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PROBABILITY: EXERCISES WITH SOLUTIONS

ADVANCED STATISTICAL METHODS
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Whenever exercises are from the textbook

SAMANIEGO, F.J. *Stochastic Modeling and Mathematical Statistics*. Chapman & Hall, Boca Raton, FL.

the numbering is the same as in the book, with the page where it can be found.

Moment generating functions

Exercise 2.8.1 (p. 111)
Determine the expression of the m.g.f. of a uniform discrete random variable X on the integers $\{1, \dots, n\}$, namely X has p.m.f.

$$p_X(x) = \frac{1}{n} \mathbb{1}_{\{1,\dots,n\}}(x)$$

and we recall that $\mathbb{1}_A$ is the indicator function of set A, meaning that $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$.

Solution.
Recall that for any q

$$\sum_{x=0}^{k} q^x = \frac{1 - q^{k+1}}{1 - q}.$$

From this one has

$$m_X(t) = \sum_{x=1}^{n} \mathrm{e}^{tx} \frac{1}{n} = \frac{\mathrm{e}^t}{n} \sum_{x=0}^{n-1} \mathrm{e}^{tx} = \frac{\mathrm{e}^t}{n} \frac{1 - \mathrm{e}^{tn}}{1 - \mathrm{e}^t}$$

where we have used the formula above with $q = \mathrm{e}^t$ and $k = n - 1$.

Exercise 2.8.3 (p. 111)
Let m_X be the moment generating function (m.g.f.) of X. If one defines the function $R = \log m_X$, show that $R'(0) = \mathbb{E}X$ and $R''(0) = \mathrm{Var}(X)$.

Solution.
Recall that if X is such that $\mathbb{E}X < \infty$, $\mathbb{E}X^2 < \infty$ and its m.g.f. exists in a neighbourhood of $t = 0$, one has

$$m'_X(0) = \mathbb{E}X, \qquad m''_X(0) = \mathbb{E}X^2.$$

The function R is also known as *cumulants generating function* and

$$R'(t) = \frac{m'_X(t)}{m_X(t)} \quad \Longrightarrow \quad R'(0) = \frac{m'_X(0)}{m_X(0)} = \mathbb{E}X$$

as $m_X(0) = 1$. Similarly

$$R''(t) = \frac{m''_X(t) m_X(t) - (m'_X(t))^2}{m_X^2(t)}$$

which implies

$$R''(0) = \frac{m''_X(0) m_X(0) - (m'_X(0))^2}{m_X^2(0)} = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathrm{Var}(X).$$

Exercise 2.8.5 (p. 111)
Determine the distribution of the random variable X whose m.g.f. is

$$m_X(t) = 0.3\,\mathrm{e}^t + 0.4\,\mathrm{e}^{3t} + 0.3\,\mathrm{e}^{5t}$$

Solution.
It is apparent that X is a discrete random variable taking values in $\{1,3,5\}$. Its p.m.f. is thus given by

$$p_X(x) = 0.3\,\mathbb{1}_{\{1\}}(x) + 0.4\,\mathbb{1}_{\{3\}}(x) + 0.3\,\mathbb{1}_{\{5\}}(x)$$

or, alternatively,

$$p_X(x) = \begin{cases} 0.3 & \text{if } x = 1 \\ 0.4 & \text{if } x = 3 \\ 0.3 & \text{if } x = 5 \end{cases}$$

Exercise 2.9.62. (p. 119)
Let $X \sim \mathrm{Po}(\lambda)$ and $f : \mathbb{R} \to \mathbb{R}$ any function such that $\mathbb{E}|f(X)| < \infty$. Show that

$$\mathbb{E}Xf(X-1) = \lambda\,\mathbb{E}f(X).$$

Solution.
Note that, by definition, one has

$$\begin{aligned}\mathbb{E}Xf(X-1) &= \sum_{x=0}^{\infty} x\, f(x-1)\, \frac{\lambda^x\, \mathrm{e}^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x\, f(x-1)\, \frac{\lambda^x\, \mathrm{e}^{-\lambda}}{x!} \\ &= \sum_{y=0}^{\infty} (y+1)\, f(y)\, \frac{\lambda^{y+1}\, \mathrm{e}^{-\lambda}}{(y+1)!} \\ &= \sum_{y=0}^{\infty} f(y)\, \frac{\lambda^{y+1}\, \mathrm{e}^{-\lambda}}{y!} = \lambda \sum_{y=0}^{\infty} f(y)\, \frac{\lambda^{y}\, \mathrm{e}^{-\lambda}}{y!} \\ &= \lambda\, \mathbb{E}f(X).\end{aligned}$$

For example, if $f(x) = x$, since $\mathbb{E}X = \lambda$ one would have

$$\mathbb{E}X(X-1) = \lambda^2$$

which implies that $\mathrm{Var}(X) = \mathbb{E}X^2 - \lambda^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Exercise 2.9.67. (p. 120)
Let X be a discrete random variable taking values on the non–negative integers. The function

$$t \mapsto g_X(t) = \mathbb{E}t^X = \sum_{x=0}^{\infty} t^x\, p_X(x)$$

is termed probability generating function. Show that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} g_X(t)\Big|_{t=0} = n!\, p_X(n) \qquad n = 0, 1, 2, \ldots$$

and determine g_X for the following: (a) $X \sim \mathrm{Bin}(n, p)$; (b) $X \sim \mathrm{G}(p)$; (c) $X \sim \mathrm{Po}(\lambda)$.

Solution.
Note that

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}t^n} g_X(t)\Big|_{t=0} &= \frac{\mathrm{d}^n}{\mathrm{d}t^n} \sum_{x=0}^{\infty} t^x\, p_X(x)\Big|_{t=0} = \sum_{x=0}^{\infty} \frac{\mathrm{d}^n}{\mathrm{d}t^n} t^x\, p_X(x)\Big|_{t=0} \\ &= \sum_{x=n}^{\infty} (x-n+1)(x-n+2)\, \cdots\, x\, t^{x-n}\, p_X(x)\, \Big|_{t=0} \\ &= n!\, p_X(n).\end{aligned}$$

(a) If $X \sim \mathrm{Bin}(n, p)$, then

$$g_X(t) = \sum_{x=0}^{n} t^x \binom{n}{x} p^x (1-p)^{n-x} = (pt + 1 - p)^n$$

(b) If $X \sim \mathrm{G}(p)$, for any $t < 1/(1-p)$

$$g_X(t) = \sum_{x=0}^{\infty} t^x \, p\,(1-p)^x = \frac{p}{1-t(1-p)}$$

(c) If $X \sim \mathrm{Po}(\lambda)$, then

$$g_X(t) = \sum_{x=0}^{\infty} t^x \, \frac{\lambda^x \, \mathrm{e}^{-\lambda}}{x!} = \mathrm{e}^{\lambda(t-1)}$$

Exercise 2.9.71 (p. 120)
A random variable X is said to have a *logarithmic series distribution* with parameter $\theta \in (0,1)$ if

$$p_X(x) = \frac{\theta^x}{-x\,\log(1-\theta)} \, \mathbb{1}_{\{1,2,\ldots\}}(x).$$

In symbols $X \sim \mathrm{LS}(\theta)$.

(a) Determine $m_X(t)$, if it exists.

(b) Determine expected value and variance of X.

Solution (a).
Since $\theta \in (0,1)$, one has that $-\log(1-\theta) = |\log(1-\theta)| > 0$ and

$$\mathbb{E}\mathrm{e}^{tX} = \frac{1}{|\log(1-\theta)|} \sum_{x=1}^{\infty} \frac{(\mathrm{e}^t\,\theta)^x}{x} < \infty \quad \text{if and only if} \quad t < -\log(\theta).$$

Hence, the moment generating function exists. Recall that for any q such that $|q| < 1$ one has

$$\sum_{x=1}^{\infty} \frac{q^x}{x} = -\log(1-q),$$

which implies that, for any $t < -\log(\theta)$,

$$m_X(t) = \frac{1}{|\log(1-\theta)|} \sum_{x=1}^{\infty} \frac{(\mathrm{e}^t\,\theta)^x}{x} = \frac{\log(1-\theta\,\mathrm{e}^t)}{\log(1-\theta)}$$

Solution (b).
As for the expected value

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} m_X(t)\Big|_{t=0} &= \frac{\theta}{|\log(1-\theta)|} \, \frac{\mathrm{e}^t}{(1-\theta\,\mathrm{e}^t)}\Big|_{t=0} \\ &= \frac{\theta}{(1-\theta)\,|\log(1-\theta)|} = \mathbb{E}X. \end{aligned}$$

As for the variance, first note that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} m_X(t)\Big|_{t=0} = \frac{\theta}{|\log(1-\theta)|}\,\frac{\mathrm{e}^t}{(1-\theta\,\mathrm{e}^t)^2}\Big|_{t=0}$$

$$= \frac{\theta}{(1-\theta)^2\,|\log(1-\theta)|} = \mathbb{E}X^2$$

and, then,

$$\begin{aligned}\operatorname{Var}(X) &= \mathbb{E}X^2 - (\mathbb{E}X)^2 \\ &= \frac{\theta}{(1-\theta)^2\,|\log(1-\theta)|} - \frac{\theta^2}{(1-\theta)^2\,|\log(1-\theta)|^2} \\ &= \frac{\theta}{(1-\theta)^2\,|\log(1-\theta)|^2}\Big\{|\log(1-\theta)| - \theta\Big\}\end{aligned}$$

Exercise 2.9.73 (p. 121)
A fair coin is tossed repeatedly and independently until the pair HT occurs. Let X be the number of trials required to obtain HT.

(a) Show that $p_X(x) = (x-1)\,2^{-x}\,\mathbb{1}_{\{2,3,\ldots,\}}(x)$.

(b) Determine the moment generating function of X, if it exists.

Solution (a).
Each sequence of outcomes that is obtained by tossing x times a fair coin has probability $1/2^x$. So, we have to count how many of them have the pair HT appearing for the first time as the final pair of outcomes of the sequence of x trials. Hence, $X = x$, for $x \in \{2,3,\ldots\}$ if and only if any of the following sequence of outcomes occur: the sequence starting with H, namely (HHH ... HT), and the sequences starting with T, namely

$$\underbrace{\text{(THHHH...HT), (TTHHH...HT), (TTTHH...HT)}, \ldots, \text{(TTT...TTHT)}}_{x-2 \text{ sequences}}$$

Hence, there are overall $(x-1)$ possible outcomes that yield $\{X = x\}$. This implies that

$$p_X(x) = (x-1)\,\frac{1}{2^x}\,\mathbb{1}_{\{2,3,\ldots\}}(x)$$

Solution (b).
Since X is discrete

$$m_X(t) = \mathbb{E}\,\mathrm{e}^{-tX} = \sum_{x=2}^{\infty} \mathrm{e}^{tx}\,(x-1)\,\frac{1}{2^x} = \sum_{x=2}^{\infty}(x-1)\left(\frac{\mathrm{e}^t}{2}\right)^x$$

and the series converges if and only if $(e^t/2) < 1$, namely if and only if $t < \log 2$, thus including a neighbourhood of the origin. The moment generating function exists and, for any $t < \log 2$, it equals

$$
\begin{aligned}
m_X(t) &= \sum_{x=2}^{\infty}(x-1)\left(\frac{e^t}{2}\right)^x = \sum_{y=1}^{\infty} y\left(\frac{e^t}{2}\right)^{y+1} = \left(\frac{e^t}{2}\right)^2 \sum_{y=1}^{\infty} y\left(\frac{e^t}{2}\right)^{y-1} \\
&= \left(\frac{e^t}{2}\right)^2 \sum_{y=1}^{\infty} \frac{d}{d\lambda}\lambda^y \Big|_{\lambda=e^t/2} = \left(\frac{e^t}{2}\right)^2 \frac{d}{d\lambda} \sum_{y=1}^{\infty} \lambda^y \Big|_{\lambda=e^t/2} \\
&= \frac{e^{2t}}{4} \frac{d}{d\lambda} \frac{\lambda}{1-\lambda} \Big|_{\lambda=e^t/2} = \frac{e^{2t}}{4} \frac{4}{(2-e^t)^2} \\
&= \frac{e^{2t}}{(2-e^t)^2}
\end{aligned}
$$

Exercise 2.9.74 (p. 121)
Let $X \sim \mathrm{NB}(r; p)$ and define $Y = X - r$. Determine the m.g.f. of Y investigate its behaviour as $r \nearrow \infty$ and $p \nearrow 1$ in such a way that $r(1-p) \to \lambda > 0$.

Solution.
Since

$$p_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \, \mathbb{1}_{\{r,r+1,\ldots\}}(x)$$

then

$$p_Y(y) = \binom{r+y-1}{r-1} p^r (1-p)^y \, \mathbb{1}_{\{0,1,2,\ldots\}}(x)$$

and Y is the number of failures one experiences before gaining the rth success in a sequence of independent and identically distributed Bernoulli trials, each with probability of success p. In order to determine that m.g.f. of Y we will make use of the following series representation: *for any $a > 0$ and z such that $|z| < 1$ one has*

$$\sum_{i=0}^{\infty} \frac{a(a+1)\cdots(a+i-1)}{i!} z^i = \frac{1}{(1-z)^a}$$

where we further agree that $a(a+1)\cdots(a+i-1) = 1$ if $i = 0$. Using this, one has

$$
\begin{aligned}
m_Y(t) &= \sum_{y=0}^{\infty} e^{ty} p_Y(Y) = \sum_{y=0}^{\infty} e^{ty} \binom{r+y-1}{r-1} p^r (1-p)^y \\
&= p^r \sum_{y=0}^{\infty} \frac{r(r+1)\cdots(r+y-1)}{y!} \{e^t(1-p)\}^y \\
&= \frac{p^r}{\{1-e^t(1-p)\}^r}
\end{aligned}
$$

if $(1-p)e^t < 1$, namely if $t < -\log(1-p)$. If $r = 1$, then m_Y is the m.g.f. of a random variable with a $G(p)$ distribution.
As for the behaviour of Y as $r \nearrow \infty$ and $p \nearrow 1$ in such a way that $r(1-p) \to \lambda$. This assumption on asymptotics implies that with r and p large enough $(1-p) \approx (\lambda/r)$. Hence

$$p^r = \{1-(1-p)\}^r \approx \left\{1 - \frac{\lambda}{r}\right\}^r \to e^{-\lambda}$$

and

$$\left\{1 - e^t(1-p)\right\}^r \approx \left\{1 - e^t \frac{\lambda}{r}\right\}^r \to e^{-\lambda e^t}.$$

Combining these two limiting results, one obtains

$$m_Y(t) = \frac{p^r}{\{1-e^t(1-p)\}^r} \to \frac{e^{-\lambda}}{e^{-e^t\lambda}} = e^{\lambda(e^t-1)}$$

which is the m.g.f. of a Poisson random variable with parameter $\lambda > 0$.
Hence, as $r \nearrow \infty$ and $p \nearrow 1$ such that $r(1-p) \to \lambda$, then

$$Y \xrightarrow{D} X \sim \text{Po}(\lambda)$$

Exercise (midterm exam in 2018).
Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $X_n \sim \text{Po}(n)$, for any $n \geq 1$, namely

$$\mathbb{P}[X_n = x] = \frac{n^x e^{-n}}{x!} \mathbb{1}_{\{0,1,\ldots\}}(x) \qquad \forall n \geq 1.$$

(a) Determine the moment generating function of

$$Y_n = \frac{X_n - n}{\sqrt{n}}$$

(b) State the *Continuity Theorem* to highlight the connection between convergence in distribution and moment generating functions.

(c) Use the theorem in (b) to show that

$$Y_n \xrightarrow{D} Z$$

where $Z \sim N(0,1)$.

Solution.
Since $m_{X_n}(t) = \exp\{n(e^t - 1)\}$, then

$$m_{Y_n}(t) = e^{-t\sqrt{n}} m_{X_n}(t/\sqrt{n}) = e^{-t\sqrt{n}} e^{n(e^{t/\sqrt{n}}-1)}$$

and this answers (a).
As for (b), just need to recall the

Theorem. Let $(X_n)_{n\geq 1}$ be a sequence of rv's having mgf's, i.e. there exists $t_0 > 0$ such that for any $n \geq 1$

$$m_n(t) = \mathbb{E}e^{tX_n} < \infty \qquad \forall t \in (-t_0, t_0).$$

(i) If $X_n \xrightarrow{\mathrm{D}} X$, then $m_n(t) \to m(t)$ as $n \to \infty$ where

$$m(t) = \mathbb{E}e^{tX} < \infty \qquad \forall t \in (-t_0, t_0)$$

(ii) If $m_n(t) \to m(t)$, as $n \to \infty$, for any t in a neighbourhood of 0 and $m(t)$ is the mgf of some random variable X, then $X_n \xrightarrow{\mathrm{D}} X$.

In (c), one applies the theorem in (b). Indeed, by Taylor's expansion, $e^{t/\sqrt{n}} - 1 = (t/\sqrt{n}) + (t^2/(2n)) + o(1/n)$ as $n \to \infty$. Hence,

$$m_{Y_n}(t) = e^{-t\sqrt{n}+t\sqrt{n}+t^2/2+o(1)} \to e^{\frac{t^2}{2}}$$

as $n \to \infty$, which is the moment generating function of the $\mathrm{N}(0,1)$ random variable.

Exercise 5.5.30. (p. 251)
Let $(X_n)_{n\geq 1}$ be a sequence of iid random variables from a $\mathrm{E}(1)$ distribution. Use a CLT type of argument onvlving $(X_n)_{n\geq 1}$ to prove the well–known Stirling approximation of the Gamma function

$$\Gamma(n) = (n-1)! \simeq \sqrt{2\pi}\, n^{n-\frac{1}{2}}\, e^{-n}$$

for n large.

Solution.
The pdf of each r.v. of the sequence is

$$f_{X_1}(x) = e^{-x}\, \mathbb{1}_{(0,+\infty)}(x).$$

Moreover, using the mgf method one can show that $S_n = X_1 + \cdots + X_n \sim \mathrm{Ga}(n, 1)$, i.e.

$$f_{S_n}(x) = \frac{1}{\Gamma(n)}\, x^{n-1}\, e^{-x}\, \mathbb{1}_{[0,+\infty)}(x)$$

The pdf of $\bar{X}_n = S_n/n$ can be easily deduced through the change of variable technique with $g(x) = x/n$, so that

$$g^{-1}(y) = ny, \quad \text{Jacobian } = \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) = n.$$

Hence, the density of $\bar{X}_n$ is

$$f_{\bar{X}_n}(y) = f_{S_n}(ny)n = \frac{n^n}{\Gamma(n)}\, y^{n-1}\, e^{-ny}\, \mathbb{1}_{[0,+\infty)}(y)$$

namely $\bar{X}_n \sim \mathrm{Ga}(n,n)$. Since

$$\mu = \mathbb{E}X_1 = 1, \qquad \sigma^2 = \mathrm{Var}(X_1) = 1$$

for any $z \in \mathbb{R}$ and for n large enough the CLT yields the following approximation

$$\begin{aligned}\mathbb{P}\Big[\sqrt{n}(\bar{X}_n - 1) \le z\Big] &= \mathbb{P}\Big[\bar{X}_n \le 1 + \frac{z}{\sqrt{n}}\Big] \\ &= \int_0^{1+\frac{z}{\sqrt{n}}} \frac{n^n}{\Gamma(n)}\, y^{n-1}\, \mathrm{e}^{-ny}\, \mathrm{d}y \\ &\simeq \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{s^2}{2}}\, \mathrm{d}\end{aligned}$$

If one differentiates both sides of the "approximate equality", with respect to z, the following is obtained

$$\frac{1}{\sqrt{n}}\, \frac{n^n}{\Gamma(n)} \left(1 + \frac{z}{\sqrt{n}}\right)^{n-1} \mathrm{e}^{-n\left(1+\frac{z}{\sqrt{n}}\right)} \simeq \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{z^2}{2}}$$

If one considers the previous "approximate equality" at $z = 0$

$$\frac{n^{n-\frac{1}{2}}}{\Gamma(n)}\, \mathrm{e}^{-n} \simeq \frac{1}{\sqrt{2\pi}}$$

and the Stirling formula follows!

Random variables, random vectors and their transformations

> **Exercise**
> A fair coin is tossed repeatedly. Let X be random variable denoting the number of tosses that are needed for "Head" (H) to appear for the first time, while Y is the same random number for "Tail" (T). Determine the joint distribution of X and Y and their respective marginal distributions.

Solution.
It is apparent that (X,Y) is a discrete vector and we, then, have to determine its probability mass function (pmf). To this end, note that $X = 1$ means that the first outcome is H and $Y = 1$ is eqivalent to saying that the first time we toss the coin we observe T. This implies that

$$p_{X,Y}(x,y) > 0 \quad \text{if and only if } (x,y) = (1,n) \ \text{ or } \ (x,y) = (m,1)$$

for some $n \geq 1$ and $m \geq 1$. Moreover,

$$p_{X,Y}(1,n) = \frac{1}{2^n} \qquad p_{X,Y}(m,1) = \frac{1}{2^m}.$$

As for the marginal distributions, they are the same since the coin is fair and, with no loss of generality, we can focus on p_X. In this case

$$p_X(1) = \sum_{n\geq 2} \frac{1}{2^n} = \frac{1}{2}, \qquad p_X(m) = \frac{1}{2^m} \quad m = 2, 3, \dots.$$

Hence,

$$p_X(n) = p_Y(n) = \frac{1}{2^n} \mathbb{1}_{\{1,2,\dots\}}(n)$$

> **Exercise**
> Let (X, Y) be a discrete random vector with pmf
>
> $$p_{X,Y}(x,y) = \begin{cases} \frac{1}{2^{y+1}} & \text{if } x = 1,2,\dots \ \ \& \ \ y = x, x+1, \dots \\ 0 & \text{otherwise} \end{cases}$$
>
> Determine the marginal distributions of X and Y.

Solution.
In order to determine the marginal distribution of X at a given integer x, we need to sum over all $y \in \{x, x+1, \dots\}$ as follows

$$p_X(x) = \mathbb{1}_{\{1,2,\dots\}}(x) \sum_{y=x}^{\infty} \frac{1}{2^{y+1}} = \frac{1}{2^x} \mathbb{1}_{\{1,2,\dots\}}(x)$$

namely $X \sim \text{Geom}(1/2)$. As for the marginal distribution of Y, at a given integer y, we need to sum over the values of $x\{1, \dots, y\}$ as follows

$$p_Y(y) = \mathbb{1}_{\{1,2,\dots\}}(y) \sum_{x=1}^{y} \frac{1}{2^{y+1}} = \frac{y}{2^{y+1}} \mathbb{1}_{\{1,2,\dots\}}(y)$$

> **Exercise**
> Let $C = \{(x,y) : \ x^2 + y^2 \leq 1\}$ be the circle of radius 1 and
>
> $$f_{X,Y}(x,y) = k \mathbb{1}_C(x,y).$$
>
> Determine
>
> (a) The constant k that makes $f_{X,Y}$ a pdf
>
> (b) The marginal pdf's f_X and f_Y. Are X and Y independent?

Solution.
In order to determine k, we need to impose

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1.$$

In the specific case

$$\begin{aligned} 1 &= \int_C k\,\mathrm{d}x\,\mathrm{d}y = k \int_C \mathrm{d}x\,\mathrm{d}y \\ &= k \times (\text{Area of the circle of radius } 1) = k\pi \end{aligned}$$

and, hence, $k = 1/\pi$.
As for (b), note that

$$\begin{aligned} f_X(x) &= \mathbb{1}_{[-1,1]}(x)\,\frac{1}{\pi} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y \\ &= \frac{2}{\pi}\sqrt{1-x^2}\,\mathbb{1}_{[-1,1]}(x). \end{aligned}$$

Due to the symmetry of $f_{X,Y}$, i.e. $f_{X,Y}(x,y) = f_{X,Y}(y,x)$, the marginal pdf of Y, f_Y, equal f_X. Finally, as expected, $f_{X,Y} \neq f_X f_Y$ so that X and Y are not independent. this can be further seen by taking a look at the conditional densities. Indeed,

- For $y \in [-1,1]$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{\frac{1}{\pi}\,\mathbb{1}_{[-\sqrt{1-y^2},\sqrt{1-y^2}]}(x,y)}{\frac{2}{\pi}\sqrt{1-y^2}} \\ &= \frac{1}{2\sqrt{1-y^2}}\,\mathbb{1}_{[-\sqrt{1-y^2},\sqrt{1-y^2}]}(x) \neq f_X(x) \end{aligned}$$

  and, hence, $X|Y = y \sim \mathrm{U}(-\sqrt{1-y^2}, \sqrt{1-y^2})$.

- Similarly, symmetry of $f_{X,Y}$ yields, for any $x \in [-1,1]$,

$$f_{Y|X}(y|x) = \frac{1}{2\sqrt{1-x^2}}\,\mathbb{1}_{[-\sqrt{1-x^2},\sqrt{1-x^2}]}(y) \neq f_Y(y)$$

  and $Y|X = x \sim \mathrm{U}(-\sqrt{1-x^2}, \sqrt{1-x^2})$.

Exercise 4.8.3 (page 217).
Let (X,Y) have density

$$f_{X,Y}(x,y) = kx^2(1-y)\,\mathbb{1}_{(0,1)}(x)\,\mathbb{1}_{(0,1)}(x).$$

(a) Determine the value of k that makes $f_{X,Y}$ a probability density function.

(b) Evaluate the expected value of $W = Y/X$.

(c) Evaluate $\mathbb{P}[X + Y \leq 1]$.

Solution.
The value of k in (a) follows from

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$$

and in our specific case this becomes

$$1 = k \int_0^1 \int_0^1 x^2(1-y)\,\mathrm{d}x\,\mathrm{d}y = k\,\frac{1}{3}\,\frac{1}{2},$$

from which one deduces that $k = 6$. Moreover, it is easy to note that X and Y are independent.
As for (b), note that

$$\begin{aligned}
\mathbb{E}W = \mathbb{E}\frac{Y}{X} &= \int_0^1 \int_0^1 \frac{y}{x}\,6x^2(1-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 6 \int_0^1 x\,\mathrm{d}x \int_0^1 y(1-y)\,\mathrm{d}y = 6\,\frac{1}{2}\,\frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} \\
&= \frac{1}{2}
\end{aligned}$$

Finally, the evaluation of the probability in (c) is as follows

$$\begin{aligned}
\mathbb{P}[X + Y \leq 1] &= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\,6\,x^2\,(1-y) \\
&= 6 \int_0^1 x^2 \left(\int_0^{1-x} (1-y)\,\mathrm{d}y \right) \mathrm{d}x \\
&= 6 \int_0^1 x^2 \left(\frac{1}{2} - \frac{x^2}{2} \right) \mathrm{d}x = 3 \int_0^1 (x^2 - x^4)\,\mathrm{d}x \\
&= 3 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2}{5}
\end{aligned}$$

Exercise
Show that

$$F(x,y) = \begin{cases} 0 & \text{if } x + y < 1 \\ 1 & \text{if } x + y \geq 1 \end{cases}$$

is not a bivariate c.d.f..

Solution.
This is shown if we are able to find points (x_1, y_1) and (x_2, y_2), with $x_1 \leq x_2$ and $y_1 \leq y_2$ such that

$$F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1) < 0.$$

For example, if $(x_1, y_1) = (1/4, 1/4)$ and $(x_2, y_2) = (1, 1)$, one has $F(x_2, y_2) = F(x_1, y_2) = F(x_2, y_1) = 1$ and $F(x_1, y_1) = 0$. Hence,

$$F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1) = -1 < 0$$

and F cannot be the c.d.f. of any random vector (X, Y).

Exercise
Suppose (X, Y) is a random vector such that

$$f_{X,Y}(x, y) = \begin{cases} \mathrm{e}^{-y} & \text{if } 0 \leq x \leq y < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Determine the marginal distributions of X and Y.

Solution.
The marginals f_X and f_y are easily determined from:

$$\begin{aligned} f_X(x) &= \mathbb{1}_{(0,\infty)}(x) \int_x^\infty \mathrm{e}^{-y}\,\mathrm{d}y = \mathrm{e}^{-x}\,\mathbb{1}_{(0,\infty)}(x) \\ f_Y(y) &= \mathbb{1}_{(0,\infty)}(y) \int_0^y \mathrm{e}^{-y}\,\mathrm{d}x = y\,\mathrm{e}^{-y}\,\mathbb{1}_{(0,\infty)}(y). \end{aligned}$$

Hence $X \sim \mathrm{E}(1)$ and $Y \sim \mathrm{Gamma}(2, 1)$.

Exercise.
Suppose (X, Y) is such that

$$f_{X,Y}(x, y) = \begin{cases} Kx^{\alpha-1}(y-x)^{\beta-1}\,\mathrm{e}^{-y} & \text{if } 0 < x < y < +\infty \\ 0 & \text{otherwise} \end{cases}$$

Identify the value of K that makes $f_{X,Y}$ a density function and determine the marginal densities f_X and f_Y.

Solution.
Recall that for any $\alpha, \beta > 0$

$$\int_0^{+\infty} x^{\alpha-1}\,\mathrm{e}^{-\beta x}\,\mathrm{d}x = \frac{\Gamma(\alpha)}{\beta^\alpha} \qquad \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

The constant K must be such that $\int f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$. In other terms

$$\int_0^\infty \mathrm{d}y \int_0^y \mathrm{d}x\, x^{\alpha-1}(y-x)^{\beta-1}\,\mathrm{e}^{-y} = \frac{1}{K}.$$

The simple change of variable

$$w = \frac{x}{y} \quad\Longrightarrow\quad x = wy \quad\Longrightarrow\quad \mathrm{d}x = y\mathrm{d}w$$

shows that the previous is equivalent to

$$\int_0^\infty y^{\alpha+\beta-1}\,\mathrm{e}^{-y}\,\mathrm{d}y \int_0^1 w^{\alpha-1}(1-w)^{\beta-1}\,\mathrm{d}w = \frac{1}{K}$$

and, then

$$K = \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}.$$

As for the marginal distributions, one has

$$\begin{aligned} f_X(x) &= \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\,x^{\alpha-1}\int_x^\infty (y-x)^{\beta-1}\,\mathrm{e}^{-y}\,\mathrm{d}y \\ &= \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\,x^{\alpha-1}\,\mathrm{e}^{-x}\int_0^\infty y^{\beta-1}\,\mathrm{e}^{-y}\,\mathrm{d}y \\ &= \frac{1}{\Gamma(\alpha)}\,x^{\alpha-1}\,\mathrm{e}^{-x} \end{aligned}$$

and

$$\begin{aligned} f_Y(y) &= \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\,\mathrm{e}^{-y}\int_0^y x^{\alpha-1}\,(y-x)^{\beta-1}\,\mathrm{d}x \\ &= \frac{1}{\Gamma(\alpha+\beta)}\,y^{\alpha+\beta-1}\,\mathrm{e}^{-y} \end{aligned}$$

This amounts to $X \sim \mathrm{Gamma}(\alpha,1)$ and $Y \sim \mathrm{Gamma}(\alpha+\beta,1)$.

Exercise

Let (X,Y) have a uniform distribution over the triangle $x \geq 0$, $y \geq 0$ and $x+y \leq 2$. Determine

(a) the probability density function of (X,Y)

(b) the probability density function of X

(c) $\mathbb{E}[Y|X=x]$

Solution.

If we denote as

$$\Delta_2 = \{(x,y):\ \ x \geq 0,\ y \geq 0,\ x+y \leq 2\}$$

the triangle, one has that the probabioity ensiuty function of (X, Y) must be a positive constant over Δ_2 and equal to o outside Δ_2. Since we are supposing uniformity, then

$$f_{X,Y}(x,y) = c\,\mathbb{1}_{\Delta_2}(x,y)$$

where $c > 0$ is some constant such that

$$\begin{aligned} 1 &= \int_{\Delta_2} f_{X,Y}(x,y)\,\mathrm{d}x\mathrm{d}y = c\int_0^2\int_0^{2-x}\mathrm{d}y\,\mathrm{d}x = c\int_0^2 (2-x)\,\mathrm{d}x \\ &= 2c \end{aligned}$$

which implies $c = 1/2$. Hence, the answer to (a) is

$$f_{X,Y}(x,y) = \frac{1}{2}\,\mathbb{1}_{\Delta_2}(x,y).$$

In order to determine the marginal density f_X of X one may proceed as follows

$$f_X(x) = \mathbb{1}_{[0,2]}(x)\int_0^{2-x}\frac{1}{2}\,\mathrm{d}y = \frac{2-x}{2}\,\mathbb{1}_{[0,2]}(x)$$

and this answers (b). Finally, as for (c), we first determine the conditional density function of $Y|X = x$, for any $x \in [0,2]$

$$f_{Y|X}(y|x) = \frac{\frac{1}{2}}{\frac{2-x}{2}}\,\mathbb{1}_{[0,2-x]}(y) = \frac{1}{2-x}\,\mathbb{1}_{[0,2-x]}(y).$$

This means that $Y|X = x \sim \mathrm{Unif}([0,2-x])$ for any $x \in [0,2]$. Hence,

$$\mathbb{E}[Y|X = x] = \int_0^{2-x} y\,\frac{1}{2-x}\,\mathrm{d}y = \frac{2-x}{2}$$

for any $x \in [0,2]$.

Exercise.
The joint density of the vector (X, Y, Z) is

$$f_{X,Y,Z}(x,y,z) = 8xyz\,\mathbb{1}_{(0,1)}(x)\,\mathbb{1}_{(0,1)}(y)\,\mathbb{1}_{(0,1)}(z).$$

Evaluate $\mathbb{P}[X < Y < Z]$.

Solution.
One way of solving this is through integration, namely

$$\begin{aligned} \mathbb{P}[X < Y < Z] &= \int_0^1 \mathrm{d}z\int_0^z \mathrm{d}y\int_0^y \mathrm{d}x\,(8xyz) \\ &= 4\int_0^1 z\left(\int_0^z y^3\,\mathrm{d}y\right)\mathrm{d}z = \int_0^1 z^5\,\mathrm{d}z \\ &= \frac{1}{6} \end{aligned}$$

This could have been evaluated without using the above integration. Indeed, it is enough to notice that $f_{X,Y,Z}(x,y,z)$ is invariant with respect to permutations of its arguments, namely

$$f_{X,Y,Z}(x,y,z) = f_{X,Y,Z}(y,x,z) = f_{X,Y,Z}(y,z,x) = \cdots$$

There are overall $3! = 6$ permutations of (x,y,z) and implies

$$\begin{aligned}\mathbb{P}[X < Y < Z] &= \mathbb{P}[Y < X < Z] = \mathbb{P}[Y < Z < X] = \\ &= \mathbb{P}[Z < Y < X] = \mathbb{P}[Z < X < Y] = \mathbb{P}[X < Z < Y].\end{aligned}$$

Since the sum of those probabilities is 1, then each must have probability $1/6$. Hence $\mathbb{P}[X < Y < Z] = 1/6$.

Exercise 3.7.53 (p. 166)
Let $X \sim \text{Pareto}(\alpha, 1)$. Determine

(a) the moment of order k of X, i.e. $\mathbb{E}X^k$, for $k < \alpha$

(b) the probability distribution of $Y = \log X$.

Solution.
Recall that, if $X \sim \text{Pareto}(\alpha, 1)$, then

$$f_X(x) = \frac{\alpha}{x^{\alpha+1}} \mathbb{1}_{(1,+\infty)}(x).$$

As for (a), note that

$$\mathbb{E}X^k = \int_{-\infty}^{+\infty} x^k f_X(x)\,\mathrm{d}x = \int_1^{\infty} x^k \frac{\alpha}{x^{\alpha+1}}\,\mathrm{d}x = \alpha \int_1^{\infty} \frac{1}{x^{\alpha-k+1}}\,\mathrm{d}x.$$

The integral above is finite if $\alpha - k > 0$, whereas it diverges if $\alpha - k \leq 0$. Hence

$$\mathbb{E}X^k \begin{cases} = +\infty & \text{if } k \geq \alpha \\ < +\infty & \text{if } k < \alpha \end{cases}$$

For any $k < \alpha$ one has

$$\mathbb{E}X^k = \alpha \frac{1}{\alpha - k} \frac{1}{x^{\alpha-k}} \Big|_1^{+\infty} = \frac{\alpha}{\alpha - k}$$

As for (b), note that $g(x) = \log x$ is invertible on $\mathbb{R}^+$ and

$$g^{-1}(y) = \mathrm{e}^y \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) = \mathrm{e}^y.$$

Hence, the change of variable technique yields

$$f_Y(y) = f_X(\mathrm{e}^y)\,\mathrm{e}^y$$

and $f_X(\mathrm{e}^y) > 0$ whenever $\mathrm{e}^y > 1$, namely $y > 0$, while $f_X(\mathrm{e}^y) = 0$ if $y \leq 0$. We can, then, conclude that

$$f_Y(y) = \mathbb{1}_{(0,+\infty)}(y)\, \frac{\alpha}{(\mathrm{e}^y)^{\alpha+1}}\, \mathrm{e}^y = \alpha\, \mathrm{e}^{-\alpha y}\, \mathbb{1}_{(0,+\infty)}(y)$$

so that $Y \sim \mathrm{E}(\alpha)$.

Exercise
Suppose that v organisms in a water basin at time $t = 0$ grow at an exponential random rate X, namely at time t the number of organisms will be $N_t = v \exp(Xt)$. If X has density function

$$f_X(x) = 3(1 - x^2)\, \mathbb{1}_{(0,1)}(x)$$

determine the density of N_t if $v = 10$ and $t = 5$.

This example is taken from the book DeGroot & Schervish (2012). *Probability & Statistics*. Pearson.

Solution.
Take $g(x) = v\, \mathrm{e}^{xt} = 10\, \mathrm{e}^{5x}$ and note that it is monotone on $(0, 1)$. Moreover, $g^{-1}(y) = 5^{-1} \log(y/10)$ which implies

$$\frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} = \frac{1}{5y}$$

Hence

$$\begin{aligned} f_Y(y) &= \frac{1}{5y}\, f_X\left(\frac{1}{5} \log \frac{y}{10}\right) = \frac{1}{5y}\, 3 \left(1 - \frac{1}{25} \log^2 \frac{y}{10}\right) \mathbb{1}_{(0,1)}\left(\frac{1}{5} \log \frac{y}{10}\right) \\ &= \frac{1}{5y}\, 3 \left(1 - \frac{1}{25} \log^2 \frac{y}{10}\right) \mathbb{1}_{(10,10\,\mathrm{e}^5)}(y). \end{aligned}$$

Exercise
Suppose $X \sim \mathrm{N}(0, 1)$ and $Y = (\log X)\, \mathbb{1}_{(0,+\infty)}(X) + 4\, \mathbb{1}_{(-\infty,0]}(X)$. Determine the cdf F_Y of Y.

This exercise is a bit more challenging.

Solution.
In this case, one cannot apply the change of variable technique because the function $g(x) = (\log x)\, \mathbb{1}_{(0,+\infty)}(x) + 4\, \mathbb{1}_{(-\infty,0]}(x)$ is not invertible. We proceed determining directly the cumulative distribution function

and note that for $y < 0$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\{\log X \leq y\} \cap \{X > 0\}) = \mathbb{P}(0 \leq X \leq \mathrm{e}^y)$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^{\mathrm{e}^y} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

On the other hand, if $y \in [0, 4)$, then

$$F_Y(y) = \mathbb{P}(0 < X \leq 1) + \mathbb{P}(\{\log X \leq y\} \cap \{X > 1\})$$
$$= \Phi(1) - \frac{1}{2} + \mathbb{P}(1 < X \leq \mathrm{e}^y)$$
$$= \Phi(1) - \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_1^{\mathrm{e}^y} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

Finally, for $y \geq 4$,

$$F_Y(y) = \mathbb{P}(X \leq 0) + \Phi(1) - \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_1^{\mathrm{e}^y} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x$$
$$= \Phi(1) + \frac{1}{\sqrt{2\pi}} \int_1^{\mathrm{e}^y} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

This reveals that the probability distribution of Y is a mixture, with equal weights of two components: a discrete distribution having mass concentrated at $y = 4$ and an absolutely continuous distribution with density

$$f(y) = \frac{1}{\sqrt{2\pi}} \exp\left\{y - \frac{\mathrm{e}^{2y}}{2}\right\}$$

In other terms

$$F_Y(y) = \frac{1}{2}\,\mathbb{1}_{[4,+\infty)}(y) + \frac{1}{2}\,f(y)$$

This shows that a transformation of a random variable with an absolutely contiuous cumulative distribution function does not necessarily have an absolutely continuous cumulative distribution function.

Exercise 4.1.5 (p. 175)
Let X and Y be independent gamma random variables with

$$X \sim \mathrm{Gamma}(1,1) \qquad Y \sim \mathrm{Gamma}(1, 1/2).$$

Compute $\mathbb{P}[X > Y]$.

Solution.
Note first that the marginal density functions of X and Y are

$$f_X(x) = \mathrm{e}^{-x}\,\mathbb{1}_{(0,+\infty)}(x), \qquad f_Y(y) = \frac{1}{2}\,\mathrm{e}^{-\frac{y}{2}}\,\mathbb{1}_{(0,+\infty)}(y).$$

Note also that both X and Y have negative-exponential distributions and one may alternatively write $X \sim \mathrm{E}(1)$ and $Y \sim \mathrm{E}(1/2)$. Because of independence

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{2}\, \mathrm{e}^{-x-\frac{y}{2}}\, \mathbb{1}_{(0,+\infty)}(x)\, \mathbb{1}_{(0,+\infty)}(y).$$

One can, now, valuate the required probability by integrating over the region $R = \{(x,y) \in (0,+\infty)^2 : \; x > y\}$, namely

$$\begin{aligned}\mathbb{P}[X > Y] &= \int_0^{+\infty} \mathrm{d}x \int_0^x \mathrm{d}y\, \frac{1}{2}\, \mathrm{e}^{-x-\frac{y}{2}} = \int_0^{+\infty} \mathrm{e}^{-x} \Big(\int_0^x \frac{1}{2}\, \mathrm{e}^{-\frac{y}{2}}\, \mathrm{d}y \Big)\, \mathrm{d}x \\ &= \int_0^{+\infty} \mathrm{e}^{-x} \left(1 - \mathrm{e}^{-\frac{x}{2}}\right) \mathrm{d}x = \int_0^{+\infty} \mathrm{e}^{-x}\, \mathrm{d}x - \int_0^{+\infty} \mathrm{e}^{-\frac{3}{2}x}\, \mathrm{d}x \\ &= 1 - \frac{2}{3} = \frac{1}{3}.\end{aligned}$$

Exercise 4.1.6 (p. 175)
Let X and Y be random variables taking values in $(0,1)$ and such that their joint p.d.f. is

$$f_{X,Y}(x,y) = k\, x^2 y\, \mathbb{1}_{(0,1)}(x)\, \mathbb{1}_{(0,1)}(y)$$

where is a constant and does not depend on x and/or y.

(a) Determinre the value of k that makes $f_{X,Y}$ a density function on $\mathbb{R}^2$.

(b) Determine $\mathbb{P}[Y > X]$

Solution.
As for (a), note that k must be such that

$$f_{X,Y} \geq 0 \quad \& \quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, \mathrm{d}x\, \mathrm{d}y = 1.$$

The former implies that $k > 0$. As for the latter, it becomes

$$1 = \int_0^1 \int_0^1 kx^2 y\, \mathrm{d}x\, \mathrm{d}y = k \Big(\int_0^1 x^2\, \mathrm{d}x \Big) \Big(\int_0^1 y\, \mathrm{d}y \Big) = k\, \frac{1}{3}\, \frac{1}{2}.$$

Hence, from the equality $(k/6) = 1$, one obtains $k = 6$ and the actual density of (X,Y) is

$$f_{X,Y}(x,y) = 6\, x^2 y\, \mathbb{1}_{(0,1)}(x)\, \mathbb{1}_{(0,1)}(y).$$

Moving on to (b), we have that

$$\mathbb{P}[Y > X] = \int_0^1 \mathrm{d}y \int_0^y \mathrm{d}x\, 6\, x^2\, y = 6 \int_0^1 y \left(\int_0^y x^2\, \mathrm{d}x \right) \mathrm{d}y$$
$$= 6 \int_0^1 y\, \frac{y^3}{3}\, \mathrm{d}y = 2 \int_0^1 y^4\, \mathrm{d}y = \frac{2}{5}$$

Exercise (midterm exam in 2018).
Suppose X and Y are independent uniform random variables on the intervals $[-1, 0]$ and $[0, 1]$, respectively. Hence

$$f_X(x) = \mathbb{1}_{[-1,0]}(x), \qquad f_Y(y) = \mathbb{1}_{[0,1]}(y)$$

(a) Identify the joint density function $f_{X,Y}$ of (X, Y)

(b) Evaluate $\mathbb{P}[X^2 < Y]$

(c) Determine the moment generating function of $X + Y$.

Solution.
The solution to (a) s trivial as independence implies

$$f_{X,Y}(x, y) = f_X(x)\, f_Y(y) = \mathbb{1}_{[-1,0]}(x)\, \mathbb{1}_{[0,1]}(y)$$

As for (b), it may be worth drawing a plot in order to identify the correct region of integration. In particular, since X takes on only negative values in $(-1, 0)$, then

$$\mathbb{P}[X^2 < Y] = \mathbb{P}[-\sqrt{Y} < X < \sqrt{Y}] = \mathbb{P}[-\sqrt{Y} < X < 0]$$
$$= \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^0 \mathrm{d}x = \int_0^1 \sqrt{y}\, \mathrm{d}y = \frac{2}{3}.$$

Finally, the answer to (c) easily follows from independence, since

$$m_{X*Y}(t) = m_X(t) m_Y(t) = \frac{(1 - \mathrm{e}^{-t})}{t}\, \frac{(\mathrm{e}^t - 1)}{t} = \mathrm{e}^{-t} \left(\frac{\mathrm{e}^t - 1}{t} \right)^2$$

Exercise (from midterm exam in 2019).
Let (X, Y) be a random vector whose density function is

$$f_{X,Y}(x, y) = \begin{cases} 2\, \mathrm{e}^{-2x+y} & 0 < y < x < +\infty \\ 0 & \text{elsewhere} \end{cases}$$

(a) Determine the probablity density functions f_X of the random variable X and f_W of the transformed r.v. $W = \mathrm{e}^{-X}$.

(b) Compute $\mathbb{P}[X > 2Y]$

(c) Identify the probability distribution of $X - Y$

Solution.

As for (a), note that for any given $x \in (0, +\infty)$ one has theat y ,ust be integrated out between 0 and x to obtain the density of X at x, i.e.

$$\begin{aligned} f_X(x) &= 2\int_0^x \mathrm{e}^{-2x+y}\,\mathrm{d}y\,\mathbb{1}_{(0,1)}(x) = 3(1-x)^2\,\mathbb{1}_{(0,+\infty)}(x) \\ &= 2\,\mathrm{e}^{-2x}\left(\mathrm{e}^{x} - 1\right)\mathbb{1}_{(0,+\infty)}(x) \\ &= 2\,\mathrm{e}^{-x}\left(1 - \mathrm{e}^{-x}\right)\mathbb{1}_{(0,+\infty)}(x) \end{aligned}$$

As for f_W, since $w = g(x) = \mathrm{e}^{-x}$, one has $x = g^{-1}(w) = -\log w$ and $(\mathrm{d}/\mathrm{d}w)g^{-1}(w) = -1/w$. Moreover, if $x \in (0, +\infty)$, one has $w \in (0, 1)$. Then

$$\begin{aligned} f_W(w) &= f_X(g^{-1}(w))\left|\frac{\mathrm{d}}{\mathrm{d}w}g^{-1}(w)\right| = 2\,\mathrm{e}^{\log w}\,(1 - \mathrm{e}^{\log w})\frac{1}{w}\,\mathbb{1}_{(0,1)}(w) \\ &= 2(1-w)\,\mathbb{1}_{(0,1)}(w) \end{aligned}$$

and $W \sim \mathrm{Beta}(1, 2)$.

The evaluation of the probability in (b) is better performed by drawing a picture and noting that for any given $y \in (0, +\infty)$, x is integrated out over the interval $(2y, +\infty)$ as displayed next

$$\begin{aligned} \mathbb{P}[X > 2Y] &= \int_0^\infty \mathrm{d}y \int_{2y}^\infty \mathrm{d}x\,2\,\mathrm{e}^{-2x+y} = \int_0^\infty \mathrm{e}^{y}\int_{2y}^\infty 2\,\mathrm{e}^{-2x}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^\infty \mathrm{e}^{y}\,\mathrm{e}^{-4y}\,\mathrm{d}y = \frac{1}{3} \end{aligned}$$

As for the final point (c), set $T = g_1(X, Y) = X - Y$ and $U = g_2(X, Y) = Y$. From this one has

$$x = t + u = g_1^{-1}(t, u), \qquad y = u = g_2^{-1}(t, u)$$

and if J is the jacobian of the transformation it can be seen that $|\det(J)| = 1$. Hence

$$\begin{aligned} f_{T,U}(t, u) &= f_{X,Y}(t+u, u)\,|\det(J)| = 2\,\mathrm{e}^{-2(t+u)+u}\,\mathbb{1}_{(0,\infty)}(t)\,\mathbb{1}_{(0,\infty)}(u) \\ &= 2\,\mathrm{e}^{-2t}\,\mathbb{1}_{(0,\infty)}(t)\,\mathrm{e}^{-u}\,\mathbb{1}_{(0,\infty)}(u) \end{aligned}$$

which shows that U and T are independent and the density of T is $f_T(t) = 2\,\mathrm{e}^{-2t}\,\mathbb{1}_{(0,+\infty)}(t)$. The conclusion is $X - Y \sim \mathrm{E}(2)$.

Exercise

Suppose X and Y are independent and identically distributed. Their common distribution is the so–called Laplace distribution

$$f(x) = \frac{1}{2}\,\mathrm{e}^{-|x|} \qquad x \in \mathbb{R}.$$

Hence, the joint probability density function is

$$f_{X,Y}(x,y) = f_X(x)\,f_Y(y) = \frac{1}{4}\,\mathrm{e}^{-|x|-|y|} \qquad x,y \in \mathbb{R}.$$

(a) Determine the probability density function of $(X-Y, X+Y)$ and the respective marginal densities.

(b) Show that $X-Y$ and $X+Y$, though not being independent, are uncorrelated.

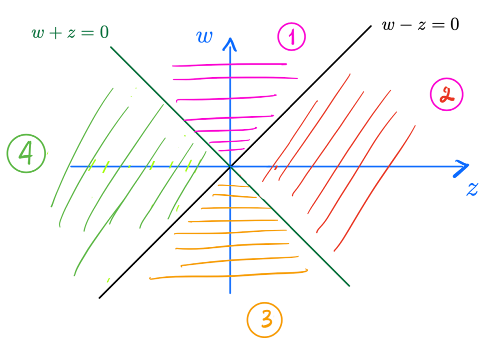
This exercise is slightly more complicated and you may skip it.

Solution.

As for (a), let $W = X - Y$ and $Z = X + Y$. The change of variable formula yields, for any $w, z \in \mathbb{R}$

$$f_{W,Z}(w,z) = f_{X,Y}\Big(\frac{w+z}{2}, \frac{w-z}{2}\Big)\,\frac{1}{2} = \frac{1}{8}\,\mathrm{e}^{-\frac{|w+z|}{2}-\frac{|w-z|}{2}}.$$

This shows that W and Z are not independent as the previous density cannot be factorized. Moreover, since $f_{W,Z}(w,z) = f_{W,Z}(z,w)$ one has $W \stackrel{\mathrm{d}}{=} Z$. Hence, it is enough to determine the marginal density function of W.



1 $w > |z|$ 3 $w < -|z|$

2 $|w| < z$ 4 $|w| < -z$

To this end, one can refer to the Figure and note that

(1) when $w > |z|$, i.e. the area labeled as 1,

$$f_{W,Z}(w,z) = \frac{1}{8} e^{-w}$$

(2) when $z > 0$ and $|w| < z$, namely the red shaded area labeled as 2 in the Figure, one has

$$f_{W,Z}(w,z) = \frac{1}{8}\, e^{z}$$

(3) when $w < -|z|$ as in the orange shaded area labeled as 3, one has

$$f_{W,Z}(w,z) = \frac{1}{8}\, e^{w}$$

(4) when $z < 0$ and $|w| < -z$ as in the green shaded area labeled as 4, one has

$$f_{W,Z}(w,z) = \frac{1}{8}\, e^{z}$$

When marginalizing with respect to Z, one has that

$$f_W(w) = \int_{-\infty}^{-w} \frac{1}{8}\, e^{z}\, dz + \int_{-w}^{w} \frac{1}{8}\, e^{-w}\, dz + \int_{w}^{+\infty} \frac{1}{8}\, e^{-z}\, dz \qquad \forall w > 0$$
$$f_W(w) = \int_{-\infty}^{w} \frac{1}{8}\, e^{z}\, dz + \int_{w}^{-w} \frac{1}{8}\, e^{w}\, dz + \int_{-w}^{+\infty} \frac{1}{8}\, e^{-z}\, dz \qquad \forall w < 0$$

All this can be concisely reported as

$$f_W(w) = \frac{1}{4}\, e^{|w|}\, (1 + |w|) \qquad w \in \mathbb{R}.$$

As for question (b) it is enough to show that $\mathrm{Cov}(W,Z) = 0$. Indeed, from

$$\mathbb{E}WZ = \mathbb{E}(X-Y)(X+Y) = \mathbb{E}X^2 - \mathbb{E}Y^2 = 0$$

since X and Y are identically distributed. Moreover, $\mathbb{E}W = \mathbb{E}(X - Y) = \mathbb{E}X - \mathbb{E}Y = 0$ still because of identity in distribution. ThEse imply that

$$\mathrm{Cov}(W,Z) = \mathbb{E}WZ - (\mathbb{E}W)(\mathbb{E}Z) = 0$$

and W and Z are uncorrelated random variables, though they are dependent since $f_{W,Z} \neq f_W\, f_Z$.

Exercise.
Let $X, Y \overset{\text{iid}}{\sim} \mathrm{N}(0,1)$. Show that the random variables

$$U = X^2 + Y^2 \qquad V = \frac{X}{\sqrt{X^2 + Y^2}}$$

are independent and give a geometric interpretation of the result.

Solution.
The assumption on (X, Y) entails

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\,\mathrm{e}^{-\frac{x^2}{2}-\frac{y^2}{2}} \qquad \forall x, y \in \mathbb{R}.$$

Since, one has $x = v\sqrt{u}$ and $y = \sqrt{u(1-v^2)}$, the Jacobian of the transformation is $|J| = 1/\{2\sqrt{1-v^2}\}$ and, hence,

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi}\,\mathrm{e}^{-\frac{u}{2}}\,\frac{1}{2\sqrt{1-v^2}}\,\mathbb{1}_{(0,+\infty)}(u)\,\mathbb{1}_{(-1,1)}(v) \\ &= \frac{1}{2}\,\mathrm{e}^{-\frac{u}{2}}\,\mathbb{1}_{(0,+\infty)}(u)\,\frac{1}{2\pi\sqrt{1-v^2}}\,\mathbb{1}_{(-1,1)}(v) \\ &= f_U(u)\,f_V(v) \end{aligned}$$

so that U and V are actually independent. A geometric interpretation is available if transforming into polar coordinates

$$X = \rho\,\cos\theta \qquad Y = \rho\,\sin\theta$$

with $\rho \in (0, +\infty)$ and $\theta \in (0, 2\pi]$. In this case, $U = \rho^2$ and $V = \cos\theta$. Hence, the independence property on (U, V) implies that ρ and θ are independent and one can determine their distributions. Indeed, since the Jacobian of the transformation is $|J| = \rho$ and

$$\begin{aligned} f_{\rho,\theta}(\rho,\theta) &= \frac{1}{2\pi}\,\mathrm{e}^{-\frac{\rho^2}{2}}\,\rho\,\mathbb{1}_{(0,+\infty)}(\rho)\,\mathbb{1}_{(0,2\pi]}(\theta) \\ &= \rho\,\mathrm{e}^{-\frac{\rho^2}{2}}\,\mathbb{1}_{(0,+\infty)}(\rho)\,\frac{1}{2\pi}\,\mathbb{1}_{(0,2\pi]}(\theta). \end{aligned}$$

Hence, $\rho^2 \sim$ neg-exp$(1/2)$ and $\theta \sim$ Unif$((0, 2\pi])$. Hence, the angle θ formed by the segment S joining (X, Y) and the origin with the x–axis and the length ρ of S are independent.

Exercise 4.8.12 (p. 219)
Let (X, Y) be such that their joint p.d.f. is

$$f_{X,Y}(x,y) = 4xy\,\mathbb{1}_{(0,1)}(x)\,\mathbb{1}_{(0,1)}(y)$$

Determine $\mathbb{P}[X + Y \le 1]$.

Solution.
Notice that X and Y are independent and identically distributed, each with a Beta$(2, 1)$ distribution. This fact could be used to determine the c.d.f. of $X + Y$, through a well–known formula (known as *convolution*) according to which

$$F_{X+Y}(z) = \mathbb{P}[X + Y \le z] = \int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{z-x} f_Y(y)\,\mathrm{d}y\Big)\,f_X(x)\,\mathrm{d}x$$

Equivalently, one can rely on a direct calculation that benefits from depicting the plot of the region of integration. We will do so and note that

$$\begin{aligned}\mathbb{P}[X+Y \le 1] &= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, 4xy = \int_0^1 2x \left(\int_0^{1-x} 2y\, \mathrm{d}y \right) \mathrm{d}x \\ &= 2\int_0^1 x(1-x)^2\, \mathrm{d}x = 2\,\frac{\Gamma(2)\,\Gamma(3)}{\Gamma(5)} = 2\,\frac{2}{4!} \\ &= \frac{1}{6}.\end{aligned}$$

Exercise 20 & 21 (p. 221)
Let (X, Y) be such that their joint p.d.f. is

$$f_{X,Y}(x,y) = (x+y)\,\mathbb{1}_{(0,1)}(x)\,\mathbb{1}_{(0,1)}(y)$$

(a) Determine $\mathbb{P}[X > \sqrt{Y}]$.

(b) Determine $\mathbb{P}[X^2 < Y < X]$

Solution.
As for (a), this can be easily evaluated if one draws the plot of the region of integration in $(0,1)^2$. In this case one has

$$\begin{aligned}\mathbb{P}[X > \sqrt{Y}] &= \int_0^1 \left(\int_{\sqrt{y}}^1 (x+y)\, \mathrm{d}x \right) \mathrm{d}y \\ &= \int_0^1 \left(\frac{1}{2} - \frac{y}{2} + y - y^{\frac{3}{2}} \right) \mathrm{d}y = \frac{7}{20}\end{aligned}$$

Can one evaluate, without carrying out any computation, $\mathbb{P}[X^2 < Y < X]$? It is enough to note that, by virtue of the symmetry of $f_{X,Y}$ with respect to the line $y = x$, one has $\mathbb{P}[X > Y] = \mathbb{P}[X < Y] = 1/2$. It, then, turns out that

$$\begin{aligned}\mathbb{P}[X^2 < Y < X] &= \mathbb{P}[X > Y] - \mathbb{P}[Y < X^2] = \mathbb{P}[X > Y] - \mathbb{P}[\sqrt{Y} < X] \\ &= \frac{1}{2} - \frac{7}{20} = \frac{3}{20}.\end{aligned}$$

Moving on to (b), still with the help of the picture of the region of integration, one finds out that

$$\begin{aligned}\mathbb{P}[X^2 < Y < X] &= \int_0^1 \left(\int_{x^2}^{x} (x+y)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^1 \left(x^2 - x^3 + \frac{x^2}{2} - \frac{x^4}{2} \right) \mathrm{d}x \\ &= \frac{1}{3} - \frac{1}{4} + \frac{1}{6} - \frac{1}{10} = \frac{3}{20}\end{aligned}$$

Exercise 4.8.47 (p. 224)
Let $\boldsymbol{X} = (X_1, \ldots, X_{k-1}) \sim \mathrm{Mult}_{k-1}(n; p_1, \ldots, p_k)$, namely for any $(x_1, \ldots, x_{k-1})$ such that $x_1, \ldots, x_{k-1} \geq 0$ and $\sum_{i=1}^{k-1} x_i \leq n$, the probability mass function of the vector is

$$p_{X_1,\ldots,X_{k-1}}(x_1, \ldots, x_{k-1}) = \frac{n!}{x_1! \cdots x_{k-1}!(n - \sum_{i=1}^{k-1} x_i)!} \times p_1^{x_1} \cdots p_{k-1}^{x_{k-1}} p_k^{n - \sum_{i=1}^{k-1} x_i}$$

For any $i \neq j$ evaluate $\mathbb{E}[X_i | X_j = x_j]$.

Solution
In order to evaluate the conditionalexpectation, it is worth noting that by properties of the multinomial distribution one has

$$(X_i, X_j) \sim \mathrm{Mult}_2(n; p_i, p_j, 1 - p_i - p_j) \quad \& \quad X_j \sim \mathrm{binom}(n; p_j).$$

In view of this, one can determine the conditional probability mass function of X_j, given X_j. Indeed, for any $x_j \in \{0, 1, \ldots, n\}$, one has

$$p_{X_i|X_j}(x_i|x_j) = \binom{n - x_j}{x_i} \frac{p_i^{x_i}(1 - p_j - p_i)^{n - x_j - x_i}}{(1 - p_j)^{n - x_j}} \mathbb{1}_{\{0,1,\ldots,n-x_j\}}(x_i)$$

For any $x_j \in \{0, 1, \ldots, n\}$ one can now evaluate the conditional expectation

$$\begin{aligned}\mathbb{E}[X_i|X_j = x_j] &= \sum_{x_i} x_i\, p_{X_i|X_j}(x_i|x_j) \\ &= \sum_{i=0}^{n-x_j} x_i \binom{n - x_j}{x_i} \frac{p_i^{x_i}(1 - p_j - p_i)^{n - x_j - x_i}}{(1 - p_j)^{n - x_j}} \\ &= \frac{p_i(n - x_j)}{(1 - p_j)^{n - x_j}} \sum_{y=0}^{n - x_j - 1} \binom{n - x_j - 1}{y} p_i^y (1 - p_j - p_i)^{n - x_j - 1 - y} \\ &= \frac{p_i(n - x_j)}{1 - p_j}\end{aligned}$$

Exercise 70 (p. 227)
Let (X, Y) be such that their joint p.d.f. is

$$f_{X,Y}(x,y) = (x+y)\,\mathbb{1}_{(0,1)}(x)\,\mathbb{1}_{(0,1)}(y)$$

Determine the p.d.f. of $U = XY$.

Solution
In order to determine the distribution of U, we introduce an additional random variable, say $V = X$, that is simply needed to apply the general formula for transformations of random vectors. So, we have

$$\begin{pmatrix} V \\ U \end{pmatrix} = \begin{pmatrix} g_1(X,Y) \\ g_2(X,Y) \end{pmatrix} = \begin{pmatrix} X \\ XY \end{pmatrix}$$

From this we deduce

$$x = v = g_1^{-1}(u,v) \quad \& \quad y = \frac{u}{v} = g_2^{-1}(u,v)$$

and the partial derivatives of g_1^{-1} and g_2^{-1} yield the jacobian of the transformation

$$J = \begin{pmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{pmatrix} \quad \Longrightarrow \quad |J| = \frac{1}{|v|} = \frac{1}{v}.$$

From this one obtains

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(v,\, u/v)\,\frac{1}{v} = \left(v + \frac{u}{v}\right)\mathbb{1}_{(0,1)}(v)\,\mathbb{1}_{(0,1)}(u/v)\,\frac{1}{v} \\ &= \left(1 + \frac{u}{v^2}\right)\mathbb{1}_{(0,1)}(v)\,\mathbb{1}_{(0,v)}(u). \end{aligned}$$

The p.d.f. of U can now be determined by integrating out V as follows

$$\begin{aligned} f_U(u) &= \mathbb{1}_{(0,1)}(u) \int_u^1 \left(1 + \frac{u}{v^2}\right) \mathrm{d}v \\ &= \left\{1 - u + u\left(\frac{1}{u} - 1\right)\right\}\mathbb{1}_{(0,1)}(u) \\ &= 2(1-u)\,\mathbb{1}_{(0,1)}(u) \end{aligned}$$

and, hence, $U \sim \text{Beta}(1,2)$.

Exercise.
Let

$$f_{X,Y}(x,y) = 4xy\,\mathbb{1}_{(0,1)}(x)\,\mathbb{1}_{(0,1)}(y).$$

Determine the density function of the vector $(W, Z) = (X/Y,\, XY)$.

Solution

Since $w = g_1(x,y) = x/y$ and $z = g_2(x,y) = xy$, one can invert and obtain

$$g_1^{-1}(w,z) = \sqrt{wz}, \qquad g_2^{-1}(w,z) = \sqrt{\frac{z}{w}}.$$

Hence, the jacobian is

$$J = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{z}{w}} & \frac{1}{2}\sqrt{\frac{w}{z}} \\ -\frac{\sqrt{z}}{2w^{3/2}} & \frac{1}{2\sqrt{wz}} \end{pmatrix}$$

and it is easily seen that $|J| = 1/(2w)$. From this, one has

$$\begin{aligned} f_{W,Z}(w,z) &= f_{X,Y}\left(\sqrt{wz}, \sqrt{\frac{z}{w}}\right) \frac{1}{2w} \\ &= 4\sqrt{wz}\sqrt{\frac{z}{w}}\,\frac{1}{2w}\,\mathbb{1}_{(0,1)}(\sqrt{wz})\,\mathbb{1}_{(0,1)}\left(\sqrt{\frac{z}{w}}\right). \end{aligned}$$

In order to identify more explicitly the support of (W,Z), note that the density is different from 0 when both indicators are equal to 1, namely when

$$0 < \sqrt{wz} < 1 \quad \& \quad 0 < \sqrt{\frac{z}{w}} < 1$$

and these inequalities hold true if and only if

$$0 < w < \frac{1}{z} \quad \& \quad 0 < z < w.$$

If we denote as $R = \{(w,z) \in \mathbb{R}^+ \times (0,1) : \; 0 < z < w < 1/z\}$, one has

$$f_{W,Z}(w,z) = 2\,\frac{z}{w}\,\mathbb{1}_R(w,z).$$

We can easily determine the two marginal distributions, at this point. Indeed

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{+\infty} f_{W,Z}(w,z)\,\mathrm{d}w = 2\int_z^{1/z} \frac{z}{w}\,\mathrm{d}w \;\; \mathbb{1}_{(0,1)}(z) \\ &= -4\,z\,(\log z)\,\mathbb{1}_{(0,1)}(z). \end{aligned}$$

Analogously

$$\begin{aligned} f_W(w) &= \mathbb{1}_{(0,1)}(w)\,2\int_0^w \frac{z}{w}\,\mathrm{d}z + \mathbb{1}_{[1,\infty)}(w)\,2\int_0^{1/w} \frac{z}{w}\,\mathrm{d}z \\ &= w\,\mathbb{1}_{(0,1)}(w) + \frac{1}{w^3}\,\mathbb{1}_{[1,\infty)}(w). \end{aligned}$$

Conditional densities and expectations

Exercise.
Given a random vector (X, Y) with density

$$f_{X,Y}(x,y) = \frac{2}{(1+x+y)^3}\,\mathbb{1}_{(0,+\infty)}(x)\,\mathbb{1}_{(0,+\infty)}(y),$$

determine

(a) The joint cdf $F_{X,Y}(x,y) = \mathbb{P}[X \le x,\ Y \le y]$

(b) The marginal pdf f_X

(c) The conditional pdf $f_{Y|X}(y|x)$.

Solution.
Recall that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \Big(\int_{-\infty}^{y} f_{X,Y}(s,t)\,\mathrm{d}t \Big)\,\mathrm{d}s = \int_{-\infty}^{y} \Big(\int_{-\infty}^{x} f_{X,Y}(s,t)\,\mathrm{d}s \Big)\,\mathrm{d}t.$$

Note that if either $x < 0$ or $y < 0$, then $F_{X,Y}(x,y) = 0$. On the contrary, for any $x > 0$ and $y > 0$ one has

$$\begin{aligned} F_{X,Y}(x,y) &= 2\int_{0}^{x} \Big(\int_{0}^{y} \frac{1}{(1+s+t)^3}\,\mathrm{d}t \Big)\,\mathrm{d}s \\ &= \int_{0}^{x} \Big\{ \frac{1}{(1+s)^2} - \frac{1}{(1+s+y)^2} \Big\}\,\mathrm{d}s \\ &= 1 - \frac{1}{1+x} - \frac{1}{1+y} + \frac{1}{1+x+y}. \end{aligned}$$

Hence, to sum up

$$F_{X,Y}(x,y) = \Big\{ 1 - \frac{1}{1+x} - \frac{1}{1+y} + \frac{1}{1+x+y} \Big\}\,\mathbf{1}_{(0,+\infty)}(x)\,\mathbf{1}_{(0,+\infty)}(y).$$

As for (b),

$$f_X(x) = \mathbb{1}_{(0,+\infty)}(x) \int_{0}^{\infty} \frac{2}{(1+x+y)^3}\,\mathrm{d}y = \frac{1}{(1+x)^2}\,\mathbb{1}_{(0,+\infty)}(x)$$

As for (c), for any $x \in (0, +\infty)$ one has

$$f_{Y|X}(y|x) = \frac{2/(1+x+y)^3}{1/(1+x)^2}\,\mathbb{1}_{(0,+\infty)}(y) = \frac{2(1+x)^2}{(1+x+y)^3}\,\mathbb{1}_{(0,+\infty)}(y)$$

Exercise.
Let (X, Y) have pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{8}(x^2 - y^2)\,\mathrm{e}^{-x} & x > 0,\ |y| < x \\ 0 & \text{elswhere} \end{cases}$$

Determine

(a) The marginal pdf's f_X and f_Y.

(b) The conditional pdf's $f_{Y|X}$ and $f_{X|Y}$.

Solution.
As for (a), note that for any $x > 0$

$$\begin{aligned} f_X(x) &= \int_{-x}^{x} \frac{1}{8}(x^2 - y^2)\,\mathrm{e}^{-x}\,\mathrm{d}y = \frac{1}{8}\,\mathrm{e}^{-x} \int_{-x}^{x} \left(x^2 - y^2\right) \mathrm{d}y \\ &= \frac{1}{8}\,\mathrm{e}^{-x} \left[x^2\,y - \frac{1}{3}\,y^3\right]_{y=-x}^{y=x} \\ &= \frac{1}{8}\,\mathrm{e}^{-x} \left\{x^3 - \frac{1}{3}\,x^3 + x^3 - \frac{1}{3}\,x^3\right\} \\ &= \frac{1}{6}\,x^3\,\mathrm{e}^{-x}. \end{aligned}$$

Hence

$$f_X(x) = \frac{1}{6}\,x^3\,\mathrm{e}^{-x}\,\mathbb{1}_{(0,+\infty)}(x)$$

and $X \sim \text{Gamma}(4, 1)$. One similarly obtains, for any $y \in \mathbb{R}$

$$f_Y(y) = \int_{|y|}^{+\infty} \frac{1}{8}(x^2 - y^2)\,\mathrm{e}^{-x}\,\mathrm{d}x$$

integration by parts

$$\begin{aligned} &= \frac{1}{8}\Big\{\Big[-(x^2 - y^2)\,\mathrm{e}^{-x}\Big]_{|y|}^{\infty} + \int_{|y|}^{\infty} 2x\,\mathrm{e}^{-x}\,\mathrm{d}x\Big\} \\ &= \frac{1}{4}\Big\{\Big[-x\,\mathrm{e}^{-x}\Big]_{|y|}^{\infty} + \int_{|y|}^{\infty} \mathrm{e}^{-x}\,\mathrm{d}x\Big\} \\ &= \frac{1}{4}\left\{|y|\,\mathrm{e}^{-|y|} + \mathrm{e}^{-|y|}\right\} = \frac{1}{4}\,\mathrm{e}^{-|y|}\,(1 + |y|) \end{aligned}$$

As for (b), for any $x > 0$

$$f_{Y|X}(y|x) = \frac{\frac{1}{8}(x^2 - y^2)\,\mathrm{e}^{-x}\,\mathbb{1}_{(-x,x)}(y)}{\frac{1}{6}\,x^3\,\mathrm{e}^{-x}} = \frac{3}{4}\Big(\frac{x^2 - y^2}{x^3}\Big)\,\mathbb{1}_{(-x,x)}(y).$$

while, for any $y \in \mathbb{R}$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{\frac{1}{8}(x^2 - y^2)\, \mathrm{e}^{-x}\, \mathbb{1}_{(|y|,+\infty)}(x)}{\frac{1}{4}\, \mathrm{e}^{-|y|}\, (1+|y|)} \\ &= \frac{1}{2(1+|y|)}(x^2 - y^2)\, \mathrm{e}^{-(x-|y|)}\, \mathbb{1}_{(|y|,+\infty)}(x). \end{aligned}$$

Exercise (midterm exam in 2018).
Let (X, Y) be a random vector whose density function is

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x) & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Determine the marginal densities of X and Y.

(b) How would you simulate realizations of Y?

(c) Determine the conditional density $f_{Y|X}$ of Y, given X, and evaluate $\mathbb{E}[Y|X = x]$, for any x in $(0, 1)$.

Solution.
The marginal densities in (a) are determined through

$$\begin{aligned} f_X(x) &= 6\int_x^1 (y-x)\, \mathrm{d}y\, \mathbb{1}_{(0,1)}(x) = 3(1-x)^2\, \mathbb{1}_{(0,1)}(x) \\ f_Y(y) &= 6\int_0^y (y-x)\, \mathrm{d}x\, \mathbb{1}_{(0,1)}(y) = 3y^2\, \mathbb{1}_{(0,1)}(y) \end{aligned}$$

so that $X \sim \text{Beta}(1, 2)$ and $Y \sim \text{Beta}(2, 1)$.
As for (b), since

$$F_Y(y) = \mathbb{1}_{[0,1)}(y)\int_0^y 3s^2\, \mathrm{d}s + \mathbb{1}_{[1,+\infty)}(y) = y^3 \mathbb{1}_{[0,1)}(y) + \mathbb{1}_{[1,+\infty)}(y)$$

the random variable Y can be simulated as follows

- `Sample` $U = u$ `from a Unif`$(0, 1)$
- `Set` $Y = y = F_y^{-1}(u) = u^{1/3}$.

Finally, the solution to (c) is achieved upon noting that for any $x \in (0, 1)$

$$f_{Y|X}(y|x) = 2\,\frac{(y-x)}{(1-x)^2}\, \mathbb{1}_{(x,1)}(y).$$

Hence, for any $x \in (0, 1)$,

$$\begin{aligned} \mathbb{E}[Y|X = x] &= \frac{2}{(1-x)^2}\int_x^1 y(y-x)\, \mathrm{d}y \\ &= \frac{1}{3(1-x)^2}\left\{2 - 3x + x^3\right\} = \frac{x+2}{3} \end{aligned}$$

Exercise (from midterm exam in 2019).
Suppose X and Y are random variables taking values in the set $\Delta_2 = \{(x,y) : x \geq 0, y \geq 0, x+y \leq 1\}$ and with joint density function

$$f_{X,Y}(x,y) = 6(1-x-y)\,\mathbb{1}_{\Delta_2}(x,y)$$

(a) Identify the conditional density of Y, given X, namely $f_{Y|X}(y|x)$.

(b) Determine $\mathbb{E}[Y|X=x]$ and use the "Adam's rule" to evaluate $\mathbb{E}Y$.

(c) Can one establish whether the correlation coefficient $\rho(X,Y)$ is zero, positive or negative, without actually computing it? Motivate the answer.

Solution.
As for (a), the marginal density of X is obtained by noting that for any given $x \in (0,1)$, the variable $y \leq 1-x$ in Δ_2 and, hence, y is integrated over the interval $(0, 1-x)$ as exemplified below

$$f_X(x) = \mathbb{1}_{(0,1)}(x)\, 6 \int_0^{1-x} (1-x-y)\, \mathrm{d}y = 3(1-x)^2\, \mathbb{1}_{(0,1)}(x)$$

and for any $x \in (0,1)$

$$f_{Y|X}(y|x) = \frac{6(1-x-y)}{3(1-x)^2}\,\mathbb{1}_{(0,1-x)}(y) = \frac{2(1-x-y)}{(1-x)^2}\,\mathbb{1}_{(0,1-x)}(y).$$

Moving on to (b), for any $x \in (0,1)$

$$\begin{aligned}
\mathbb{E}[Y \mid X = x] &= \frac{2}{(1-x)^2} \int_0^{1-x} y\,(1-x-y)\, \mathrm{d}y \\
&= \frac{2}{(1-x)^2} \left\{ \frac{(1-x)^3}{2} - \frac{(1-x)^3}{3} \right\} \\
&= \frac{1-x}{3}
\end{aligned}$$

Moreover,

$$\mathbb{E}Y = \mathbb{E}\,\mathbb{E}[Y \mid X] = \int_0^1 \frac{1-x}{3}\, f_X(x)\, \mathrm{d}x = \int_0^1 (1-x)^3\, \mathrm{d}x = \frac{1}{4}$$

Finally, the answer to (c) is immediate. Indeed, the linear correlation coefficient $\rho(X,Y)$ will have a negative sign since the constraint $x+$

$y \le 1$ implies that as one of the variable increases the other one must decrease. The exact calculation (not required here) would give

$$\rho(X, Y) = -\frac{1}{3}.$$

Exercise.
Suppose

$$f_{X,Y}(x, y) = \frac{x+y}{8}\, \mathbb{1}_{[0,2]}(x)\, \mathbb{1}_{[0,2]}(y).$$

Determine the conditional densities $f_{X|Y}$ and $f_{Y|X}$.

Solution.
The marginal distribution of X is

$$f_X(x) = \mathbb{1}_{[0,2]}(x) \int_0^2 \frac{x+y}{8}\, \mathrm{d}y = \left\{ \frac{x}{4} + \frac{1}{4} \right\} \mathbb{1}_{[0,2]}(x)$$

Due to symmetry, a similar expression arises for f_Y, namely

$$f_Y(y) = \frac{y+1}{4}\, \mathbb{1}_{[0,2]}(y).$$

We can, now, determine the conditional density of X, given Y, and the conditional density of Y, given X. We confine ourselves to determining only the former and obtain

$$f_{X|Y}(x|y) = \frac{(x+y)/8}{(y+1)/4}\, \mathbb{1}_{[0,2]}(x) = \frac{x+y}{2(1+y)}\, \mathbb{1}_{[0,2]}(x)$$

for any $y \in [0, 2]$. A simmlar expression can be obtained for $f_{Y|X}(y|x)$ for any $x \in [0, 2]$.

Exercise 4.2.4 (p. 184)
Let (X, Y) have p.d.f.

$$f_{X,Y}(x, y) = y^2\, \frac{\mathrm{e}^{-\frac{x}{3}}}{3x^2}\, \mathbb{1}_{(0,+\infty)}(x)\, \mathbb{1}_{(0,x)}(y).$$

Determine

(a) The marginal density f_X of X

(b) The conditional density of Y, given $X = x$

(c) $\mathbb{E}[Y|X]$, $\mathbb{E}Y$ and $\mathrm{Var}(Y)$.

Solution.
As for (a), the definition of $f_{X,Y}$ yields

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \mathbb{1}_{(0,+\infty)}(x) \int_0^x y^2\,\frac{\mathrm{e}^{-\frac{x}{3}}}{3x^2}\,\mathrm{d}y \\
&= \frac{\mathrm{e}^{-\frac{x}{3}}}{3x^2}\left(\int_0^x y^2\,\mathrm{d}y\right)\mathbb{1}_{(0,+\infty)}(x) = \frac{\mathrm{e}^{-\frac{x}{3}}}{3x^2}\,\frac{x^3}{3}\,\mathbb{1}_{(0,+\infty)}(x) \\
&= \frac{x}{9}\,\mathrm{e}^{-\frac{x}{3}}\,\mathbb{1}_{(0,+\infty)}(x) = \frac{(1/3)^2}{\Gamma(2)}\,x^{2-1}\,\mathrm{e}^{-\frac{x}{3}}\,\mathbb{1}_{(0,+\infty)}(x)
\end{aligned}$$

and it turns out that $X \sim \mathrm{Gamma}(2, 1/3)$.
The answer to (b) is straightforward, as one only needs to consider the ratio $f_{X,Y}/f_X$. Hence, for any $x > 0$ (which is needed to have a non–null denominator)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{y^2\,\frac{\mathrm{e}^{-x/3}}{3x^2}\,\mathbb{1}_{(0,x)}(y)}{\frac{x}{9}\,\mathrm{e}^{-x/3}} = \frac{3y^2}{x^3}\,\mathbb{1}_{(0,x)}(y)$$

It is interesting to note that if $W = Y/X$, the conditional density function of W, given $X = x$, can be determined through a variable transformation by resorting to $f_{Y|X}$. Indeed, $g^{-1}(w) = wx$ and the jacobian of the transformation is $|J| = x$. This entails that

$$f_{W|X}(w|x) = f_{Y|X}(wx|x)\,x = \frac{3(wx)^2}{x^3}\,x\,\mathbb{1}_{(0,x)}(wx) = 3\,w^2\,\mathbb{1}_{(0,1)}(w).$$

Hence, given $X = x$, one has $(Y/X) \sim \mathrm{Beta}(3,1)$.
As for (c), for any $x > 0$ we have

$$\begin{aligned}
\mathbb{E}[Y|X = x] &= \int_{-\infty}^{+\infty} y\,f_{Y|X}(y|x)\,\mathrm{d}y = \int_0^x y\,\frac{3y^2}{x^3}\,\mathrm{d}y \\
&= \frac{3}{x^3}\int_0^x y^3\,\mathrm{d}y = \frac{3}{4}\,x.
\end{aligned}$$

In order to determine $\mathbb{E}Y$, one may rely on the iterated conditional expectation (or Adam's rule, in the textbook's terminology)

$$\begin{aligned}
\mathbb{E}Y &= \mathbb{E}\,\mathbb{E}[Y|X] = \int_0^{+\infty} \mathbb{E}[Y|X = x]\,f_X(x)\,\mathrm{d}x = \int_0^{+\infty} \frac{3}{4}\,x\,\frac{x}{9}\,\mathrm{e}^{-\frac{x}{3}}\,\mathrm{d}x \\
&= \frac{1}{12}\int_0^{+\infty} x^2\,\mathrm{e}^{-\frac{x}{3}}\,\mathrm{d}x = \frac{1}{12}\,\frac{\Gamma(3)}{(1/3)^3} = \frac{27}{12}\,2 = \frac{9}{2}
\end{aligned}$$

where, for any $\alpha, \beta > 0$, we have used the well-known fact

$$\int_0^{+\infty} x^{\alpha-1}\,\mathrm{e}^{-\beta x}\,\mathrm{d}x = \frac{\Gamma(\alpha)}{\beta^\alpha}.$$

In our case we have $\alpha = 3$ and $\beta = 1/3$.
Finally, we will determine $\mathrm{Var}(Y)$ by using the relationship

$$\mathrm{Var}(Y) = \mathbb{E}\,\mathrm{Var}(Y|X) + \mathrm{Var}\Big(\mathbb{E}[Y|X]\Big).$$

First note that

$$\mathbb{E}[Y^2|X=x] = \int_0^x y^2\,\frac{3y^2}{x^3}\,\mathrm{d}y = \frac{3}{x^3}\int_0^x y^4\,\mathrm{d}y = \frac{3}{x^3}\,\frac{x^5}{5} = \frac{3x^2}{5}.$$

This implies that

$$\mathrm{Var}(Y|X=x) = \mathbb{E}[Y^2|X=x] - \Big(\mathbb{E}[Y|X=x]\Big)^2 = \frac{3x^2}{5} - \frac{9x^2}{16} = \frac{3x^2}{80}$$

and

$$\begin{aligned}\mathbb{E}\,\mathrm{Var}(Y|X) &= \mathbb{E}\,\frac{3X^2}{80} = \frac{3}{80}\,\mathbb{E}X^2 = \frac{3}{80}\int_0^{+\infty} x^2\,\frac{x}{9}\,\mathrm{e}^{-\frac{x}{3}}\,\mathrm{d}x \\ &= \frac{3}{720}\int_0^{+\infty} x^3\,\mathrm{e}^{-\frac{x}{3}}\,\mathrm{d}x = \frac{3}{720}\,\frac{\Gamma(4)}{(1/3)^4} = \frac{81}{40}.\end{aligned}$$

Finally, since

$$\begin{aligned}\mathrm{Var}(X) &= \int_0^{+\infty} x^2\,\frac{x}{9}\,\mathrm{e}^{-\frac{x}{3}}\,\mathrm{d}x - \Big(\int_0^{+\infty} x\,\frac{x}{9}\,\mathrm{e}^{-\frac{x}{3}}\,\mathrm{d}x\Big)^2 \\ &= \frac{1}{9}\int_0^{+\infty} x^3\,\mathrm{e}^{-\frac{x}{3}}\,\mathrm{d}x - \frac{1}{81}\Big(\int_0^{+\infty} x^2\,\mathrm{e}^{-\frac{x}{3}}\,\mathrm{d}x\Big)^2 \\ &= \frac{1}{9}\,\frac{\Gamma(4)}{(1/3)^4} - \frac{1}{81}\Big(\frac{\Gamma(3)}{(1/3)^3}\Big)^2 \\ &= 54 - 36 = 18\end{aligned}$$

one has

$$\mathrm{Var}\Big(\mathbb{E}[Y|X]\Big) = \mathrm{Var}\Big(\frac{3}{4}\,X\Big) = \frac{9}{16}\,\mathrm{Var}(X) = \frac{9}{16}\,18 = \frac{81}{8}.$$

To conclude, we have

$$\mathrm{Var}(Y) = \frac{81}{40} + \frac{81}{8} = \frac{243}{20}$$

Exercise 4.8.1 (p. 217)
Let (X, Y) be a discrete random vector with probability mass function (pmf)

$$p_{X,Y}(x,y) = \frac{x+y}{27}\,\mathbb{1}_{\{0,1,2\}}(x)\,\mathbb{1}_{\{1,2,3\}}(y)$$

Determine

(a) The marginal pmf's of Y and Y and the conditional pmf's of $X|Y=y$ and of $Y|X=x$.

(b) Evaluate $\mathbb{E}X$, $\mathbb{E}XY$ and $\mathbb{E}[X|Y=2]$.

Solution.
The marginal of X is

$$p_X(x) = \sum_{y=1}^{3} p_{X,Y}(x,y) = \mathbb{1}_{\{0,1,2\}}(x) \sum_{y=1}^{3} \frac{x+y}{27} = \frac{3x+6}{27} \mathbb{1}_{\{0,1,2\}}(x)$$
$$= \frac{x+2}{9} \mathbb{1}_{\{0,1,2\}}(x)$$

Similarly

$$p_Y(y) = \sum_{x=0}^{2} p_{X,Y}(x,y) = \mathbb{1}_{\{1,2,3\}}(y) \sum_{x=0}^{2} \frac{x+y}{27} = \frac{3y+3}{27} \mathbb{1}_{\{1,2,3\}}(y)$$
$$= \frac{y+1}{9} \mathbb{1}_{\{1,2,3\}}(x)$$

The conditional pmf's are now easily determined and one has:
for any $y \in \{1,2,3\}$

$$p_{X|Y}(x|y) = \frac{(x+y)/27}{(y+1)/9} \mathbb{1}_{\{0,1,2\}}(x) = \frac{x+y}{3(y+1)} \mathbb{1}_{\{0,1,2\}}(x)$$

and for any $x \in \{0,1,2\}$

$$p_{Y|X}(y|x) = \frac{(x+y)/27}{(x+2)/9} \mathbb{1}_{\{1,2,3\}}(y) = \frac{x+y}{3(x+2)} \mathbb{1}_{\{1,2,3\}}(y)$$

As for (b), one has

$$\mathbb{E}X = \sum_{x=0}^{2} x \frac{x+2}{9} = 1$$
$$\mathbb{E}XY = \sum_{x=0}^{2} \sum_{y=1}^{3} xy \frac{x+y}{27} = \frac{8}{3}$$
$$\mathbb{E}[X|Y=2] = \sum_{x=0}^{2} x\, p_{X|Y}(x|2) = \sum_{x=0}^{2} x \frac{x+2}{9} = 1$$

Exercise 4.8.2 (p. 217)
Let X be a random variable with probability density function

$$f_X(x) = 20x^3(1-x)\, \mathbb{1}_{(0,1)}(x)$$

and let Y be another random variable such that, for any $x \in (0,1)$

$$f_{Y|X}(y|x) = \frac{1}{1-x} \mathbb{1}_{(0,1-x)}(y).$$

Determine

(a) The marginal density f_Y of Y and the conditional density $f_{X|Y}$.

(b) Obtain an expression for $\mathbb{E}[X|Y=y]$.

Solution.
In order to address (a), one needs to identify the joint density $f_{X,Y}$ and by definition this equals

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x) = 20\, x^3\, \mathbb{1}_{(0,1)}(x)\, \mathbb{1}_{(0,1-x)}(y).$$

From this, one now easily obtains

$$f_Y(y) = \mathbb{1}_{(0,1)}(y) \int_0^{1-y} 20\, x^3\, \mathrm{d}x = 5(1-y)^4\, \mathbb{1}_{(0,1)}(y),$$

i.e. $Y \sim \text{Beta}(1,5)$. As for $f_{X|Y}$, for any $y \in (0,1)$, one similarly has

$$f_{X,Y}(x|y) = \frac{20\, x^3\, \mathbb{1}_{(0,1-y)}(x)}{5(1-y)^4} = \frac{4\, x^3\, \mathbb{1}_{(0,1-y)}(x)}{(1-y)^4}.$$

The solution to (b) is now straightforward, as for any $y \in (0,1)$

$$\begin{aligned}\mathbb{E}[X|Y=y] &= \int_0^{1-y} x\, \frac{4\, x^3}{(1-y)^4}\, \mathrm{d}x = \frac{4}{(1-y)^4} \int_0^{1-y} x^4\, \mathrm{d}x \\ &= \frac{4}{5}\,(1-y).\end{aligned}$$

Exercise 4.8.5 (page 218).
Let $Y \sim \text{Gamma}(2, 1/\theta)$ and $X|Y=y \sim \text{U}(0,y)$. This entails that the joint density function is

$$f_{X,Y}(x,y) = \frac{1}{\theta^2}\, \mathrm{e}^{-\frac{y}{\theta}}\, \mathbb{1}_{(0,+\infty)}(y)\, \mathbb{1}_{(0,y)}(x).$$

(a) Determine the marginal density f_X of X.

(b) Evaluate $\mathbb{E}[Y|X=x]$.

Solution.
It can be seen that

$$\begin{aligned}f_X(x) &= \mathbb{1}_{(0,+\infty)}(x) \int_x^{+\infty} \frac{1}{\theta^2}\, \mathrm{e}^{-\frac{y}{\theta}}\, \mathrm{d}y = \frac{1}{\theta^2} \Big[-\theta\, \mathrm{e}^{-\frac{y}{\theta}} \Big]_{y=0}^{y=+\infty} \mathbb{1}_{(0,+\infty)}(x) \\ &= \frac{1}{\theta}\, \mathrm{e}^{-\frac{x}{\theta}}\, \mathbb{1}_{(0,+\infty)}(x).\end{aligned}$$

and, hence, $X \sim \text{E}(1/\theta)$.
As for (b), since for any $x > 0$

$$f_{Y|X}(y|x) = \frac{\frac{1}{\theta^2}\, \mathrm{e}^{-y/\theta}\, \mathbb{1}_{(0,+\infty)}(y)\, \mathbb{1}_{(0,y)}(x)}{\frac{1}{\theta}\, \mathrm{e}^{-x/\theta}} = \frac{1}{\theta}\, \mathrm{e}^{-\frac{y-x}{\theta}}\, \mathbb{1}_{(x,+\infty)}(y).$$

Hence, for any $x > 0$ one obtains

$$\mathbb{E}[Y|X=x] = \int_x^{+\infty} y\,\frac{1}{\theta}\,\mathrm{e}^{-\frac{y-x}{\theta}}\,\mathrm{d}y$$

by the change of variable $w = y - x$

$$\begin{aligned} &= \int_0^{+\infty} (w+x)\,\frac{1}{\theta}\,\mathrm{e}^{-\frac{w}{\theta}}\,\mathrm{d}w = \int_0^{+\infty} w\,\frac{1}{\theta}\,\mathrm{e}^{-\frac{w}{\theta}}\,\mathrm{d}w + x\int_0^{+\infty} \frac{1}{\theta}\,\mathrm{e}^{-\frac{w}{\theta}}\,\mathrm{d}w \\ &= \theta + x. \end{aligned}$$

Exercise 4.8.6 (page 218).
Let (X, Y) have density function

$$f_{X,Y}(x,y) = \begin{cases} 10xy^2 & \text{if } 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Evaluate $\mathbb{E}X$, $\mathbb{E}Y$, $\mathbb{E}XY$ and $\mathbb{E}(X/Y)$.

(b) Evaluate $\rho(X, Y)$.

Solution.
As for (a), note that the marginal densities of X and Y are

$$\begin{aligned} f_X(x) &= \mathbb{1}_{(0,1)}(x)\int_x^1 10xy^2\,\mathrm{d}y = \frac{10}{3}\,x(1-x^3)\,\mathbb{1}_{(0,1)}(x) \\ f_Y(y) &= \mathbb{1}_{(0,1)}(y)\int_0^y 10xy^2\,\mathrm{d}y = 5y^4\,\mathbb{1}_{(0,1)}(y) \end{aligned}$$

Given these, one easily determines

$$\begin{aligned} \mathbb{E}X &= \int_0^1 x\,\frac{10}{3}x(1-x^3)\,\mathrm{d}x = \frac{10}{3}\Big\{\int_0^1 x^2\,\mathrm{d}x - \int_0^1 x^5\,\mathrm{d}x\Big\} = \frac{5}{9} \\ \mathbb{E}Y &= \int_0^1 y\,5y^4\,\mathrm{d}y = 5\int_0^1 y^5\,\mathrm{d}y = \frac{5}{6} \end{aligned}$$

and, unsurprisingly, $\mathbb{E}X < \mathbb{E}Y$. Finally

$$\begin{aligned} \mathbb{E}XY &= \int_0^1\int_0^y xy\,10xy^2\,\mathrm{d}x\,\mathrm{d}y = 10\int_0^1 y^3\int_0^y x^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{10}{3}\int_0^1 y^6\,\mathrm{d}y = \frac{10}{21} \\ \mathbb{E}\frac{X}{Y} &= \int_0^1\int_0^y \frac{x}{y}\,10xy^2\,\mathrm{d}x\,\mathrm{d}y = 10\int_0^1 y\int_0^y x^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{10}{3}\int_0^1 y^4\,\mathrm{d}y = \frac{2}{3} \end{aligned}$$

As for the evaluation of $\rho(X,Y)$, one has $\mathrm{Cov}(X,Y) = (10/21) - (5/9)(5/6) = 1/63 \approx 0.016$. Moreover, from

$$\mathrm{Var}(X) = \int_0^1 x^2\,\frac{10}{3}\,x(1-x^3)\,\mathrm{d}x - \left(\frac{5}{9}\right)^2 = \frac{5}{14} - \frac{25}{81} = \frac{55}{1134} \approx 0.048$$
$$\mathrm{Var}(Y) = \int_0^1 y^2\,5y^4\,\mathrm{d}x - \left(\frac{5}{6}\right)^2 = \frac{5}{7} - \frac{25}{36} = \frac{5}{252} \approx 0.02$$

and

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} \approx 0.025$$

Exercise 4.8.8 (page 218).
Suppose $Y \sim \mathrm{Ga}(\alpha,\lambda)$, namely

$$f_Y(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\,y^{\alpha-1}\,\mathrm{e}^{-\lambda x}\,\mathbb{1}_{(0,+\infty)}(y).$$

Further, for any $y > 0$, let $X|Y = y \sim$ neg-exp(y).

(a) Determine the marginal density f_X of X.

(b) Determine $f_{Y|X}$.

(c) Evaluate $\mathbb{E}[Y \mid X = x]$.

Solution.
As for (a), note that the density of the vector (X,Y) is

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\,f_Y(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\,y^\alpha\,\mathrm{e}^{-(\lambda+y)x}\,\mathbb{1}_{(0,+\infty)}(x)\,\mathbb{1}_{(0,+\infty)}(y)$$

Hence, the marginal density of X is

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \frac{\lambda^\alpha}{\Gamma(\alpha)}\,\mathbb{1}_{(0,+\infty)}(x)\int_0^{+\infty} y^\alpha\,\mathrm{e}^{-(\lambda+x)y}\,\mathrm{d}y$$
$$= \frac{\alpha\,\lambda^\alpha}{(\lambda+x)^{\alpha+1}}\,\mathbb{1}_{(0,+\infty)}(x)$$

As for (b), we will use the result in (a) and for any $x > 0$ obtain

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(\lambda+x)^{\alpha+1}}{\alpha\,\Gamma(\alpha)}\,y^\alpha\,\mathrm{e}^{-(\lambda+x)y}\,\mathbb{1}_{(0,+\infty)}(y)$$

and one easily notes that $Y|X = x \sim \mathrm{Ga}(\alpha+1,\lambda+x)$. This corresponds to a specific type of updating that will be discussed more at length during the part of the course on Bayesian statistics.

Recall that $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$ for any $\alpha > 0$

Finally, for any $x > 0$

$$\mathbb{E}[Y|X = x] = \int_{-\infty}^{+\infty} y\, f_{Y|X}(y|x)\,\mathrm{d}y = \frac{(\lambda + x)^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^{+\infty} y^{\alpha+1}\,\mathrm{e}^{-(\lambda+x)y}\,\mathrm{d}y$$
$$= \frac{\alpha+1}{\lambda+x}$$

Exercise 4.8.9 (page 218).
Let $\Delta = \{(x,y) : \; x > 0,\, y > 0,\, x + y < 1\}$, and (X,Y) a random vector taking values in Δ with joint density function

$$f_{X,Y}(x,y) = 24\,x\,y\,\mathbb{1}_{\Delta}(x,y).$$

(a) Determine the marginal density f_Y of Y.

(b) Determine $f_{X|Y}$.

(c) Obtain an expression for $\mathbb{E}[1/X \mid Y = y]$.

Solution.
In order to determine f_Y, note that for any $y \in (0,1)$ we need to integrate x between 0 and $1 - y$, i.e.

$$f_Y(y) = \mathbb{1}_{(0,1)}(y) \int_0^{1-y} 24\,x\,y\,\mathrm{d}y = 12y(1-y)^2\,\mathbb{1}_{(0,1)}(y)$$

and $Y \sim \mathrm{Beta}(2,3)$.
As for (b), for any $y \in (0,1)$

$$f_{X|Y}(x|y) = \frac{24\,x\,y\,\mathbb{1}_{\Delta}(x,y)}{12\,y(1-y)^2} = \frac{2\,x}{(1-y)^2}\,\mathbb{1}_{(0,1-y)}(x).$$

One can, now, address (c) and obtain for any $y \in (0,1)$

$$\mathbb{E}\Big[\frac{1}{X}\,\Big|\,Y = y\Big] = \int_{-\infty}^{+\infty} \frac{1}{x}\, f_{X|Y}(x|y)\,\mathrm{d}x = \int_0^{1-y} \frac{1}{x}\,\frac{2x}{(1-y)^2}\,\mathrm{d}x$$
$$= 2\int_0^{1-y} \frac{1}{(1-y)^2}\,\mathrm{d}x = \frac{2}{1-y}$$

Exercise 4.8.10 (page 218).
Let $\theta > 0$ and (X,Y) has density function

$$f_{X,Y}(x,y) = \frac{1}{\theta^2}\,\mathrm{e}^{-\frac{y}{\theta}}\,\mathbb{1}_{(0,+\infty)}(x)\,\mathbb{1}_{(x,+\infty)}(y).$$

(a) Determine the marginal density f_Y of Y.

(b) Determine $f_{X|Y}$.

(c) Evaluate $\mathbb{P}[Y < 2X]$.

Solution.
As for (a), note that for any $y > 0$, one has $x \in (0, y)$. Hence,

$$f_Y(y) = \mathbb{1}_{(0,+\infty)}(y) \int_0^y \frac{1}{\theta^2} \, \mathrm{e}^{-\frac{y}{\theta}} \, \mathrm{d}x = \frac{y}{\theta^2} \, \mathrm{e}^{-\frac{y}{\theta}} \, \mathbb{1}_{(0,+\infty)}(y)$$

which entails that $Y \sim \mathrm{Ga}(2, 1/\theta)$.

As for (b), one has that for any $y > 0$

$$f_{X|Y}(x|y) = \frac{\theta^{-2} \, \mathrm{e}^{-y/\theta}}{y \, \theta^{-2} \, \mathrm{e}^{-y/\theta}} \, \mathbb{1}_{(0,y)}(x) = \frac{1}{y} \, \mathbb{1}_{(0,x)}(y)$$

which means that $X|Y = y \sim \mathrm{Unif}(0, y)$.

finally, as for (c), one has

$$\begin{aligned}
\mathbb{P}[Y < 2X] &= \int_0^{+\infty} \mathrm{d}x \int_x^{+\infty} \mathrm{d}y \, \frac{1}{\theta^2} \, \mathrm{e}^{-\frac{y}{\theta}} \\
&= \frac{1}{\theta} \int_0^{+\infty} \left(\int_x^{2x} \frac{1}{\theta} \, \mathrm{e}^{-\frac{y}{\theta}} \, \mathrm{d}y \right) \mathrm{d}x \\
&= \frac{1}{\theta} \int_0^{+\infty} \left(\mathrm{e}^{-\frac{x}{\theta}} - \mathrm{e}^{-\frac{2x}{\theta}} \right) \mathrm{d}x \\
&= \frac{1}{\theta} \left(\theta - \frac{\theta}{2} \right) = \frac{1}{2}
\end{aligned}$$

Exercise 4.8.17 (page 220).
Let X repesent the number of car accidents a person experiences over a year and let us assume that $X|\lambda \sim \mathrm{Po}(\lambda)$ where $\lambda > 0$ is a parameter denoting the person's propensity for accidents. It is further assumed that such a propensity is random and $\lambda \sim \mathrm{Ga}(\alpha, \beta)$.

(a) Determine the probability mass function p_X of X.

(b) Determine $f_{\lambda|X}$.

Solution.

As for (a), this follows from

$$
\begin{aligned}
p_X(x) &= \mathbb{1}_{\{0,1,2,\dots\}}(x) \int_0^{+\infty} \frac{\lambda^x\, \mathrm{e}^{-\lambda}}{x!} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1}\, \mathrm{e}^{-\beta\lambda}\, \mathrm{d}\lambda \\
&= \mathbb{1}_{\{0,1,2,\dots\}}(x) \frac{\beta^\alpha}{x!\,\Gamma(\alpha)} \int_0^{+\infty} \lambda^{\alpha+x-1}\, \mathrm{e}^{-(\beta+1)\lambda}\, \mathrm{d}\lambda \\
&= \left(\frac{\beta}{\beta+1}\right)^\alpha \frac{1}{(\beta+1)^x\, x!} \frac{\Gamma(\alpha+x)}{\Gamma(\alpha)}\, \mathbb{1}_{\{0,1,2,\dots\}}(x)
\end{aligned}
$$

where we let $\prod_{i=0}^{-1} \equiv 1$.

The solution of (b) is obtained by applying he definition of conditional density. Hence, for any $x \in \{0, 1, 2, \dots\}$ one has

$$
\begin{aligned}
f_{\lambda|X}(\lambda|x) &= \frac{\frac{\lambda^x\, \mathrm{e}^{-\lambda}}{x!} \frac{\beta^\alpha}{\Gamma(\alpha)}}{\left(\frac{\beta}{\beta+1}\right)^\alpha \frac{\Gamma(\alpha+x)}{(\beta+1)^x\, x!\Gamma(\alpha)}}\, \mathbb{1}_{(0,+\infty)}(\lambda) \\
&= \frac{(\beta+1)^{\alpha+x}}{\Gamma(\alpha+x)} \lambda^{\alpha+x-1}\, \mathrm{e}^{-(\beta+1)\lambda}\, \mathbb{1}_{(0,+\infty)}(\lambda)
\end{aligned}
$$

and, hence, $\lambda|x \sim \mathrm{Ga}(\alpha+x,\, \beta+1)$. The fact that both λ and $\lambda|x$ have a gamma distribution, though with different parameters, is referred to as *conjugacy* in Bayesian statistics.

Exercise 4.8.18 (page 220).
Let $Y \sim \mathrm{Ga}(\alpha, \lambda)$ and suppose that $X_1, \dots, X_n | Y = y \overset{\text{iid}}{\sim}$ neg-exp(y). Show that

$$
Y|X_1 = x_1, \dots, X_n = x_n \sim \mathrm{Ga}(\alpha+n, \lambda + \sum_{i=1}^n x_i).
$$

Solution.
From the definition of conditional density

$$
f_{Y|X_1,\dots,X_n}(y|x_1,\dots,x_n) = \frac{f_{X_1,\dots,X_n,Y}(x_1,\dots,x_n,y)}{f_{X_1,\dots,X_n}(x_1,\dots,x_n)}
$$

As for the denominator, one may use the assumption of conditional independent and identity in distribution of the X_i's, given Y,

$$
f_{X_1,\dots,X_n|Y}(x_1,\dots,x_n|y) = \prod_{i=1}^n f_{X_i|Y}(x_i|y) = \prod_{i=1}^n y\, \mathrm{e}^{-y\, x_i} = y^n\, \mathrm{e}^{-y \sum_{i=1}^n x_i}
$$

and the marginal density of $(X_1, \dots, X_n)$ is obtained as follows

$$\begin{aligned} f_{X_1,\dots,X_n}(x_1,\dots,x_n) &= \int_{-\infty}^{+\infty} f_{X_1,\dots,X_n|Y}(x_1,\dots,x_n|y)\, f_Y(y)\,\mathrm{d}y \\ &= \int_0^{+\infty} y^n\, \mathrm{e}^{-y\sum_{i=1}^n x_i}\, \frac{\lambda^\alpha}{\Gamma(\alpha)}\, y^{\alpha-1}\, \mathrm{e}^{-\lambda y}\,\mathrm{d}y \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} y^{\alpha+n-1}\, \mathrm{e}^{-(\lambda+\sum_{i=1}^n x_i)y}\,\mathrm{d}y \\ &= \frac{\lambda^\alpha}{(\lambda+\sum_{i=1}^n x_i)^{\alpha+n}}\, \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \end{aligned}$$

The numerator can be evaluated as follows

$$\begin{aligned} f_{X_1,\dots,X_n,Y}(x_1,\dots,x_n,y) &= f_{X_1,\dots,X_n|Y}(x_1,\dots,x_n|y)\, f_Y(y) \\ &= y^n\, \mathrm{e}^{-y\sum_{i=1}^n x_i}\, \frac{\lambda^\alpha}{\Gamma(\alpha)}\, y^{\alpha-1}\, \mathrm{e}^{-\lambda y}. \end{aligned}$$

By considering the ratio, one has

$$f_{Y|X_1,\dots,X_n}(y|x_1,\dots,x_n) = \frac{\Gamma(\alpha+n)}{(\lambda+\sum_{i=1}^n x_i)^{\alpha+n}}\, y^{\alpha+n-1}\, \mathrm{e}^{-(\lambda+\sum_{i=1}^n x_i)y}\, \mathbb{1}_{(0,+\infty)}(y)$$

which shows that $Y|X_1 = x_1, \dots, X_n = x_n \sim \mathrm{Ga}(\alpha+n, \lambda+\sum_{i=1}^n x_i)$.

Exercise 64 (p. 226)
Suppose the density function of (X, Y) is

$$f_{X,Y}(x,y) = 30(y-x)^4\, \mathbb{1}_{(0,1)}(x)\, \mathbb{1}_{(x,1)}(y).$$

(a) Determine the marginal density f_y of Y.

(b) Determine the conditional density $f_{X|Y}$

(c) Determine the conditional density of $U = X/Y$, given $Y = y$.

(d) Determine $\mathbb{E}[U|Y=y]$ and from this deduce $\mathbb{E}[X|Y=y]$.

Solution.
For any $y \in (0,1)$, the margional density of Y at y can be obtained by integrating out x over the interval $(0, y)$. Hence

$$\begin{aligned} f_Y(y) &= \mathbb{1}_{(0,1)}(y) \int_0^y 30(y-x)^4\,\mathrm{d}x = \Big[-6(y-x)^5\Big]_{x=0}^{x=y} \mathbb{1}_{(0,1)}(y) \\ &= 6\,y^5\, \mathbb{1}_{(0,1)}(y). \end{aligned}$$

Hence, $Y \sim Beta(6,1)$.
As for (b), for any $y \in (0,1)$, one has

$$f_{X|Y}(x|y) = \frac{30(y-x)^4\, \mathbb{1}_{(0,y)}(x)}{6\,y^5} = \frac{5}{y^5}\,(y-x)^4\, \mathbb{1}_{(0,y)}(x)$$

The solution to (c) follows from applying the variable transformation $U = X/y$ to the above conditional density function of X, given $Y = y$. Indeed, in this case $U = g(X) = X/y$, from which $g^{-1}(u) = yU$ and $|J| = y$. Hence, the change of variable technique yields

$$\begin{aligned} f_{U|Y}(u|y) &= f_{X|Y}(g^{-1}(u)|y)\,|J| = f_{X|Y}(yu|y)\,y \\ &= \frac{5}{y^5}\,(y - yu)^4\,y\,\mathbb{1}_{(0,y)}(yu) \\ &= 5(1-u)^4\,\mathbb{1}_{(0,1)}(u) \end{aligned}$$

which implies that $U = X/Y \sim \text{Beta}(1,5)$, conditionally on $Y = y$. Interestingly, this does not depend on $Y = y$.

Finally, the solution to (d) is from

$$\begin{aligned} \mathbb{E}[U|Y=y] &= \int_0^1 u\,5(1-u)^4\,\mathrm{d}u = 5\int_0^1 u(1-u)^4\,\mathrm{d}u \\ &= 5\,\frac{\Gamma(2)\,\Gamma(5)}{\Gamma(7)} = 5\,\frac{2!\,4!}{6!} = \frac{1}{3} \end{aligned}$$

and

$$\mathbb{E}[X|Y=y] = \mathbb{E}[Y\,U|Y=y] = y\,\mathbb{E}[U|Y=y] = \frac{1}{3}\,y.$$

for any $y \in (0,1)$.

Order statistics

Exercise 18 (p. 249)
Let $X_1, \ldots, X_n$ be a sample of i.i.d. random variables from a $\mathrm{U}(0,1)$ distribution. Prove that

(a) $X_{(1)} \xrightarrow{\mathrm{P}} 0$

(b) $U_n = nX_{(1)} \xrightarrow{\mathrm{D}} V$, where $V \sim \mathrm{Ga}(1,1)$ or, equivalently, $V \sim \mathrm{E}(1)$.

Solution.
We first need to show that

$$\lim_{n\to\infty} \mathbb{P}[|X_{(1)}| \geq \varepsilon] = 0 \qquad \forall\varepsilon > 0.$$

Since the pdf and the cdf of the random variables of the sample are

$$f(x) = \mathbb{1}_{(0,1)}(x), \qquad F(x) = x\,\mathbb{1}_{[0,1)}(x) + \mathbb{1}_{[1,+\infty)}(x),$$

the cdf of $X_{(1)}$ is

$$F_{X_{(1)}}(x) = 1 - \Big(1 - F(x)\Big)^n = 1 - \Big(1 - x\,\mathbb{1}_{[0,1)}(x) - \mathbb{1}_{[1,+\infty)}(x)\Big)^n$$
$$= 1 - (1-x)^n\,\mathbb{1}_{[0,1)}(x).$$

Hence, for any $\varepsilon \in (0,1)$ and $n \geq 1$

$$\mathbb{P}[|X_{(1)}| \geq \varepsilon] = \mathbb{P}[X_{(1)} \geq \varepsilon] = 1 - F_{X_{(1)}}(\varepsilon) = (1-\varepsilon)^n.$$

This holds true also in the limit and one then has

$$\lim_{n\to\infty} \mathbb{P}[|X_{(1)}| \geq \varepsilon] = \lim_{n\to\infty} (1-\varepsilon)^n = 0.$$

If $\varepsilon \geq 1$, one rivally has $\mathbb{P}[|X_{(1)}| \geq \varepsilon] = 0$ for any n. Hence $X_{(1)} \xrightarrow{\text{p}} 0$. As for (b), note that the cdf of U_n is

$$\mathbb{P}[U_n \leq u] = \mathbb{P}[nX_{(1)} \leq u] = \mathbb{P}\Big[X_{(1)} \leq \frac{u}{n}\Big] = F_{X_{(1)}}\Big(\frac{u}{n}\Big)$$
$$= \Big\{1 - \Big(1 - \frac{u}{n}\Big)^n\Big\}\,\mathbb{1}_{[0,+\infty)}(u)$$

By recalling the definition of the exponential function as a limit, one has

$$\lim_{n\to\infty} F_{U_n}(u) = \Big(1 - \mathrm{e}^{-u}\Big)\,\mathbb{1}_{[0,+\infty)}(u) \qquad \forall u \in \mathbb{R}$$

and the limit is the cdf of a random variable having a negative–exponential distribution with parameter 1.

Exercise 3.6.5 (p. 159)
Let $X_1, \ldots, X_n$ be a sample of i.i.d. random variables from a common cumulative distribution function F and let $X_{(1)} = \min\{X_1, \ldots, X_n\}$ be the first order statistic. If the common distribution is Pareto with parameters $(\alpha, 1)$, show that the distribution of $X_{(1)}$ is still Pareto, with updated parameters. Additionally, show that $X_{(1)}$ converges in probability to 1.

Solution.
Recall that

$$F_{X_{(1)}}(x) = \mathbb{P}[X_{(1)} \leq x] = 1 - \mathbb{P}[X_{(1)} > x] = 1 - \mathbb{P}[X_1 > x, \ldots, X_n > x]$$
$$= 1 - \Big\{\mathbb{P}[X_1 > x]\Big\}^n = 1 - \Big\{1 - F(x)\Big\}^n$$

and, if continuous and with p.d.f. equal to f, the p.d.f. of $X_{(1)}$ is

$$f_{X_{(1)}}(x) = n\,f(x)\,\Big\{1 - F(x)\Big\}^{n-1}.$$

In the specific case of the exercise, since $X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Pareto}(\alpha, 1)$, one has

$$f(x) = \frac{\alpha}{x^{\alpha+1}} \mathbb{1}_{(1,+\infty)}(x)$$

and

$$F(x) = \mathbb{1}_{(1,+\infty)}(x) \int_1^x \frac{\alpha}{s^{\alpha+1}} \,\mathrm{d}s = \left\{1 - \frac{1}{x^\alpha}\right\} \mathbb{1}_{(1,+\infty)}(x).$$

Hence, the p.d.f. of $X_{(1)}$ is

$$\begin{aligned} f_{X_{(1)}}(x) &= n \, \frac{\alpha}{x^{\alpha+1}} \, \mathbb{1}_{(1,+\infty)}(x) \left\{1 - \left(1 - \frac{1}{x^\alpha}\right) \mathbb{1}_{(1,+\infty)}(x)\right\}^{n-1} \\ &= \frac{n\alpha}{x^{\alpha+1}} \, \frac{1}{x^{(n-1)\alpha}} \, \mathbb{1}_{(1,+\infty)}(x) \\ &= \frac{n\alpha}{x^{n\alpha+1}} \, \mathbb{1}_{(1,+\infty)}(x) \end{aligned}$$

and, then, $X_{(1)} \sim \text{Pareto}(n\alpha, 1)$. Moreover, its c.d.f. is

$$\begin{aligned} F_{X_{(1)}}(x) &= 1 - \left\{1 - \left(1 - \frac{1}{x^\alpha}\right) \mathbb{1}_{(1,+\infty)}(x)\right\}^n \\ &= \left\{1 - \frac{1}{x^{n\alpha}}\right\} \mathbb{1}_{(1,+\infty)}(x). \end{aligned}$$

this is useful for studying convergence in probability of $X_{(1)}$ to 1 as $n \to \infty$, as we have to prove that

$$\lim_{n\to\infty} \mathbb{P}[|X_{(1)} - 1| < \varepsilon] = 0 \qquad \forall \varepsilon > 0.$$

Note that

$$\mathbb{P}[|X_{(1)} - 1| < \varepsilon] = \mathbb{P}[1 - \varepsilon < X_{(1)} < 1 + \varepsilon] = F_{X_{(1)}}(1+\varepsilon) - F_{X_{(1)}}(1-\varepsilon).$$

Since $F_{X_{(1)}}(1-\varepsilon) = 0$ and $F_{X_{(1)}}(1+\varepsilon) = 1 - (1+\varepsilon)^{-n\alpha}$, one has

$$\lim_{n\to\infty} \mathbb{P}[|X_{(1)} - 1| < \varepsilon] = \lim_{n\to\infty} \left\{1 - \frac{1}{(1+\varepsilon)^{n\alpha}}\right\} = 1$$

for any $\varepsilon > 0$ and this proves that

$$X_{(1)} \xrightarrow{\text{p}} 1.$$

The same arguments apply if $X_1, \dots, X_n$ are iid from a Pareto distribution with parameters (α, θ), namely

$$f(x) = \frac{\alpha \, \theta^\alpha}{x^{\alpha+1}} \, \mathbb{1}_{(\theta,+\infty)}(x).$$

In this case, one may show that $X_{(1)}$ is Pareto with parameters $(n\alpha, \theta)$.

Exercise 4.7.1.4 (p. 217)
Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathrm{U}(\theta-1, \theta+1)$. Determine

(a) $f_{X_{(1)}, X_{(n)}}$

(b) $\mathbb{P}[X_{(1)} < \theta < X_{(n)}]$

Solution.
In this case, one has

$$f(x) = \frac{1}{2}\,\mathbb{1}_{(\theta-1,\theta+1)}(x), \quad F(x) = \frac{x-\theta+1}{2}\,\mathbb{1}_{(\theta-1,\theta+1)}(x) + \mathbb{1}_{[\theta+1,+\infty)}(x).$$

Using the general result on the density function of $(X_{(1)}, X_{(n)})$, one has

$$\begin{aligned}
f_{X_{(1)}, X_{(n)}}(x_1, x_n) &= n(n-1)\, f(x_1) f(x_2)\, \{F(x_n) - F(x_1)\}^{n-2}\,\mathbb{1}_{(x_1,+\infty)}(x_n) \\
&= n(n-1)\,\frac{1}{2}\,\mathbb{1}_{(\theta-1,\theta+1)}(x_1)\,\frac{1}{2}\,\mathbb{1}_{(\theta-1,\theta+1)}(x_n) \\
&\qquad \times \left(\frac{x_n - x_1}{2}\right)^{n-2} \mathbb{1}_{(x_1,+\infty)}(x_n) \\
&= \frac{n(n-1)}{2^n}\,(x_n - x_1)^{n-2}\,\mathbb{1}_{(\theta-1,\theta+1)}(x_1)\,\mathbb{1}_{(x_1,\theta+1)}(x_n)
\end{aligned}$$

As for (b), one has

$$\begin{aligned}
\mathbb{P}[X_{(1)} < \theta < X_{(n)}] &= \int_{\theta-1}^{\theta} \mathrm{d}x_1 \int_{\theta}^{\theta+1} \mathrm{d}x_n\,\frac{n(n-1)}{2^n}\,(x_n - x_1)^{n-2} \\
&= \frac{n}{2^n}\int_{\theta-1}^{\theta}\left(\int_{\theta}^{\theta+1}(n-1)(x_n-x_1)^{n-2}\,\mathrm{d}x_n\right)\mathrm{d}x_1 \\
&= \frac{1}{2^n}\int_{\theta-1}^{\theta}\left\{n(\theta+1-x_1)^{n-1} - n(\theta-x_1)^{n-1}\right\}\mathrm{d}x_1 \\
&= \frac{1}{2^n}\,\{2^n - 1 - 1\} = 1 - \frac{1}{2^{n-1}}.
\end{aligned}$$

It is worth noting that

$$\lim_{n\to\infty} \mathbb{P}[X_{(1)} < \theta < X_{(n)}] = 1.$$

Exercise (midterm exam in 2019).
Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathrm{E}(\lambda)$, i.e. they all have density function

$$f(x) = \lambda\,\mathrm{e}^{-\lambda x}\,\mathbb{1}_{(0,+\infty)}(x)$$

for some $\lambda > 0$.

(a) Determine the density functions of the order statistics

$$X_{(1)} = \min\{X_1, \ldots, X_n\} \qquad \& \qquad X_{(n)} = \max\{X_1, \ldots, X_n\}.$$

(b) Identify the distribution of $Y_n = n\,X_{(1)}$ and describe how would you simulate pseudo–random realizations of Y_n?

(c) Determine the cumulative distribution function of $W_n = \lambda X_{(n)} - \log n$. Can you say anything about the convergence in distribution of W_n, as $n \to \infty$?

Solution.
As for (a), start noting that $f(x) = \lambda\,\mathrm{e}^{-\lambda x}\,\mathbb{1}_{(0,+\infty)}(x)$ and

$$F(x) = \mathbb{1}_{(0,+\infty)}(x) \int_0^x \lambda\,\mathrm{e}^{-\lambda s}\,\mathrm{d}s = \left\{1 - \mathrm{e}^{-\lambda x}\right\} \mathbb{1}_{(0,+\infty)}(x).$$

From general formulae for the density functions of $X_{(1)}$ and of $X_{(n)}$, one has

$$f_{X_{(1)}}(x) = n\,f(x)\,\{1 - F(x)\}^{n-1} = n\lambda\,\mathrm{e}^{-n\lambda x}\,\mathbb{1}_{(0,+\infty)}(x)$$

$$f_{X_{(n)}}(x) = n\,f(x)\,\{F(x)\}^{n-1} = n\lambda\,\mathrm{e}^{-\lambda x}\{1 - \mathrm{e}^{-\lambda x}\}^{n-1}\,\mathbb{1}_{(0,+\infty)}(x)$$

As for (b), since $Y_n = g(X_{(1)}) = nX_{(1)}$, one then has $X_{(1)} = g^{-1}(Y_n) = Y_n/n$ and $(\mathrm{d}/\mathrm{d}y)g^{-1}(y) = 1/n$. Hence

$$f_{Y_n}(y) = f_{X_{(1)}}(y/n)\frac{1}{n} = \frac{1}{n}\,n\lambda\,\mathrm{e}^{-n\lambda\,\frac{y}{n}}\mathbb{1}_{(0,+\infty)}(y) = \lambda\,\mathrm{e}^{-\lambda\,y}\mathbb{1}_{(0,+\infty)}(y)$$

Hence, $Y_n \sim \mathrm{E}(\lambda)$. To simulate Y_n, one can proceed as follows

- Generate $U = u$ from a $\mathrm{Unif}(0,1)$
- Set $y = -\lambda^{-1}\,\log(1-u)$
- Return $Y = y$

In order to determine convergence in distribution in (c), recall that

$$F_{X_{(n)}}(x) = \{F(x)\}^n = \{1 - \mathrm{e}^{-\lambda x}\}\,\mathbb{1}_{(0,+\infty)}(x)$$

For any $w \in \mathbb{R}$ one can take n large enough so that $(w + \log n)/\lambda > 0$ and

$$\begin{aligned} F_{W_n}(w) &= \mathbb{P}[W_n \le w] = \mathbb{P}[\lambda X_{(n)} - \log n \le w] = F_{X_{(n)}}((w + \log n)/\lambda) \\ &= \left(1 - \mathrm{e}^{-\lambda\,\frac{w+\log n}{\lambda}}\right)^n = \left(1 - \frac{\mathrm{e}^{-w}}{n}\right)^n \\ &\to \mathrm{e}^{-\mathrm{e}^{-w}} \qquad (\text{as} \quad n \to \infty) \end{aligned}$$

which is a c.d.f. on $\mathbb{R}$ that identifies the limiting random variable, in distribution.