \\ 0 & a_2 & b_3 & c_4 \\ 0 & 0 & a_3 & b_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} \tag{1}$$

We can solve this with Gaussian elimination in two general steps: [1] Forward substitution and [2] Backward substitution.

Forward substitution

The goal here is to get an upper triangular matrix, that is we want to do the follwoing:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

So we want to get rid of all a_i entries in the matrix equation (1). We will number and denote the rows with roman numerals.

$$II = II - \frac{a_2}{b_1} I \tag{2}$$

The diagonal new entries will be denoted $\tilde{b_i}$

$$III = III - \frac{a_3}{\tilde{b_2}} II \tag{3}$$

$$IV = IV - \frac{a_4}{\tilde{b_3}} III \tag{4}$$

$$\tag{5}$$

This gives us the following algorithm for precdure

$$\tilde{b_1} = b_1 \tag{6}$$
$$\tilde{b_i} = b_i - \frac{a_i}{\tilde{b_{i-1}}} c_{i-1} \tag{7}$$
$$\tilde{g_1} = g_1 \tag{8}$$
$$\tilde{g_i} = g_i - \frac{a_i}{\tilde{b_{i-1}}} \tilde{g_{i-1}} \tag{9}$$
$$\tag{10}$$

This will be for $i = 2, 3, ..., n$ for a general tridiagonal matrix.

Backward substitution

Now that we have done the forward substitution we want to find the solution to all v_i from the following equation:

$$\begin{bmatrix} \tilde{b_1} & c_2 & 0 & 0 \\ 0 & \tilde{b_2} & c_3 & 0 \\ 0 & 0 & \tilde{b_3} & c_4 \\ 0 & 0 & 0 & \tilde{b_4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} \tilde{g_1} \\ \tilde{g_2} \\ \tilde{g_3} \\ \tilde{g_4} \end{bmatrix} \tag{11}$$

we quickly see that $v_4 = \frac{\tilde{g_4}}{\tilde{b_4}}$ and to get the remaining v_i we move up with the following algorithm (we include the case of v_n to generalize the algorithm):

$$v_n = \frac{\tilde{g_n}}{\tilde{b_n}} \tag{12}$$
$$v_i = \frac{\tilde{g_i} - c_i v_{i-1}}{\tilde{b_i}} \tag{13}$$
$$\tag{14}$$

for $i = n - 1, n - 2, ..., 2, 1$.

FLOPs

We can see that algorithm 6 requires $2(n - 1)$ floating point operations. $n - 1$ FLOPs for equation (8) and equation (10) each. From equation 12 we can count $3(n - 1) + 1$ FLOPs. This gives us a total of $5n - 4$ FLOPs to solve $A\vec{v} = \vec{g}$ with a tridiagonal matrix of dimentions $n \times n$.