Independence

The two events are independent if any one of the following equivalent statements is true:

1. $P(A|B) = P(A)$
2. $P(B|A) = P(B)$
3. $P(A \cap B) = P(A).P(B)$

Ex Suppose a day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Suppose two parts are selected from the batch, but the first part is replaced before the second part is selected. What is the probability that the second part is defective (denoted as B) given that the first part is defective (denoted as A)? The probability needed can be expressed as $P(B|A)$.

Because the first part is replaced prior to selecting the second part, the batch still contains 850 parts, of which 50 are defective. Therefore, the prob. of B does not depend on whether or not the first part was defective.

$$P(B|A) = P(B) = 50/850$$

The prob. of both parts are defective is

$$P(A \cap B) = P(B|A).P(A)$$
$$= P(B).P(A) = \frac{50}{850} \cdot \frac{50}{850} = 0.0035$$

Ex

		Surface flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	2	18	20
	No	38	342	380
	Total	40	360	400

$P(D|F) = 2/40 = 0.05$ and $P(D) = 20/400 = 0.05$

→ That is, the probability that the part is defective does not depend on whether it has surface flaws.

$P(F|D) = 2/20 = 0.10$ and $P(F) = 40/400 = 0.10$

→ So, the prob. of a surface flaw does not depend on whether the part is defective.

$$P(F \cap D) = P(D|F) \cdot P(F) = P(D) \cdot P(F)$$

$$= \frac{2}{40} \cdot \frac{2}{20} = \frac{1}{200}$$

The Multiplication Rule

$$P(A \cap B) = P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$$

Ex: The prob. that an automobile battery subject to high engine compartment temperature suffers low charging current is 0.7. The prob. that a battery is subject to high engine compartment temperature is 0.05.

Let C denote the event that a battery suffers low charging current, and let T denote the event that a battery is subject to high engine compartment temperature.

The probability that a battery is subject to low charging current and high engine compartment temperature is

$$P(C \cap T) = P(C|T) \cdot P(T) = 0.7 \times 0.05 = 0.035$$

Ex: We have 3 green and 2 red balls in a bag. We pick balls one by one till we find a red ball. What is the prob. that we don't pick more than 2 balls? (Finding the red on the first or second trial)

$P(\text{Finding at first}) = \frac{2}{5}$

$P(\text{Finding at second}) = P(\text{First green, second red})$

$= P(G_1 \cap R_2) = P(G_1) \cdot P(R_2|G_1)$

$= \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{5} \cdot \frac{1}{2} = \frac{3}{10}$

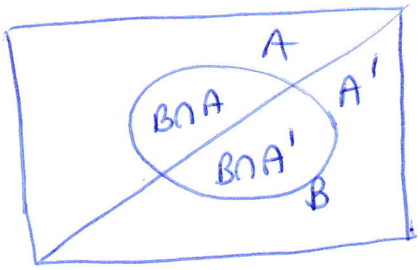
4 balls left after picking one

$P(\text{Finding the red on the first or second}) = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}$

(From the lecture notes of H.S. Sozak and B. AYTAGOĞLU)

Total Probability Rule

For any two events A and B,



Partitioning an event into two mutually exclusive subsets.

$$P(B) = P(B \cap A) + P(B \cap A')$$
$$= P(B|A) . P(A) + P(B|A') . P(A')$$

Because A and A' are mutually exclusive, $A \cap B$ and $A' \cap B$ are also mutually exclusive.

Example: Suppose that in semiconductor manufacturing the probability is 0.10 that a chip that is subjected to high levels of contamination during manufacturing causes a product failure. The probability is 0.005 that a chip that is not subjected to high contamination levels during manufacturing causes a product failure. In a particular production run, 20% of the chips are subject to high levels of contamination. What is the probability that a product using one of these chips fails?

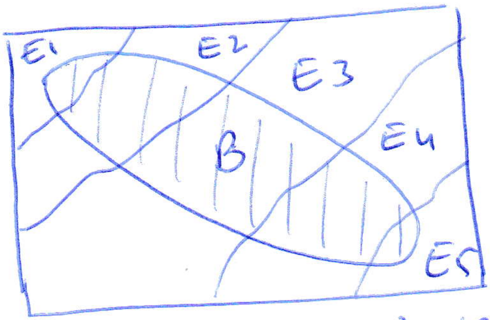
Let F denote the event that the product fails, let H denote the event that the chip is exposed to high levels of contamination. The requested probability is P(F).

$$P(F|H) = 0.10 \quad \text{and} \quad P(F|H') = 0.005$$
$$P(H) = 0.20 \quad \text{and} \quad P(H') = 0.80$$

Then;

$$P(F) = P(F|H) . P(F) + P(F|H') . P(H')$$
$$= 0.10 \times 0.20 + 0.005 \times 0.80$$
$$= 0.0235$$

Total Probability Rule (Multiple Events)



$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4) \cup (B \cap E_5)$

Partitioning an event into several mutually exclusive subsets.

Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then,

$$P(B) = P(B \cap E_1) + P(B \cap E_2) + \ldots + P(B \cap E_k)$$
$$= P(B|E_1) \cdot P(E_1) + P(B|E_2) \cdot P(E_2) + \ldots + P(B|E_k) \cdot P(E_k)$$

<u>Example</u>: Semiconductor manufacturing example, assume the following probabilities for product failure subject to levels of contamination in manufacturing:

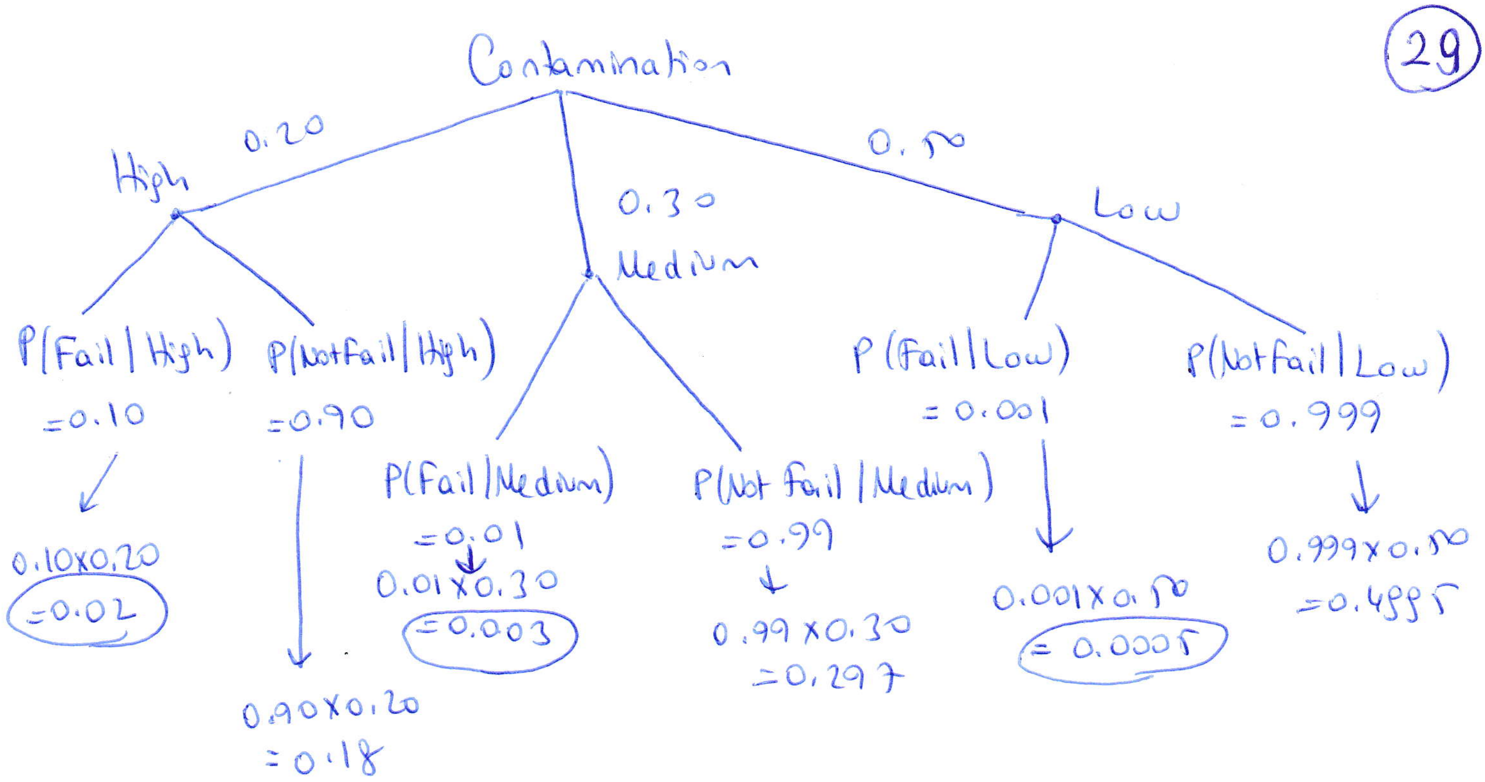
Probability	Level of Contamination
0.10	High
0.01	Medium
0.001	Low

In a particular run, 20% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 50% to low levels of contamination. What is the probability that a product using one of these chips fails?

H → event that a chip is exposed to high levels of contamination.
M → " " " " " " medium " " "
L → " " " " " " low " " "

Then,

$$P(F) = P(F|H) \cdot P(H) + P(F|M) \cdot P(M) + P(F|L) \cdot P(L)$$
$$= 0.10 \times 0.20 + 0.01 \times 0.30 + 0.001 \times 0.50 = 0.0235$$



$$P(\text{Fail}) = 0.02 + 0.003 + 0.0005 = 0.0235$$

Bayes' Theorem

If $E_1, E_2, \ldots, E_k$ are k mutually and exhaustive events and B is any event,

$$P(E_1 | B) = \frac{P(B|E_1).P(E_1)}{P(B|E_1).P(E_1) + P(B|E_2).P(E_2) + \ldots + P(B|E_k).P(E_k)}$$

for $P(B) > 0$

or

if we deal with n events $(E_1, E_2, \ldots, E_n)$

$$P(E_k | B) = \frac{P(B|E_k).P(E_n)}{\sum_{i=1}^{n} P(B|E_i).P(E_i)}, \quad P(B) > 0$$

Ex Only one in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed. We know that

$$P(\text{Test result is "+"} \mid \text{Ind. has the disease}) = 0.99$$
$$P(\text{Test result is "+"} \mid \text{ind. has no disease}) = 0.02$$

If a randomly selected individual is tested and the test is positive what is the probability that the individual has the disease?

$$P(\text{Ind. has disease} \mid \text{Test is "+"}) = ?$$

D → the event that you have the illness.
S → " " " the test result is positive.

$$P(D \mid S) = ?$$

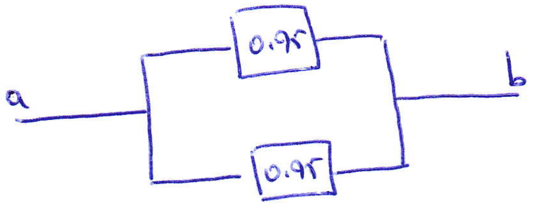
$$P(D \mid S) = \frac{P(D \cap S)}{P(S)} = \frac{P(S \mid D).P(D)}{P(S)}$$

$$P(S) = P(S \cap D) + P(S \cap D')$$
$$= P(S \mid D).P(D) + P(S \mid D').P(D')$$
$$= 0.99 \times 0.001 + 0.02 \times 0.999$$
$$= 0.02097$$

Then,

$$P(D \mid S) = \frac{0.99 \times 0.001}{0.02097} = \frac{0.00099}{0.02097} = 0.047$$

Example: The following circuit operates if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates.
The prob. that the circuit operates is;

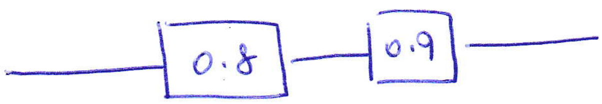
$$P(T \text{ or } B) = 1 - P[(T \text{ or } B)']$$
$$= 1 - P(T' \text{ and } B')$$

$$P(T' \text{and } B') = P(T').P(B') = (1-0.95)^2 = 0.05^2$$

Then;

$$P(T \text{ or } B) = 1 - 0.05^2 = 0.9975$$

Example



L → event that left device operates
R → " " right " "
If both operates, system operates so;

$P(L \text{ and } R)$ → prob. that the circuit operates

$$P(L \cap R) = P(L).P(R) = 0.8 \times 0.9 = 0.72 //$$