Experiment title

MEC2003 – Determining the gravitational constant (Cavendish Experiment)

Team members

Emre Berna, Marcus Waterson, Nils Thiessen

Instructors

Chris Pawley, Lorenzo Reverberi, Susan du Pree

Date

17/04/2024

Introduction

The idea of this lab was to replicate the Cavendish Experiment using a gravitational torsion balance, a setup which allows the determination of the gravitational constant G.

While a relationship of the force between two masses had been established by Newton, direct measurements of the proportionality constant would have meant knowing Earth's mass. The main historic achievement of this experiment, as performed by Henry Cavendish in 1798, was that it circumvented this problem, by using two smaller known masses and negating the effect of Earth's gravity. It took a while for the accuracy of the value for G to be improved, and due to this experiment, people were finally able to determine Earth's mass. [3]

Theory

Newton's Law of Universal Gravity dictates that masses m_1 and m_2, independent of their size, will affect each other according to $F = \frac{Gm_1m_2}{r^2}$, where r is the distance between their centres. Nowadays, $G = (6.67408 \pm 0.00031) * 10^{-11}\ \frac{m^3}{kgs^2}$ is the most precise value found in CODATA 2014. [1]

The apparatus uses a torsion balance: Onto a rod, two small masses are attached to either side, while a torsion band connected to the centre of the rod leaves the balance hanging. Due to the symmetry of the masses around the centre, the Earth's gravity is negated. Two larger masses can be placed onto a rotatable surface, on the same level as the larger ones. The apparatus is grounded, so to avoid forces due to charges on the surface.

As the small force of gravity causes this type of pendulum to rotate and thereby twists the band accordingly, the band will provide a counteracting torque depending on the angle of rotation, until these forces balance out.

The laser is pointing towards a mirror on the pendulum, turning with the balance and thus reflecting the light at a certain angle. Thereby, the light beam is reflected onto a wall an appropriate distance away from the setup. One can use the measurement of the light beam's displacement on the wall to deduct the gravitational constant G.

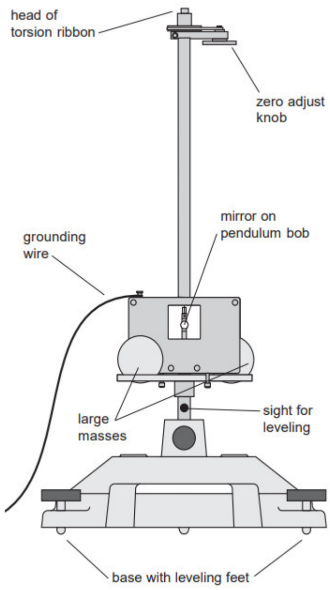

Fig.1: Structure of the Gravitational Torsion Balance apparatus. [2]

With the setup in a crowded lab with surrounding masses potentially influencing the outcome of the experiment, yielding a precise measurement will not be probable. Additionally, measurement and human errors can compound and lead to bigger deviations.

However, measures can be taken to minimize these effects. If one can trust the PASCO setup's manual, 5% accuracy is feasible under isolated conditions [2], so a prediction of the result lying within 10% of the actual gravitational constant can be made here.

Materials & Methods

The apparatus was mounted onto a support base sitting on a stable, horizontal table, with the pendulum's mirror facing a wall around $L = 4.85\ m$ away. This would help in identifying the slight rotation of the balance, which was measured through a metric scale. Through a mirror below the pendulum bob, it was possible to adjust the feet of the base until the bob was centred, meaning that it now directly faced the wall.
At a slightly lower height than the pendulum's mirror, the laser was placed facing away from the wall towards the mirror, so that the beam would be reflected onto the projection surface. A fainter one, casted off the glass window's surface and therefore pointing orthogonally away from setup was used as a reference point. By turning the adjust knob, one could then vertically align the two beams. However, this would take time, since after unscrewing the locking mechanisms, the pendulum would rotate around its equilibrium position. To circumvent this, the peaks were measured on the metric scale and the equilibrium position was thereby determined and marked, also introducing errors.

After determining the big masses $m_{1a} = (1.50083 \pm 0.000005)\ \text{kg}$ and $m_{1b} = (1.49118 \pm 0.000005)\ \text{kg}$, they were placed onto the rotatable surface, so that they were touching the box, now referred to as Position I. With the smaller masses' radius being $r = 9.55 * 10^{-3}\ m$, their and the bigger masses' centres were a distance $b = 4.22 * 10^{-2}\ m$ apart. After some time, the new equilibrium position was measured 13 times in the same manner as before and then averaged. The masses were then shifted into Position II, where the same procedure was repeated.

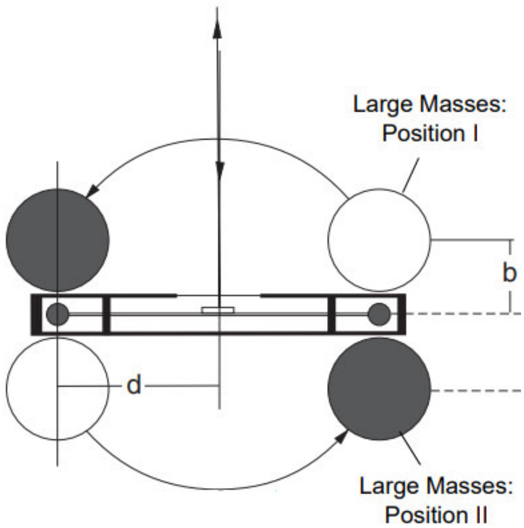

Fig.2: Arrangement of the masses on the rotatable surface. [2]

Finding the constant G through Newton's Law of Universal Gravity could be achieved through finding the torque of the wire that was counteracting this. Establishing the corresponding torsion constant κ required determination of the period T of oscillation unique to this setup. Since the beam was oscillating back and forth between its left and right peaks, the time in between accounted for half a period. Again, these were collected through repeated measurements and averaging.

Results & Analysis

For the half period, 21 measurements were taken and averaged, resulting in $T_{1/2} \approx 32.38381\ s$ being utilized in the calculations. However, due to human error, slight variations did occur. This is especially crucial since noting down the time in the correct moments is highly dependent on judgement. To take these effects into account, the standard deviation $\sigma_T = \sqrt{\frac{\sum_{i=1}^{n}(T_i - T)^2}{n}}$ was implemented, with one standard deviation being $\sigma_{T\frac{1}{2}} \approx 1.0106\ s$. Although the deviations below the average mostly compensated those above it, as seen in Fig.3, a safety margin of 1σ (within 68.27%) was established, thereby considering the propagated error when calculating G.

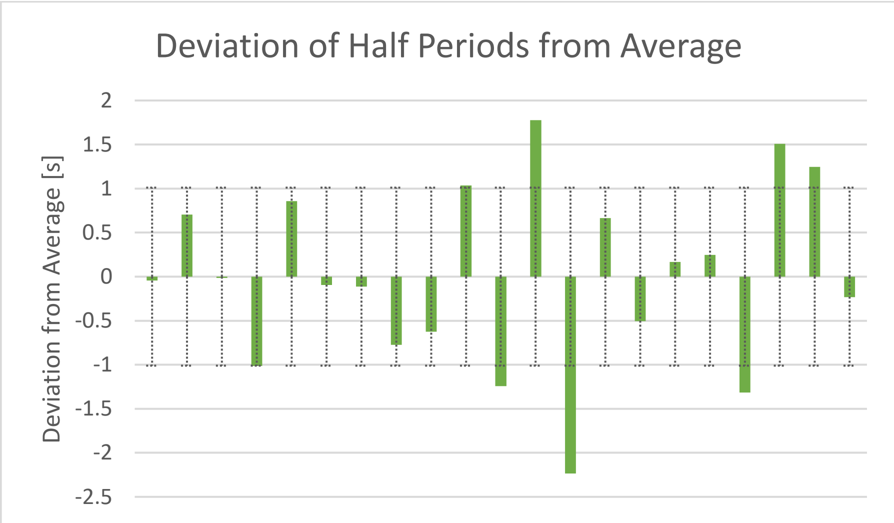

Fig.3: Visualization of each value's deviation from the average in seconds. The black error bars represent the accepted value for $T_{1/2}$

The torsion constant κ, key in deducing the torque exerted by the torsion band the mirror was attached to, settled the period of each oscillation according to:

$$T^2 = \frac{4\pi^2 I}{\kappa} \quad (1)$$

This is highly depended on the moment of inertia of the system attached to it. Here, with two masses $m_2 = 0.0383\ kg$ of radius $r = 9.55 * 10^{-3}\ m$, a distance $d = 5 * 10^{-2}\ m$ away from each other and symmetrically placed about the centre of rotation, the related moment of inertia can be calculated with $I = 2m_2\left(d^2 + \frac{2}{5}r^2\right) \approx 0.000194\ kgm^2$. Solving (1) for κ:

$$\kappa = \frac{4\pi^2 I}{T^2} \approx 1.8285 * 10^{-6}\ Nm$$

Knowledge of how much the mirror had rotated depended on the shift in equilibrium points between Position I and II. After setting up the bigger masses in Position I, in 13 distinct measurements, the left and right peak of the oscillating beam were recorded, then averaged and compiled in Table 1.

Left Peak [m]	0.206	0.213	0.21	0.213	0.217	0.216	0.218	0.22	0.22	0.222	0.224	0.224	0.226
Right Peak [m]	0.394	0.392	0.388	0.384	0.387	0.383	0.381	0.381	0.379	0.377	0.377	0.375	0.373
Equilibrium Position I [m]	0.3	0.3025	0.299	0.2985	0.302	0.2995	0.2995	0.3005	0.2995	0.2995	0.3005	0.2995	0.2995

Table 1: Overview of the metric scale readings for the left and right peak of oscillation, as well as the resulting equilibrium position I.

Is this method accurate?

Now, an assumption was made here. By averaging the first value for the left peak $P_{0,left}$ and the first value of the right peak $P_{0,right}$ in order to receive a value for the equilibrium position P_0, one assumes that these peaks are at the same distance towards the actual equilibrium point. However, over time, the motion of the beam and thereby the amplitudes of the peaks are damped. $P_{0,right}$ was measured half a period later than $P_{0,left}$, meaning that taking the average would not accurately describe the equilibrium position at that time. However, this deviation from one peak to the next one is rather miniscule.

One way of showing this is to compare the equilibrium position as calculated from $P_{n,left}$ and $P_{n,right}$ to that of $P_{n,left}$ and $P_{(n+1),right}$, where n refers to the n^{th} oscillation. Depending on their similarity, one can argue that the left and consecutive right peak do in fact approximately describe the same equilibrium position, which means that this method would be a good approximation for the equilibrium point.

A_N: Average of $P_{(n,+1),right}$ & $P_{n,left}$	B_N: Average of $P_{n,right}$ & $P_{n,left}$	Deviation A_N from B_N	Deviation B_N from A_{N+1}
-	0.3	-	0.001
0.299	0.3025	0.0035	0.002
0.3005	0.299	0.0015	0.002
0.297	0.2985	0.0015	0.0015
0.3	0.302	0.002	0.002
0.3	0.2995	0.0005	0.001
0.2985	0.2995	0.001	0
0.2995	0.3005	0.001	0.001
0.2995	0.2995	0	0.001
0.2985	0.2995	0.001	0
0.2995	0.3005	0.001	0.001
0.2995	0.2995	0	0.001
0.2985	0.2995	0.001	-

Table 2: Comparison of equilibrium points derived from consecutive peaks. The deviation columns summarize how each equilibrium point deviates from the next. The deviations are in absolute values.

In Table 2, the deviations are numerically small, meaning that one can correlate two consecutive peaks in this arrangement. The average deviation of approximately $0.854\ mm$ will cause problems later. However, when collecting all the derived equilibrium point and averaging them, the aforementioned deviation will be taken into account with the corresponding standard deviation.
The average values of A_N and B_N result in an equilibrium position I of $S_1 = 0.2996\ m$, with a standard deviation of $\sigma \approx 0.001086\ m$. A safety margin of 1σ, as applied for the period error, can be used here. This, in combination with the measurement error of $0.0005\ m$ due to the metric scale, yields a total error of $\epsilon_{S1} \approx 0.001586\ m$.

After switching the bigger masses to Position 2, the new readings can be compiled in Table 2:

Left Peak [m]	0.227	0.227	0.2305	0.231	0.233	0.234	0.236	0.237	0.2375	0.239	0.241	0.2415	0.2425
Right Peak [m]	0.368	0.367	0.364	0.363	0.362	0.3615	0.361	0.3595	0.358	0.357	0.355	0.3535	0.3535

Table 2: Overview of the metric scale readings for the left and right peaks of oscillation.

Conducting the same steps as elaborated before, the average deviation between consecutive peaks is $0.6\ mm$, it can thus be proceeded as before. The average equilibrium position II becomes $S_2 = 0.29739\ m$, with a standard deviation of $\sigma \approx 0.0006\ m$. Together with the measurement error, the total error generated for position II is $\epsilon_{S2} \approx 0.0011\ m$.

The difference between S_1 and S_2 shows the distance ΔS the beam moved between Position I and II, with $\Delta S = 0.00221\ m$. As the errors during this form of error propagation add up, the absolute error is $\epsilon_{\Delta S} = 0.002686\ m$.

The angle θ, describing the angular distance the torsion band is twisted from Position 0 due to the gravitational force, can be derived from $L * \tan(2\theta) = \frac{\Delta S}{2}$. The angle is doubled due to the reflection upon the mirror and $\frac{\Delta S}{2}$ corresponds to the distance on the wall from Position 0 to I, where the rotation was caused by gravity. With the small angle approximation $\lim_{\varphi \to 0}(\tan(\varphi)) \approx \varphi$, the angle θ can be calculated with:

$\theta = \frac{|S_2 - S_1|}{4L} = \frac{0.00221\ m}{4*4.85\ m} \approx 0.0001139.$

Each big mass m_1 exerts a force onto the smaller ones, which the torsion band must resist to, with $\tau_{band} = \kappa\theta = F_a d + F_b d$. Due to m_{1a} and m_{1b} possessing distinct values, there is also a difference in the ensuing force. Averaging them and thereby averaging the force simplifies the expression: $\kappa\theta = 2F_{avg}d$, with $m_{avg} = \frac{1}{2}(m_{1a} + m_{1b}) \approx 1.496001kg$.

Ultimately applying Newton's Law of Universal Gravity $F_{avg} = G\frac{m_{avg}m_2}{b^2}$ and solving for G, the equation becomes:

$$G = \frac{\kappa\theta b^2}{2dm_{avg}m_2} \quad (2)$$

Inserting the derived values into equation (2) yields $G_{exp} \approx 6.4752 * 10^{-11}\frac{m^3}{kgs^2}$.

The percentage error in G_{exp} is then:

$$100 * \frac{\Delta G_{exp}}{G_{exp}} = 100 * \sqrt{\left(\frac{\epsilon_{\Delta S}}{\Delta S}\right)^2 + 2 * \left(\frac{\sigma_{T\frac{1}{2}}}{T_{\frac{1}{2}}}\right)^2}$$

$$= 100 * \sqrt{\left(\frac{0.002686\ m}{0.00221\ m}\right)^2 + 2 * \left(\frac{1.0106\ s}{32.38381\ s}\right)^2} \approx 121.62\%.$$

With errors, one can write out G_{exp} as $G_{exp} \approx (6.4752 \pm 7.875) * 10^{-11}\frac{m^3}{kgs^2}$.

Although other quantities such as the mass m_{avg} or the distance L towards the projection surface had their own measurement errors, they were negligible compared to their own size.

Discussion

Whilst varying measurements of G have been conducted over the last few years, the expected result for the gravitational constant, in agreement with CODATA 2014 [1], was set at $G = (6.67408 \pm 0.00031) * 10^{-11}\ \frac{m^3}{kgs^2}$. One can compare this value to the actual experimentally gathered G_{exp} and calculate the percentage error $\frac{(G_{exp} - G)}{G} * 100$, which gives an error of (-2.9831%). This small discrepancy is rather surprising considering the circumstances of the experiment. Although the PASCO manual promised a 5% accuracy, the result could also have appeared by chance. This can be highlighted through the sensitivity of the experiment: In the end, the dimensions of the distance ΔS between the two equilibrium points allowed for small variations to lead to significant changes, as seen in the percentage error of G_{exp}. The problem here is the uncertainty of the result, as even for such an accurate value, one cannot be certain about its validity.

The method for finding the equilibrium points was one of the main factors influencing this variation. Normally, when the big masses are put into the two positions, the measurements are taken a few hours later. In this way, the oscillation would be damped until eventually, it would only slowly spin around the equilibrium position. However, due to time constraints, it was chosen to mark the positions of two consecutive peaks and take the equilibrium points as the average of these two values. As already elaborated on in the corresponding section, whereas the moderate speed at which the oscillation was damped permitted two successive peaks to approximately describe the same equilibrium position, the scale of ΔS made even this approximation significantly effect the end result. This could have been avoided by modelling the position of the beam as a function of a damped harmonic oscillator, or for an even more precise measurement, by pointing another laser onto the mirror, such that both left and right peak could be read at the same time.

A major problem in general was that due to the sensitivity of the experiment, small amounts of background noises could considerably impact the recorded values. The environment therefore should have prevented these as much as possible. While the procedure was undertaken in a crowded lab, the apparatus was stabilized on a table heavy enough to inhibit such influences. Additionally, masses in proximity (that is, horizontal proximity) were removed. Since the Force in Newton's Law of Universal Gravity decreases with the inverse-square law ($\frac{1}{b^2}$), other objects' masses would be trivial due to their distance towards the apparatus. That being said, the data for the right peak in Table 2 shows that even though the damping should have decreased the value consistently, certain oscillations appear with a higher amplitude. Background noise hence must have increased the system's energy.

An uncertainty that has not been commented yet is the form of the laser beam as projected onto the wall, which took the shape of a circle. This means that one had to estimate the location of the circle's centre, which might have caused additional human errors.

Lastly, a simple way to reduce the period error would have been to record the time for a large number of oscillations, and then dividing by the number of oscillations. Since there should be no deviation in the time between two peaks, one could now disregard the uncertainty thereof.

Conclusion

The aim of this experiment was to find an experimental value for the gravitational constant G with the PASCO Gravitational Torsion Balance apparatus. This meant setting up two bigger masses in two positions close to smaller balls, which were attached to a torsion balance. A laser was reflected off of a mirror attached to this pendulum, which meant that one could deduce the distance the equilibrium position had shifted. From this, finding the angle θ through which the torsion band was twisted lead to the determination of the Gravitational constant G:

$$G_{exp} \approx (6.4752 \pm 7.875) * 10^{-11} \frac{m^3}{kgs^2}$$

The value itself is quite close to the one found in CODATA 2014, as hypothesized in the introduction. However, the error caused by the inaccuracy of the experiment requires further research. In an improved experiment, it would be important to further isolate the experimental setup form background noise. Additionally, a different approach to taking measurements of the equilibrium points would need to be taken. Ideas reach from leaving the pendulum to settle down, thereby achieving an accurate measurement without the need to measure the left and right peaks, or to reflect a second laser onto the projection surface. This way, one could measure both peaks at the same time, meaning that they would refer to the same equilibrium point.

References

[1] C. Xue *et al.*, “Precision measurement of the Newtonian gravitational constant,” *National Science Review*, vol. 7, no. 12, pp. 1803–1817, Jul. 2020, doi: https://doi.org/10.1093/nsr/nwaa165.

[2] PASCO, “Instruction Manual and Experiment Guide for the PASCO scientific Model AP-8215,” Nov. 1998. http://www.hep.fsu.edu/~wahl/phy3802/expinfo/cavendish/pasco-06802.pdf

[3] “The Cavendish Experiment and the Quest to Determine the Gravitational Constant | Encyclopedia.com,” *www.encyclopedia.com*, Apr. 15, 2024. https://www.encyclopedia.com/science/encyclopedias-almanacs-transcripts-and-maps/cavendish-experiment-and-quest-determine-gravitational-constant

[4] Lab coordinators et al., “MECH203 – Cavendish Experiment”, 2024