Fundamental Robotics
a comprehensive summary of robotics

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1 Introduction

2 Fundamental Algebra

2.1 Matrix Classes

2.1.1 Rotation Matrix

$R = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

- Orthogonal: Matrix times its transpose equals the identity $RR^T = I$
- Special orthogonal: Determinant is $+1$
- Closed under multiplication: The product of any two rotation matrices is another rotation matrix
- The inverse of a rotation matrix is also a rotation matrix

2.1.2 Matrix Derivatives

$A(t)$

$$\dot{A}(t) = \begin{bmatrix} \frac{dR_{11}(t)}{dt} & \frac{dR_{12}(t)}{dt} \\ \frac{dR_{21}(t)}{dt} & \frac{dR_{22}(t)}{dt} \end{bmatrix}$$

$\frac{d}{dt}(A \pm B) = \dot{A} \pm \dot{B}$

$\frac{d}{dt}(AB) = \dot{A}B + A\dot{B}$

$\frac{d}{dt}(A(\theta(t))) = \frac{dA}{d\theta}\dot{\theta}$

2.1.3 Jacobian

2.1.4 Hessian

2.1.5 Eigendecomposition

3 Fundamental Analysis

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5.1 Coordinate Frames

5.1.1 Inertia Frame of Reference

An inertial frame is a frame with constant acceleration i.e. $\mathbf{a} = 0$. A body with zero net force acting upon it is not accelerating within the inertial frame; that is, such a body is at rest or it is moving at a constant speed in a straight line.

5.2 Sources

5.3 Definitions

Transformation from coordinate frame B to frame A

$^A\mathbf{A}_B,^A\mathbf{R}_B$

5.3.1 Rigid Body Transformation

- length is preserved $\|g(p) - g(q)\| = \|p - q\|$
- cross product is preserved $g_*(v) \times g_*(w) = g_*(v \times w)$
- inner product is preserved $g_*(v) \cdot g_*(w) = v \cdot w$

5.4 Rotation Group SO(3)

$SO(3) = \{\mathbf{R} \in \mathbb{R} | \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}, det\mathbf{R} = 1\}$

5.4.1 Rotation Matrices

5.4.2 Euler Angles

5.4.3 Axis Angle Parametrization

Rodrigues' formula

5.4.4 Quaternions

5.5 Time Derivatives of Rotations

$R^T(t)R(t) = I \frac{d}{dt}(.)\dot{R}^T R + R^T\dot{R} = 0$

$R(t)R^T(t) = I \frac{d}{dt}(.)R\dot{R}^T + \dot{R}R^T = 0$

$R^T\dot{R}$ and $\dot{R}R^T$ are skew symmetric

5.6 Skew Symmetric

5.6.1 Transformation vs Displacement

5.7 Momentum

5.7.1 Linear Momentum

5.7.2 Angular Momentum

${}^{A}\mathbf{H}_{C}^{B} = \mathbf{I}_{C} \cdot^{A} \omega^{B}$

Angular momentum $\mathbf{H}$ with inertia tensor $\mathbf{I}$ with C as the origin and angular velocity ω of the body B in coordinate frame A.

5.7.3 Rate of Change of Angular Momentum

The rate of change of angular momentum of the rigid body B relative to C in A is equal to the resultant moment of all external forces $\mathbf{M}$ acting on the body B relative to C.

$$\frac{{}^{A}d^{A}\mathbf{H}_{C}^{B}}{dt} = \mathbf{M}_{C}^{B}$$

Simplification

$$\frac{{}^{B}d^{A}\mathbf{H}_{C}^{B}}{dt} +^{A} \omega^{B} \times \mathbf{H}_{C} = \mathbf{M}_{C}^{B}$$

$$\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} M_{C,1} \\ M_{C,2} \\ M_{C,3} \end{bmatrix}$$

The second term on the left hand side, ${}^{A}\omega^{B} \times \mathbf{H}_{C}$, is zero when angular velocity ${}^{A}\omega^{B}$ is perpendicular to $\mathbf{H}_{C}$. This is the case if the body rotates around a **principal axis of inertia**.

5.8 Moment of Inertia

The moment of inertia $\mathbf{I}$ is a tensor that determines the torque needed for a desired angular acceleration about a rotational axis; similar to how mass determines the force needed for a desired acceleration. It depends on the body's mass distribution and the axis chosen, with larger moments requiring more torque to change the body's rotation. It is an **additive** property: for a point mass the moment of inertia is just the mass m times the square of the distance r to the rotation axis.

$$\mathbf{I} = mr^2$$

For bodies free to rotate in 3D, their moments can be described bz a szmmetric 3x3 matrix, with a set of mutually perpendicular **principal axes** for which this matrix is diagonal and torques around the axes act **independently** of each other.

5.8.1 Principal Axis of Inertia

Measured in the body frame the inertia matrix is a constant real symmetric matrix. A real symmetric matrix has the eigendecomposition into the product of a rotation matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{\Lambda}$.

$$\mathbf{I}_{\mathbf{C}}^{B} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathsf{T}}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

The columns of the rotation matrix $\mathbf{Q}$ define the directions of the principal axes of the body, and the constant I_1, I_2 and I_3 are called the **principal moments of inertia**.

6 Fundamental Control

7 Fundamental Machine Learning

8 Fundamental Computer Vision