APPLIED STATISTICS

WEEK 5

Hypothesis Testing (Single Sample)

PART I

OUTLINE

- Single Sample Hypothesis Testing for Means, Proportions and Variances

Statistical Inference

Accuracy of our estimation is crucial !!

Estimation

(Estimation of Population Parameters)

→ Construct Confidence Intervals

$< \mu < \rightarrow \bar{X} \pm ME$

$< p < \rightarrow \hat{p} \pm ME$

Tests for Hypothesis

(we do not attempt to estimate a parameter, but instead we try to arrive at a correct decision about a prestated hypothesis

* Accuracy of our decision is important !!

What is a Hypothesis?

* A hypothesis is a claim (assumption) about a population parameter

Examples: Population Mean

→ The mean monthly cell phone bill of this city is $\mu = \$42$

Population Proportion:

→ The proportion of adults in this city with cell phones is $P = 0.68$

Ho:
H1: or Ha

What is a Hypothesis?

Hypothesis

Null Hypothesis, Ho

* Begin with the assumption that the null hypothesis is true. (Currently accepted value)
* Always contain "=", "≤", "≥"
* May or may not be rejected.
* Status quo

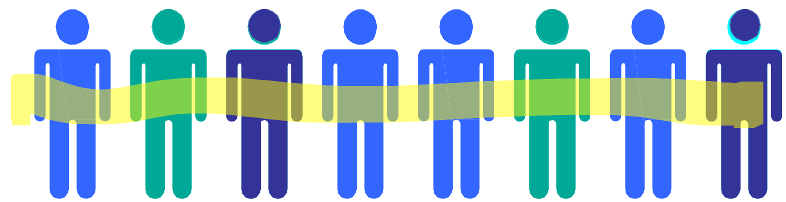
Alternative Hypothesis, H1 or Ha

* The hypothesis that the researcher is trying to support
* It is the opposite of the null hypothesis
* Never contains "=", "≤", "≥"
* May or may not be supported.

Hypothesis Testing Process

Claim: **the population mean age is 50. (Null Hypothesis:**

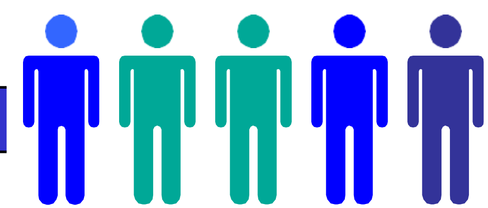
H_0: μ = 50)



Population

Now select a random sample

Is $\overline{X}$=20 likely if μ = 50?



Sample

Suppose the sample mean age is 20: $\overline{X}$ = 20

If not likely,

REJECT

Null Hypothesis

Hypothesis Testing Procedure

Level of Significance, α

- Level of Significance, α

→ It defines rejection region of the sampling distribution

→ It should be selected by the researcher at the beginning of the testing process

→ Typical values are, $\alpha = 0.05$, $\alpha = 0.10$, $\alpha = 0.01$.

→ It provides the critical value(s) of the test.

Level of Significance and the Rejection Region

Level of significance = α = 0.05

✦ **Represents critical value**

Rejection region is shaded

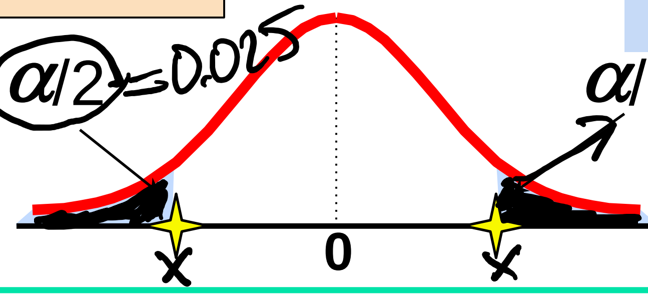
$H_0: \mu = 3$

$H_1: \mu \neq 3$

Two-tail test

α/2 = 0.025

α/2 = 0.025

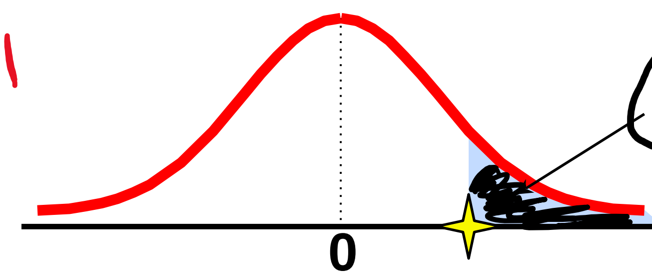


$H_0: \mu \leq 3$

$H_1: \mu > 3$

Upper-tail test

α = 0.05 Level of Significance

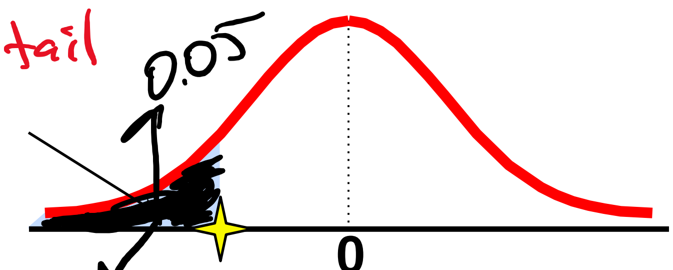


$H_0: \mu \geq 3$

$H_1: \mu < 3$

Lower-tail test

0.05



α → Level of significance

Errors in Making Decisions

Type I Error:

* Reject a true null hypothesis, H_0
* The probability of Type I Error is α

Type II Error:

* Fail to reject a false null hypothesis, H_0
  ~~Accept~~
* The probability of Type II error is β.

Possible Hypothesis Test Outcomes

Actual Situation

Statistical Decision	H_0 True	H_0 False
Do not reject H_0	No error $(1-\alpha)$	Type II error (β)
Reject H_0	Type I Error $(\alpha$	No error $(1-\beta)$

Type I & II Error Relationship

✓ Type I and Type II errors can not happen at the same time

- Type I error can only occur if H_0 is true.
- Type II error can only occur if H_0 is false

If Type I error probability (α) ↑, then Type II error probability (β) ↓

Hypothesis Tests for the Mean

Hypothesis Tests for μ

σ KNOWN

Case I

σ UNKNOWN

Case II

Test of Hypothesis for the Mean (σ Known)

- Convert sample result ($\bar{x}$) to a z value

Hypothesis Tests for μ

Case I: σ Known

σ Unknown → z-stat value

Null hypothesis

Consider the test

$H_0: \mu = \mu_0$ → Testing value

$H_1: \mu > \mu_0$

(Assume the population is normal)

→ H_1 or H_a: Alternative Hypothesis

The decision rule is:

$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_\alpha$$

Decision Rule

Critical value

Reject H_0 if $z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha$

z-statistic

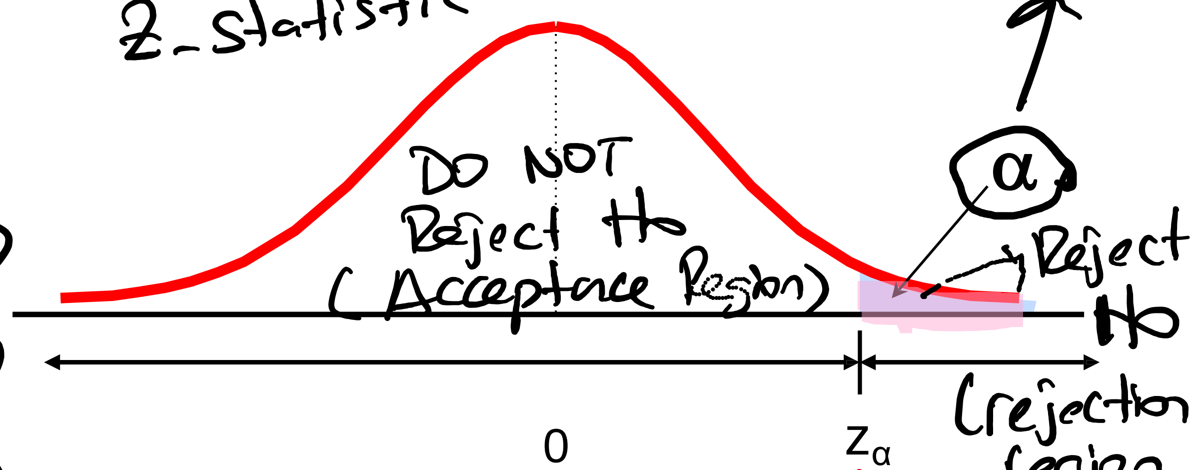
Level of significance

$H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$

Testing

- Critical Value Approach ✓
- P-value approach ✓



Critical Value

probability

p-Value Approach to Testing

* p-value : Probability of obtaining a test statistic more extreme ($\leq$ or $\geq$) than the observed sample value given H_0 is true.

* smallest value of α for which H_0 can be rejected

p-Value Approach to Testing

$$p\_value = P\left(Z > \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}, \text{ given that } H_0 \text{ is true}\right)$$

$$= P\left(Z > \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu_0\right)$$

Decision Rule:

Compare the p_value to α (level of significance)

- If p_value $< \alpha$, reject H_0
- If p_value $\geq \alpha$, do not reject H_0.

p-Value Approach to Testing

Testing

I. Critical Value Approach

II. P-value Approach

Note:

→ P-value approach uses probabilities instead of critical values for specifying decision rules.

Example: Upper-Tail Z Test for Mean (σ Known) → Case I

Question 1: A phone industry manager thinks that customer monthly cell phone bills have increased and now average over \$52 per month. The company asks you to test this claim. The population standard deviation, σ, is known to be equal to 10 from historical data. Moreover, after consulting with the manager and discussing error risk, they choose a level of significance, α, of 0.10. Their resources allow them to sample 64 sample cell phone bills and $\bar{x}$=53.1 .

$n = 64$ $\bar{X} = 53.1$

$H_0: \mu \leq 52$ (the average is not over \$52 per month)

$H_1: \mu > 52$ (the average is over \$52 per month)

(sufficient evidence exists to support the manager's claim)

$\alpha = 0.10$

$n = 64$

$\bar{X} = 53.1$

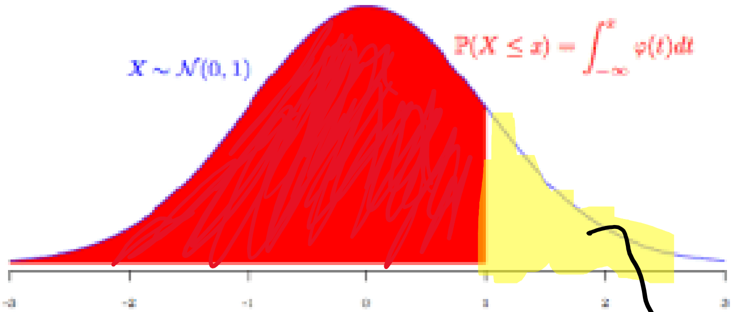
One-tail test Z_0, Z_α

I. Critical Value Approach

Decision Rule:

Reject Ho if $z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha$

z-statistic (z-stat) $= \dfrac{53.2 - 52}{10/\sqrt{64}} = \boxed{0.88}$



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

→ Standard Normal Table

$P(Z \geq 0.88)$

$= 1 - 0.8106$

$= \boxed{0.1894}$

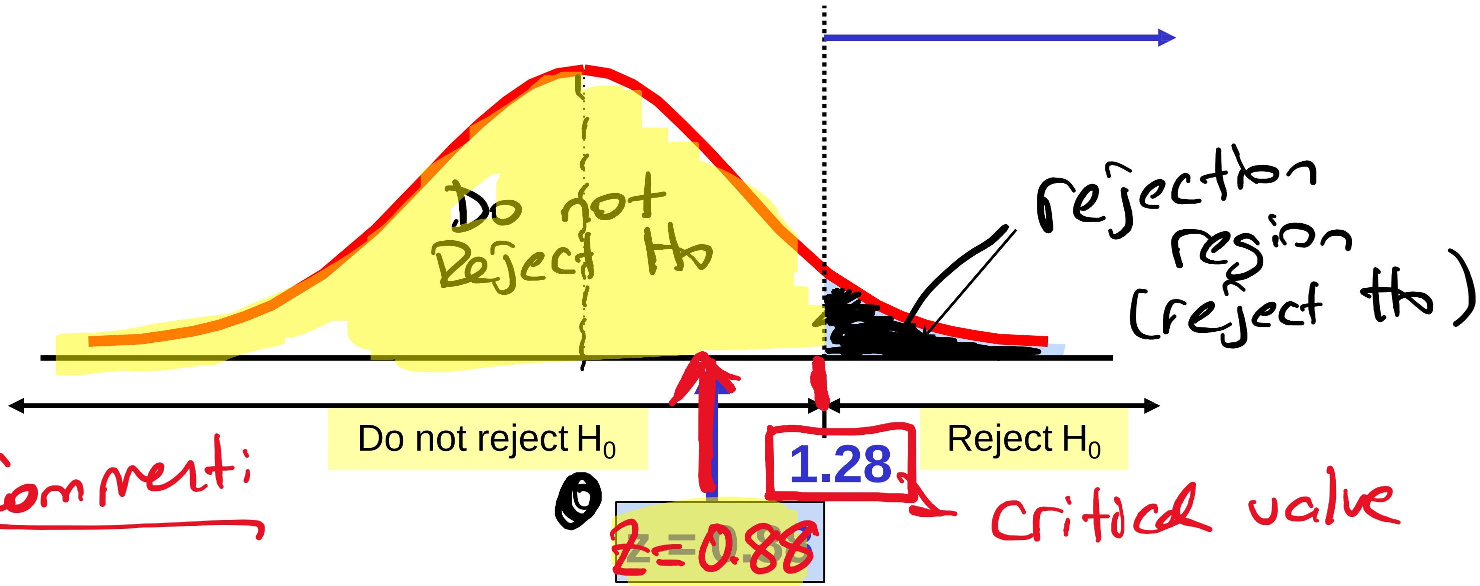
p-value

Example: Decision

$z_\alpha = ?$

$\alpha = 0.10$

Critical Value Approach



Comment:

Do not reject H_0 since z = 0.88 < 1.28

i.e.: there is not sufficient evidence that the mean bill is over $52

Example: p-Value Solution

Calculate p-value and compare to α $(\alpha = 0.10)$

$$P(\bar{X} \geq 53.1 \mid \mu = 52.0) = ?$$

$$\Rightarrow P\left(Z \geq \frac{53.1 - 52.0}{10/\sqrt{64}}\right) = ?$$

$$\Rightarrow P(Z \geq 0.88) = ?$$

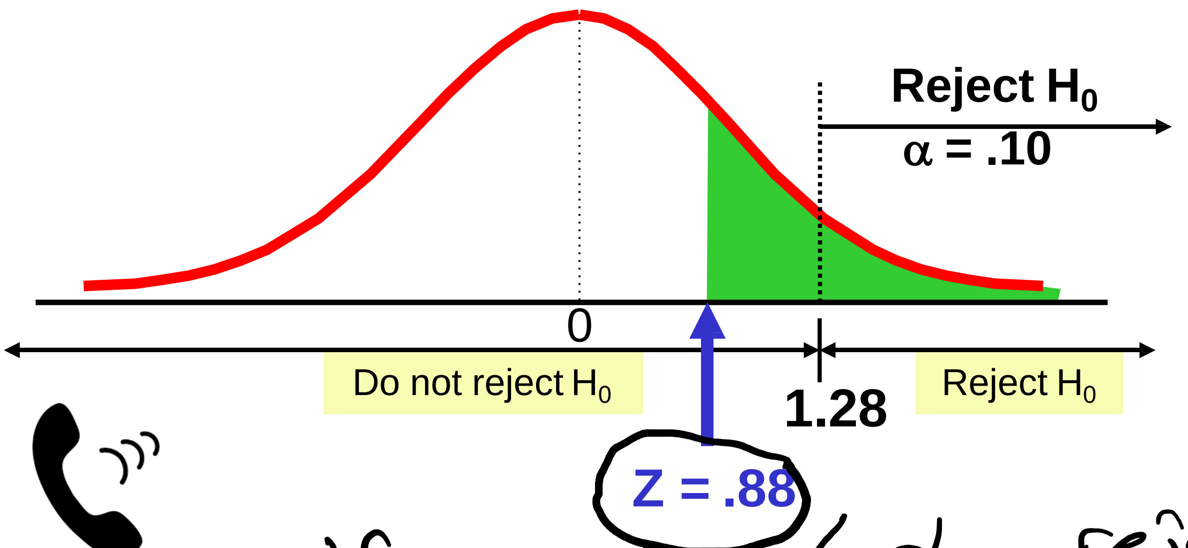
Example: p-Value Solution

Calculate the p-value and compare to α

Comment!

p-value = 0.1894 > 0.10

Do not reject Ho since p-value = 0.1894 > α = 0.10



If p-value < α, reject Ho.

If p-value $\geq$ α, do not reject Ho.

One-Tail Tests

- In many cases, the alternative hypothesis focuses on one particular direction

$H_0: \mu \leq 3$

$H_1: \mu > 3$

This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

$H_0: \mu \geq 3$

$H_1: \mu < 3$

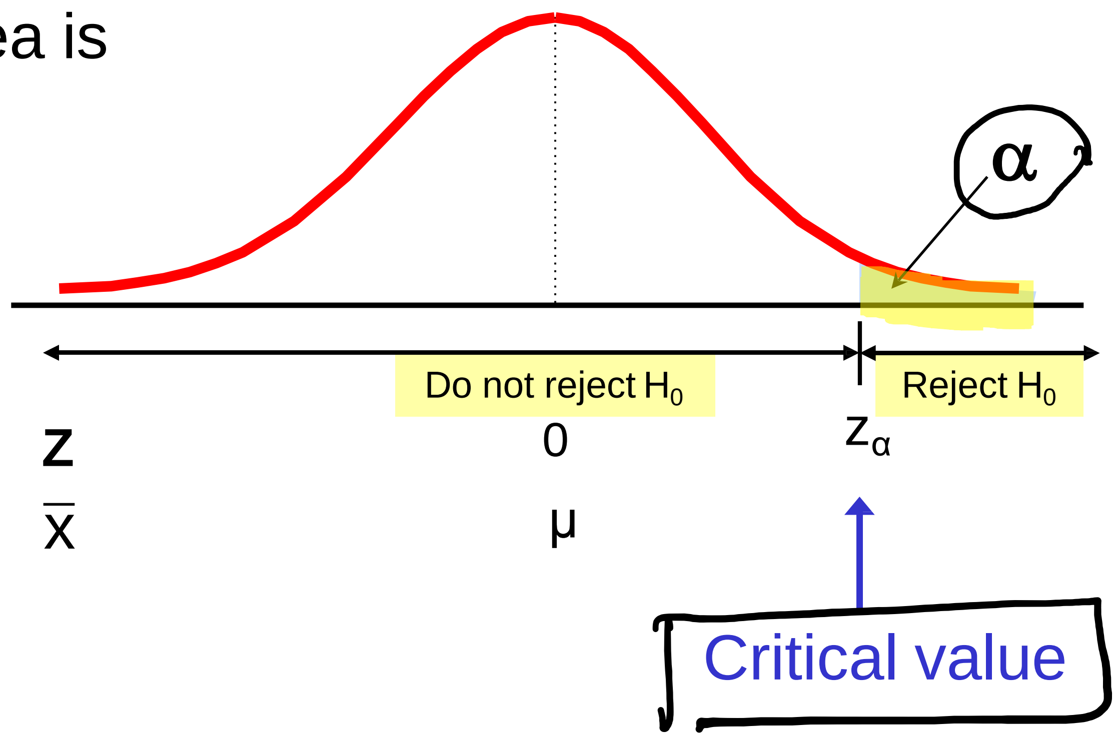
This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

H_0: $\mu \leq 3$

H_1: $\mu > 3$

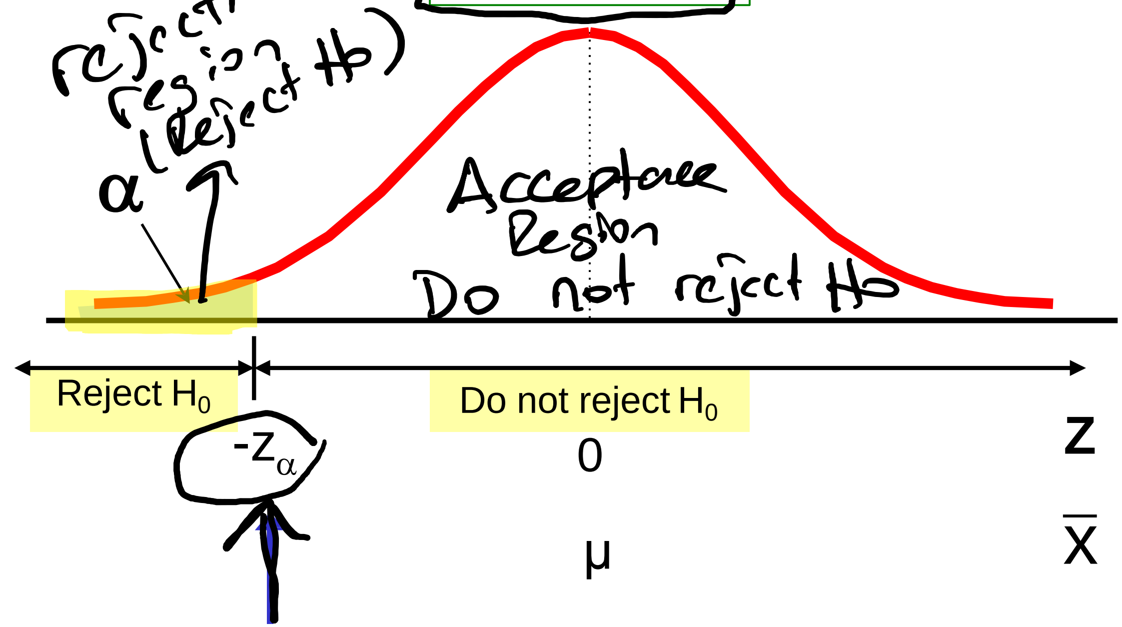


Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

H_0: μ ≥ 3

H_1: μ < 3



$-z_\alpha$

Critical value

Two-Tail Tests

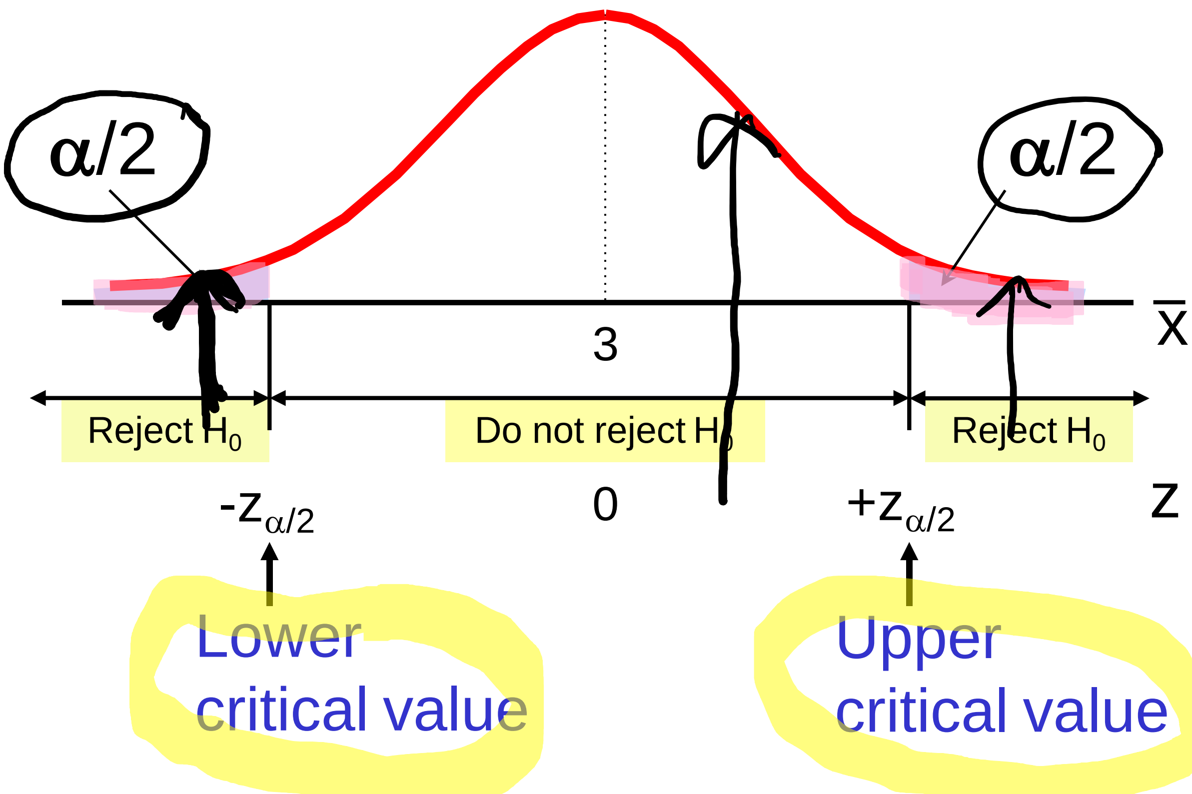
α

$\alpha/2 + \alpha/2$

- In some settings, the alternative hypothesis does not specify a unique direction

H_0: $\mu = 3$
H_1: $\mu \neq 3$

- There are two critical values, defining the two regions of rejection



$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Z-stat

Hypothesis Testing Example

<u>Question 3:</u> Test the claim that the true mean # of TV sets in US homes is equal to 3. Assume $\sigma = 0.8$. Suppose that $\alpha = .05$ is chosen for this test. Suppose a sample of size $n = 100$ is selected and $\bar{x} = 2.84$.

(σ is known)

Case I

Hypothesis Testing Example
(continued)

- Determine the appropriate technique

σ is known so this is a z-test

- Set up the critical values

For $\alpha = 0.05$ the critical z-values are ± 1.96

- Collect the data and compute the test statistic (z-stat=?)

$n = 100$, $\bar{X} = 2.84$

$$z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3.0}{\frac{0.8}{\sqrt{100}}} = \boxed{-2.0}$$

$H_0: \mu = 3$

$H_1: \mu \neq 3$

This is a two-tailed test

Hypothesis Testing Example

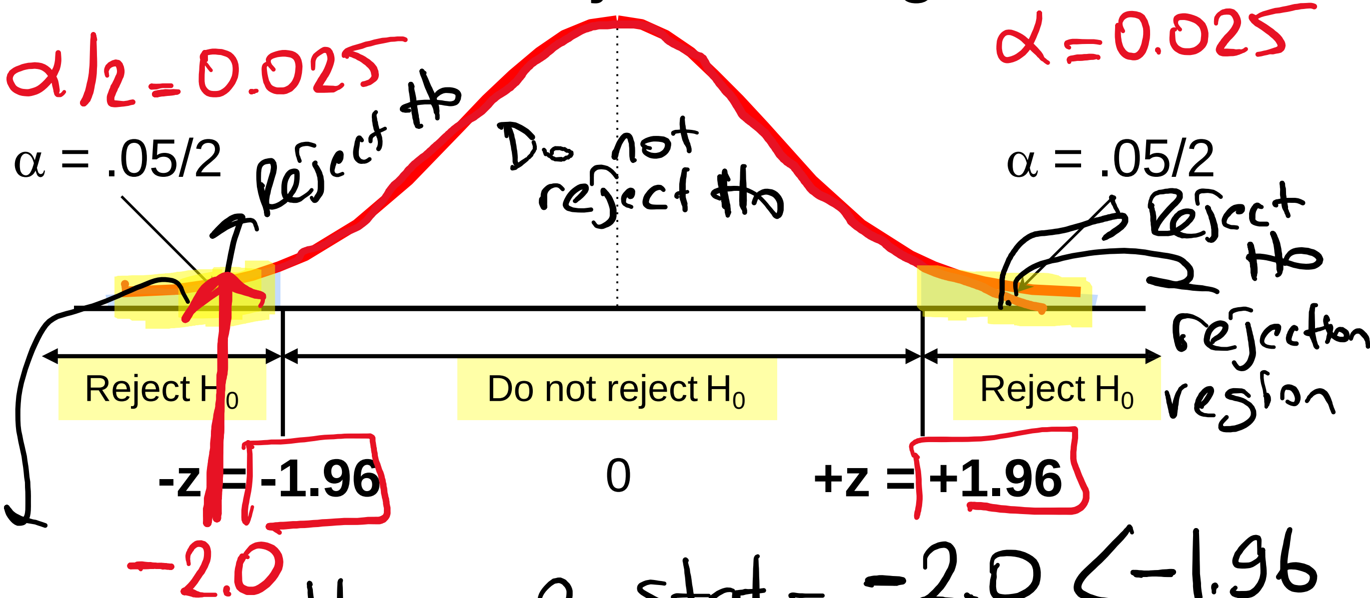
(continued)

- Is the test statistic in the rejection region?

Decision Rule:

$\alpha/2 = 0.025$ $\alpha = 0.025$

Reject H_0 if $z < -1.96$ or $z > 1.96$; otherwise do not reject H_0



Here z-stat = $-2.0 < -1.96$

So the test statistic is in the rejection region. Reject Ho.

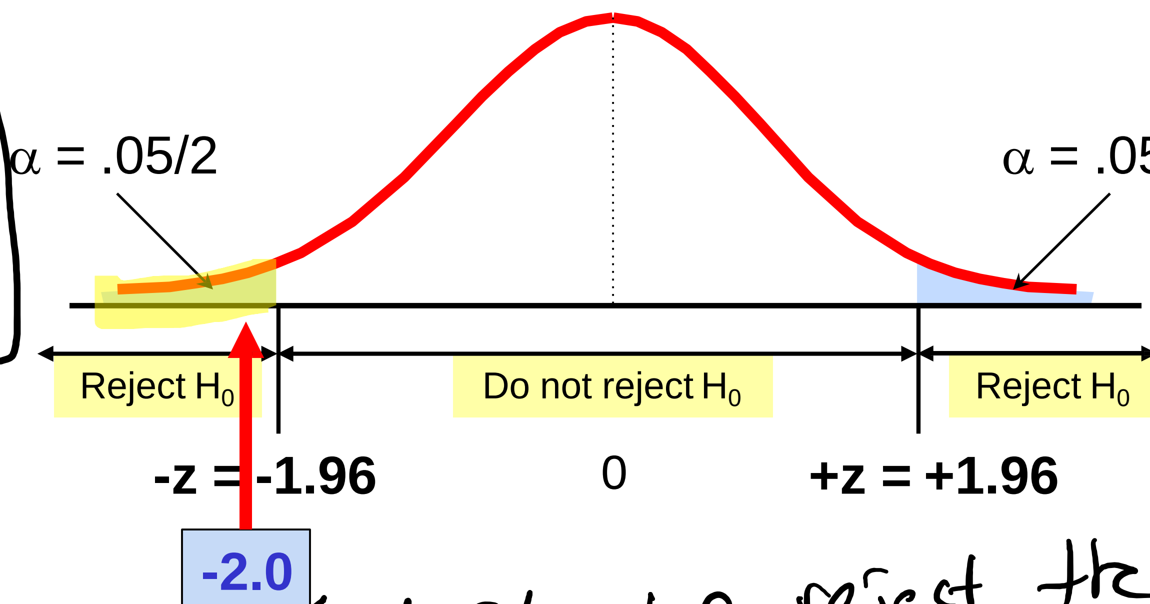
Hypothesis Testing Example

(continued)

- Reach a decision and interpret the result

$H_0: \boxed{\mu = 3}$

$H_1: \boxed{\mu \neq 3}$



Since $z = -2.0 < -1.96$, we reject the null hypothesis and conclude that there is sufficient evidence that the mean number of TV sets in US homes is not equal to 3.

t Test of Hypothesis for the Mean (σ Unknown)

- Convert sample result ($\bar{x}$) to a t test statistic

Hypothesis Tests for μ

σ Known | **σ Unknown**

upper tail test

Consider the test

$$H_0: \mu = \mu_0$$
$$H_1: \mu > \mu_0$$

Testing Value

(Assume the population is normal)

The decision rule is:

df = n-1

(df, α)

Reject H_0 if $t = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1,\alpha}$

t-statistic

critical t-value

t Test of Hypothesis for the Mean (σ Unknown)

(continued)

- For a two-tailed test:

Consider the test

$$\boxed{\begin{aligned} H_0 &: \mu = \mu_0 \\ H_1 &: \mu \neq \mu_0 \end{aligned}}$$

The decision rule is:

t-statistic

Reject H_0 if $\boxed{t = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} < -t_{n-1, \alpha/2}}$ or if $\boxed{t = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1, \alpha/2}}$

critical value

critical value

Example: Two-Tail Test (σ Unknown)- Question 4

Case II

The average cost of a hotel room in New York is said to be \$168 per night. A random sample of 25 hotels resulted in

$\bar{x}$ = \$172.50 and

s = \$15.40. Test at the

α = 0.05 level.

(Assume the population distribution is normal)

sample mean

sample standard deviation

n = 25

H_0: μ = $\boxed{168}$

H_1: $\mu \neq \boxed{168}$

Two-tail test!

$H_0: \mu = 168$

$H_1: \mu \neq 168$

$\mu_0 = 168$

Since σ is unknown, use t-statistic. (s is given!)

$n-1 = 25-1 = \boxed{24}$

$$t\_stat = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

$$t = \frac{172.50 - 168}{15.40/\sqrt{25}}$$

$$\boxed{t = 1.46}$$

t-statistic value

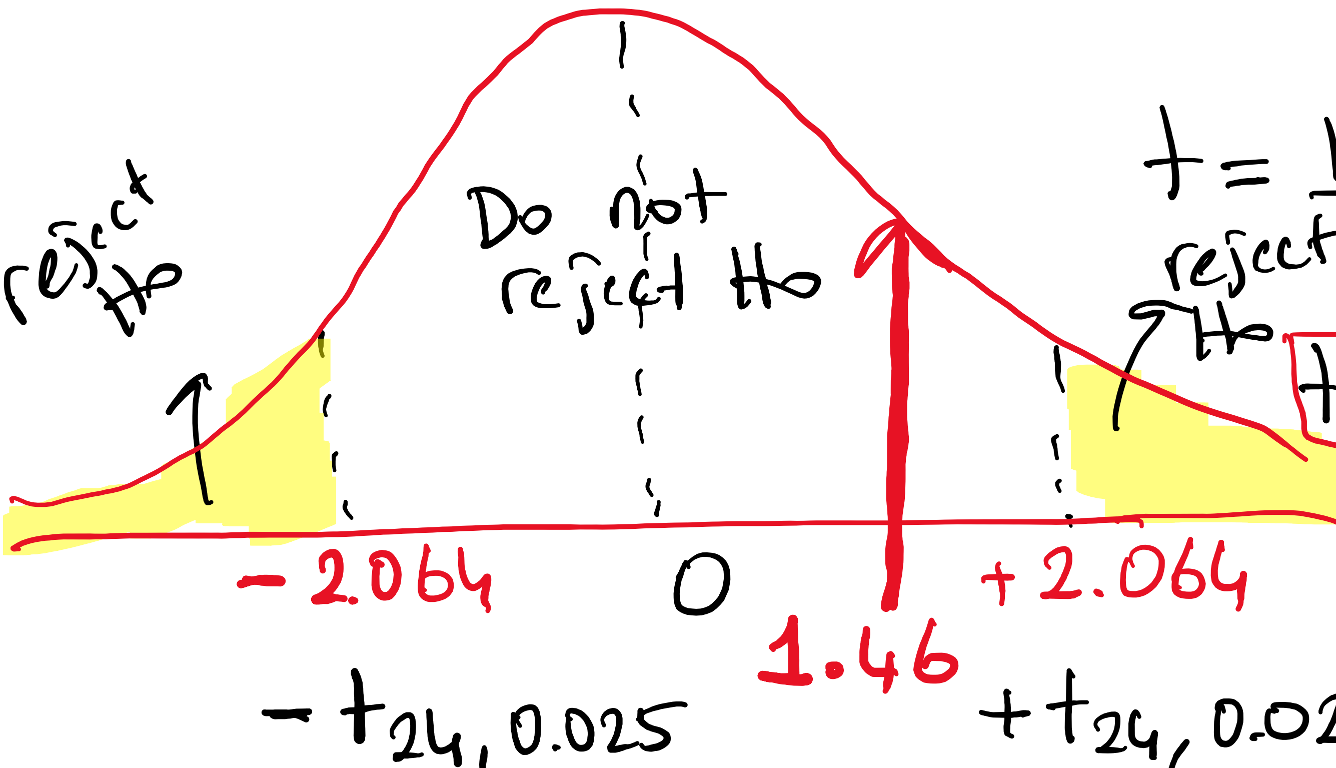


Table A.4 Critical Values of the t-Distribution values

$\nu = df$

df = degrees of freedom

v	0.40	0.30	0.20	0.15	0.10	0.05	0.025
				α			
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	$\boxed{2.064}$

critical t-value

$t_{24, 0.025} = \boxed{2.064}$

$H_0: \mu = 168$

$H_1: \mu \neq 168$

Interpret the result:

(Fail to reject H_0)

Do not reject H_0 since

$1.46 < 2.064$.

There is not sufficient evidence that true (actual) mean cost is different $168.