→ Probability is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur.

"The chance of rain today is 30%" is a statement that quantifies our feeling about the possibility of a rain.

The likelihood of an outcome is quantified by assigning a number from the interval $[0,1]$ to the outcome.

(*) Higher numbers indicate that the outcome is more likely than lower numbers. A "0" indicates an outcome will not occur. A probability of "1" indicates an outcome will occur with certainty.

→ Probability is a measure of uncertainty. The probability of event A is a numerical measure of the <u>likelihood</u> of the event occuring.

A prob. of 0.95 implies a very high confidence in the occurence of the event. However, the prob. 0.2 implies that the event is not very likely to occur.

Events that are certain to occur have probability 1.

When an event cannot occur, its prob. is 0.

→ Another interpretation of prob. is that the probability of an outcome is interpreted as the limiting value of the proportion of times that the outcome occurs in n repetitions of the random expeiment.

Theoretical explanation of Probability

$\Omega = S = \{ w_1, w_2, \dots, w_n \}$

elementary event or an outcome

If for each $w \in \Omega$, we have $P(w)$ $(0 \leq P(w) \leq 1)$

and $\sum_{w \in \Omega} P(w) = 1$, then $P(w)$ is defined in Ω.

Then $P(w)$ is a function

$$P(w) : \Omega \to [0,1]$$

And also A is an event, a set of elementery events.

$A \subset \Omega$

$A = \{ w_1, w_3, w_5 \}$

Then $P(A) = P(w_1) + P(w_3) + P(w_5)$

$$P(A) = \sum_{w \in A} P(w)$$

Also $P(A \cup B) = \sum_{w \in A \cup B} P(w) = \sum_{w \in A} P(w) + \sum_{w \in B} P(w) - \sum_{w \in A \cap B} P(w)$

$$= P(A) + P(B) - P(A \cap B)$$

$A \subseteq B \to P(A) \leq P(B)$

Ex If we deal with tossing a dice experiment.

$S = \{ 1, 2, 3, 4, 5, 6 \}$

$A = \{$ getting even numbers in tossing a dice experiment $\}$

$A = \{ 2, 4, 6 \} \subset \Omega$

If $w \in A \Rightarrow$ that means A occurs.

We say that the event occurs if the expeiment gives rise to a basic outcome belonging to the event.

Ex : The event "an ace is drawn out of a deck of cards" is the set of the all four aces within the sample space consisting of all 52 cards. This event occurs whenever one of the four aces (the basic outcome) is drawn.

Probability of event A : $P(A) = \dfrac{n(A)}{n(S)}$

The rule we use in computing probabilities, assuming equal likelihood of all basic outcomes, is as follows

$$P(A) = \frac{n(A)}{n(S)}$$

where $n(A)$ gives the number of elements in the set of the event A

$n(S)$ gives the number of elements in the sample space.

The prob. of drawing an ace is $P(A) = \dfrac{4}{52}$

Equally Likely Outcomes : Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

Example : Experiment of tossing (rolling) dice

Probability of an Event : For a discrete sample space, the prob. of an event E, $P(E)$, equals the sum of the probabilities of the outcomes in E.

Ex A random experiment can result in one of the outcomes $\{a,b,c,d\}$ with probabilities 0.1, 0.3, 0.5 and 0.1, respectively. Let A denote the event $\{a,b\}$, B the event $\{b,c,d\}$ and C the event $\{d\}$, Then,

$P(A) = 0.1+0.3=0.4$

$P(B) = 0.3+0.5+0.1=0.9$

$P(C) = 0.1$

Also, $P(A') = 0.6$, $P(B') = 0.1$ and $P(C') = 0.9$.

$A \cap B = \{b\}$, $\Rightarrow P(A \cap B) = 0.3$

$A \cup B = \{a,b,c,d\} \Rightarrow P(A \cup B) = 0.1+0.3+0.5+0.1 = 1$

$A \cap C = \emptyset \Rightarrow P(A \cap C) = 0$

Ex : Suppose that a batch contains six parts with part numbers $\{a,b,c,d,e,f\}$. Suppose that two parts are selected without replacement. Let E denote the event that the part number of the first part selected is a. Then E can be written as $E = \{ab, ac, ad, ae, af\}$. The sample space can be enumerated. It has 30 outcomes. If each outcome is equally likely, $P(E) = 5/30 = 1/6$. If E_2 denotes the event that the second part selected is a then $P(E_2) = ?$

A visual inspection of a location on wafers from a semiconductor manufacturing process resulted in the following table:

Number of Contamination Particles	Proportion of Wafers
0	0.40
1	0.20
2	0.15
3	0.10
4	0.05
5 or more	0.10

If one wafer is selected randomly from this process and the location is inspected, what is the probability that it contains no particles? If information were available for each wafer, we could define the sample space as the set of all wafers inspected and proceed as in the example with diodes. However, this level of detail is not needed in this case. We can consider the sample space to consist of the six categories that summarize the number of contamination particles on a wafer. Then, the event that there is no particle in the inspected location on the wafer, denoted as E, can be considered to be comprised of the single outcome, namely, $E = \{0\}$. Therefore,

$$P(E) = 0.4$$

What is the probability that a wafer contains three or more particles in the inspected location? Let E denote the event that a wafer contains three or more particles in the inspected location. Then, E consists of the three outcomes $\{3, 4, 5 \text{ or more}\}$. Therefore,

$$P(E) = 0.10 + 0.05 + 0.10 = 0.25$$

From the book of "Applied Statistics and Probability for Engineers" by Montgomery D.C and Runger G.C.

Axioms of Probability

If S' is the sample space and E is any event in a random experiment

1. $P(S') = 1$
2. $0 \leq P(E) \leq 1$
3. For two events, E_1 and E_2 with $E_1 \cap E_2 = \phi$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

For n events, $E_1, \ldots, E_n$ with $E_i E_j = \phi,\ i \neq j$

$$P(E_1 \cup \ldots \cup E_n) = P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) \quad n = 1, 2, \ldots, \infty$$

Some other results;

$$P(\phi) = 0$$
$$P(E^c) = 1 - P(E)$$

if event E_1 is contained in the event E_2,

$$P(E_1) \leq P(E_2)$$

Some Additional Rules

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example: The following table lists the history of 940 wafers in a semiconductor manufacturing process.

Table. Wafers Classified by Contamination and Location

	Location in Sputtering Tool		
Contamination	Center	Edge	Total
Low	514	68	582
High	112	246	358
Total	626	314	

Suppose one wafer is selected at random. Let H denote the event that the wafer contains high levels of contamination.

Then $P(H) = 358/940$

Let C denote the event that the wafer is in the center of a sputtering tool. Then, $P(C) = 626/940$.

$P(H \cap C) \rightarrow$ is the prob. that the wafer is from the center of the sputtering tool and contains high levels of contamination.

Therefore, $P(H \cap C) = 112/940$.

$H \cup C \rightarrow$ is the event that a wafer is from the center of the sputtering tool or contains high levels of contamination (or both).

$P(H \cup C) = 872/940$.

$$P(H \cup C) = P(H) + P(C) - P(H \cap C)$$
$$= 358/940 + 626/940 - 112/940 = 872/940$$

The following table shows the proportion of wafers in each category.

Table. Wafers Classified by Contamination and Location

Number of Contamination Particles	Center	Edge	Totals
0	0.30	0.10	0.40
1	0.15	0.05	0.20
2	0.10	0.05	0.15
3	0.06	0.04	0.10
4	0.04	0.01	0.05
5 or more	0.07	0.03	0.10
Totals	0.72	0.28	1.00

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What is the probability that a wafer was either at the edge or that it contains four or more particles?

Let E_1 denote the event that a wafer contains four or more particles

Let E_2 denote the event that a wafer is at the edge.

Requested probability $\rightarrow P(E_1 \cup E_2)$

$P(E_1) = 0.15$, $P(E_2) = 0.28$ $P(E_1 \cap E_2) = 0.04$

$P(E_1 \cup E_2) = 0.15 + 0.28 - 0.04 = 0.39$ //

What is the prob. that a wafer contains less than 2 particles or that it is both at the edge and contains more than 4 particles?

If A and B are mutually exclusive events,

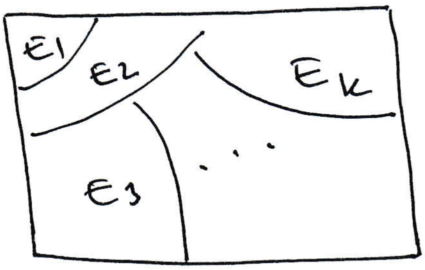
$$P(A \cup B) = P(A) + P(B)$$

Generalization

A collection of events, $E_1, E_2, \ldots, E_k$ is said to be mutually exclusive if for all pairs, $E_i \cap E_j = \phi$

For collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k)$$



If we have the events A, B and C which are not mutually exclusive then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Try to obtain this probability. It is a kind of proof.

Try to give the proof of it. And also try to generalize the related probability if we have n events such as $A_1, A_2, \ldots, A_n$

How to write $P(A_1 \cup A_2 \cup \ldots \cup A_n) = ?$

CONDITIONAL PROBABILITY

Sometimes additional information becomes available and under this condition, probabilities need to be calculated.

$P(B|A) \rightarrow$ the probability of an event B under the knowledge that the outcome will be in event A.

This is called the conditional probability of B given A.

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{for } P(A) > 0$$

prob. of event B given an event A

$$P(A) = \frac{\text{number of outcomes in } A}{n} \qquad P(A \cap B) = \frac{\text{number of outcomes in } A \cap B}{n}$$

$$P(B|A) = \frac{\text{number of outcomes in in } A \cap B}{\text{number of outcomes in } A}.$$

Therefore $P(B|A)$ can be interpreted as the relative frequency of event B among the trials that produce an outcome in event A.

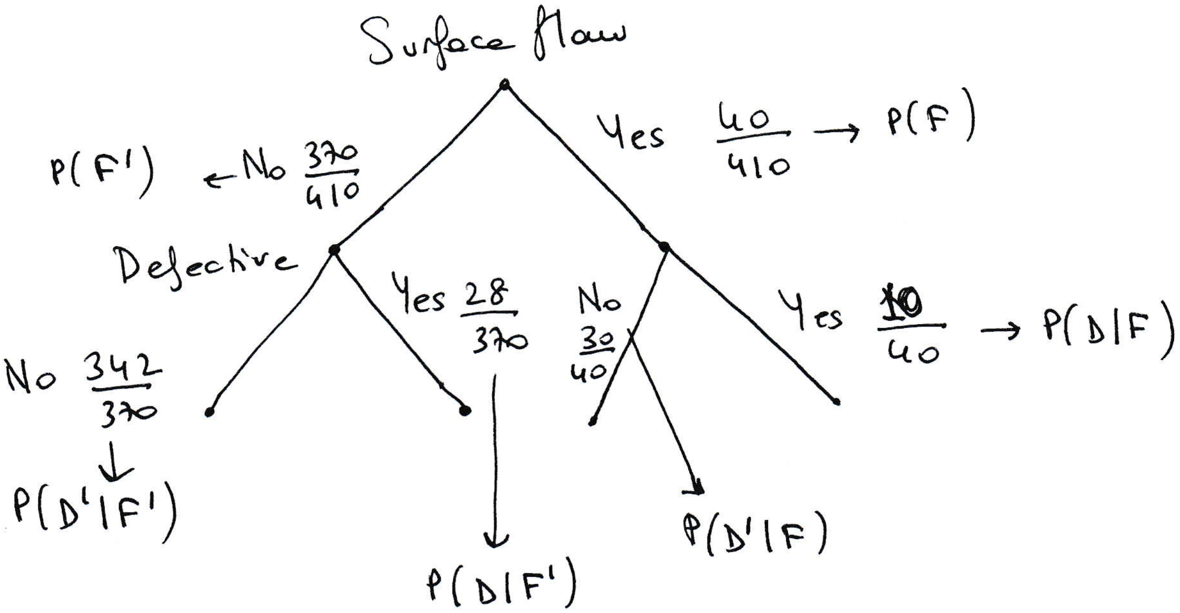
Example: Table Parts Classified

		Surface flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	28	38
	No	30	342	372
	Total	40	370	410

The above Table provides an example of 410 parts classified by surface flaws and as (functionally) defective.

The parts with surface flaws (40 parts) the number defective is 10. Then $P(D/F) = 10/40 = 0.25$

Of the parts without surface flaws (370) the number defective is 28. Then, $P(D/F') = 28/370 = 0.08$



Ex From the table $P(D|F) = ?$

$$P(D|F) = \frac{P(D \cap F)}{P(F)} = \frac{10/410}{40/410} = \frac{10}{40}$$

$P(F|D) = ?$

$$P(F|D) = \frac{P(D \cap F)}{P(D)} = \frac{10/410}{38/410} = \frac{10}{38}$$

$P(D'|F) = ?$

$$P(D'|F) = \frac{P(D' \cap F)}{P(F)} = \frac{30/410}{40/410} = \frac{30}{40}$$

Random Samples from a Batch

To select one item randomly from a batch implies that each item is equally likely. If more than one item is selected, randomly implies that each element of the sample space is equally likely.

If we have a batch of 3 items such that $\{a,b,c\}$, and if two items are selected randomly from this batch <u>without replacement</u>, each of the six outcomes in the ordered sample space has prob. $1/6$. $S_{\text{without}} = \{ab, ac, ba, bc, ca, cb\}$

If the unordered sample space is used, each of the three outcomes in $\{\{a,b\},\{a,c\},\{b,c\}\}$ has prob $1/3$.

(*) What is the conditional probability that b is selected <u>second</u> given that a is selected first?

The ordered sample space is used.

Let E_1 denote the event that the first item selected is a.
Let E_2 " " " " the second " " is b.

$E_1 = \{ab, ac\}$ and $E_2 = \{ab, cb\}$ and $E_1 \cap E_2 = \{ab\}$

$$P(E_2 | E_1) = P(E_1 \cap E_2) / P(E_1) = \frac{1/6}{2/6} = 1/2$$

<u>Example</u>: Suppose a batch contains 10 parts from tool 1 and 40 parts from tool 2. If 2 parts are selected randomly, without replacement, what is the conditional probability that a part from tool 2 is selected second given that a part from tool 1 is selected first?

There are 50 possible parts to select first and 49 to select second. Therefore, the (ordered) sample space has

$50 \times 49 = 2450$ outcomes.

Let E_1 denote the event that the first part is from tool 1.
Let E_2 " " " " the second " is " tool 2.

If a part from tool 1 were selected with the first pick, 49 item would remain, 9 from tool 1 and 40 from tool 2, and they would be equally likely to be picked.

Therefore, the prob. that a part from tool 2 would be selected with the second pick given this first pick is

$$P(E_2 | E_1) = 40/49$$

<u>Example</u>: What is the prob. that a part from tool 1 is selected with the first pick <u>and</u> a part from tool 2 is selected with the second pick? $P(E_1 \cap E_2) = P(E_2 | E_1) . P(E_1)$

$$P(E_1 \cap E_2) = 40/49 \cdot 10/50 = 0.163$$

$P(E_1)$ denotes the prob. that a part from tool 1 is selected with the first pick.

$P(E_2 | E_1)$ denotes the prob. that a part from tool 2 is selected with the second pick given that a part from tool 1 is selected first.

Example : A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected randomly without replacement from the batch. What is the probability that the second part is defective given that the first part is defective?

Let A → denote the event that the first part selected is defective.

Let B → denote the event that the second part selected is defective.

$P(B|A) = ?$ $\qquad P(B|A) = 49/849$ //

Example : Following the above example, if 3 parts are selected at random, what is the prob. that the first 2 are defective and third is not defective?

$$P(ddn) = \frac{50}{850} \cdot \frac{49}{849} \cdot \frac{800}{848} = 0.0032$$