Week 3: Random Variables

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Overview

- This week will cover Module 3
  - Module 3: Probability - Random Variables
- Topics include:
  - Random variables and probability distributions
  - Bernoulli/Binomial random variables
  - Expected Values and Variances of Discrete random variables
  - Expected Values and Variances of Sample Means
  - Normal Distribution and Table

Random Variables

What are random variables?

- **Random variables** are very similar to events.
  - They represent some collection of outcomes of interest from an experiment
  - So they are often used to represent outcomes of interest like
    - the number of heads out of 3 coin flips
    - the sum of rolling 2 dice
    - the height of students in this class
    - etc.
  - Usually denoted using a capital letter
  - Example: X = the number of Heads in 10 coin flips. X is a random variable.
- Random variables differ from events because they take the possible outcomes of an experiment and instead give us a number
- We don't know if/when they will come up, so we can talk about their probability

What are random variables?

- Random variables <u>differ</u> from events because they take the possible outcomes of an experiment and instead <u>give us a number</u>
  - Event: Flipping 2 heads in 5 coin flips
  - Event: Flipping 5 heads in 5 coin flips
  - Random variable: The number of Heads in 5 coin flips
- We don't know if/when random variables will take on specific values, so we can talk about their probability

Event: Let A be the event of rolling a 1 on a die

$P(A) = \frac{1}{6}$

R.V.: Let X be the result of rolling a die

$P(X = 1) = \frac{1}{6}$

Random Variables and Probability

- In the same way that we can talk about the probability a certain event happens, we can talk about the probability the random variable takes some value
- **Probability models/distribution** gives us a way to talk about the chances of seeing certain values
  - they describe how randomness affects what we see in an experiment
- But we do not always know what are the chances of seeing certain outcomes
  - therefore we run an experiment to **estimate** the probability of these outcomes
- When we list out all the possible values for a random variable, and all of their probabilities of happening, we get a probability distribution/model.

Properties of Random Variables

- Since random variables are similar to events, we have similar properties that can help us find probabilities that the random variable takes a certain value
- Ex: we can say X is a random variable representing the number of heads in 3 coin flips
  - X can have values 0, 1, 2, 3
  - We can also talk about the probability I get 2 or more heads, $P(X \geq 2)$
    - just like with events, this probability must be a number between 0 and 1
    - we can also talk about the complement of this probability: $P(X < 2) = 1 - P(X \geq 2)$
- If I also roll a dice and say Y represents the value of the roll, I can say that X and Y are **independent** if the random variables have no effect on each other

Types of Random Variables

- Random variables can be one of two types
- The type depends on the kinds of numbers that they can take
- **Discrete** random variables can only take distinct numbers, like integers/whole numbers
  - you can sometimes think of this as the analogue of a qualitative variable (i.e. treat these whole numbers as a bin)
- **Continuous** random variables can take any number in an interval, like a decimal number or fraction
  - again, you could think of this as the analogue of a quantitative variable (i.e. measured on a scale/interval, can be decimals)

Check your understanding!

What type of variable is “the number of times you filled your water bottle today”?

A. Discrete random variable

B. Continuous random variable

Check your understanding!

What type of variable is “the amount of water (in litres) you have drank today”?

A. Discrete random variable

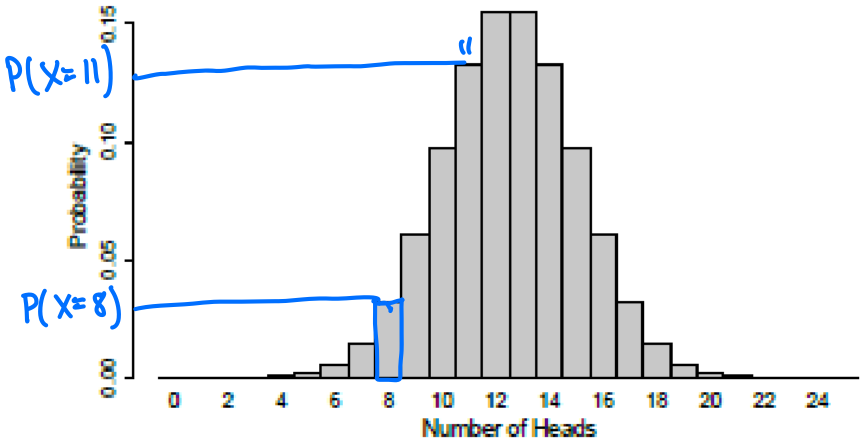
B. Continuous random variable

Types of Random Variables

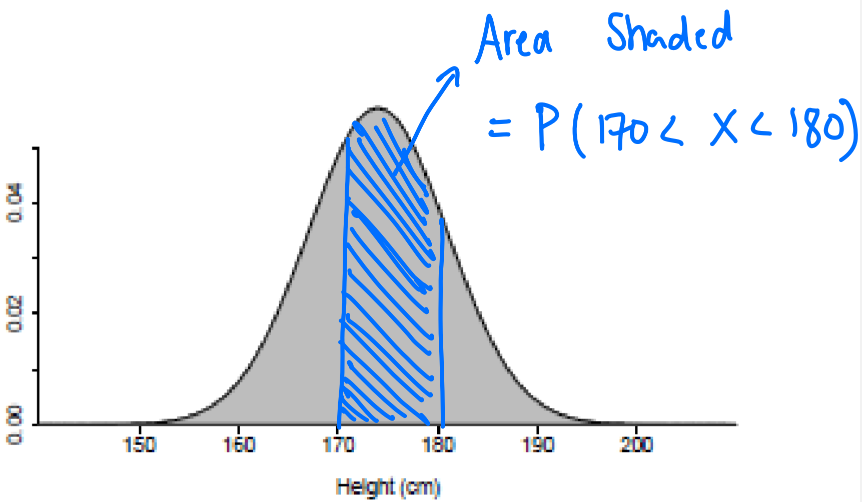
- It is important to be able to tell the difference between discrete and continuous random variables
- This is because you treat each one slightly differently
- For example, how we can visualize a random variable changes depending on if it is discrete or continuous
  - this is like quantitative versus qualitative variables
  - **Discrete** probability distributions can be graphed with a bar plot
  - **Continuous** probability distributions are often visualized using density plots
- Need to recognize which one you are dealing with because the probability model will also be different.

Discrete

Continuous



(a) The distribution of the number of heads in 25 tosses



(b) The distribution of the height of adult Canadian males

Figure 5.1: A discrete probability distribution and a continuous probability distribution.

Bernoulli/Binomial random variables

Types of Discrete Variables

- You will generally be responsible for working with 2 main types of Discrete random variables - Bernoulli and Binomial
- When we say that we are dealing with a Bernoulli or a Binomial random variable, we are actually referring to the <u>shape and properties of the probability model</u>
- So in order for us to say that we have a Bernoulli or Binomial model, we need to be able to <u>recognize certain features</u> of these probability models that tell us which one we have

Bernoulli Random Variables

- A **Bernoulli** random variable is most easily recognized as a variable that only takes on 2 possible values
  - Any situation that deals with outcomes that are Yes/No, True/False, Heads/Tails, etc. are Bernoulli variables.
  - Obviously these possible outcomes are not numerical values
    - We can easily just label Yes = 1 and No = 0, and then we have given our two outcomes a numerical value
    - Then we say that $X = 1$ when we get outcome Yes, and $X = 0$ when we get outcome No.
  - We usually call the event of interest a "success", and the complement of that a "failure"

Probability with Bernoulli's

- This is the simplest discrete random variable to find probabilities for, because we only have to find 2 probabilities, one for $\mathrm{X} = 1$, and one for $X = 0$
  - It is also helpful to notice that, because we only have 2 values, we can easily write $P(X = 1) = 1 - P(X = 0)$
- Example: I roll a dice and I'm interested in if I get a 3 or something else
  - This is a Bernoulli, because I can either get a 3 ($X = 1$) or anything else ($X = 0$)
  - Since I only have 6 possible faces to land on, $P(X = 1) = \frac{1}{6}$
  - Therefore, $\mathrm{P}(X = 0) = 1 - P(X = 1) = 1 - \frac{1}{6} = \frac{5}{6}$
- Special case: If I get that $P(X = 1) = P(X = 0)$, then we can say that we have Uniform probability.

Bernoulli Notation

"follows"

" has the distribution of "

$X \sim Bern(p)$

$p = \frac{1}{6}$

- Therefore we have a Bernoulli random variable,

$$X \sim Bern(p)$$

Where X is the random variable and p is the probability of my Bernoulli outcome of interest (success)

$p = P(X = 1)$

Eg. Let $X = 1$ if I roll a 3, and $X = 0$ otherwise

$X \sim Bern(\frac{1}{6})$

Binomial Random Variables

- If we go back to our coin flipping examples, it should be pretty clear now that the outcome from flipping a *single* coin once is a Bernoulli random variable
  - $X = 1$ if I get heads, $X = 0$ if I get tails
  - $P(X = 1) = P(X = 0) = 1/2$
- But we often saw experiments like "flip the same coin 3 times and count the number of heads"
  - each individual coin flip still has heads/tails as the only 2 options -> Bernoulli
  - each coin flip (because just flipping one coin 3 times) has the same chance of landing on heads (1/2)
  - so by flipping the coin 3 times, I am repeating my Bernoulli experiment 3 independent times

Binomial Random Variables

- This is the setup to look for to help you identify a **Binomial** random variable
  - repeating independent Bernoulli experiments/trials
    1. only two outcomes for each trial
    2. same probability of seeing the outcome of interest at each trial
    3. interested in number of times I see my outcome of interest
- So the Binomial is just a collection of independent Bernoullis!
- When we talk about what the distribution for a Binomial will look like, we need to refer to two pieces of information:
  - probability of my Bernoulli outcome of interest (p)
  - the number of Bernoulli trials I am dealing with (n)
- We state this as $Y \sim Bin(n, p)$

Example: Carriers of a disease

- If a couple are both carriers of a particular disease, a child of theirs has probability 1/4 of having the disease. If the couple has 4 children, and we are interested in the number of them with the disease, what random variable do we have?
  - for one child, you either have the disease ($X = 1$) or you don't ($X = 0$)
  - the probability one child gets the disease, $P(X = 1) = \frac{1}{4}$
  - this probability is the same for each child, because children are independent
  - and we are counting the number of outcomes of interest
  - Therefore we have a Binomial random variable,

$$Y \sim Bin(4, 1/4)$$

where Y is the number of children out the 4 children that have the disease

Calculating Probabilities for Binomial Variables

- What if you were asked to find the probability that 2 of the 4 children have the disease?
- We would want $P(Y = 2)$ where $Y \sim Bin(4, ¼)$
- <u>Option 1</u>: In general, for a Binomial random variable $X \sim Bin(n, p)$, we have that

Aside:

$3! = 3 \times 2 \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

$0! = 1$

$1! = 1$

$$P(Y = k) = \frac{n!}{k!\,(n-k)!}\, p^k\, (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Calculating Probabilities for Binomial Variables

- What if you were asked to find the probability that 2 of the 4 children have the disease?
- We would want $P(Y = 2)$ where $Y \sim Bin(4, ¼)$
- <u>Option 2:</u> Use Binomial Distribution Table

Binomial Probability Table

n=Number of trials, k=Number of successes and p=Probability of success

		p										
n	k	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
2	0	0.9801	0.9025	0.8100	0.7225	0.6400	0.5625	0.4900	0.4225	0.3600	0.3025	0.2500
	1	0.0198	0.0950	0.1800	0.2550	0.3200	0.3750	0.4200	0.4550	0.4800	0.4950	0.5000
	2	0.0001	0.0025	0.0100	0.0225	0.0400	0.0625	0.0900	0.1225	0.1600	0.2025	0.2500
3	0	0.9703	0.8574	0.7290	0.6141	0.5120	0.4219	0.3430	0.2746	0.2160	0.1664	0.1250
	1	0.0294	0.1354	0.2430	0.3251	0.3840	0.4219	0.4410	0.4436	0.4320	0.4084	0.3750
	2	0.0003	0.0071	0.0270	0.0574	0.0960	0.1406	0.1890	0.2389	0.2880	0.3341	0.3750
	3		0.0001	0.0010	0.0034	0.0080	0.0156	0.0270	0.0429	0.0640	0.0911	0.1250
4	0	0.9606	0.8145	0.6561	0.5220	0.4096	0.3164	0.2401	0.1785	0.1296	0.0915	0.0625
	1	0.0388	0.1715	0.2916	0.3685	0.4096	0.4219	0.4116	0.3845	0.3456	0.2995	0.2500
	2	0.0006	0.0135	0.0486	0.0975	0.1536	0.2109	0.2646	0.3105	0.3456	0.3675	0.3750
	3		0.0005	0.0036	0.0115	0.0256	0.0469	0.0756	0.1115	0.1536	0.2005	0.2500
	4			0.0001	0.0005	0.0016	0.0039	0.0081	0.0150	0.0256	0.0410	0.0625

Back to Example: Carriers of a disease

- What if you were asked to find the probability that 2 of the 4 children have the disease?

Let Y be the # of children with the disease

$Y \sim Bin(4, \frac{1}{4}) \equiv Bin(n=4, p=\frac{1}{4})$

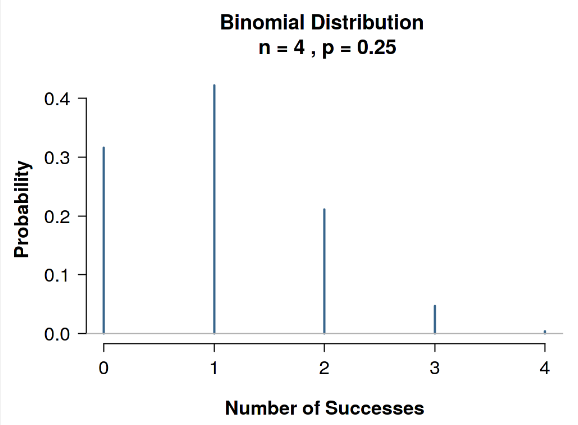
Option 1 : $P(Y=2) = \frac{4!}{2!\,2!} 0.25^2 (1-0.25)^2 = 0.2109$

Option 2 : Using Table, $P(Y=2) = 0.2109$

Probability Mass Function

- For any discrete random variable, the **probability mass function (PMF)** is $P(X = k)$ as a function of k
- The PMF can be shown in a table or a graph

Y = y	0	1	2	3	4
P(Y = y)	0.316	0.422	0.211	0.047	0.004



- View the PMF for any binomial distribution [here]

Back to Example: Carriers of a disease

- What if you were asked to find the probability that 3 or more children have the disease?
  - We need to know the probability distribution for all values of my Binomial

Y = y	0	1	2	3	4
P(Y = y)	0.316	0.422	0.211	0.047	0.004

$$P(Y \geq 3) = P(Y=3) + P(Y=4)$$

$$P(Y \geq 3) = 1 - P(Y < 3)$$

$$= 1 - [P(Y=0) + P(Y=1) + P(Y=2)]$$

Expected Values of Random Variables

Expected Values

- **Expected values** are just averages/means!
- It represents the value that you expect a random variable to take on
- We use

$$E(X) = \sum_{x_i} x_i \, P(X = x_i)$$

  - this just says that for every data point/value for X, multiply by its probability and add this up for every x.
  - what changes is how we talk about the probability of either each data point or each possible value of the random variable

Example: Dice Rolls

Let X = the result of a 6-sided die. Find the expected value of X

$$E(X) = 1 \times P(X=1) + 2 \times P(X=2) + 3 \times P(X=3) + 4 \times P(X=4) +$$
$$5 \times P(X=5) + 6 \times P(X=6)$$
$$= \left(1 \times \tfrac{1}{6}\right) + \left(2 \times \tfrac{1}{6}\right) + \left(3 \times \tfrac{1}{6}\right) + \left(4 \times \tfrac{1}{6}\right) + \left(5 \times \tfrac{1}{6}\right) + \left(6 \times \tfrac{1}{6}\right)$$
$$= 3.5$$

Findings about the Expected Value

- If I am rolling a fair die, then I know what the chances/probabilities are of landing on each side. The expected value represents the average die roll if I kept repeating the experiment over and over again
  - If I roll the die many times, I would *expect* the average of all those die rolls to be 3.5.
- The expected value of a discrete random variable doesn't need to be a valid realization
  - I expect to roll 3.5 on average, even though 3.5 isn't on the die so, $P(X = 3.5) = 0$
  - The expected value is a mean of all possible values for X–> it is a weighted mean where the weights are the probability $P(X = x)$
- The calculation of expected value using this formula requires knowing the full probability model. In other words, $P(X = x)$ for every x

Expected Values of Bernoulli

- Since the Bernoulli random variable is just an example of a discrete random variable, I can use the expected value formula to find the expected value of a Bernoulli random variable
- Ex: Dice roll example, where now my random variable is $\mathrm{X} = 1$ for if I roll a 6
  - this means $X = 0$ is if I roll not a 6
  - $P(X = 1) = 1/6$ so then $P(X = 0) = 5/6$
  - My expected value would now be:

$$E(x) = 1 \times P(X = 1) + \cancel{0 \times P(X = 0)}$$

$$= P(X = 1)$$

$$= \frac{1}{6}$$

Expected Value of Bernoulli

- In general, for $X \sim Bern(p)$,

$$E(X) = p$$

Expected Value of Binomial

If a couple are both carriers of a disease, a child of theirs has probability 1/4 of having the disease. Suppose the couple has 4 children and we are interested in the number of children with the disease. What is the expected number of children with the disease?

$Y = y$	0	1	2	3	4
$P(Y = y)$	0.316	0.422	0.211	0.047	0.004

Y is the # of children w/ disease $Y \sim Bin(4, 0.25)$

$E(Y) = 0 \times P(Y=0) + \dots + 4 \times P(Y=4)$

$= 1$

Expected Value of Binomial

- In general, for $\mathrm{Y} \sim Bin(n, p)$,

$$E(Y) = np$$

Variances of Random Variables

Variances of Discrete Variables

- Variances of random variables follow a similar logic as with expected values.
- For expected values:
  - take each possible value for the random variable and weight it by the chance it could happen
- For variances:
  - take each possible value for the random variable and see how far it is from the mean, then square it
  - now weight that squared distance by the chance the random variable could take that value
- Just like expected values, I can find the variance for any discrete random variable

Variance of Random Variable

- The formula for the variance of a random variable looks slightly different than the variance of some data
  - Random Variable:

$$Var(X) = \sum_{x_i} \left(x_i - E(X)\right)^2 P(X = x_i)$$

  - Data:

$$\text{Variance} = \frac{\sum(x-\bar{x})^2}{n-1} = \sum(x-\bar{x})^2 \times \frac{1}{n-1}$$

- You can see that when you are finding the variance of data, the $1/(n-1)$ basically just acts like the $P(X = x_i)$ for a random variable
  - it just weights the distances by how likely that data value will happen

Example: Calculate the Variance

If a couple are both carriers of a disease, a child of theirs has probability 1/4 of having the disease. Suppose the couple has 4 children and we are interested in the number of children with the disease. What is the **variance** number of children with the disease?

$Y = y$	0	1	2	3	4
$P(Y = y)$	0.316	0.422	0.211	0.047	0.004

Recall $Y \sim Bin(n = 4, p = 0.25)$ and $E(Y) = 1$

$$Var(Y) = (0-1)^2 P(Y=0) + (1-1)^2 P(Y=1) + \dots + (4-1)^2 P(Y=4)$$
$$= 0.75$$

There are simpler formulas for computing the expectation and variance of random variables that come from known distributions

Summary of Bernoulli and Binomial Distributions

- In general, for $X \sim Bern(p)$,

$$P(X = k) = p^k \times (1-p)^{n-k} = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$$

$$E(X) = p$$

$$Var(X) = p(1-p)$$

- In general, for $\mathrm{Y} \sim Bin(n, p)$,

$$P(Y = k) = \frac{n!}{k!\,(n-k)!} \times p^k \times (1-p)^{n-k}$$

$$E(Y) = np$$

$$Var(Y) = np(1-p)$$

Mean of Sample Means

$E(2+3X) = 2 + 3E(X)$

Some Helpful Properties

- There are some helpful results that we will use in our discussion today
- If I have two random variables, X and Y, and they are independent, then
  - $E(X+Y) = E(X) + E(Y)$
  - $E(a+bX) = a + bE(X)$, where a and b are just numbers (not r.v.)
  - $Var(a+bX) = b^2 Var(X)$
  - $Var(X+Y) = Var(X) + Var(Y)$
  - $Var(X-Y) = Var(X) + (-1)^2 Var(Y) = Var(X) + Var(Y)$

How to use these properties?

Suppose I have a situation where X is a random variable whose distribution has mean = 3 and standard deviation = 0.5 and Y is a random variable whose distribution has mean = 4 and standard deviation = 2. These are independent of one another.

Find the mean and standard deviation of $X + Y$.

$E(X) = 3$

$SD(X) = 0.5$

$Var(X) = 0.5^2 = 0.25$

$E(Y) = 4$

$SD(Y) = 2$

$Var(Y) = 2^2 = 4$

$E(X+Y) = E(X) + E(Y) = 3 + 4 = 7$

$Var(X+Y) = Var(X) + Var(Y) = 0.25 + 4 = 4.25$

$SD(X+Y) = \sqrt{Var(X+Y)} = \sqrt{4.25} = 2.06$

Example:

Consider a game where you flip a fair coin 5 times and win $5 every time you land on Heads. Find the expected value and variance of the dollar amount won.

Let X = # of Heads $\quad X \sim Bin(n=5, p=0.5)$

Let Y = dollar amount won $\quad Y = 5X$

$$E(Y) = E(5X) = 5E(X) = 5np = 5 \times 5 \times 0.5 = 12.5$$

$$Var(Y) = Var(5X) = 5^2 Var(X) = 25 Var(X) = 25np(1-p)$$

$$= 25 \times 5 \times 0.5 \times (1-0.5) = 31.25$$

Mean of Sample Means

- This takes a bit of time to wrap your head around, but it's actually not super crazy.
- Let's think about an easy experiment we know really well:
  - Suppose I flip three coins, and I am interested in counting the number of heads I get
  - My random variable is X = # of heads, so can take values 0, 1, 2, 3
  - The probability distribution for this experiment is

X	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

  - The expected value for this experiment can be found in the usual way

$$E(X) = 0\times\frac{1}{8} + 1\times\frac{3}{8} + 2\times\frac{3}{8} + 3\times\frac{1}{8} = 1.5$$

Means of Sample Means

- Suppose I run my experiment on day 1
  - I will call the possible results of my experiment on this day X_1
  - X_1 has the probability distribution on the previous slide
- Now I run my experiment on day 2
  - I will call the possible results of this experiment on this day X_2
  - X_2 also has the probability distribution on the previous slide
- I can keep running my experiment each day for 10 days
  - I will have my possible results for each day represented as $X_1, X_2, \ldots, X_{10}$
  - Each one has the probability distribution from the previous slide.

Means of Sample Means

- Now why would I be repeating my experiment for 10 days?
  - It's because I know that the outcome I see on one day might change on the next day due to randomness
  - So if I repeat my experiment I can compute a summary of all 10 days such as the mean.
- What does it mean to find the mean from all 10 days?
  - In this case, since I have not yet actually collected the results of my experiments, I just have an average of random variables
  - I can write this as $\frac{X_1+X_2+\cdots+X_{10}}{10}$

Means of Sample Means

- When I take my mean, I'm not using data yet, just the random variables representing possible outcomes of those experiments
- So in order for me to get some idea of the average number of heads out of my 10 experiments, I need to somehow get a value
  - The key is that when I write $\frac{X_1+X_2+\cdots+X_{10}}{10}$, I still don't have a value, just random variables
  - So to get an idea for the average number of heads, I can find what I would expect to get for an average… which is just the expected value of the average of random variables

$$E\left[\frac{X_1 + \cdots + X_{10}}{10}\right]$$

Means of Sample Means

- By using the useful properties of expected values, I can see that

$$E\left(\frac{X_1 + X_2 + \cdots + X_{10}}{10}\right) = \frac{1}{10} E(X_1 + X_2 + \cdots + X_{10})$$
$$= \frac{1}{10}[E(X_1) + E(X_2) + \cdots + E(X_{10})]$$

- But because each variable uses the same probability distribution, each expected value will be the same so

$$\frac{1}{10}[E(X_1) + E(X_2) + \cdots + E(X_{10})] = \frac{1}{10} 10\mathrm{E}(\mathrm{X}) = \mathrm{E}(\mathrm{X})$$

- So as long as we are repeating the same experiment each day, even if I average the random variables for all 10 days, I still end up getting the same expected value as if I only ran my experiment once

Variance of Sample Means

- So if I just get the same expected value for one experiment as I would for 10, why would I do 10 experiments?
- It turns out that the more times you repeat the experiment, you are actually reducing the variance for the average of your experiments compared to the variance of just one experiment:

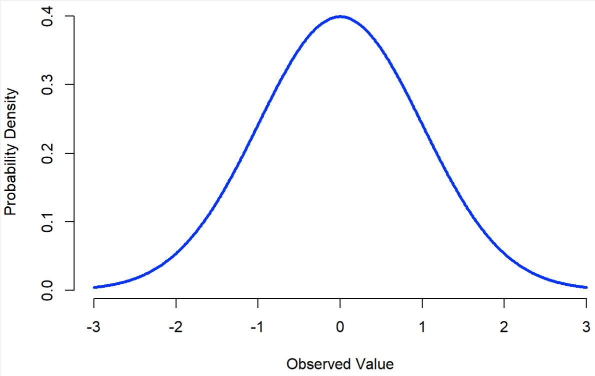
$$Var\left(\frac{X_1 + X_2 + \cdots + X_{10}}{10}\right) = \frac{1}{100} Var(X_1 + X_2 + \cdots + X_{10}) = \frac{1}{100} 10Var(X) = \frac{Var(X)}{10}$$

- So even though I will still have the same expected value in 1 experiment as in 10 experiments, I gain an advantage by being able to reduce the variation of my combined experiment
  - I am able to reduce some of the variability in my results due to randomness!

Continuous Random Variables

Continuous Random Variables

- Continuous random variables can take on any value on an interval, which means decimals/fractions
- Example: X=the height of a 6-year old male
- For discrete variables, we discussed probability mass functions (PMF)
- Equivalently, for continuous variables, we can show the probability distribution using a **probability density functions** (PDF)

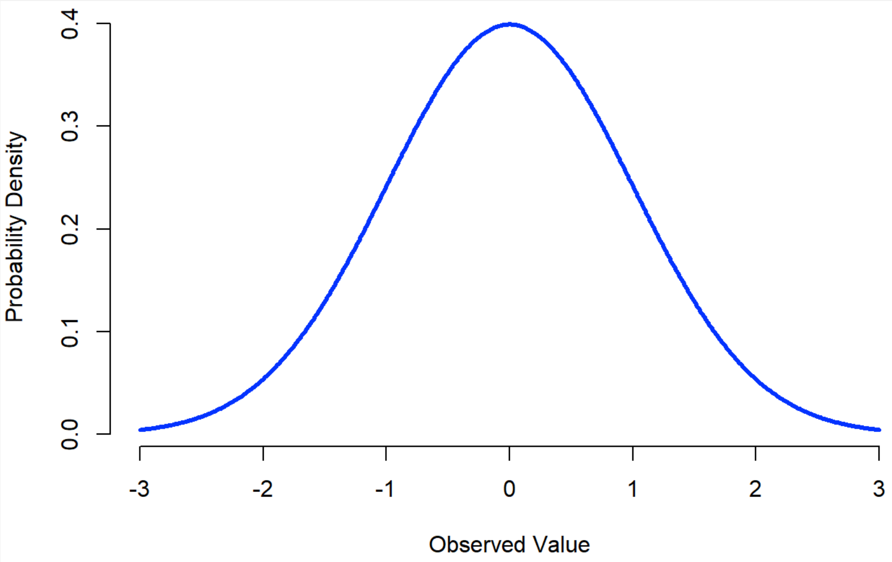


Continuous Random Variables

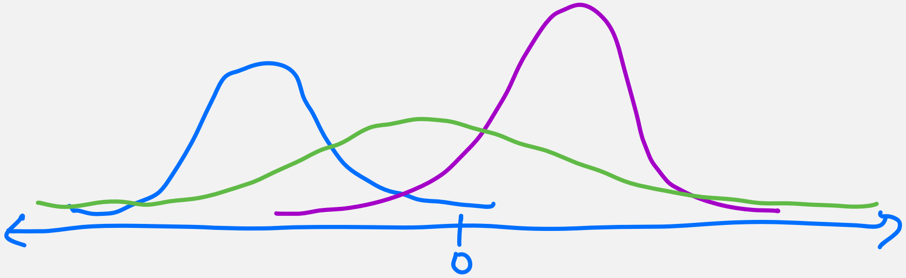
- One thing to remember is that, just because it is a continuous variable, doesn't mean that its probability distribution doesn't follow the same rules as the discrete.
  - in particular, we need to have that if I sum over all the probabilities associated with values my variable can take, I must get that my sum equals 1
  - the consequence of this property is that $P(X = k) = 0$ for all k
  - this happens because in an interval between two numbers, I can keep adding decimals places which would give me a new value X could take
    - If I keep doing this, I will eventually end up with an infinite number of values
    - this means if I add up all their probabilities, if they are not 0, I end up getting a number bigger than 1, which makes no sense.

Normal Distribution

- Now we move on to continuous random variables
- The first one we deal with is the Normal random variable
- It is bell-shaped, symmetric and unimodal



Normal Distribution



- The shape of the Normal distribution is determined by both its centre (mean) and its spread (variance or standard deviation)

$$X \sim N(\mu, \sigma^2)$$

μ = mean

σ^2 = variance $\Rightarrow$ σ = SD

- Certain specific values for these are useful, especially mean = 0 and variance = standard deviation = 1.

$$Z \sim N(0, 1)$$

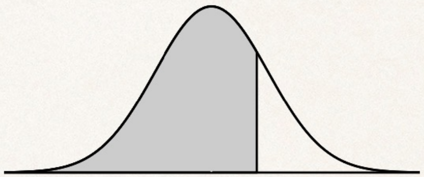
  - this gives us what we call the **standard Normal distribution**, and we know a lot about how this looks

- Since we know exactly what it's shape is and all probabilities of the form $P(X \leq x)$ then we can use this to determine whether data we have collected is actually Normally distributed.

Finding probabilities for the Std. Normal Distribution

- Let Z be a standard normal random variable. That is, $Z \sim N(0,1)$
- We know that $P(Z = k) = 0$ since it is continuous. But how would we find values for $P(Z < k)$?
- Probabilities correspond to the area under the curve

Normal probability table



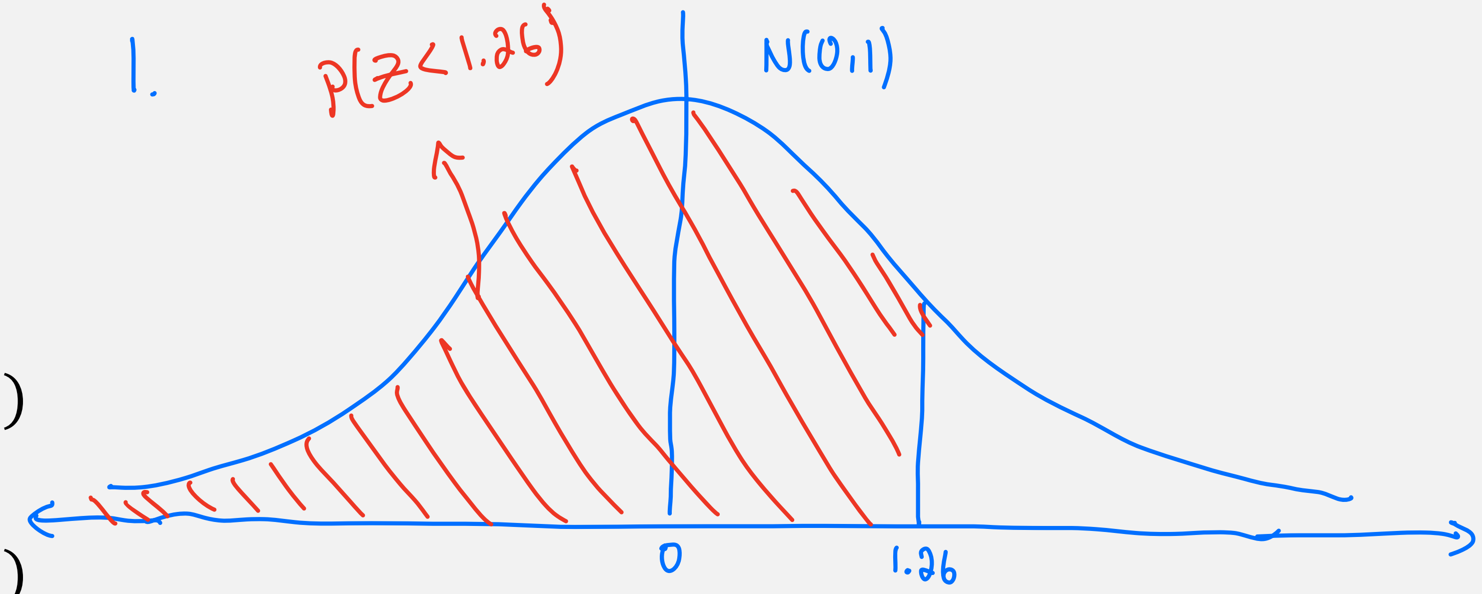
Z	Second decimal place of Z									
	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

Exercises

1.

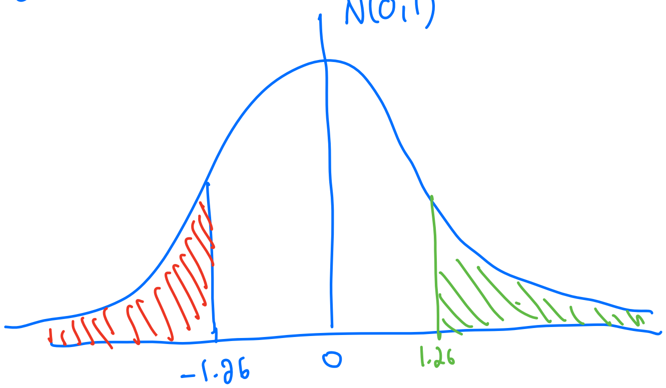
$Let\ Z \sim N(0,1)$

1. $P(Z < 1.26)$
2. $P(Z < -1.26)$
3. $P(Z = 1.26)$
4. $P(Z > -1.26)$



From table, $P(Z < 1.26) = 0.8962$

2.



$P(Z < -1.26) = 0.1038$

Alternatively, $P(Z < -1.26) = P(Z > 1.26)$ by symmetry of $N(0,1)$

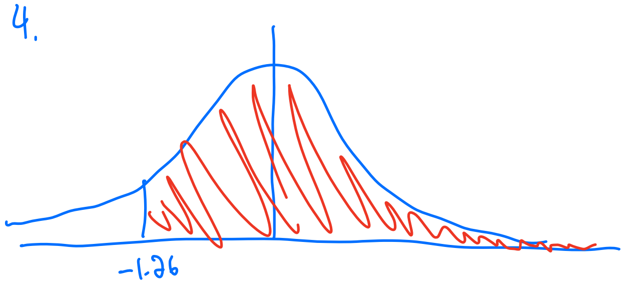
$= 1 - P(Z < 1.26)$

$= 1 - 0.8962$

$= 0.1038$

3. $P(Z = 1.26) = 0$

4.



$P(Z > -1.26)$

$= 1 - P(Z < -1.26)$

$= 1 - 0.1038$

$= 0.8962$

Finding probabilities for any Normal Distribution

- We now know how to find probabilities for $N(0,1)$, but what about all the other normal distributions?

- If $X \sim N(\mu, \sigma^2)$, and $Z = \frac{X-\mu}{\sigma}$, then $Z \sim N(0,1)$

- This means that $X \sim N(\mu, \sigma^2)$,

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right),$$

where $Z \sim N(0,1)$

Example: Birth Weight

Suppose the birth weight of an infant follows a normal distribution with mean 120 ounces and standard deviation 18 ounces.

a) What percent of infants weigh less than 96 ounces ?

b) What percent of infants weigh between 128 and 144 ounces ?

c) What is the first quartile of birth weight?

d) Given that an infant has a birthweight less than 144, what is the probability that the birth weight is less than 128?

Let X be the birthweight of an infant

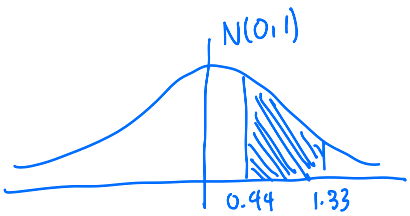
$X \sim N(120, 18^2)$

a) $P(X < 96) = P\left(\frac{X-120}{18} < \frac{96-120}{18}\right) = P(Z < -1.33)$, $Z \sim N(0,1)$

$= 0.0918$ using table

b) $P(128 < X < 144) = P\left(\frac{128-120}{18} < \frac{X-120}{18} < \frac{144-120}{18}\right)$

$= P(0.44 < Z < 1.33)$, $Z \sim N(0,1)$



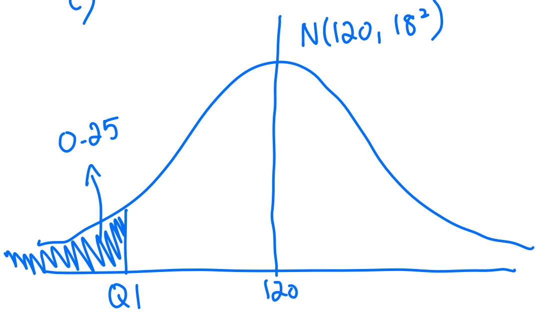
$= 1 - P(Z > 1.33) - P(Z < 0.44)$

$= 1 - P(Z < -1.33) - P(Z < 0.44)$

$= 1 - 0.0918 - 0.67$

$= 0.2382.$

c)



$P(X < Q1) = 0.25$

$\Rightarrow P\left(\frac{X-120}{18} < \frac{Q1-120}{18}\right) = 0.25$

$\Rightarrow P\left(Z < \underbrace{\frac{Q1-120}{18}}_{-0.67}\right) = 0.25$

$\Rightarrow -0.67 = \frac{Q1-120}{18}$

$\Rightarrow Q1 = 107.8$

d) $P(X < 128 \mid X < 144)$

$= \frac{P(X < 128 \text{ and } X < 144)}{P(X < 144)}$

$= \frac{P(X < 128)}{P(X < 144)}$

Recall:

$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$

$$= \frac{P(Z < 0.44)}{P(Z < 1.33)}$$

$$= \frac{0.67}{0.9082}$$

$$= 0.7377$$

Videos and Practice Problems

- In this lecture, we covered all of Module 3: Probability- Random Variables

Next Week

- Usual quiz is due on Sunday
- Next week we will discuss sampling distributions