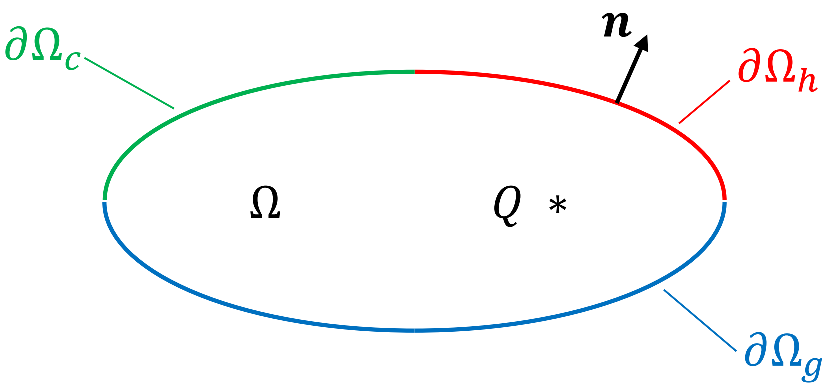
10 FE formulation heat flow



3d body occupying the domain Ω

$\boldsymbol{n}$ unit outward normal vector to the boundary $\partial\Omega$

boundary $\partial\Omega = \partial\Omega_h \cup \partial\Omega_g \cup \partial\Omega_c$

Strong form of the transient heat equation $\quad c\rho\dot{T} + div(\boldsymbol{q}) - Q = 0$

c heat capacity $\quad \rho$ mass density $\quad T$ temperature $\quad \boldsymbol{q}$ heat flux $\quad Q$ heat source

Boundary conditions

Dirichlet or essential BCs	$T = g$	given on	$\partial\Omega_g$
Neumann or natural BCs	$q_n = \boldsymbol{q}^T\boldsymbol{n} = h$	given on	$\partial\Omega_h$
Robin or convection BCs	$q_n = \alpha[T - T_\infty]$	given on	$\partial\Omega_c$
	α convection coefficient		T_∞ temperature at infinity

Initial condition $\quad T(\boldsymbol{x}, t = 0) = T_{init}$ given in Ω

10 FE formulation heat flow

Strong form → **weak form** for FE formulation, we assume here no convection BCs!

Multiply by an arbitrary weight function $v(\boldsymbol{x})$ and apply Green-Gauss theorem

$$\int_{\Omega} vc\rho\dot{T}\,dV - \int_{\Omega} [\nabla v]^T \boldsymbol{q}\,dV - \int_{\Omega} vQ\,dV + \int_{\partial\Omega} v\,\boldsymbol{q}^T\boldsymbol{n}\,dV = 0$$

Approximate weight function by $v(\boldsymbol{x}) = \boldsymbol{c}^T\boldsymbol{N}^T(\boldsymbol{x})$ and $[\nabla v]^T = \boldsymbol{c}^T\boldsymbol{B}^T(\boldsymbol{x})$

$$\boldsymbol{c}^T\left[\int_{\Omega} \boldsymbol{N}^T c\rho\dot{T}\,dV - \int_{\Omega} \boldsymbol{B}^T\boldsymbol{q}\,dV - \int_{\Omega} \boldsymbol{N}^T Q\,dV + \int_{\partial\Omega} \boldsymbol{N}^T\boldsymbol{q}^T\boldsymbol{n}\,dA\right] = 0$$

Since $\boldsymbol{c}$ can be choosen arbitrary the expression in the brackets $[\ldots] = \boldsymbol{0}$

$$\left[\int_{\Omega} \boldsymbol{N}^T c\rho\dot{T}\,dV - \int_{\Omega} \boldsymbol{B}^T\boldsymbol{q}\,dV - \int_{\Omega} \boldsymbol{N}^T Q\,dV + \int_{\partial\Omega} \boldsymbol{N}^T\boldsymbol{q}^T\boldsymbol{n}\,dA\right] = \boldsymbol{0}$$

10 FE formulation heat flow

Strong form → **weak form** for FE formulation, we assume here no convection BCs!

$$\left[\int_{\Omega} \boldsymbol{N}^T c\rho\dot{T}\, dV - \int_{\Omega} \boldsymbol{B}^T\boldsymbol{q}\, dV \;-\; \int_{\Omega} \boldsymbol{N}^T Q\, dV \;+\; \int_{\partial\Omega} \boldsymbol{N}^T \boldsymbol{q}^T\boldsymbol{n}\, dA\right] = \boldsymbol{0}$$

Approximate time-dependent temperature field by $T(\boldsymbol{x},t) = \boldsymbol{N}(\boldsymbol{x})\,\boldsymbol{a}(t)$
and introduce Fourier's law for heat conduction $\boldsymbol{q} = -\boldsymbol{D}\nabla T$

$$\int_{\Omega} \boldsymbol{N}^T c\rho\boldsymbol{N}\, dV\,\dot{\boldsymbol{a}} + \int_{\Omega} \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\, dV\,\boldsymbol{a} \;-\; \int_{\Omega} \boldsymbol{N}^T Q\, dV \;+\; \int_{\partial\Omega} \boldsymbol{N}^T q_n\, dA = \boldsymbol{0}$$

Rearrange $\qquad \boldsymbol{C}\dot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}_l + \boldsymbol{f}_b \qquad$ with

Heat capacity matrix $\qquad \boldsymbol{C} = \int_{\Omega} \boldsymbol{N}^T c\rho\boldsymbol{N}\, dV$

‚Stiffness' or conductivity matrix $\qquad \boldsymbol{K} = \int_{\Omega} \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\, dV$

Load vector $\qquad \boldsymbol{f}_l = \int_{\Omega} \boldsymbol{N}^T Q\, dV$

Boundary vector $\qquad \boldsymbol{f}_b = -\int_{\partial\Omega} \boldsymbol{N}^T q_n\, dA$

10 FE formulation heat flow

Semi-discrete equations $\boldsymbol{C}\dot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}_l + \boldsymbol{f}_b = \boldsymbol{f}$ still continuous in time

Now, discretization in time, here the θ-method.
$\theta \in [0,1]$ is a numerical parameter of this method.

Subdivide time interval $t \in [0, T_{end}]$ into $N+1$ subintervals
$[0, T_{end}] = \cup_{n=0}^{N}[t_n, t_{n+1}]$ and $\Delta t_n = t_{n+1} - t_n$

Approximate time derivative $\dot{\boldsymbol{a}} = \frac{\boldsymbol{a}(t_{n+1}) - \boldsymbol{a}(t_n)}{t_{n+1} - t_n} = \frac{\boldsymbol{a}_{n+1} - \boldsymbol{a}_n}{\Delta t_n}$

Approximate temperature $\boldsymbol{a} = \theta \boldsymbol{a}_{n+1} + [1-\theta]\boldsymbol{a}_n$

Apprimate right hand side $\boldsymbol{f} = \theta \boldsymbol{f}_{n+1} + [1-\theta]\boldsymbol{f}_n$

$$\frac{1}{\Delta t_n}\boldsymbol{C}[\boldsymbol{a}_{n+1} - \boldsymbol{a}_n] + \boldsymbol{K}[\theta \boldsymbol{a}_{n+1} + [1-\theta]\boldsymbol{a}_n] = \theta \boldsymbol{f}_{n+1} + [1-\theta]\boldsymbol{f}_n$$

$\boldsymbol{a}_{n+1}$ unknown node temperatures at time t_{n+1}

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$$\frac{1}{\Delta t_n}\boldsymbol{C}[\boldsymbol{a}_{n+1}-\boldsymbol{a}_n]+\boldsymbol{K}[\theta\boldsymbol{a}_{n+1}+[1-\theta]\boldsymbol{a}_n]=\theta\boldsymbol{f}_{n+1}+[1-\theta]\boldsymbol{f}_n$$

$\boldsymbol{a}_{n+1}$ unknown node temperatures at time t_{n+1}
$\boldsymbol{a}_n$ known node temperature from the previous time step t_n

initial conditions
At $t=0$, i.e. $n=0$, $\boldsymbol{a}_0=\mathbf{a}(\mathrm{t}=0)$ are given values!
You need to know the temperature field in the whole domain Ω at $t=0$

- $\theta=1$. This is known as an implicit scheme and is often chosen due to it's stability properties
- $\theta=1/2$. Midpoint rule or Crank-Nicholson scheme. Often used due to its accuracy properties.
- $\theta=0$. Forward scheme or explicit scheme. If additional assumption regarding the $\boldsymbol{C}$ matrix is made, this method enables a very efficient scheme on large scale clusters to be obtained.

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Global FE formulation

Integration has to be performed over the complete domain Ω or boundary $\partial\Omega$
like global matrices $\boldsymbol{K} = \int_\Omega \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\, dV$ or vectors $\boldsymbol{f}_b = -\int_{\partial\Omega} \boldsymbol{N}^T q_n\, dA$ etc.

Element FE formulation
Practical implementation: split integrals over the complete domain Ω
into a sum of integrals over each individual element e. Then **assemble in a loop** the **element matrices** and vectors to **global matrices** and vectors.

One introduces element matrices and vectors like

$$\boldsymbol{K}^e = \int_{\Omega^e} \boldsymbol{B}^{e^T}\boldsymbol{D}\boldsymbol{B}^e\, dV \quad \text{and} \quad \boldsymbol{f}_l^e = -\int_{\partial\Omega^e} \boldsymbol{N}^{eT} q_n\, dA \quad \text{and} \quad \boldsymbol{f}_b^e = -\int_{\Omega^e} \boldsymbol{N}^{eT}\, Q dA$$

e.g. for a triangular 2d element

$$\boldsymbol{N}^e = [N_1^e \quad N_2^e \quad N_3^e] \qquad\qquad \boldsymbol{B}^e = \nabla\boldsymbol{N}^e = \begin{bmatrix} \dfrac{\partial \boldsymbol{N}^e}{\partial x} \\ \dfrac{\partial \boldsymbol{N}^e}{\partial y} \end{bmatrix}$$

10 FE formulation heat flow

Element FE formulation

$\boldsymbol{K}^e = \int_{\Omega^e} \boldsymbol{B}^{e^T} \boldsymbol{D} \boldsymbol{B}^e \, dV$ and $\boldsymbol{f}_b^e = -\int_{\partial\Omega^e} \boldsymbol{N}^{eT} q_n \, dA$ and $\boldsymbol{f}_b^e = -\int_{\Omega^e} \boldsymbol{N}^{eT} \, Q dA$

$\dim \boldsymbol{K}^e = [3 \times 3]$ for a 3-node triangular element

Expanded element matrices and vectors

Define expanded element stiffness matrix by

$\boldsymbol{K}^{ee} = \int_{\Omega^e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \, dV$

Attention: e.g. 2d mesh of 3-node triangular elements with total node number n we use the global $\boldsymbol{B}$ matrix with $\dim \boldsymbol{B} = [2 \times n]$ but **integrate only over one elment** Ω^e

Thus, $\dim \boldsymbol{K}^{ee} = [n \times n]$ but only $[3 \times 3] = 9$ entries are non-zero.

We do this for all n_{el}elements in the mesh and get the global stiffness matrix

$\boldsymbol{K} = \sum_{e=1}^{n_{el}} \boldsymbol{K}^{ee}$ analog $\boldsymbol{f}_l = \sum_{e=1}^{n_{el}} \boldsymbol{f}_{\boldsymbol{l}}^{ee}$ etc.

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Example expanded element matrices and vectors

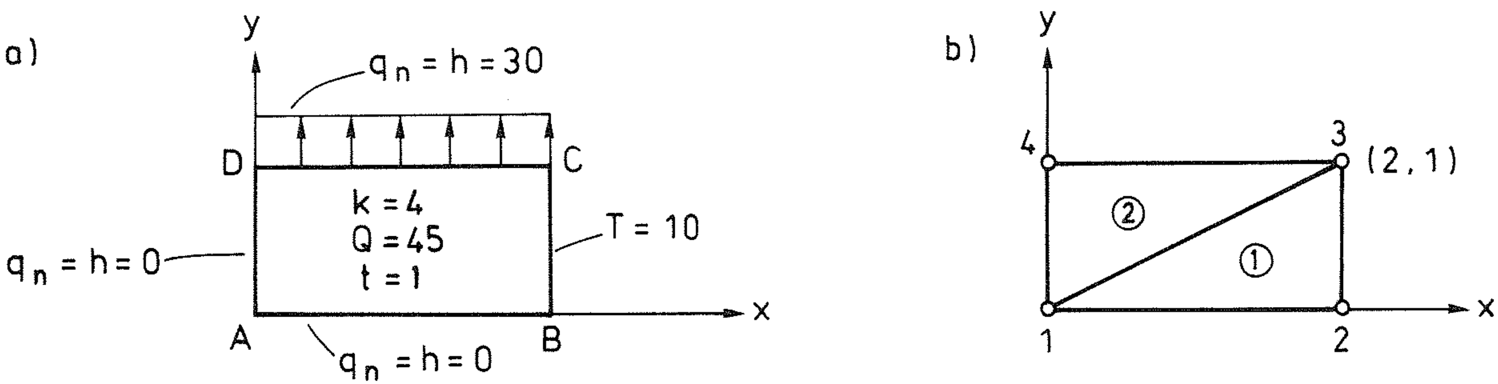

Figure 10.4 (a) Problem reformulation; (b) finite element mesh

$$\mathbf{K}^{\mathrm{e}} = k\mathbf{B}^{\mathrm{eT}}\mathbf{B}^{\mathrm{e}}tA_{\alpha}$$

$$\mathbf{B}^{\mathrm{e}} = \frac{1}{2A_{\alpha}}\begin{bmatrix} y_j - y_k & y_k - y_i & y_i - y_j \\ x_k - x_j & x_i - x_k & x_j - x_i \end{bmatrix}$$

$$\mathbf{f}_{\mathrm{l}}^{\mathrm{e}} = Qt\int_{A_{\alpha}} \mathbf{N}^{\mathrm{eT}}\,\mathrm{d}A$$

k thermal condictivity
t thickness of the plate
A_{α} area of the element α

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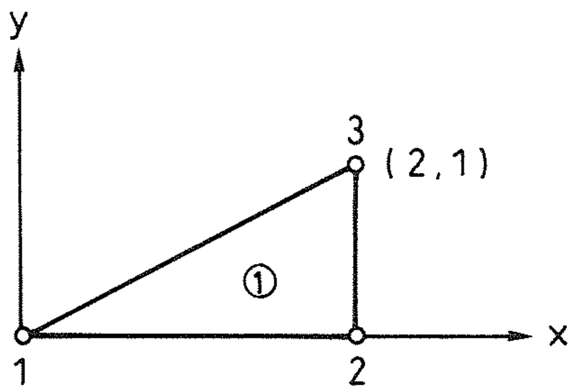


$$x_i = y_i = 0; \quad x_j = 2,\ y_j = 0; \quad x_k = 2,\ y_k = 1$$

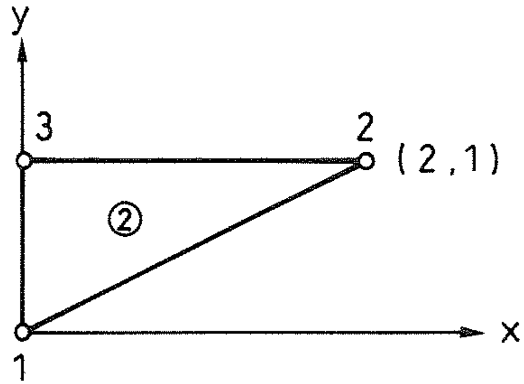
$$\mathbf{B}^e = \frac{1}{2}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\mathbf{K}^e = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 5 & -4 \\ 0 & -4 & 4 \end{bmatrix}; \quad \mathbf{f}_l^e = 15\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{K}_1^{ee} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 5 & -4 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{f}_l = \mathbf{f}_{l1}^{ee} = 15\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

10 FE formulation heat flow

b)



$$x_i = y_i = 0; \quad x_j = 2,\ y_j = 1; \quad x_k = 0,\ y_k = 1$$

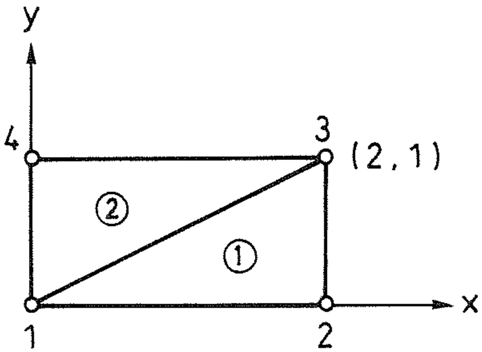
$$\mathbf{B}^e = \frac{1}{2}\begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix}$$

$$\mathbf{K}^e = \begin{bmatrix} 4 & 0 & -4 \\ 0 & 1 & -1 \\ -4 & -1 & 5 \end{bmatrix}; \quad \mathbf{f}_1^e = 15\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{K}_2^{ee} = \begin{bmatrix} 4 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -4 & 0 & -1 & 5 \end{bmatrix}; \quad \mathbf{f}_{12}^{ee} = 15\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

10 FE formulation heat flow

b)



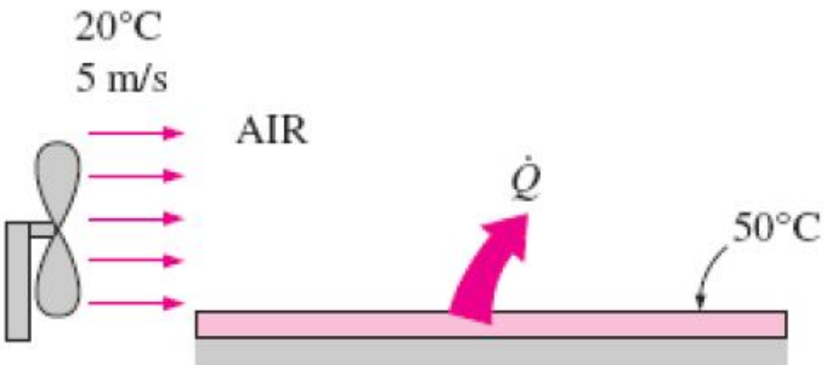
$$\mathbf{K}_1^{ee} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 5 & -4 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{f}_l = \mathbf{f}_{l1}^{ee} = 15\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{K}_2^{ee} = \begin{bmatrix} 4 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -4 & 0 & -1 & 5 \end{bmatrix}; \quad \mathbf{f}_{l2}^{ee} = 15\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

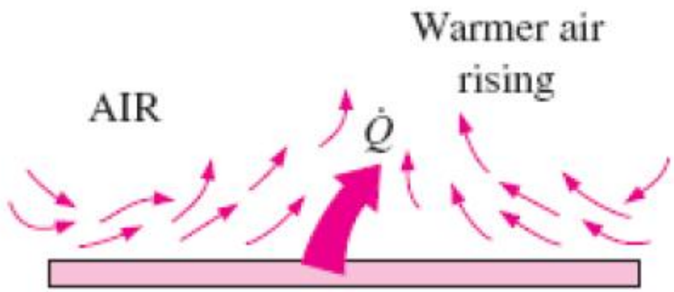
$$\boldsymbol{K} = \sum_{e=1}^{n_{el}} \boldsymbol{K}^{ee}$$

$$\boldsymbol{f}_l = \sum_{e=1}^{n_{el}} \boldsymbol{f}_l^{ee} \quad \Longrightarrow \quad \mathbf{K} = \begin{bmatrix} 5 & -1 & 0 & -4 \\ -1 & 5 & -4 & 0 \\ 0 & -4 & 5 & -1 \\ -4 & 0 & -1 & 5 \end{bmatrix}; \quad \mathbf{f}_l = 15\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Heat flow with convection



(a) Forced convection $\alpha = 12 \ \dots 120 \, W/(m^2K)$



(b) Free convection $\alpha = 5 \ \dots 25 \, W/(m^2K)$

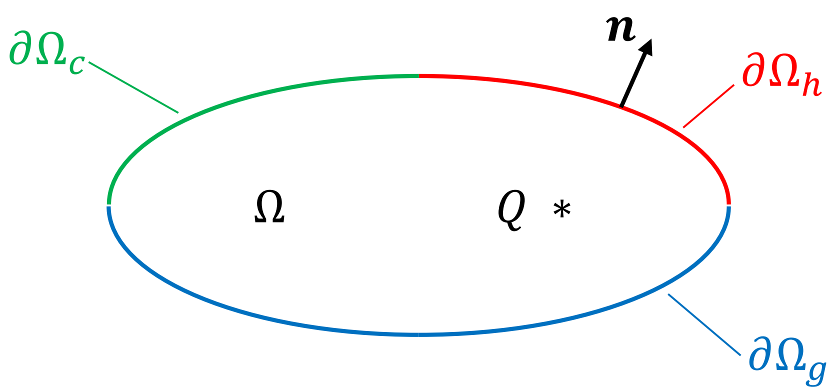
The convection coefficient α has to be measured for convective heat transfer!!!

$$q_n = \alpha[T - T_\infty]$$

Air-cooled engine

CPU-cooler

10 FE formulation heat flow



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	α convection coefficient		T_∞ temperature at infinity

Initial condition $\quad T(\boldsymbol{x}, t = 0) = T_{init}$ given in Ω

10 FE formulation heat flow

Strong form → **weak form** for FE formulation, we assume here no convection BCs!

$$\left[\int_\Omega \boldsymbol{N}^T c\rho \dot{T}\, dV - \int_\Omega \boldsymbol{B}^T \boldsymbol{q}\, dV - \int_\Omega \boldsymbol{N}^T Q\, dV + \int_{\partial\Omega} \boldsymbol{N}^T \boldsymbol{q}^T \boldsymbol{n}\, dA\right] = \boldsymbol{0}$$

Approximate time-dependent temperature field by $T(\boldsymbol{x}, t) = \boldsymbol{N}(\boldsymbol{x})\, \boldsymbol{a}(t)$
and introduce Fourier's law for heat conduction $\boldsymbol{q} = -\boldsymbol{D}\nabla T$

$$\int_\Omega \boldsymbol{N}^T c\rho \boldsymbol{N}\, dV\, \dot{\boldsymbol{a}} + \int_\Omega \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV\, \boldsymbol{a} - \int_\Omega \boldsymbol{N}^T Q\, dV + \int_{\partial\Omega} \boldsymbol{N}^T q_n\, dA = \boldsymbol{0}$$

Rearrange $\quad \boldsymbol{C}\dot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}_l + \boldsymbol{f}_b \quad$ with

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Assigment

- Project should be solved in groups of two.

- Write a technical report

- Send your project as a zipped folder to FHLF20@solid.lth.se
    - Report as a pdf-file
    - Matlab files

- Hand in a printed version of your report at the Division of Solid Mechanics
  Use the box close to our printer on the 5th floor, M-Huset

- Deadline 22nd May 2018, 16:00

- Max. 5 bonus points only valid for the exam end of May/June 2018

- Approval of your report until 12 June 2018 if it contains substantial mistakes

- Do **not** use subroutines which are included in CALFEM!
- You **can** use the matlab program skeletons provided for the Computer Exercises