Playing with Latex

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I. DERIVING THE DISCRETIZED POISSON EQUATION.

Let's start by deriving the discretized version of $\frac{d^2u}{dx^2}$. First, we taylor expand the function $u(x+h)$ around the point x. We get:

$$
\begin{aligned}
u(x+h) &= \sum_{n=0}^{\infty} \frac{u^{(n)}(x)}{n!} h^n \\
u(x+h) &= u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + O_1(h^4)
\end{aligned}
$$

where $O_1(h^2)$ is the remainder of the expansion, and $O(h) \equiv \frac{O_1(h^2)}{h}$. Now let's also Taylor expand $u(x-h)$ (again, around the point x).

$$
\begin{aligned}
u(x-h) &= \sum_{n=0}^{\infty} \frac{u^{(n)}(x)}{n!} (-h)^n \\
u(x-h) &= u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + O_2(h^4)
\end{aligned}
$$

If we now add these two expansions together, we get:

$$
\begin{aligned}
u(x+h) + u(x-h) &= u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + O_1(h^4) \\
&\qquad + u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + O_2(h^4) \\
u(x+h) + u(x-h) &= 2u(x) + u''(x)h^2 + O_1(h^4) + O_2(h^4) \\
u''(x) &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - \frac{O_1(h^4) + O_2(h^4)}{h^2} \\
u''(x) &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)
\end{aligned}
$$

This can be discretized by letting $v_i \approx u(x)$, $v_{i+1} \approx u(x+h)$ and $v_{i-1} \approx u(x-h)$. Since this is an approximation we can ignore the remainder, and our result becomes

$$
v_i'' = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}
$$

Here x is discretized as ih, where h is the step length, and i is the number of steps to reach the x value. Using this, the discretization of the forcing term $f(x) = 100e^{-10x}$ becomes $f_i = 100e^{-10ih}$. Using this together with the discretized version of $v''(x)$ we get the complete discretized Poisson equation:

$$
-\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = 100e^{-10ih} \tag{1}
$$