Discrete Probability Theory (INHN0020) | Summer Term 2023 TU Munich

Practice Exam

Dr. Jakob Piribauer
Dr. Benedikt Hoock

This document gives you an impression of how the questions in the exam might look like.

Question 1 (Discrete Probability Space, 8 points)**.** We consider a cube whose eight vertices are coloured independently. Each vertex is coloured red with probability $1/2$ or green with probability $1/2$. An edge of the cube is called *colourful* if the two vertices it connects have a different colour.
We define the following events:
E: The top face of the cube contains one red and three green vertices.
F: Exactly three of the four edges connecting the top face to the bottom face are colourful.

a) Formally describe the sample space Ω and show that all sample points $\omega \in \Omega$ have the same probability mass.

b) Compute the probability of $E \cup F$.

c) Determine whether E and F are independent.

d) Compute the expected number of colourful edges in the whole cube.

Question 2 (Discrete Random variables, 6 points)**.** Consider two tetrahedra whose four faces are labeled with natural numbers. We toss the two tetrahedra simultaneously and both land on one of their faces. For both tetrahedra, each face has the same probability of $1/4$. Let X be the sum of the numbers on which the two tetrahedra landed.

a) Suppose both tetrahedra are labeled with the numbers $1, 2, 3, 4$. Show that X has the following probability mass function:

x	2	3	4	5	6	7	8	otherwise
$f_X(x)$	1/16	1/8	3/16	1/4	3/16	1/8	1/16	0

b) Now, we label the faces of one of the tetrahedra with the numbers $1, 2, 2, 3$. Find a suitable labeling for the other tetrahedron such that X is distributed as in question a).

c) Find a labeling for a single tetrahedron such that repeatedly tossing the tetrahedron requires two tosses in expectation until the tetrahedron first lands on a prime number.

Question 3 (Continuous Random variables, 6 points)**.** Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} 1/2 & \text{if } 0 < x < 1, \\ 1/6 & \text{if } 2 < x < 5, \\ 0 & \text{otherwise.} \end{cases}$$

Define the continuous random variable Y as $Y = \frac{1}{X}$.

a) Determine the probability density function of Y.

b) Prove or disprove the existence of the expected value of Y.

Question 4 (Exponential distribution, 6 points)**.** The scientist Amadeus investigates mutations in a colony of spherical bacteria and a colony of cylindrical bacteria. In both colonies, mutations occur independently and the time between two mutations is exponentially distributed in both colonies. In the colony of spherical bacteria, a mutation occurs every four minutes in expectation. In the colony of cylindrical bacteria, a mutation occurs every 20 minutes in expectation.

a) How long does Amadeus have to wait in expectation until he can observe the first mutation?

b) How long does Amadeus have to wait in expectation until in both colonies a mutation occurred?

c) How long does Amadeus have to wait in expectation until 27 mutations occurred in total?

Question 5 (Testing hypotheses, 8 points)**.** Aida wants to buy a biased coin in a shop for magic accessories. The salesperson promises that the coin shows heads with probability at most 0.1. To demonstrate, the salesperson tosses the coin $n = 100$ times and conducts an approximate binomial test with level of significance $\alpha = 0.05$.

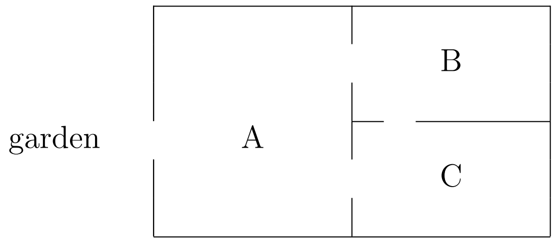
Remark: At the end of this mock exam, there is a table with the values of the cumulative distribution function of the standard normal distribution.

a) Determine the largest possible critical region $K = \{k, \ldots, n\}$ such that a number of heads in K in the trial leads to the rejection of the hypothesis of the salesperson.

b) Aida is skeptical and believes that the probability for heads is more than 0.2. Consider the null hypothesis of the salesperson and Aida's hypothesis as alternative hypothesis. Compute the probability of a type II error based on the critical region determined in question a).

   Hint: You may use that the function $f(x) = \frac{a-bx}{\sqrt{bx(1-x)}}$ with $0 < a < b$ is strictly decreasing on the interval $x \in (0, 1)$.

Question 6 (Markov chains, 6 points)**.**



Two cats Ajax and Berta are chasing each other through the depicted flat with a door to the garden. Initially, they are in room B and C, respectively. At every time step, both cats choose a door independently and move to another room (or possibly from room A to the garden). They always choose all doors in their current room with the same probability. The chase ends when both cats are in the same room or one of the cats arrives in the garden.

We describe the situation as a Markov chain $(X_t)_{t\in\mathbb{N}}$. The random variables X_t describes the position of the two cats at time point t by taking values in $\{0,1,2,3,4\}$ that represent the following situations:

state	meaning
0	The cats are in room A and B.
1	The cats are in room A and C.
2	The cats are in room B and C.
3	The cats are in the same room.
4	One cat reached the garden.

a) Draw a graphical representation with transition probabilities of the Markov chain with states $\{0,1,2,3,4\}$.

b) What is the probability that the chase ends because a cat reached the garden?

c) How long does the chase last in expectation?

Table 1: Values of the cumulative distribution function Φ of the standard normal distribution. For example, $\Phi(1.55) \approx 0,93943$.

z	0	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0	0,50000	0,50399	0,50798	0,51197	0,51595	0,51994	0,52392	0,52790	0,53188	0,53586
0,1	0,53983	0,54380	0,54776	0,55172	0,55567	0,55962	0,56356	0,56749	0,57142	0,57535
0,2	0,57926	0,58317	0,58706	0,59095	0,59483	0,59871	0,60257	0,60642	0,61026	0,61409
0,3	0,61791	0,62172	0,62552	0,62930	0,63307	0,63683	0,64058	0,64431	0,64803	0,65173
0,4	0,65542	0,65910	0,66276	0,66640	0,67003	0,67364	0,67724	0,68082	0,68439	0,68793
0,5	0,69146	0,69497	0,69847	0,70194	0,70540	0,70884	0,71226	0,71566	0,71904	0,72240
0,6	0,72575	0,72907	0,73237	0,73565	0,73891	0,74215	0,74537	0,74857	0,75175	0,75490
0,7	0,75804	0,76115	0,76424	0,76730	0,77035	0,77337	0,77637	0,77935	0,78230	0,78524
0,8	0,78814	0,79103	0,79389	0,79673	0,79955	0,80234	0,80511	0,80785	0,81057	0,81327
0,9	0,81594	0,81859	0,82121	0,82381	0,82639	0,82894	0,83147	0,83398	0,83646	0,83891
1	0,84134	0,84375	0,84614	0,84849	0,85083	0,85314	0,85543	0,85769	0,85993	0,86214
1,1	0,86433	0,86650	0,86864	0,87076	0,87286	0,87493	0,87698	0,87900	0,88100	0,88298
1,2	0,88493	0,88686	0,88877	0,89065	0,89251	0,89435	0,89617	0,89796	0,89973	0,90147
1,3	0,90320	0,90490	0,90658	0,90824	0,90988	0,91149	0,91309	0,91466	0,91621	0,91774
1,4	0,91924	0,92073	0,92220	0,92364	0,92507	0,92647	0,92785	0,92922	0,93056	0,93189
1,5	0,93319	0,93448	0,93574	0,93699	0,93822	0,93943	0,94062	0,94179	0,94295	0,94408
1,6	0,94520	0,94630	0,94738	0,94845	0,94950	0,95053	0,95154	0,95254	0,95352	0,95449
1,7	0,95543	0,95637	0,95728	0,95818	0,95907	0,95994	0,96080	0,96164	0,96246	0,96327
1,8	0,96407	0,96485	0,96562	0,96638	0,96712	0,96784	0,96856	0,96926	0,96995	0,97062
1,9	0,97128	0,97193	0,97257	0,97320	0,97381	0,97441	0,97500	0,97558	0,97615	0,97670
2	0,97725	0,97778	0,97831	0,97882	0,97932	0,97982	0,98030	0,98077	0,98124	0,98169
2,1	0,98214	0,98257	0,98300	0,98341	0,98382	0,98422	0,98461	0,98500	0,98537	0,98574
2,2	0,98610	0,98645	0,98679	0,98713	0,98745	0,98778	0,98809	0,98840	0,98870	0,98899
2,3	0,98928	0,98956	0,98983	0,99010	0,99036	0,99061	0,99086	0,99111	0,99134	0,99158
2,4	0,99180	0,99202	0,99224	0,99245	0,99266	0,99286	0,99305	0,99324	0,99343	0,99361
2,5	0,99379	0,99396	0,99413	0,99430	0,99446	0,99461	0,99477	0,99492	0,99506	0,99520
2,6	0,99534	0,99547	0,99560	0,99573	0,99585	0,99598	0,99609	0,99621	0,99632	0,99643
2,7	0,99653	0,99664	0,99674	0,99683	0,99693	0,99702	0,99711	0,99720	0,99728	0,99736
2,8	0,99744	0,99752	0,99760	0,99767	0,99774	0,99781	0,99788	0,99795	0,99801	0,99807
2,9	0,99813	0,99819	0,99825	0,99831	0,99836	0,99841	0,99846	0,99851	0,99856	0,99861
3	0,99865	0,99869	0,99874	0,99878	0,99882	0,99886	0,99889	0,99893	0,99896	0,99900
3,1	0,99903	0,99906	0,99910	0,99913	0,99916	0,99918	0,99921	0,99924	0,99926	0,99929
3,2	0,99931	0,99934	0,99936	0,99938	0,99940	0,99942	0,99944	0,99946	0,99948	0,99950
3,3	0,99952	0,99953	0,99955	0,99957	0,99958	0,99960	0,99961	0,99962	0,99964	0,99965
3,4	0,99966	0,99968	0,99969	0,99970	0,99971	0,99972	0,99973	0,99974	0,99975	0,99976
3,5	0,99977	0,99978	0,99978	0,99979	0,99980	0,99981	0,99981	0,99982	0,99983	0,99983
3,6	0,99984	0,99985	0,99985	0,99986	0,99986	0,99987	0,99987	0,99988	0,99988	0,99989
3,7	0,99989	0,99990	0,99990	0,99990	0,99991	0,99991	0,99992	0,99992	0,99992	0,99992
3,8	0,99993	0,99993	0,99993	0,99994	0,99994	0,99994	0,99994	0,99995	0,99995	0,99995
3,9	0,99995	0,99995	0,99996	0,99996	0,99996	0,99996	0,99996	0,99996	0,99997	0,99997
4	0,99997	0,99997	0,99997	0,99997	0,99997	0,99997	0,99998	0,99998	0,99998	0,99998