Evolutionary Dynamics of Mixed Rings of Coordinators and Anticoordinators

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***Abstract*— Decision-making based on peers' actions often results in one of the following two groups: *coordinators* who tend to blend in with the population and make trending decisions, and *anticoordinators* who act against the majority. Only mixed networks of both coordinators and anticoordinators are capable of not reaching an equilibrium state, where every individual is satisfied with her choice. However, the conditions for non-equilibration, and more challengingly, the characterization of the non-equilibrium limit set remain concealed. We answer these problems for ring networks. We show that a mixed ring of coordinators and anticoordinators equilibrates if and only if it does not contain a particular arrangement of consecutive agents, and never equilibrates if and only if it contains a particular arrangement of consecutive agents with particular actions. As a result, a ring may admit both equilibrium and non-equilibrium limit sets. We further investigate the stability of the equilibrium states of the resulting network dynamics.**

I. INTRODUCTION

Decision-makers often decide based on the actions of their peers. When joining a riot, for example, individuals often wait for enough others to join and will leave the riot if less than enough others stay [1]. These decision-makers are called *coordinators* [2] as they coordinate their decisions to the action chosen by a considerable number of their fellows. Coordinators appear in many social contexts, such as the spread of innovation [3], fashion [4], dynamic norms [5], and hashtags in Twitter [6]. The opposite type of decision-makers are the so-called *anticoordinators* who take an action if a considerable number of their fellows have chosen another action. For example, when faced with a decision between two queues, individuals often choose the less crowded one. Avoiding labor congestion [7], traffic jams [8], and fashion norms [9] are examples of when individuals tend to be anti-coordinators. While both coordinators and anticoordinators choose the myopically most beneficial actions, they may not be satisfied with their decisions in the long run [10]. A fundamental question is, hence, whether a population of coordinators and anticoordinators will eventually reach a state of satisfactory decisions, i.e., an equilibrium [11], [12].

This question has been investigated in the context of *evolutionary game theory* [13]–[18], particularly *network games* [19]–[21], representing a network of interacting individuals, where each individual plays games with her neighbors by choosing one of the available strategies, earns payoffs based on her payoff matrix and neighbors' choices and correspondingly revises her strategy. Under the *best-response* update rule [22]–[26], individuals update their strategies to the one with the highest payoff, which in case of binary strategies, results in two types of individuals based on their payoff matrices: those who choose a strategy if the portion of their neighbors choosing that strategy exceeds (resp. falls short of) a certain threshold [27], [28], yielding the coordinating and anticoordinating types of individuals [19]. The decision-making dynamics of coordinators and anticoordinators have been studied under different assumptions, such as stochastic strategy updating [29]–[32], (payoff matrix) heterogeneity [33]–[35], and co-existence with other types of agents [2], [36], [37]. We particularly know that exclusive populations of coordinators and exclusive populations of anticoordinators equilibrate, regardless of the network structure [38]. The conditions for a mixed network of coordinators and anticoordinators to equilibrate remain an open problem. Furthermore, while characterization of the equilibria that the resulting decision-making dynamics admit is usually straightforward, characterization of the non-equilibrium limit sets which are in the form of a non-singleton positively invariant set is challenging, even for unstructured populations [39].

We approach these problems for one of the simplest networks: the ring. We answer the following questions: Given a mixed network of coordinators and anticoordinators, what is the necessary and sufficient condition on the arrangement of the two types of agents that guarantees the existence of an equilibrium? Given the mixed network, what is the necessary and sufficient condition for the existence of a non-equilibrium limit set? And finally, are the equilibrium states stable?

II. ASYNCHRONOUS THRESHOLD DYNAMICS

Consider the undirected network $\mathbb{G} = (\mathcal{V}, \mathcal{E}), \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, of agents $\mathcal{V} = \{1, \dots, n\}$, who decide on the strategies A and B over the time sequence $k = 0, 1, \dots$. Each agent is either a *coordinator* or an *anticoordinator* and is associated with the same threshold $\tau \in [0, 1]$. A coordinator tends to choose (or *play*) strategy A if the ratio of her neighbors who have chosen A exceeds her threshold, and the opposite holds for an anticoordinator. More specifically, denote the strategy of agent i by $x_i \in \{A, B\}$ and the number of her neighbors who play A by n_i^A, both being a function of time. At each time $k \geq 0$, a single agent i becomes active to revise her strategy at time $k+1$ based on the *(linear) threshold dynamics*. If

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the active agent is a coordinator, she updates her strategy to

$$x_i(k+1) = \begin{cases} A & n_i^A(k) \geq \tau \deg_i \\ B & n_i^A(k) < \tau \deg_i \end{cases}, \tag{1}$$

where $\deg_i$ is the *degree of agent* i, that is, the number of neighbors of agent i, and if the agent is an anticoordinator, she updates her strategy to

$$x_i(k+1) = \begin{cases} A & n_i^A(k) \leq \tau \deg_i \\ B & n_i^A(k) > \tau \deg_i \end{cases}. \tag{2}$$

We stack the strategies of all of the agents to obtain the n-dimensional *state* of the network, denoted by $\boldsymbol{x}(t) = [x_i(t)]$. The agents update asynchronously as a single agent becomes active at each time. The sequence of agents that become active over time are referred to as the *(agents') activation sequence*. We make the mild assumption that the activation sequence is persistent: every agent becomes active infinitely many times. We also often consider a *random activation sequence*, where a random agent becomes active at every time step, with every agent having a nonzero activation probability. The activation sequence together with update rules 1 and 2 define the *network dynamics*.

Given the finite-size discrete state space, every solution trajectory will be contained in a *positively invariant set*, defined as $\mathcal{X} \in \{A, B\}^n$, where

$$\boldsymbol{x}(k) \in \mathcal{X} \Rightarrow \boldsymbol{x}(k+1) \in \mathcal{X} \quad \forall k \in \mathbb{Z}_{\geq 0},$$

under every activation sequence. The complete state space $\{A, B\}^n$ is a trivial example of a positively invariant set. A positively invariant set is *minimal* if no subset of it is positively invariant under every activation sequence. If a minimal positively invariant set is singleton, it results in an *equilibrium state* $\boldsymbol{x}^*$. It follows that the dynamics either reach an equilibrium or a non-singleton minimal positively invariant set, which we refer to as a *fluctuation set*. Every state of a fluctuation set will be visited infinitely often under activation sequences where agents are activated randomly, resulting in perpetual "fluctuations" of the state. Namely, unlike in equilibrium states, where agents are satisfied with and do not intend to switch their strategies, in fluctuation sets, they never settle on their strategies. Our goal is to determine for an arbitrary initial state, whether the dynamics reach a fluctuation set or equilibrate, i.e., reach an equilibrium. The problem is solved for exclusive networks of coordinators and exclusive networks of anticoordinators.

Proposition 1 ([38]): Every network of all coordinators or all anticoordinators equilibrates after a finite number of strategy switches.

However, the case where a mixture of both exists in the population remains unsolved. This is the only case where perpetual oscillations can happen. We approach this problem by investigating the dynamics on a particular type of network: A *ring network* defined as $(\mathcal{V}, \mathcal{E})$ with the edge set $\mathcal{E} = \{(1,2), (2,3), \ldots, (n-1,n), (n,1)\}$, resulting in the *ring (threshold-)dynamics*. We assume that $n \geq 3$. We assume that the ring consists of a mixture of coordinators and anticoordinators:

Assumption 1 (mixed population): There is at least one coordinator and one anticoordinator in the population.

III. Equilibrium States

We determine the form of equilibrium states and fluctuation sets and provide the circumstances under which they exist. In the subsequent analyses, we assume the agents' thresholds to be a non-negative number less than half. The results for the case with a greater-than-half threshold are similar, and we ignore the trivial cases $\tau = 0$ and $\tau = 1$.

Assumption 2: $\tau \in (0, 1/2)$.

As each agent has two neighbors in the ring, she will update her strategy according to Table I. Notice that coordinators switch their strategies to B only if they have no A-playing neighbors. On the opposite, anticoordinators switch to B only if both of their neighbors play A. This restricts the possible strategy patterns for the agents at equilibrium.

TABLE I

Update map for agents with two neighbors

Number of A-playing neighbors	0	1	2
A coordinator's tendentious strategy	B	A	A
An anticoordinator's tendentious strategy	A	A	B

Proposition 2 (form of equilibrium states): State $\boldsymbol{x}$ is an equilibrium if and only if

1) every coordinator plays A and is adjacent with another A-player, and
2) for every anticoordinator i, one of the strategy sequences $\boldsymbol{x}_{i-1} = (x_{i-2,\ldots,i})$, $\boldsymbol{x}_i = (x_{i-1,\ldots,i+1})$, or $\boldsymbol{x}_{i+1} = (x_{i,\ldots,i+2})$ equals (A, B, A).

Proof: Necessity. Part 1) According to Table I, a B-playing coordinator is satisfied with her strategy only if both of her neighbors are playing B. On the other hand, those neighbors cannot be anticoordinators as then both of their neighbors must play A. So both of the neighbors of the coordinator are also B-playing coordinators. By induction, it follows that all of the agents in the population are coordinators, which contradicts the Assumption 1. So every coordinator plays A at equilibrium and must be adjacent with another A-player to remain satisfied with her choice. Part 2) The case $x_i = B$ results in $\boldsymbol{x}_i = (A, B, A)$ in view of Table I. If instead, $x_i = A$, then at least one of her neighbors, say $i-1$ must play B. This agent may not be a coordinator in view of Part 1). Hence, both of the neighbors of agent $i-1$ must be A-players, resulting in $\boldsymbol{x}_{i-1} = (A, B, A)$.

Sufficiency. Every coordinator at $\boldsymbol{x}$ is adjacent with another A-player, and hence, does not tend to switch strategies. On the other hand, every anticoordinator is either playing B and adjacent with two A-players or playing A and adjacent with at least one B-player. In either case, the anticoordinator does not tend to switch strategies. So the dynamics remain at $\boldsymbol{x}$ at the next time step, regardless of the activation sequence. ■

Does a ring always admit an equilibrium? The answer is negative if it contains a consecutive sequence of agents that no strategy assignment to the ring makes them all satisfied. In what follows, we formalize this idea and show that the reverse also holds.

Define a *component* as a sequence of consecutive agents in the ring. If the beginning and end of the sequence match, the component represents the whole ring. We often denote a component by a sequence of '$+$' and '$-$'s, representing coordinators and anticoordinators, respectively. For example, the component $+-+$ represents a coordinating agent i that is adjacent with an anticoordinating agent $i+1$ that is adjacent with a coordinating agent $i+2$, for some $i \in \{1, \dots, n-2\}$. If in addition to the coordinating and anticoordinating types, the strategy of some of the agents are represented, we refer to the sequence as a *(strategy) pattern*. For example, the pattern $\underset{+-}{AB}$ represents an A-playing coordinator adjacent to a B-playing anticoordinator, and the pattern $\underset{+-}{A*}$ represents an A-playing coordinator adjacent to an anticoordinator whose strategy is not specified. Given a state $\boldsymbol{x} \in \mathcal{X}$, we say a component is *at equilibrium* if every agent in the component is satisfied with her strategy in $\boldsymbol{x}$. We say a component is *fluctuating* if it is never at equilibrium under any strategy assignment to the ring. A fluctuating component is *minimal* if it does not contain a smaller fluctuating component.

Lemma 1: The ring dynamics admit an equilibrium if and only if the ring does not include a minimal fluctuating component.

Proof: The necessity is trivial. For the sufficiency, note that if no strategy assignment brings the ring to equilibrium, then the whole ring is a fluctuating component, which contains a minimal fluctuating component, that can be itself. ■

What is the form of a minimal fluctuating component? Define

$$\mathcal{Y}_\alpha = +\,(-)_\alpha +, \quad \alpha \in \mathbb{Z}_{\geq 0},$$

where $(-)_\alpha$ compactly denotes α consecutive repetitions of '$-$', e.g., $(-)_2 = --$. Any component with more than one coordinator includes $\mathcal{Y}_\alpha$ for some $\alpha \geq 0$. What values of α are "eligible" for $\mathcal{Y}_\alpha$ to be part of a minimal fluctuating component? We say a pattern is *fixed* if every agent in the component is satisfied with her strategy regardless of the strategies of the agents outside the component; namely, it is at equilibrium even if some of the agents outside the component change their strategies. A component is *fixable* if there is a strategy assignment to the agents in the component that makes the resulting pattern fixed. We will show that a minimal fluctuating component does not contain any fixable component $\mathcal{Y}_\alpha$, determined in the following lemma.

Lemma 2: $\mathcal{Y}_\alpha$ is fixable if and only if $\alpha \in \{0, 3\} \cup \mathbb{Z}_{\geq 5}$.

Proof: *Necessity.* We prove by contradiction that $\mathcal{Y}_\alpha$ is not fixable for $\alpha = 1, 2, 4$. Assume on the contrary, $\mathcal{Y}_\alpha$ is fixable for one of these values of α, implying the existence of a strategy assignment that makes all agents in $\mathcal{Y}_\alpha$ satisfied, regardless of the strategy of the other agents. Therefore, each of the ending coordinators and their adjacent anticoordinators in $\mathcal{Y}_\alpha$ must play A; otherwise, the coordinators tends to switch to B if their adjacent neighbors that are out of $\mathcal{Y}_\alpha$ play B. So the only possible patterns are

$$\underset{+-+}{AAA}, \quad \underset{+--+}{AAAA}, \quad \underset{+----+}{AA**AA}.$$

The first two are clearly not fixed. As with the third, if any of the $*$'s is A, it results in three consecutive anticoordinators playing A, and the middle anticoordinator is unsatisfied. So both $*$'s are B. However, then both of the anticoordinators playing B are unsatisfied according to Table I.

Sufficiency. Two adjacent A-playing coordinators are clearly fixed, solving the problem for $\alpha = 0$. For $\alpha = 3$ and $\alpha \geq 5$, it holds that $\alpha = 3\gamma$, $3\gamma + 2$, or $3\gamma + 4$ for some $\gamma \geq 1$, each of which results in a fixable $\mathcal{Y}_\alpha$ according to the following left, middle, and right patterns, respectively:

$$\underset{+}{A}\underset{---}{(ABA)}_\gamma \underset{+}{A}, \quad \underset{+}{A}\underset{---}{(ABA)}_\gamma \underset{--+}{BAA}, \quad \underset{+--}{AAB}\underset{---}{(ABA)}_\gamma \underset{--+}{BAA},$$

which are all fixed according to Table I. ■

For the case of $\alpha \in \{2, 4\}$ the component $\mathcal{Y}_\alpha$ can still be at equilibrium.

Lemma 3: Assume that every coordinator plays A. Then if a component $\mathcal{Y}_2$ is at equilibrium, it takes either of the following patterns:

$$\underset{+--+}{ABAA}, \quad \underset{+--+}{AABA},$$

and if a component $\mathcal{Y}_4$ is at equilibrium, it takes one of the following patterns:

$$\underset{+----+}{AABABA}, \quad \underset{+----+}{ABABAA}, \quad \underset{+----+}{ABAABA}.$$

Proof: the proof is trivial in view of Table I. ■

Lemma 4: All minimal fluctuating components are in the following form:

$$+-+(-)_{\beta_1} + (-)_{\beta_2} + \dots (-)_{\beta_r} + - + \tag{3}$$

where $r \geq 1$ and

$$\beta_i \in \{2, 4\} \; \forall i \in \{1, \dots, r\},$$

and the endings $+$ or $+-+$ may coincide, resulting in the whole ring.

Proof: First, we prove by contradiction that (3) is fluctuating. Assume on the contrary that there exists some strategy assignment $\boldsymbol{x} \in \mathcal{X}$ that equilibrates the component. Similar to Proposition 2, it can be shown that then the coordinators must play A and be adjacent to another A-player. Hence, all coordinators in (3) must play A, resulting in the most left and right anticoordinators playing B:

$$\underset{+-+}{ABA}\underset{-}{(*)}_{\beta_1}\underset{+}{A}\underset{-}{(*)}_{\beta_2}\underset{+}{A}\dots\underset{-}{(*)}_{\beta_r}\underset{+-+}{ABA}.$$

Excluding the ending coordinators, there are $r+1$ coordinators in the component that each need to be adjacent with an A-player. On the other hand, for the anticoordinator components $(-)_{\beta_i}$ to be at equilibrium, at most one of their ending anticoordinators can play A according to Lemma 3. Hence, they can provide at most r A-playing neighbors for the $r+1$ coordinators. So one coordinator will remain unsatisfied, a contradiction.

Second, we prove that if a fluctuating component $\mathcal{C} = (i_1, i_2, \ldots, i_l)$ is minimal, it equals (3). Component $\mathcal{C}$ satisfies one of the following cases:

Case 1. Component $\mathcal{C}$ does not include component $\mathcal{Y}_1$. Then $\mathcal{C}$ is in the following form:

$$(-)_{\beta_0} + (-)_{\beta_1} + \ldots (-)_{\beta_r} + (-)_{\beta_{r+1}},$$

where $\beta_0, \beta_{r+1} \in \mathbb{Z}_{\geq 0}$ and $\beta_i \in \mathbb{Z}_{\geq 0} \setminus \{1\}$, for all $i \in \{1, \ldots, r\}$. Assign the coordinators strategy A. Assign each of the anticoordinating components $(-)_{\beta_i}$, $i \in \{0, \ldots, r+1\}, \beta_i > 0$, the strategies A and B from left to right and starting with A, consecutively, resulting in

$$(\underset{-}{A}\underset{-}{B}\ldots)\underset{+}{A}(\underset{-}{A}\underset{-}{B}\ldots)\underset{+}{A}\ldots(\underset{-}{A}\underset{-}{B}\ldots)\underset{+}{A}(\underset{-}{A}\underset{-}{B}\ldots\underset{-}{A}\underset{-}{B}).$$

Finally, assign strategy A to the left neighbor of i_1 and right neighbor of i_l, which may be the same agent in the ring. This makes all agents in $\mathcal{C}$ satisfied with their strategies. Note that this assignment works for the special cases $\beta_0 = 0$, $\beta_{r+1} = 0$, $r = -1$, $r = 0$, and where $(-)_{\beta_0}$ and $(-)_{\beta_{r+1}}$ coincide, resulting in the ring. So this case does not result in a fluctuating component.

Case 2. Component $\mathcal{C}$ includes exactly one component $\mathcal{Y}_1$. Then $\mathcal{C}$ is in the following form:

$$(-)_{\alpha_0} + \ldots (-)_{\alpha_s} + - + (-)_{\beta_1} \ldots + (-)_{\beta_{r+1}}, \tag{4}$$

where $\alpha_0, \beta_{r+1} \in \mathbb{Z}_{\geq 0}$ and $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0} \setminus \{1\}$, for all $i \in \{1, \ldots, r\}$. Assign the coordinators strategy A, and the anticoordinators in the $(-)_{\beta_i}$ components the same strategies as in the Case 1, and do the opposite for the anticoordinators in the $(-)_{\alpha_i}$ components; that is, assign strategies A and B from right to left and starting from A, consecutively, resulting in

$$(\ldots\underset{-}{B}\underset{-}{A})\underset{+}{A}\ldots(\ldots\underset{-}{B}\underset{-}{A})\underset{+}{A}\underset{-}{B}\underset{+}{A}(\underset{-}{A}\underset{-}{B}\ldots)\ldots\underset{+}{A}(\underset{-}{A}\underset{-}{B}\ldots) \tag{5}$$

Moreover, assign strategy A to the left neighbor of i_1 and right neighbor of i_l. This makes $\mathcal{C}$ at equilibrium; unless, $(-)_{\alpha_0}$ and $(-)_{\beta_{r+1}}$ coincide. Then if all α_i and β_i belong to $\{2, 4\}$, then $\mathcal{C}$ equals to (3). Otherwise, there exists a $\mathcal{Y}_\gamma$, $\gamma \in \{0, 3\} \cup \mathbb{Z}_{\geq 5}$, which we fix according to Lemma 3, with its ending coordinators playing A. Then if we present the ring in the form of (4) such that $(-)_{\alpha_0}$ and $(-)_{\beta_{r+1}}$ coincide and together with their right and left coordinators form $\mathcal{Y}_\gamma$, the same strategy assignment as in (5) brings $\mathcal{C}$ to equilibrium.

Case 3. Component $\mathcal{C}$ includes at least two $\mathcal{Y}_1$ components. Then the components starting from and ending with two consecutive $\mathcal{Y}_1$'s will be in the form of (3). Denote these components by $\mathcal{Z}_1, \ldots, \mathcal{Z}_s$. If the β_i's in any of these components belong to $\{2, 4\}$, then the component is the same as in (3) and will be fluctuating according to the first part of this proof. Moreover, no smaller component of it will be fluctuating according to the results of Case 1 and 2. So it will be minimally fluctuating, and hence, $\mathcal{C}$ must be equal to this component. So assume that in every component $\mathcal{Z}_i$, there exists some β_j that does not equal 2 or 4. It does not equal 1 either. Hence, it is either 0, 3, 5, or more. Then according to Lemma 2, there exists a component $\mathcal{Y}_j$ in $\mathcal{Z}_i$ that we can fix while forcing its ending coordinators playing A. What remains from $\mathcal{C}$ is a number of components in the form of (4) in Case 2, which was shown to be at equilibrium for a particular strategy assignment, given that the neighbors of the ending agents play A. These neighbors will now be either the ending agents in $\mathcal{Y}_j$'s which are coordinators playing A, or the two not necessarily different agents out of $\mathcal{C}$ which can be assigned A. Hence, $\mathcal{C}$ will be at equilibrium under this assignment, which is impossible.

Finally, the results of Case 1 and 2 imply the minimality of the component in (3), completing the proof. ■

So a minimal fluctuating component includes $\mathcal{Y}_1$ at its ends and $\mathcal{Y}_2$ and $\mathcal{Y}_4$ in its interior. If such a sequence does not take place in a ring, it is "equilibrable," meaning that the ring dynamics admit an equilibrium, as stated in the main result of this section:

Theorem 1 (equilibrium existence): The ring dynamics admit an equilibrium if and only if the ring does not contain the component

$$+ - + (-)_{\beta_1} + (-)_{\beta_2} + \ldots (-)_{\beta_r} + - +$$

where $r \geq 1$, $\beta_i \in \{2, 4\}$ for all $i \in \{1, \ldots, r\}$, and the endings $+$ or $+ - +$ may coincide, resulting in the whole ring.

Proof: The proof follows Lemmas 1 and 4. ■

IV. Stability

Due to the discrete nature of the dynamics, the well-known notion of stability can be shown to be reduced to the following as discussed in [2]. The equilibrium state $\boldsymbol{x}^*$ is *stable* if starting from any initial condition $\boldsymbol{x}(0)$ whose first-norm distance to $\boldsymbol{x}^*$ is one, i.e., $\|\boldsymbol{x}(0) - \boldsymbol{x}^*\| = 1$, the solution trajectory will always remain within a distance of one to $\boldsymbol{x}^*$, $\|\boldsymbol{x}(t) - \boldsymbol{x}^*\| \leq 1$, under all activation sequences. Given agent i with strategy x_i, denote her complement strategy by $\bar{x}_i$, i.e., $\{x_i, \bar{x}_i\} = \{A, B\}$. Correspondingly, define the state $\boldsymbol{y} = \bar{\boldsymbol{x}}^i$ to be the same as $\boldsymbol{x}$ but where the strategy of agent i is flipped, i.e., $y_i = \bar{x}_i$ and $y_j = x_j$ for all $j \neq i$.

Theorem 2: All of the equilibrium states of the ring dynamics are unstable.

Proof: The proof is based on Table I, and the fact that all coordinators play A at equilibrium according to Proposition 2. Let $\boldsymbol{y}$ be an equilibrium state. Either of the following cases may hold for an agent i at $\boldsymbol{y}$:

Case 0: Agent i is playing B and has at least one B-playing neighbor. This case is impossible as then agent i is not satisfied with her strategy according to Table I.

Case 1: Agent i is playing A and has at least one B-playing neighbor. At the state $\boldsymbol{x}(0) = \bar{\boldsymbol{y}}^i$, where agent i plays B, her B-playing neighbor tends to switch strategies. Hence, under the activation sequence, where the neighbor becomes active at time 0, we obtain the state $\boldsymbol{x}(1)$ where two agents are playing the opposite strategies compared to $\boldsymbol{y}$, yielding $\|\boldsymbol{x}(1) - \boldsymbol{y}\| = 2$, making $\boldsymbol{y}$ unstable.

Case 2: Agent i is playing A and both of her neighbors are playing A. According to Proposition 2, agent i is satisfied with her strategy only if she is a coordinator. If at least one of agent i's A-playing neighbors, say agent $i+1$, is an anticoordinator, then agent $i+1$ must have at least one B-playing neighbor, that is agent $i+2$, in order to be satisfied with her strategy. However, then agents $i+1$ and $i+2$ fall into the category of Case 1. So both agent i's neighbors are coordinators. Yet there is at least one anticoordinator in the ring, who is satisfied with her strategy only if she has at least one B-playing neighbor according to Table I, which again leads to Case 1, completing the proof. ■

V. Fluctuation sets

The inclusion of a fluctuating component, characteristics of which are stated in Lemma 4, does not determine if a ring enters a fluctuation set or not. The absence of a component in the form of (3) does not imply the absence of a fluctuating set. There are some non-fluctuating components that do not equilibrate if their marginal neighbors are fixed. Namely, the component equilibrates at specific states, but those states may be avoided if the ring starts from a particular initial state. Define a *fluctuating pattern* as a pattern starting from which, the corresponding component never equilibrates under any activation sequence, independent on the strategies of the exterior agents. More specifically, pattern $\boldsymbol{y}_{\mathcal{C}}$ for component $\mathcal{C}$ is fluctuating if under every activation sequence,

$$\boldsymbol{x}_{\mathcal{C}}(0) = \boldsymbol{y}_{\mathcal{C}} \Rightarrow \mathcal{U}(t) \cap \mathcal{C} \neq \emptyset \forall t \geq 0,$$

where $\mathcal{U}(t)$ is the set of agents who are unsatisfied with their strategies at time t. Namely, if the ring dynamics start from an initial condition that assigns to the fluctuating component the same strategies as in the fluctuating pattern, then the component never equilibrates. A fluctuating pattern is *minimal* if it does not contain a smaller pattern that is fluctuating. The component corresponding to a (minimal) fluctuating pattern is called a *(minimal) fluctuatable component* as it may become fluctuating only under a specific strategy assignment.

Lemma 5: The ring dynamics admit a fluctuation set if and only if the ring includes a fluctuatable component.

Proof: The sufficiency is trivial. For the necessity, note that if the ring admits a fluctuation set, then any state in that set is a fluctuating pattern, making the whole ring a fluctuatable component. The fact that every fluctuatable component admits a minimal fluctuatable component completes the proof. ■

The following proposition is the main result of this section.

Proposition 3 (fluctuation existence): The ring dynamics admit a fluctuation set if and only if the ring contains the component

$$\mathcal{Z}_0 - +(-)_{\beta_1} + (-)_{\beta_2} + \dots (-)_{\beta_r} + - \mathcal{Z}_{r+1}$$

where

$$\mathcal{Z}_0 \in \{+(+--)_{\alpha_1}, ---(+--)_{\alpha_1}\},$$

$$\mathcal{Z}_{r+1} \in \{(--+)_{\alpha_2}+, (--+)_{\theta_2}---\},$$

and where $r \geq 1$, and

$$\beta_i \in \{2,4\} \forall i \in \{1,\dots,r\}, \alpha_i \in \mathcal{Z}_{\geq 0},$$

and the two ends $\mathcal{Z}_0$ and $\mathcal{Z}_{r+1}$ or part of them may coincide, resulting in the whole ring.

We skip the proof due to the space limit.

VI. Example

We present a mixed ring with an initial state that never enters any of the equilibrium states under a certain activation sequence, but reaches an equilibrium under many other sequences. Consider a ring with four agents, with the structure $(+--+)$; that is, two neighboring coordinators that neighbor two neighboring anticoordinators. Consider the initial condition, where the coordinators play B and the anticoordinators play A, i.e., $\boldsymbol{x}_L(0) = (B,A,A,B)$. Consider the two activation sequences $(y_k)_{k=0}^{\infty}$ and $(z_k)_{k=0}^{\infty}$, defined as

$$y_k = \begin{cases} 1, & k \equiv 0 \text{ or } 2 \mod 8 \\ 2, & y_{k-1} = 1 \\ 3, & y_{k-1} = 4 \\ 4, & k \equiv 4 \text{ or } 6 \mod 8 \end{cases}$$

$$z_k = (y_k + 2 \mod 4) + 1.$$

The ring threshold dynamics under both sequences are indicated in Table II for $k = 0,1,\dots,8$. Under $(y_k)_{k=0}^{\infty}$, the ring does not equilibrate since $\boldsymbol{x}_L(0) = \boldsymbol{x}_L(8) \neq \boldsymbol{x}_L(1)$ and y_k repeats itself with a period of eight. However, the ring equilibrates under $(z_k)_{k=0}^{\infty}$: by $k=2$, both coordinators play A, so they will not change to B; activating the anticoordinators leaves $\boldsymbol{x}_L$ fixed for all $k \geq 5$ as well.

k	$\boldsymbol{x}_L(k)$ with y_k	y_k	$\boldsymbol{x}_L(k)$ with z_k	z_k
0	(B,A,A,B)	1	(B,A,A,B)	4
1	(A,A,A,B)	2	(B,A,A,A)	1
2	(A,B,A,B)	1	(A,A,A,A)	4
3	(B,B,A,B)	2	(A,A,A,A)	1
4	(B,A,A,B)	4	(A,A,A,A)	3
5	(B,A,A,A)	3	(A,A,B,A)	2
6	(B,A,B,A)	4	(A,A,B,A)	3
7	(B,A,B,B)	3	(A,A,B,A)	2
8	(B,A,A,B)	1	(A,A,B,A)	4

TABLE II

Two activation sequences for an equilibrable mixed ring.

Therefore, existence of an equilibrium and non-existence of fluctuation sets does not imply equilibration for mixed rings. However, in this case, the periodicity of the first activation sequence in the example is necessary to prevent the ring from equilibration. Should the sequence be generated by a random process, e.g., a random agent becomes active at each time, any ring that admits equilibrium and possesses no fluctuation sets, will equilibrate in the long run with probability one.

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