

Playing with Latex

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(Dated: August 31, 2022)

I. DERIVING THE DISCRETIZED POISSON EQUATION.

Let's start by deriving the discretized version of $\frac{d^2 u}{dx^2}$. First, we Taylor expand the function $u(x+h)$ around the point x . We get:

$$u(x+h) = \sum_{n=0}^{\infty} \frac{u^{(n)}(x)}{n!} h^n$$
$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + O_1(h^4)$$

where $O_1(h^2)$ is the remainder of the expansion, and $O(h) \equiv \frac{O_1(h^2)}{h}$. Now let's also Taylor expand $u(x-h)$ (again, around the point x).

$$u(x-h) = \sum_{n=0}^{\infty} \frac{u^{(n)}(x)}{n!} (-h)^n$$
$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + O_2(h^4)$$

If we now add these two expansions together, we get:

$$u(x+h) + u(x-h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + O_1(h^4)$$
$$+ u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + O_2(h^4)$$
$$u(x+h) + u(x-h) = 2u(x) + u''(x)h^2 + O_1(h^4) + O_2(h^4)$$
$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - \frac{O_1(h^4) + O_2(h^4)}{h^2}$$
$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

This can be discretized by letting $v_i \approx u(x)$, $v_{i+1} \approx u(x+h)$ and $v_{i-1} \approx u(x-h)$. Since this is an approximation we can ignore the remainder, and our result becomes

$$v_i'' = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}$$

Here x is discretized as ih , where h is the step length, and i is the number of steps to reach the x value. Using this, the discretization of the forcing term $f(x) = 100e^{-10x}$ becomes $f_i = 100e^{-10ih}$. Using this together with the discretized version of $v''(x)$ we get the complete discretized Poisson equation:

$$-\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = 100e^{-10ih} \quad (1)$$