Visual Computing I:

Interactive Computer Graphics and Vision

2D Transforms

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Quiz

Detect the red balloons in this photo.

Which color model is best to work in?

Why?

Original Image

Red Channel

Greyscale

Hue from HSV

2D Geometry

Points, Lines, etc.

- A point is a 2D location $(u,\ v)$
- Two points define a line

$$\left(u_0,\ v_0\right),\ (u_1, v_1)$$

- A polyline with k segments is a sequence of $k + 1$ points

$$\left(u_0, v_0\right),\ \left(u_1, v_1\right),\ \ldots,\ \left(u_k,\ v_k\right)$$

- A polygon is a polyline where

$$\left(u_0, v_0\right) = \left(u_k,\ v_k\right)$$

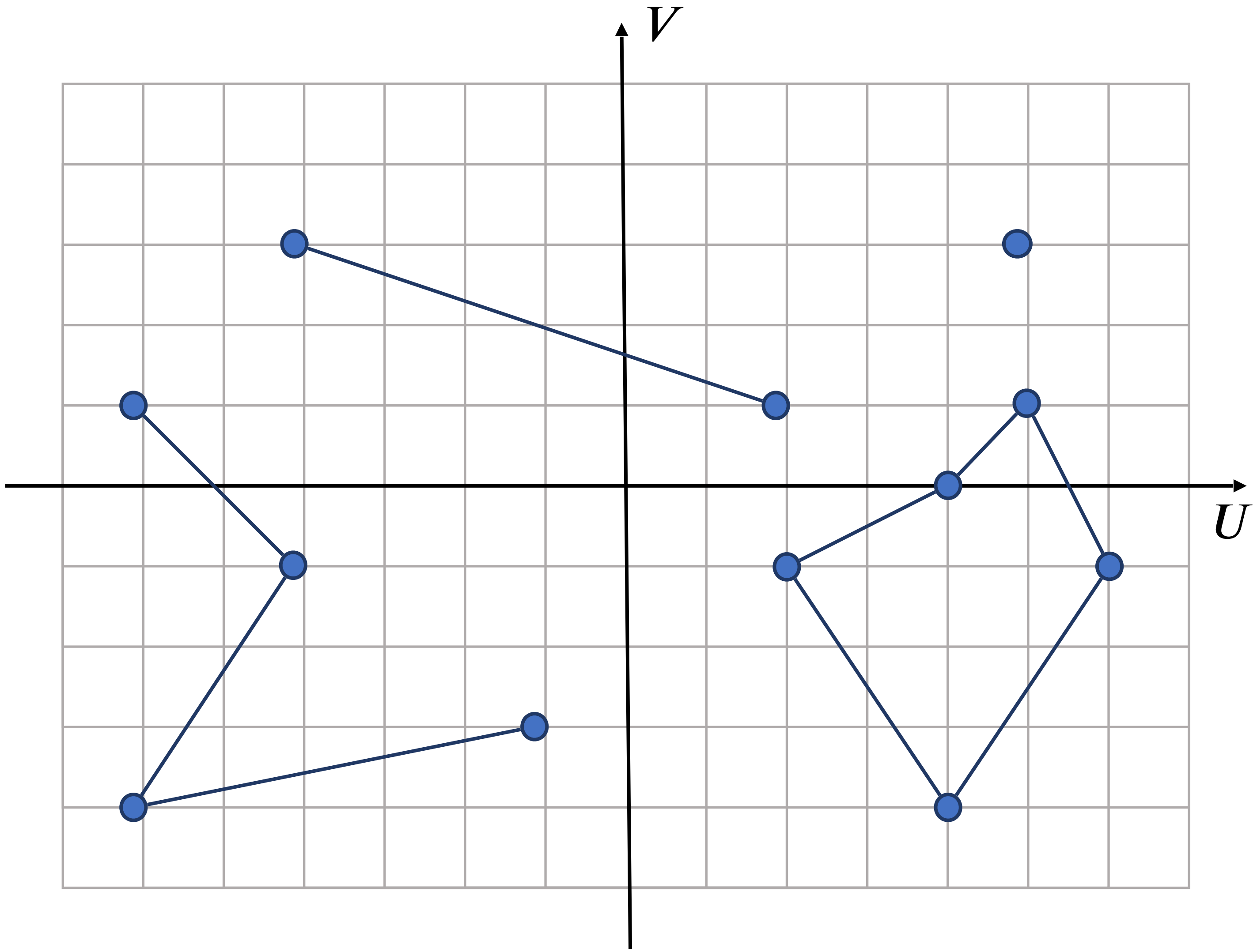
- We often omit the duplicate point

Points, Lines, etc.

- A point, $\left(5, 3\right)$
- A line, $(-4, 3),\ (2, 1)$
- A polyline,

  $\left(-6, 1\right),\ (-4,\ -1),$
  $\left(-6,\ -4\right),\ (-1,\ -3)$
- A polygon,

  $(4, 0),\ \left(5, 1\right),\ \left(6,\ -1\right),$
  $(4,\ -4),\ (2,\ -1)$

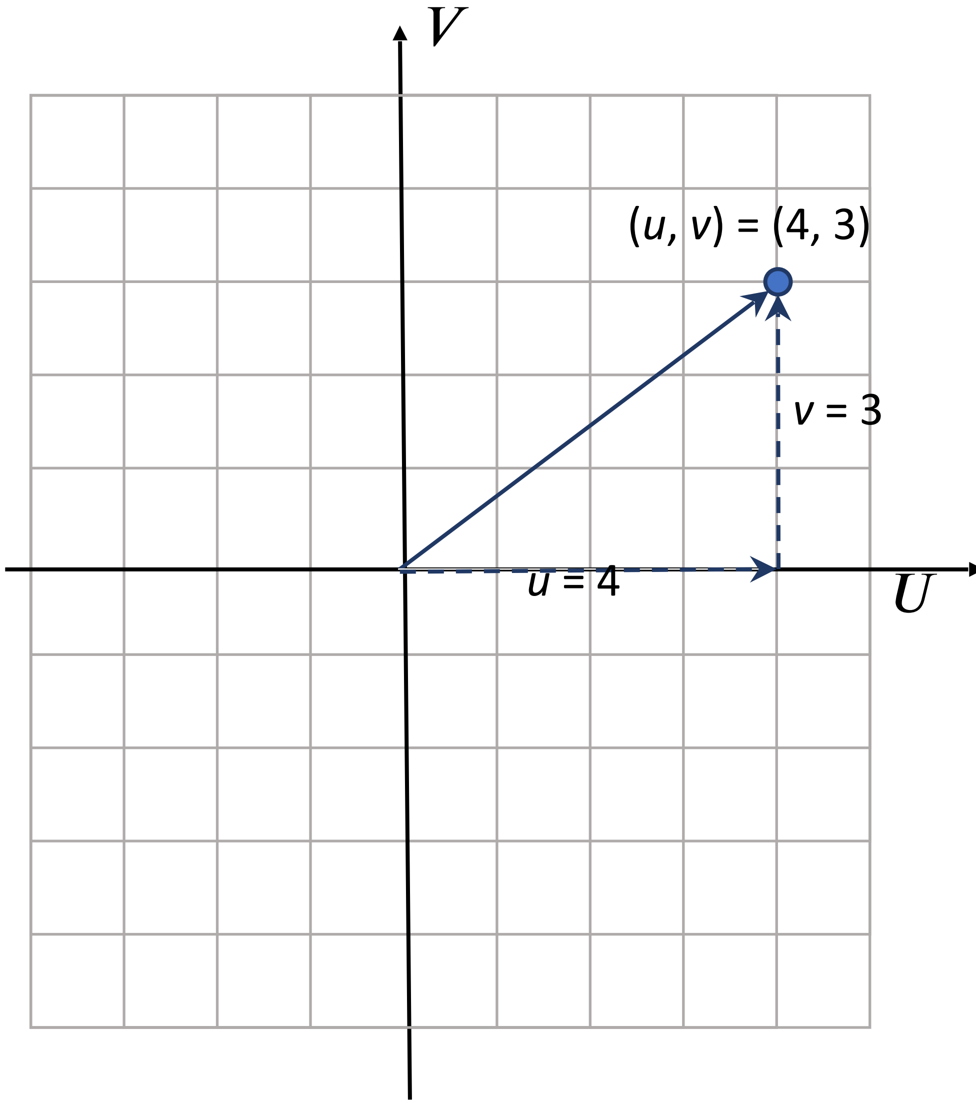


Points as Vectors

- We represent points as vectors

$$(u,\ v) \rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = [u \quad v]^T$$

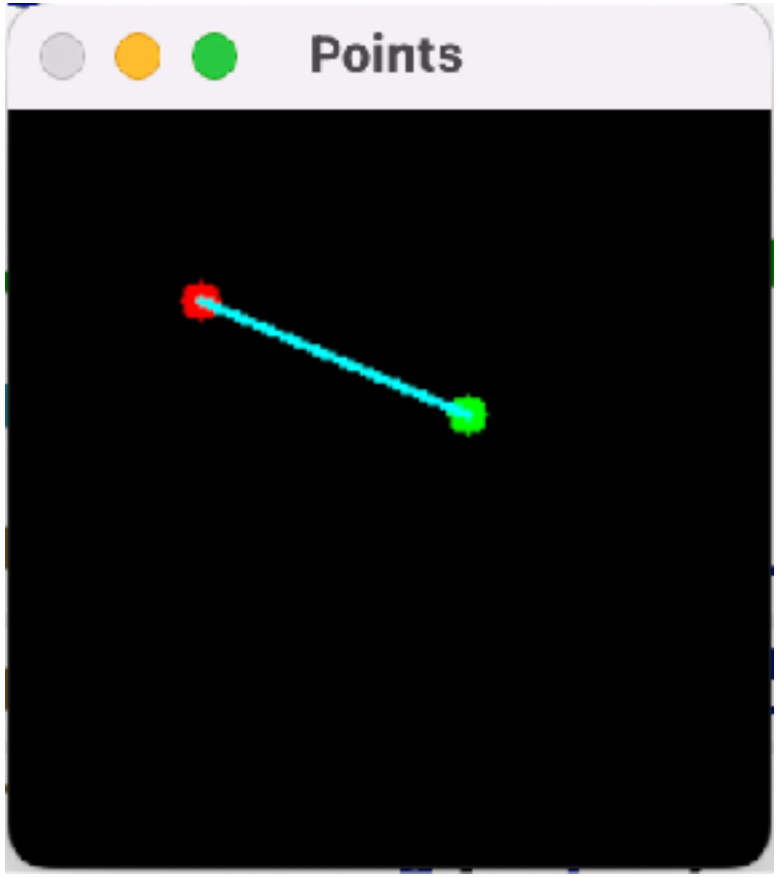
- We can think of this vector as:
    - The point $(u,\ v)$
    - A direction, or step of u units along one axis and v units along the other
    - The point is where you end up if you step along the vector from the origin



Example: 2D Points in OpenCV

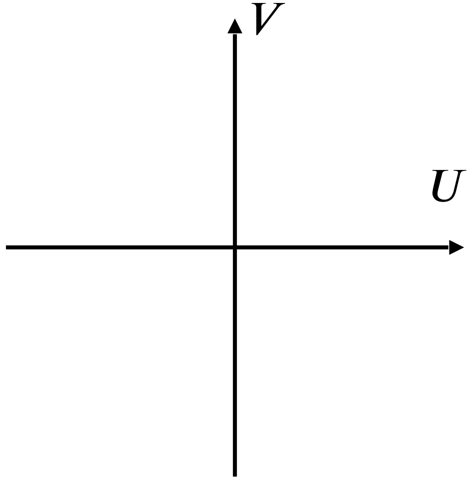
- In OpenCV, cv::Point (or cv::Point2f for floats) represents a 2D point

```
#include <opencv2/opencv.hpp>
#include <iostream>
using namespace std;

int main() {
    // Create some points
    cv::Point p1(50, 50);           // integer coordinates
    cv::Point2f p2(120.5f, 80.3f); // float coordinates

    // Print points
    cout << "p1 = " << p1 << endl;
    cout << "p2 = " << p2 << endl;

    // Visualize on a blank image
    cv::Mat img = cv::Mat::zeros(200, 200, CV_8UC3);
    circle(img, p1, 5, cv::Scalar(0, 0, 255), cv::FILLED);  // red dot
    circle(img, p2, 5, cv::Scalar(0, 255, 0), cv::FILLED);  // green dot
    line(img, p1, p2, cv::Scalar(255, 255, 0), 2);      // yellow line

    cv::imshow("Points", img);
    cv::waitKey(0);
    return 0;
}
```



Choice of Coordinate Systems

Mathematical



- U: left -> right
- V: bottom -> top

Image-based

- U: left -> right
- V: top -> bottom

Matrix-based

C

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

R

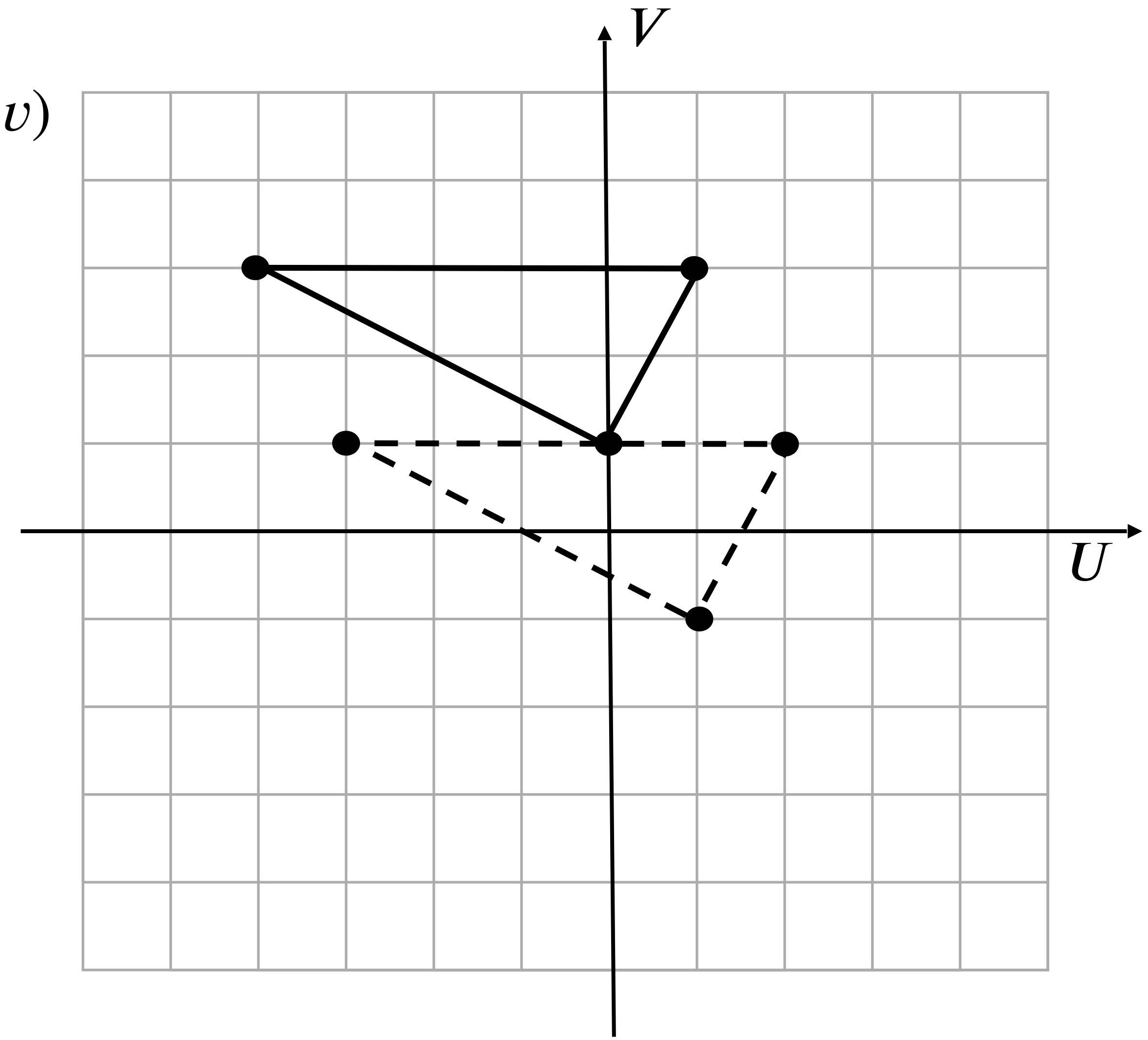
- R: top -> bottom
- C: left -> right

Transformations - Translation

- Shifting a point, $(u,\ v)$, by some offset, $(\Delta u,\ \Delta v)$
  $(u,\ v) \rightarrow (u + \Delta u,\ v + \Delta v)$

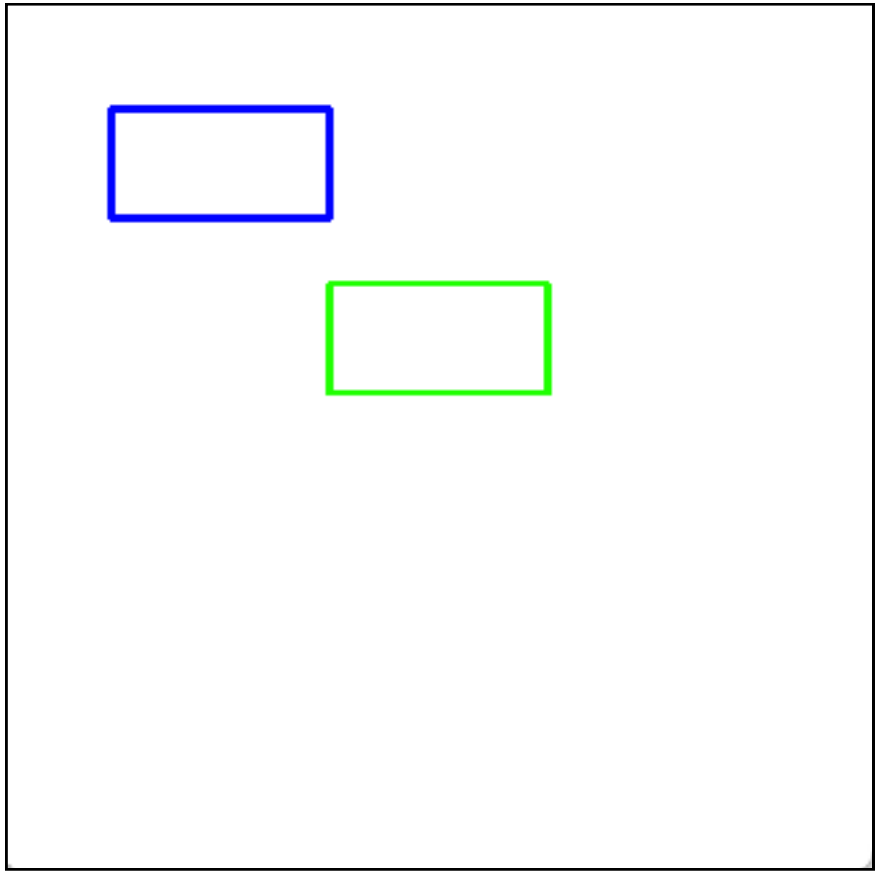
- In vector form:
  $$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$

- Example:
  $$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



Transformations - Translation

```
cv::Mat img2(400, 400, CV_8UC3, cv::Scalar(255, 255, 255)); // white BGR

// Original rectangle points
std::vector<cv::Point> pts = { {50,50}, {150,50}, {150,100}, {50,100} };

// Draw original rectangle (blue)
 cv::polylines(img2, std::vector<std::vector<cv::Point>>{pts}, true,
 cv::Scalar(255,0,0), 2);
cv::imshow("2DTransform", img2);
cv::waitKey(0);

// Translation vector
int tx = 100, ty = 80;

// Apply translation
std::vector<cv::Point> translated_pts;
for (auto &p : pts) {
    translated_pts.push_back(cv::Point(p.x + tx, p.y + ty));
}

// Draw translated rectangle (green)
 cv::polylines(img2, std::vector<std::vector<cv::Point>>{translated_pts},
 true, cv::Scalar(0,255,0), 2);

cv::imshow("2DTransform", img2);
cv::waitKey(0);
```

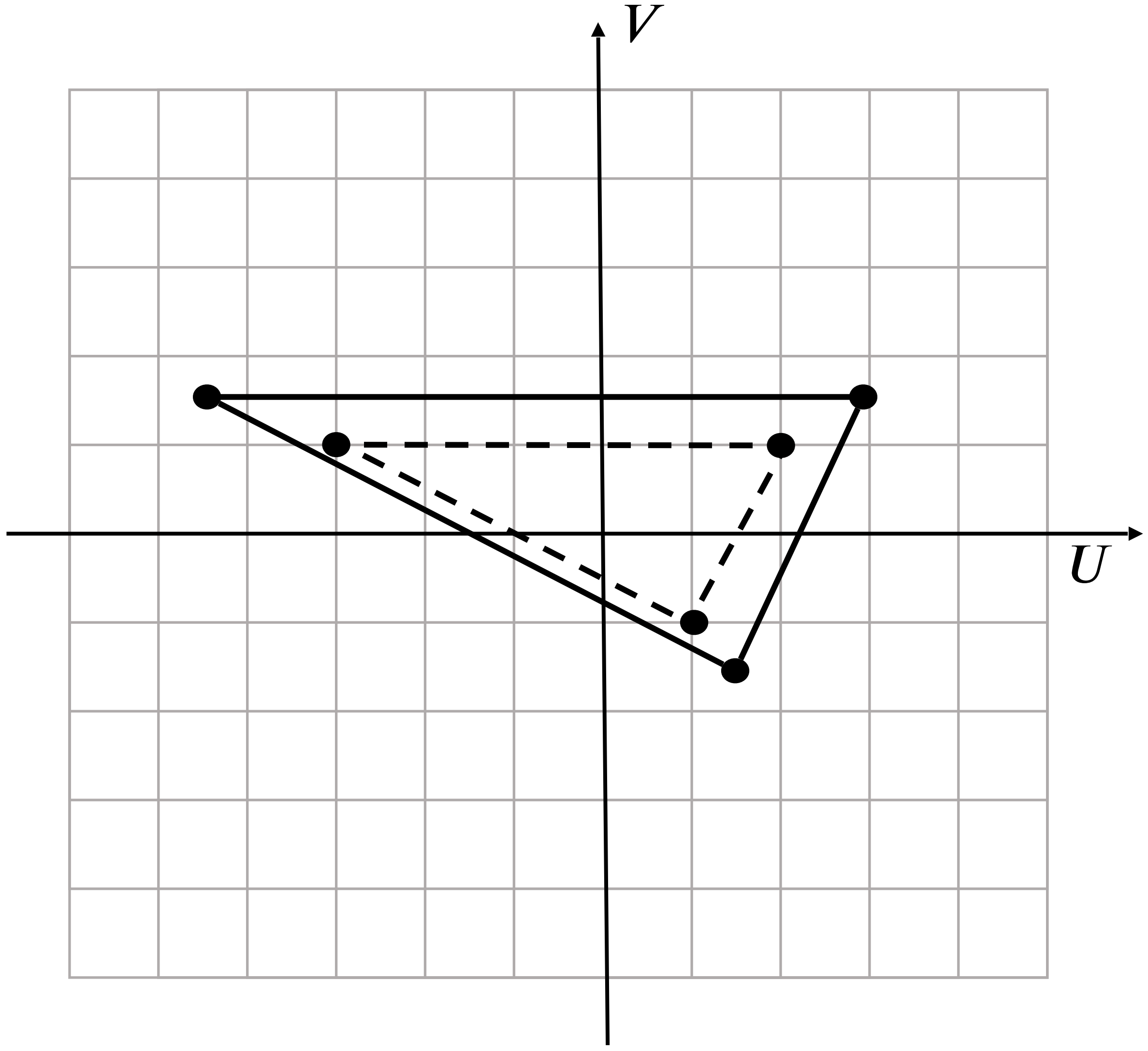


Transformations - Scaling

- Scaling points by factor, s
  $(u, v) \rightarrow (su,\ sv)$

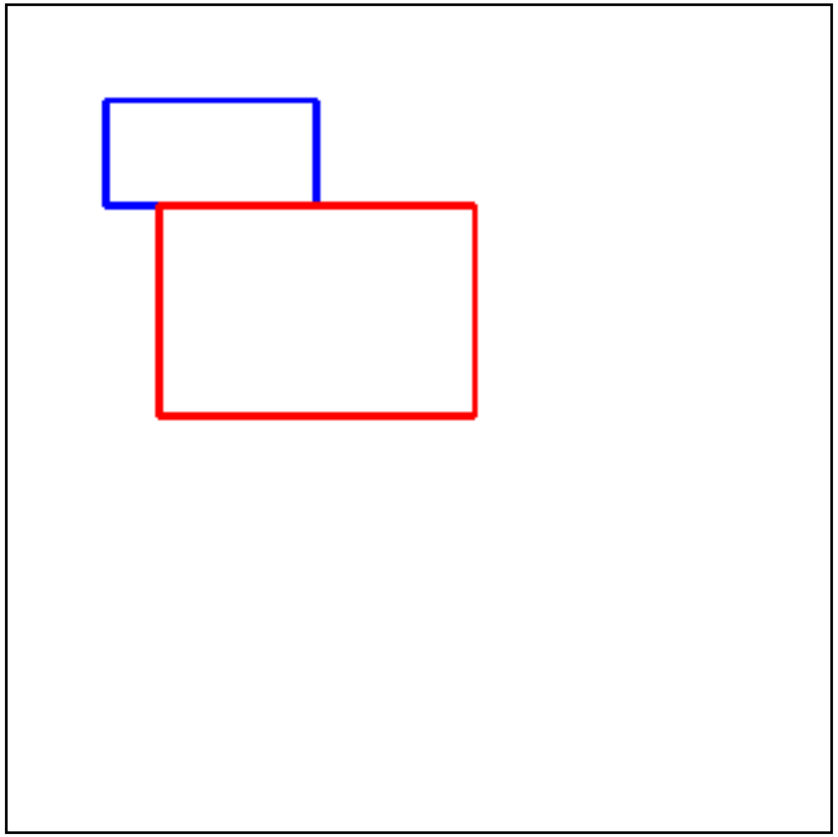
- In vector terms
  $$\begin{bmatrix} u' \\ v' \end{bmatrix} = s \begin{bmatrix} u \\ v \end{bmatrix}$$

- Example:
  $$\begin{bmatrix} u' \\ v' \end{bmatrix} = 1.5 \begin{bmatrix} u \\ v \end{bmatrix}$$



Transformations - Scaling

```
cv::Mat canvas(400, 400, CV_8UC3, cv::Scalar(255, 255, 255)); // white BGR

std::vector<cv::Point2f> pts = { {50,50}, {150,50}, {150,100}, {50,100} };

// Draw original rectangle (blue)
std::vector<cv::Point> pts_int;
 for(auto &p: pts) pts_int.push_back(cv::Point(cvRound(p.x),
 cvRound(p.y)));
cv::polylines(canvas, pts_int, true, cv::Scalar(255,0,0), 2);
cv::imshow("Scale", canvas);
cv::waitKey(0);
float sx = 1.5f, sy = 2.0f;

std::vector<cv::Point> scaled_pts;
for(auto &p: pts) {
    scaled_pts.push_back(cv::Point(cvRound(p.x * sx), cvRound(p.y * sy)));
}

cv::polylines(canvas, scaled_pts, true, cv::Scalar(0,0,255), 2);
cv::imshow("Scale", canvas);
cv::waitKey(0);
```



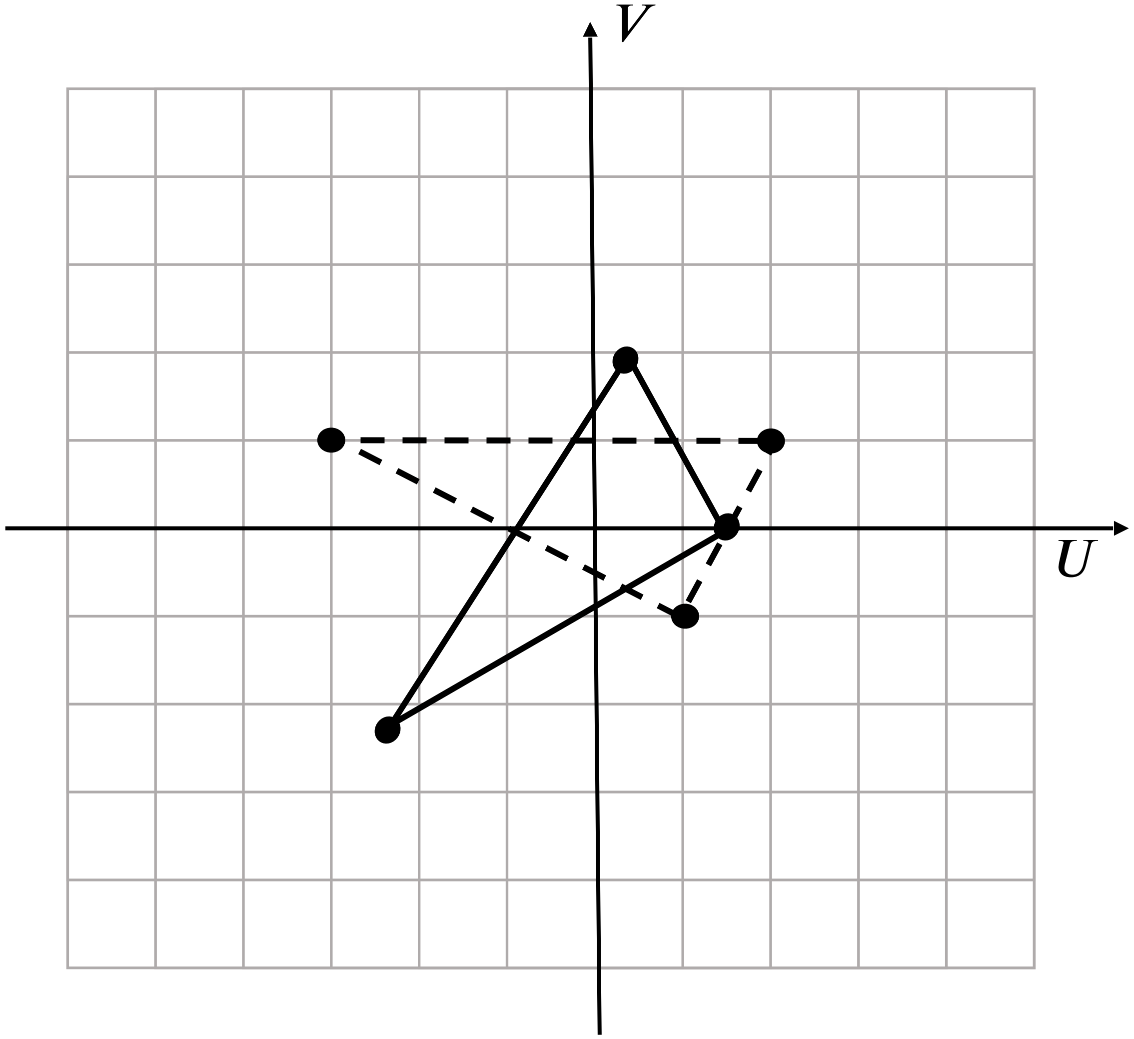
Transformations - Rotation

- Rotate by some angle, θ, about the origin

- Rotation from the U axis towards the V axis
    - NOT clockwise/anticlockwise

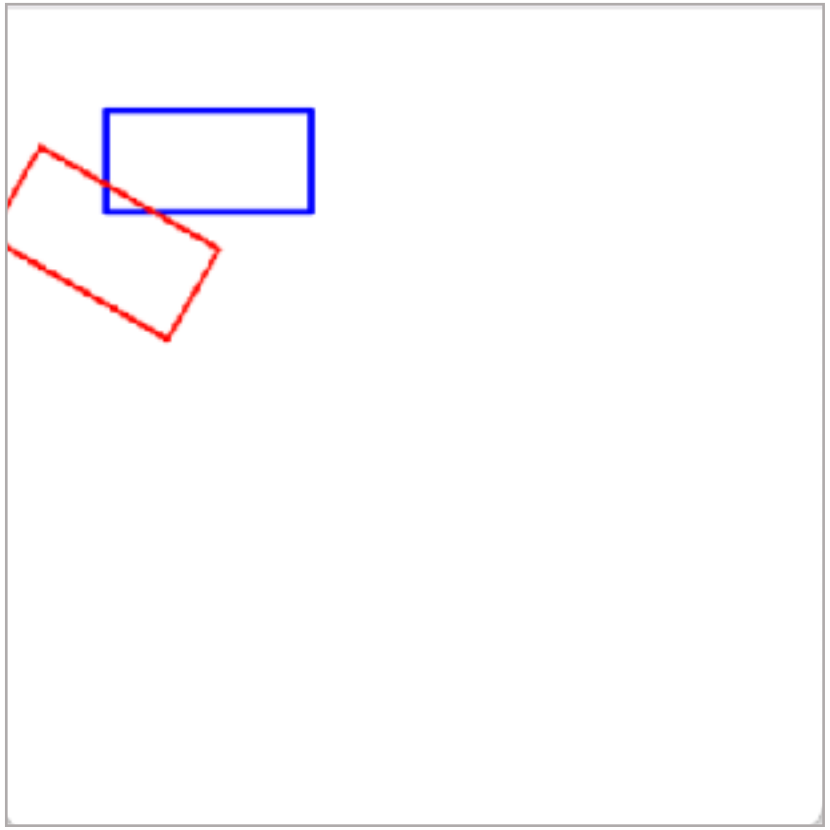
- In vector terms

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$



Transformations - Rotation

```
std::vector<cv::Point2f> pts = { {50,50}, {150,50}, {150,100},
{50,100} };

// Draw original rectangle (blue)
std::vector<cv::Point> pts_int;
for(auto &p: pts) pts_int.push_back(cv::Point(cvRound(p.x),
cvRound(p.y)));
cv::polylines(canvas, pts_int, true, cv::Scalar(255,0,0), 2);

// Rotation angle
double theta = CV_PI / 6; // 30 degrees

std::vector<cv::Point> rotated_pts;
for(auto &p: pts) {
    float x_new = p.x * cos(theta) - p.y * sin(theta);
    float y_new = p.x * sin(theta) + p.y * cos(theta);
    rotated_pts.push_back(cv::Point(cvRound(x_new),
    cvRound(y_new)));
}

cv::polylines(canvas, rotated_pts, true, cv::Scalar(0,0,255), 2);
```



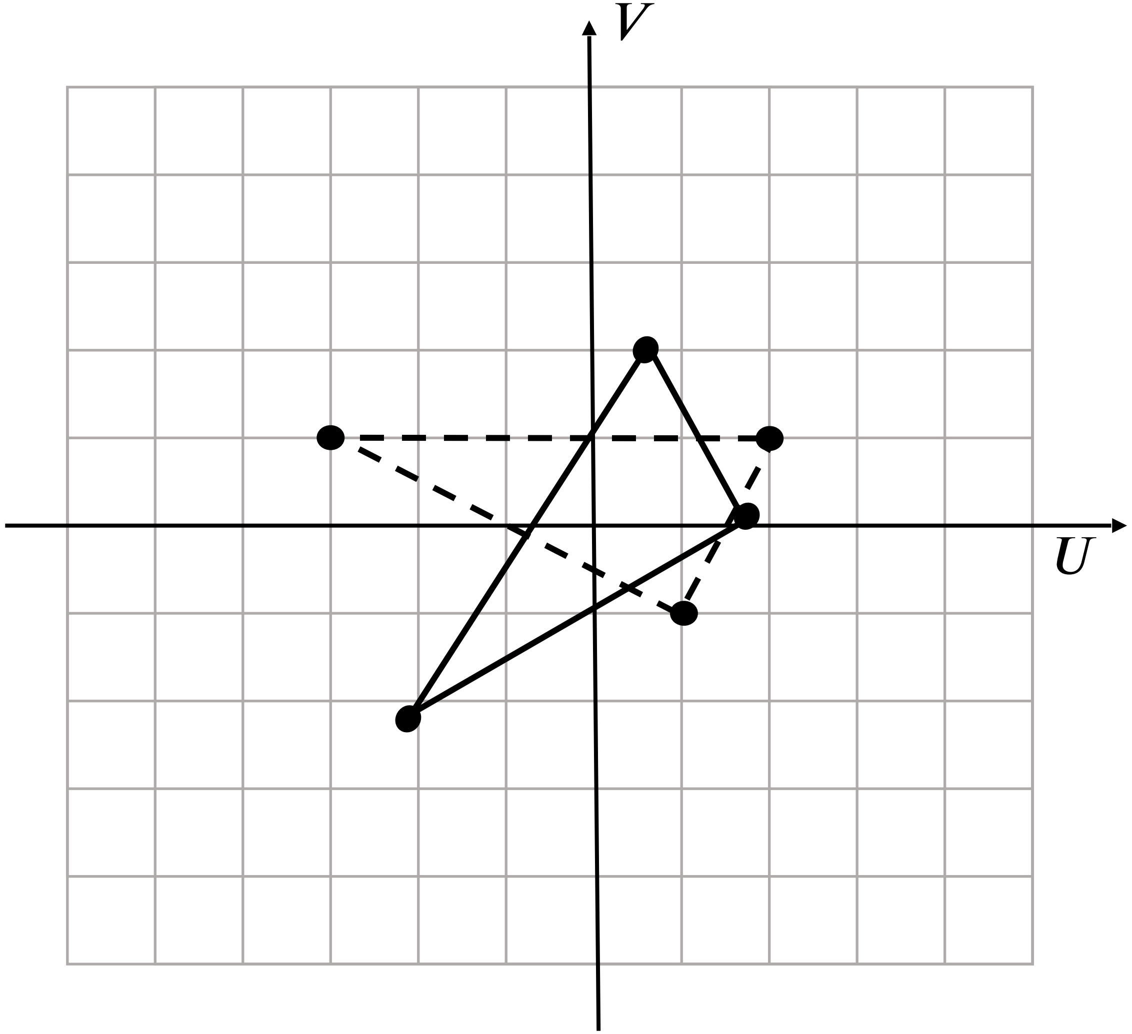
Transformations - Rotation

- Example rotation around 45 degrees

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R\left(\frac{\pi}{4}\right) = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot 2 + (-\frac{1}{\sqrt{2}}) \cdot 1 \\ \frac{1}{\sqrt{2}} \cdot 2 + \frac{1}{\sqrt{2}} \cdot 1 \end{bmatrix}$$



Transformations - Rotation

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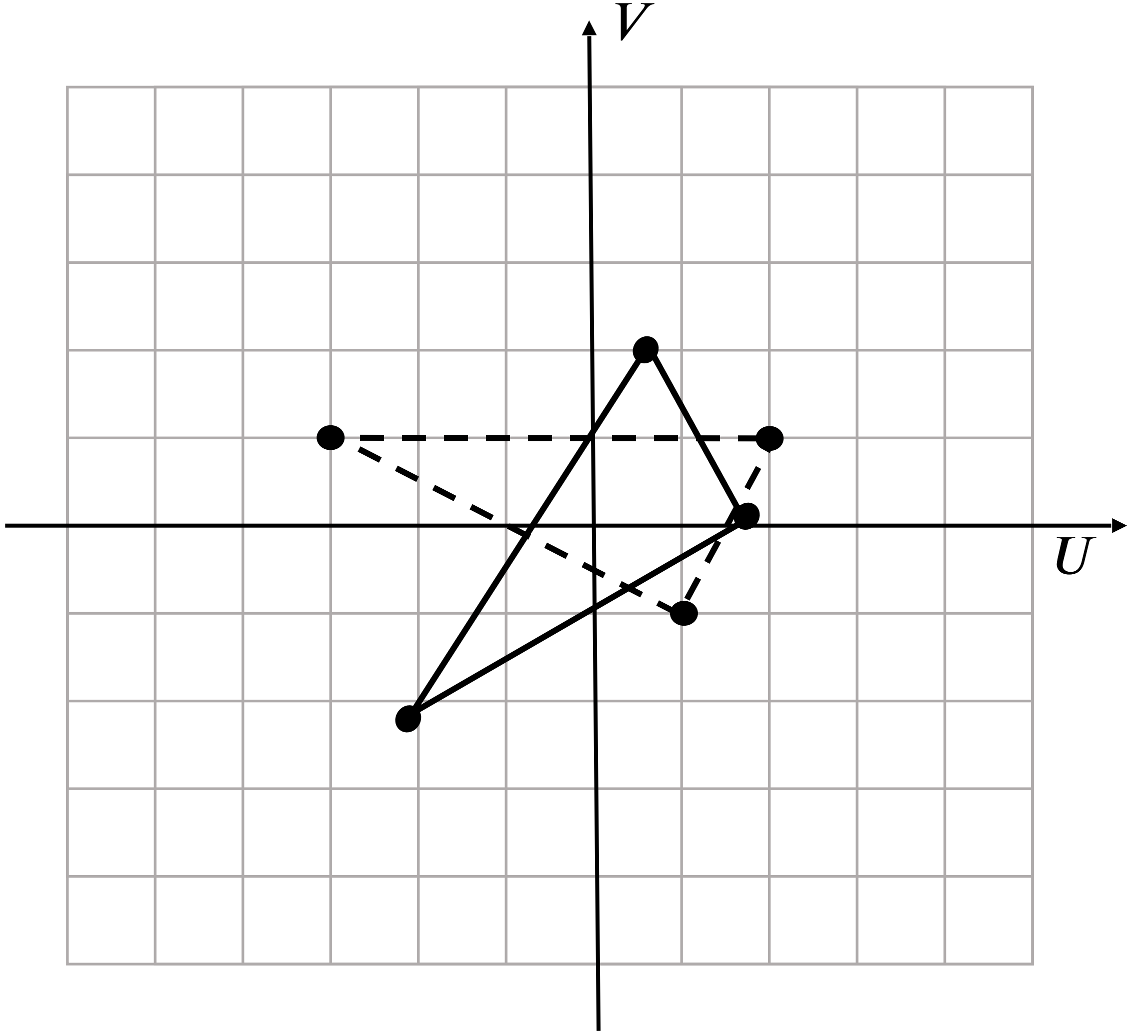
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.7071 \\ 2.1213 \end{bmatrix}$$



Transformations - Rotation

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$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix}$$

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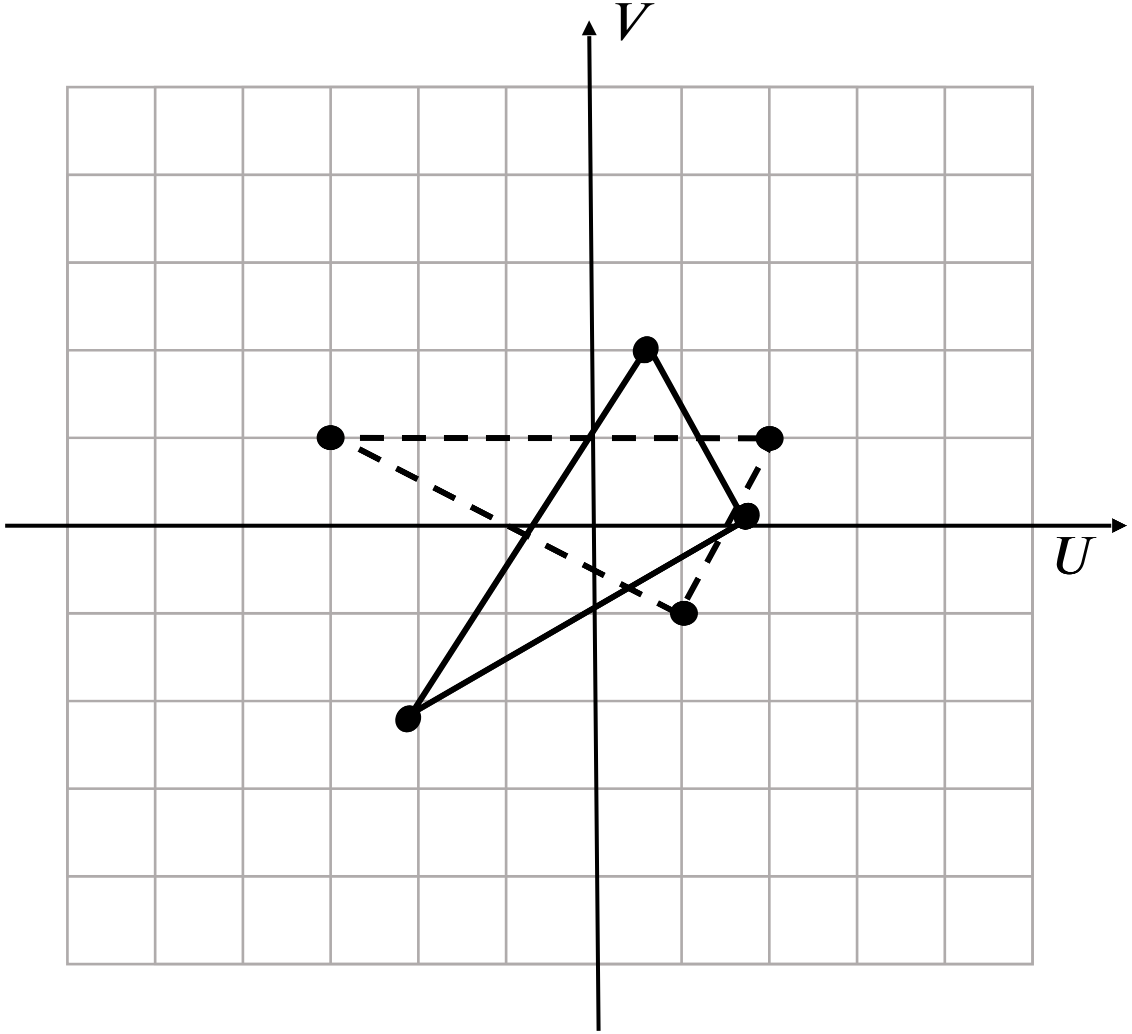


Transformations - Inverse Transforms

- Transforms can be undone geometrically
  - Inverse of shifting by $(\Delta u, \Delta v)$ is shifting by $(-\Delta u, \ -\Delta v)$
  - Inverse of scaling by s is scaling by $\frac{1}{s}$
  - Inverse of rotating by θ is rotating by $-\theta$

- Inverse of a rotation matrix is its transpose

$$R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = R_\theta^T$$

  - This is a special property of rotations – not of matrices in general

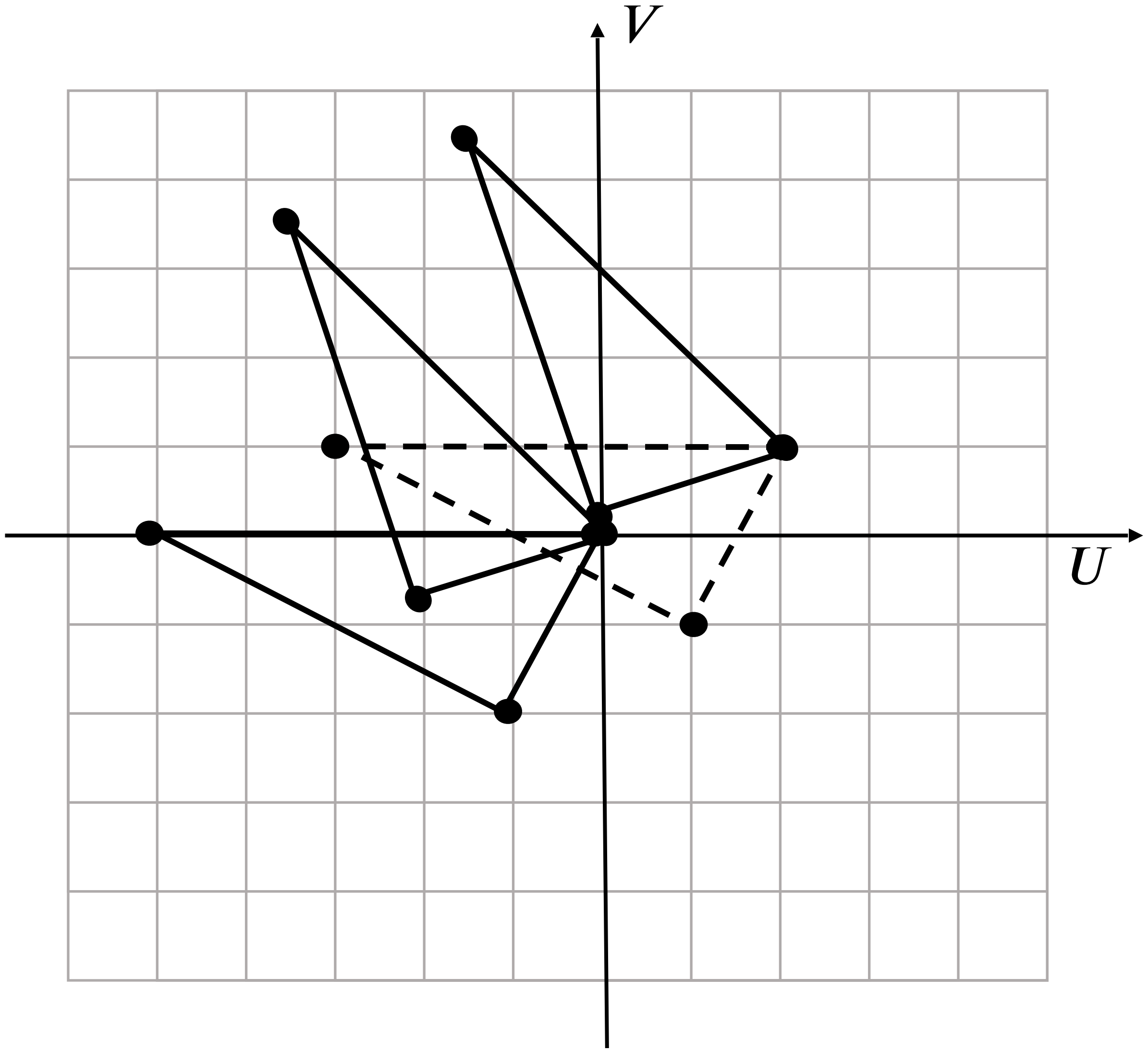
Transformations - Combining Transforms

- Rotate $-45°$ about $(2,1)$
  1. Shift by $(-2, -1)$
  2. Rotate by $-45°$
  3. Shift by $(2, 1)$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = R_{-45°}\left(\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right)$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = R_{-45°}\left(\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Easier Way?

Homogenous Coordinates

- The solution is counter-intuitive

- 2D points become sets of 3-vectors

$$\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow k \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}, \quad \forall k \neq 0$$

- The vector $[a \quad b \quad c]^T$ corresponds to the point $\left(\frac{a}{c}, \frac{b}{c} \right)$

- These are homogeneous coordinates

- All (linear) transformations become 3×3 matrices

Homogenous Coordinates

https://szollmann.github.io/LectureExamples/tutorials/transformations/Transformations2DTutorial

Homogenous Transforms

- Translation by $(\Delta u,\ \Delta v)$ becomes

$$\begin{bmatrix} 1 & 0 & \Delta u \\ 0 & 1 & \Delta v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \equiv \begin{bmatrix} u + \Delta u \\ v + \Delta v \\ 1 \end{bmatrix}$$

- Rotation by an angle θ becomes

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \equiv \begin{bmatrix} \cos(\theta)u - \sin(\theta)v \\ \sin(\theta) + \cos(\theta)v \\ 1 \end{bmatrix}$$

Homogenous Transforms

- Scaling by a factor of s becomes

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \equiv \begin{bmatrix} su \\ sv \\ 1 \end{bmatrix} \equiv \begin{bmatrix} u \\ v \\ \frac{1}{s} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

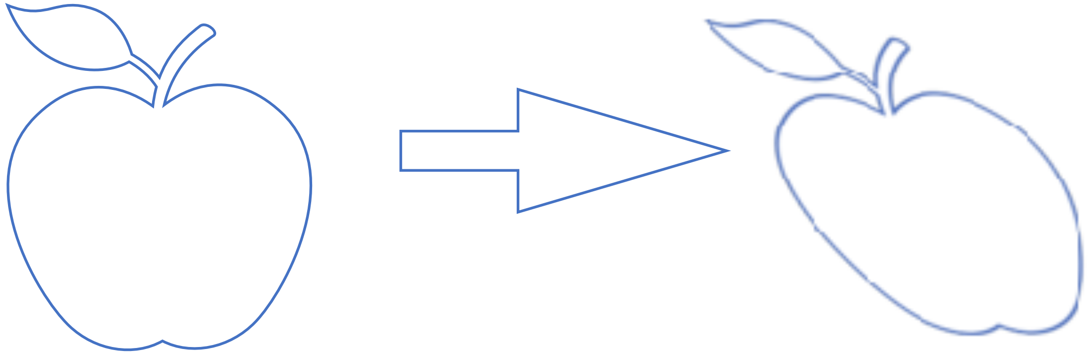
- Can also do non-uniform scaling:

$$\begin{bmatrix} s_u & 0 & 0 \\ 0 & s_v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \equiv \begin{bmatrix} s_u u \\ s_v v \\ 1 \end{bmatrix}$$

Homogenous Transforms - Shearing

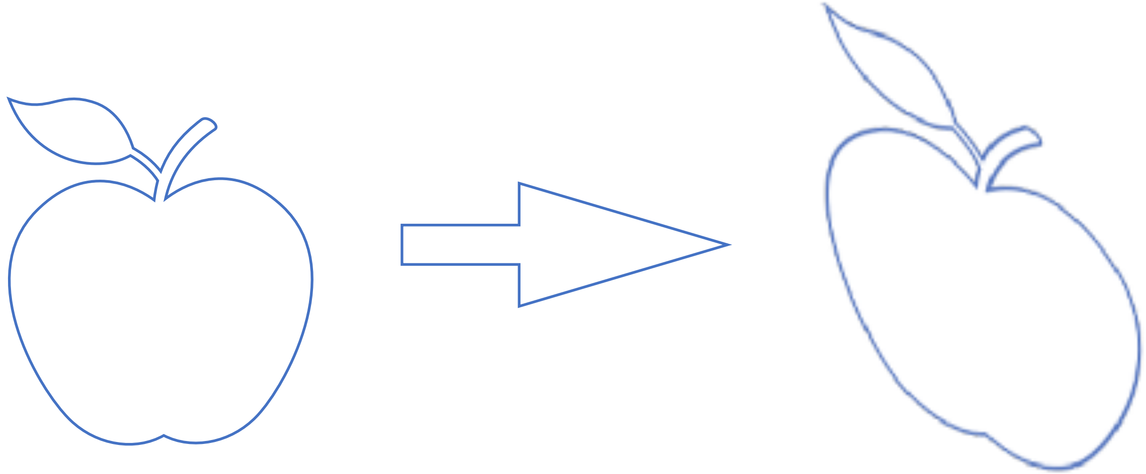
- Distorts the shape by shifting one axis relative to the other
- Example: Turning a rectangle into a parallelogram
- Shearing by a factor of sh_x in x direction:

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Shearing by a factor of sh_y in your direction:

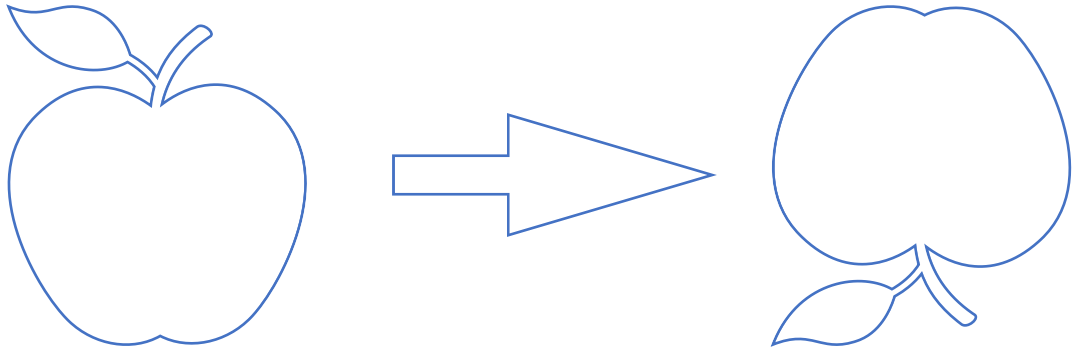
$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Homogenous Transforms - Reflection

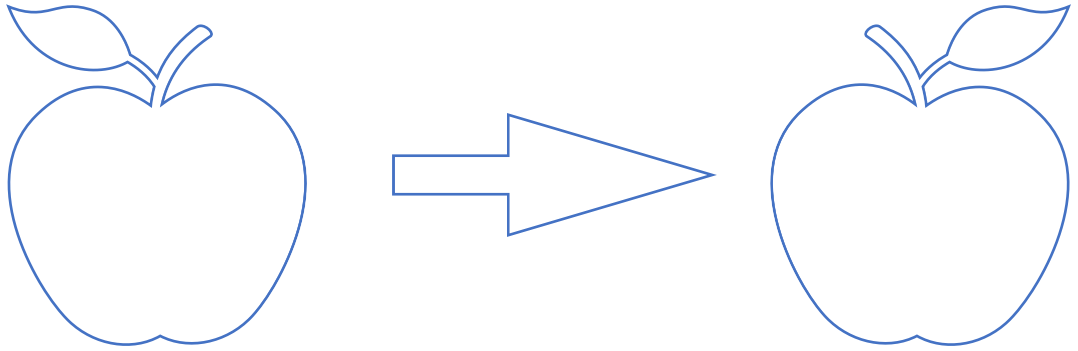
- Flips points across a line (e.g., the x-axis, y-axis, or an arbitrary line)
- Reflection across the x-axis:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Reflection across the y-axis:

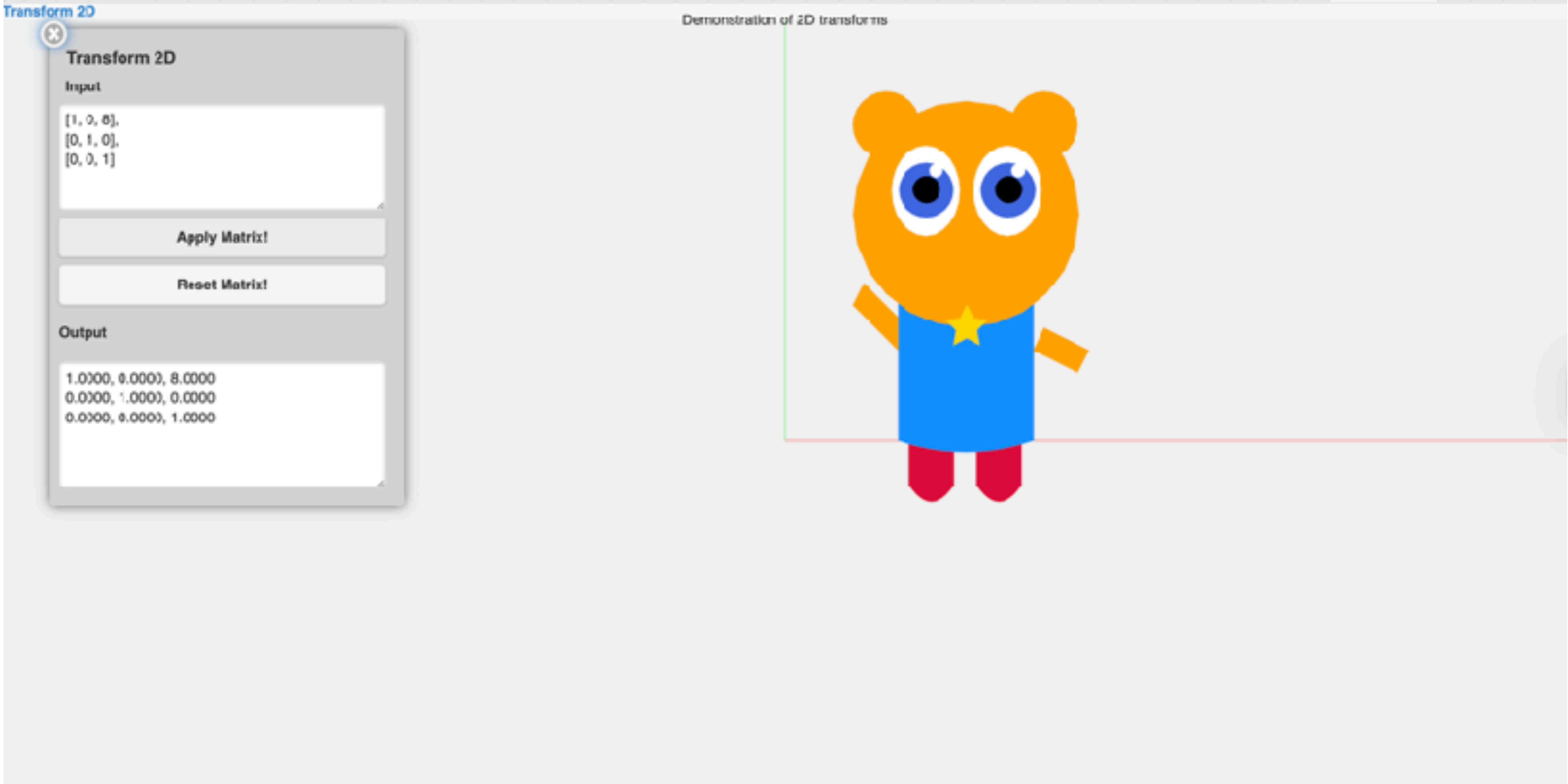
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Combining Transforms

- Because all transforms are matrices they combine easily
- Suppose we have a series of transforms, $T_1, T_2, \ldots, \ T_k$
- Applying them to a point, $\mathbf{p}$, in order gives
  $\mathbf{p}' = T_k\left(T_{k-1}\left(\ldots\left(T_2\left(T_1\mathbf{p}\right)\right)\right)\right)$
- Because matrix multiplication is associative, we have
  $\mathbf{p}' = \left(T_kT_{k-1}\ldots T_2T_1\right)\mathbf{p}$
- Can combine transforms once and apply to a set of points

Try yourself



https://szollmann.github.io/LectureExamples/tutorials/transformations/Transformations2DTutorial

Additional Examples Combining Transforms

Inverse Transforms

- Inverse of shifting by $(\Delta u, \Delta v)$ is shifting by $(-\Delta u, \ - \Delta v)$

$$\begin{bmatrix} 1 & 0 & \Delta u \\ 0 & 1 & \Delta v \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -\Delta u \\ 0 & 1 & -\Delta v \\ 0 & 0 & 1 \end{bmatrix}$$

- Inverse of scaling by s is scaling by $\frac{1}{s}$

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/s & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$$

Inverse Transforms

- Inverse of rotating by θ is rotating by $-\theta$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- If we have a sequence of transforms, $T_k T_{k-1} \ldots T_2 T_1$, then

$$\left(T_k T_{k-1} \ldots T_2 T_1\right)^{-1} = T_1^{-1} T_2^{-1} \ldots T_{k-1}^{-1} T_k^{-1}$$

Recap: What have we learned so far?

Transforms

- Translation
- Rotation
- Scaling
- Shear
- Reflection
- These can all be written as matrices in homogeneous coordinates
- Instead of thinking of each transform separately, we can compose them

Affine Transforms

- General form (homogeneous 3×3 matrix, bottom row fixed):

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- Includes: translation, rotation, scaling, shear, reflection
- Properties:
    - Preserve straightness of lines
    - Preserve parallelism
    - Ratios of lengths along the same line are preserved
- Examples: skewing a photo, mapping a rectangle to a parallelogram

Limits of Affine Transforms

- Cannot capture perspective effects
- Example: a photo of a square table looks like a trapezoid
- In reality, parallel edges seem to meet at a vanishing point
- Affine transformations cannot model this

Projective Transforms

- To capture perspective, we generalize to Projective Transforms
- Affine = special case where: $h_{31} = 0, h_{32} = 0, h_{33} = 0$
- Properties:
  - Lines stay straight
  - Parallelism not preserved
  - Can map any quadrilateral to any other quadrilateral
  - Models perspective projection

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.5 & 0.3 & 1 \end{bmatrix},$$

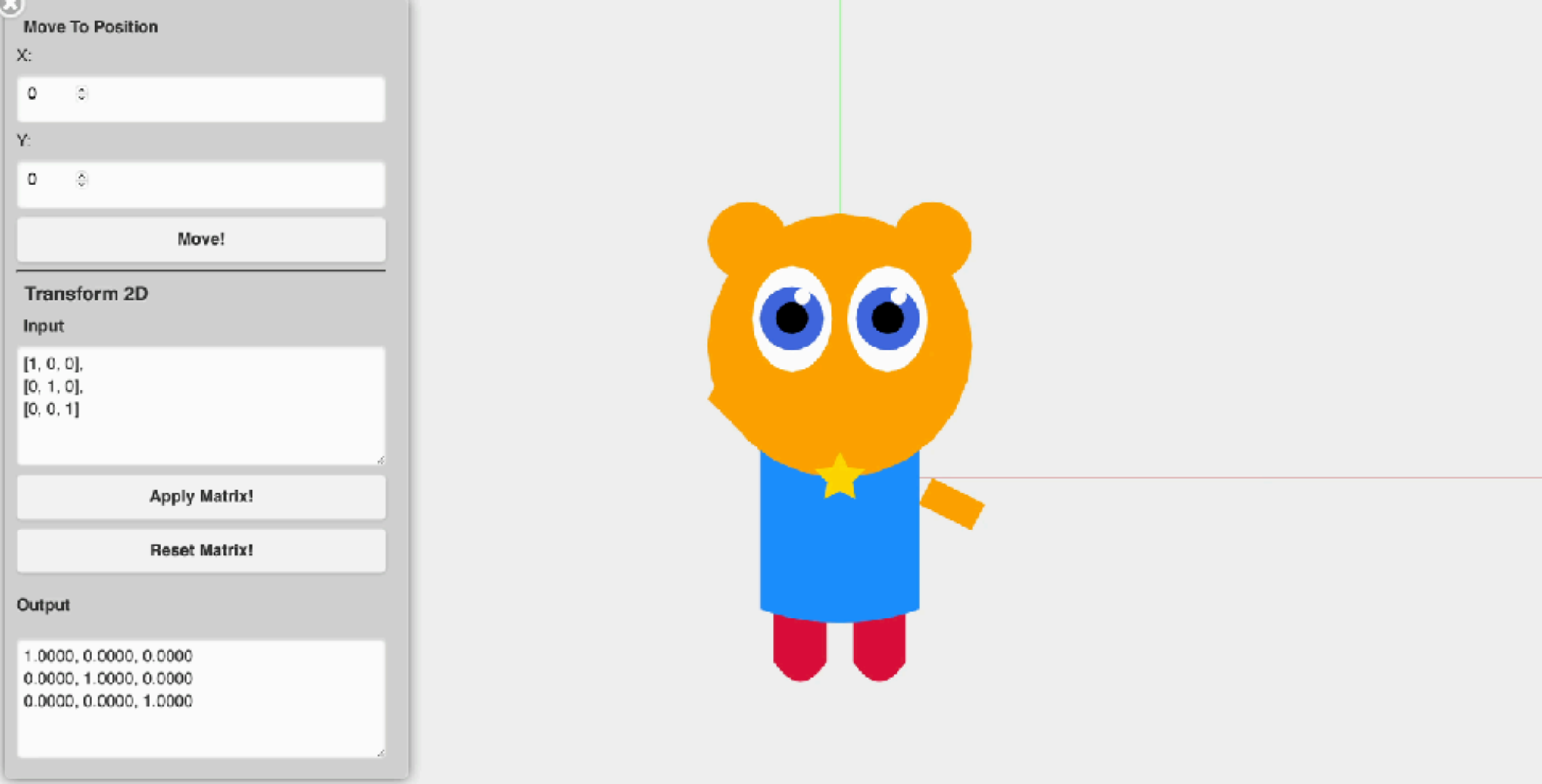
Applications

- Affine:
  - Geometric modeling
  - Simple graphics transforms
- Projective:
  - Image stitching
  - AR plane tracking
  - Perspective correction in images

2D Animations

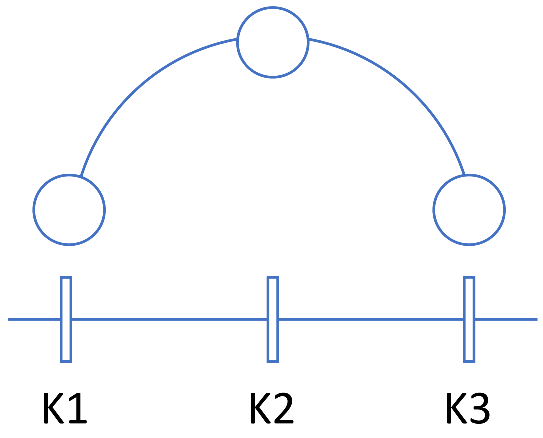
2D Animations

- Animation = applying transformations over time
- Common techniques:
  - Translation over time -> moving an object
  - Scaling over time -> growing/shrinking
  - Rotation over time -> spinning
- Achieved by updating transformation parameters frame by frame



https://szollmann.github.io/LectureExamples/tutorials/transformations/2DAnimations

Principles 2D Animations



- Keyframes: define positions/transform states at specific times
- Interpolation (tweening): smooth transitions between keyframes
- Combining transforms: e.g., rotate + translate for circular motion
- Applications:
  - Game character motion
  - UI transitions
  - Data visualization (dynamic charts)

Example 2D Animation

```
// Animation loop
    for (int i = 0; i < total_frames; i++) {
        double t = (double)i / (total_frames - 1);
        // goes 0 → 1
        int dx = (int)(t * shift_x);
        int dy = (int)(t * shift_y);

        // Create a black frame
        cv::Mat frame = cv::Mat::zeros(h, w, img.type());

        // Copy pixels manually with translation
        for (int y = 0; y < h; y++) {
            for (int x = 0; x < w; x++) {
                int nx = x + dx; // new x position
                int ny = y + dy; // new y position
                if (nx >= 0 && nx < w && ny >= 0 && ny < h) {
                    frame.at<cv::Vec3b>(ny, nx) = img.at<cv::Vec3b>(y, x);
                }
            }
        }
```

Example 2D Animation Composite

```
// Animation loop
for (int i = 0; i < total_frames; i++) {
    double t = (double)i / (total_frames - 1); // 0 → 1

    // Background
    cv::Mat frame = cv::Mat::zeros(height, width, CV_8UC3);

    // Circle parameters
    int x = (int)lerp(50, width - 50, t);                   // move left → right
    int y = height / 2 + (int)(50 * std::sin(t * 6.28 * 2)); // wiggle up/down
    int r = (int)lerp(20, 80, t);                           // radius grows

    cv::circle(frame, cv::Point(x, y), r, cv::Scalar(0, 255, 255), -1);

    // Show and save
    cv::imshow("Circle Animation", frame);
    if (cv::waitKey(30) == 27) break; // ESC to stop

    writer.write(frame);
}
```

Summary: 2D geometry manipulation

How about image content manipulation?

Next time :)

The end!