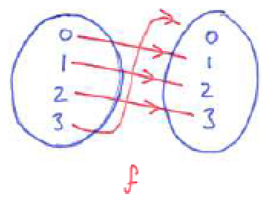
Solutions - Practice Exam Questions - Tutorial 5

1.

Q1

(a)



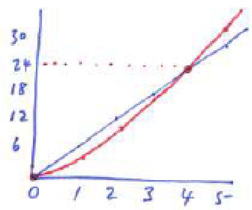
i.e. $f(0)=1$

$f(1)=2$

$f(2)=3$

$f(3)=0$

(b) Let's sketch this to get a feeling for what is going on.



$6x$ in blue

$(x+1)^2-1$ in ~~black~~ red.

They crossover at $x=4$,

So it looks like $c=4$.

That is: $f(x) = \begin{cases} 6x & \text{if } 0 \le x \le 4 \quad \text{SD1} \\ (x+1)^2-1 & \text{if } x > 4 \quad \text{SD2} \end{cases}$

Let's prove this!

Let's prove injective.

Assume $f(x)=f(y)$. [We need to prove that $x=y$].

Observe that if x is from SD1, $0 \le f(x) \le 24$ and if y is from SD2,

$f(x) > 24$. ~~(Since~~ These regions are disjoint, so under the assumption

that $f(x)=f(y)$, it must hold that both x and y are from SD1, or both from SD2.

both from SD1

$f(x)=f(y)$

$\Rightarrow 6x = 6y$

$\Rightarrow x=y$ ✓

both from SD2

$f(x)=f(y)$

$\Rightarrow (x+1)^2-1 = (y+1)^2-1$

$\Rightarrow (x+1)^2 = (y+1)^2$

$\Rightarrow x+1 = y+1$ (because $x+1$ and $y+1$ are both >0, which is a consequence of the fact that they are both in SD2).

$\Rightarrow x=y$ ✓

So it is injective.

Let's prove <u>surjectivity</u>.

For $0 \le y \le 24$ take $x = \frac{y}{6}$. (Why? $y = 6x \Rightarrow x = \frac{y}{6}$)

For $y > 24$ take $x = \sqrt{y+1} - 1$ (Why? $(x+1)^2 - 1 = y \Rightarrow x = \sqrt{y+1} - 1$)

take the positive root.

Need to check this really works. (Yes, you <u>do</u> need this step).

- For $0 \le y \le 24$, $0 \le \frac{y}{6} \le 4$ so $f\left(\frac{y}{6}\right) = 6\left(\frac{y}{6}\right) = y$ ✓

- for $y > 24$, $\sqrt{y+1} - 1 > \sqrt{25} - 1 = 4$.

$$\text{So } f\left(\sqrt{y+1} - 1\right) = \left[\sqrt{y+1} - 1 + 1\right]^2 - 1$$
$$= \left[\sqrt{y+1}\right]^2 - 1$$
$$= y + 1 - 1 = y \checkmark$$

So it <u>is</u> surjective.

So it is injective & surjective, and thus bijective (and thus invertible).

Inverse: $f^{-1}(x) = \begin{cases} \frac{x}{6} & \text{if } 0 \le x \le 24 \\ \sqrt{x+1} - 1 & \text{if } x > 24 \end{cases}$

make sure you get these right!

2.

x	1	2	3	4	5	6	7	8	9	10	11	12	13	...	24	25	26
$f(x)$	1	4	9	16	25	36	49	64	81	100	23	25	27		49	51	53
$g(x)$	3	~~5~~6	11	18	27	38	51	192	243	300	25	27	29		51	153	154
	1							2			3					4	

So the function behaves like this:

$$(g \circ f)(x) = \begin{cases} x^2 + 2 & 1 \le x \le 7. & (1) \\ 3x^2 & 8 \le x \le 10 & (2) \\ 2x+3 & 11 \le x \le 24. & (3) \\ 6x+3 & x > 24. & (4) \end{cases}$$

3.

(b) $f(3) = 3^2 - 10 = 9 - 10 = -1.$

So $g(f(3)) = 4(-1) + 7 = \underline{3}$

$f(4) = 4^2 - 10 = 16 - 10 = 6$

So $g(f(4)) = 5 \times 6^3 =$

$= \underline{1080}.$

(a) $\underline{c = 33}.$ [Not needed in answer, but why? It's the solution to $6^2 = \sqrt{6+3} + c$
$\Rightarrow 36 = \sqrt{9} + c$
$\Rightarrow 36 = 3 + c$
$\Rightarrow c = 33.$]

f:

x^2 if $0 \leq x \leq 6$ "SD1"

$+\sqrt{x+3} + 33$ if $x > 6$ "SD2"

I need to prove injectivity & surjectivity.

(i) Injectivity.

Note that for $0 \leq x \leq 6$, $0 \leq f(x) \leq 36$, but for $x > 6$, $f(x) > \sqrt{6+3} + 33 = 36.$

In other words, the two subdomains have disjoint output spaces.

So in the injection proof we can focus on $f(x) = f(y)$ when x and y are both from the same subdomain.

Let x and y be arbitrary numbers $0 \leq x, y \leq 6$ such that $f(x) = f(y)$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y$ (because both x, y assumed to be ≥ 0) ✓

Let x and y be arbitrary numbers such that $x, y > 6$ and $f(x) = f(y)$

$\Rightarrow +\sqrt{x+3} + 33 = +\sqrt{y+3} + 33$

$\Rightarrow +\sqrt{x+3} = +\sqrt{y+3}$

$\Rightarrow x = y$ ✓

So injective!!

Surjective?

- Let y be an arbitrary element of the co-domain $\mathbb{R}^+$, i.e. $y \geq 0$.

– If $y > 36$, take $x = (y-33)^2 - 3$.

Check: for such y, $y - 33 > 3$,

so $(y-33)^2 > 9$,

so $9 - 3 > 6$,

so $x > 6$,

$$\text{So } f\left((y-33)^2 - 3\right) = {}^{+}\sqrt{(y-33)^2 - 3 + 3} \quad + 33$$

$$= {}^{+}\sqrt{(y-33)^2} \quad + 33$$

$$= y - 33 \quad + 33$$

$$= y \checkmark$$

– if $0 \leq y \leq 36$,
take $x = {}^{+}\sqrt{y}$.

Check: for such y, ${}^{+}\sqrt{y}$ is in the interval $[0, 6]$.

$$\text{So } f({}^{+}\sqrt{y}) = \left[{}^{+}\sqrt{y}\right]^2 = y. \checkmark$$

So surjective!

$$\text{Inverse: } f^{-1}(y) = \begin{cases} (y-33)^2 - 3 & \text{if } y > 36 \\ {}^{+}\sqrt{y} & \text{if } 0 \leq y \leq 36. \end{cases}$$

4.

I "guess" $c=6$. I will verify this by showing that for this choice the function is invertible.

Injective: Assume $f(x)=f(y)$ and prove $x=y$.

Note that for $0 \le x < 6$, $0 \le f(x) \le \frac{1}{2}(6)^2 = 18$.

for $x \geq 6$, $f(x) = 2x+c = 2x+6 \geq 18$.

ie. outputs of the 2 subdomains are disjoint

So, if $f(x)=f(y)$, then x and y are both $0 \le x,y < 6$ or are both ≥ 6.

case $0 \le x,y < 6$:

$f(x) = f(y)$

$\Rightarrow \frac{1}{2}x^2 = \frac{1}{2}y^2$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y$ (because $x,y \geq 0$) ✓

case $x,y \geq 6$:

$f(x) = f(y)$

$\Rightarrow 2x+6 = 2y+6$

$\Rightarrow x = y$ ✓

So it is injective.

Sujective: for $0 \le y < 18$, take $x = +\sqrt{2y}$.

for $y \geq 18$, take $x = \frac{y-6}{2}$.

Observe that in this area, $0 \le 2y < 36$

$\Rightarrow 0 \le \sqrt{2y} < 6$

So $f\left(\sqrt{2y}\right) = \frac{1}{2}\left[\sqrt{2y}\right]^2 = \frac{1}{2} \times 2y = y$ ✓

Observe that here, $\frac{y-6}{2} \geq 6$, So $f\left(\frac{y-6}{2}\right) = 2\left[\frac{y-6}{2}\right] + 6 = y$ ✓ So it is surjective.

Inverse: $$f^{-1}(x) = \begin{cases} +\sqrt{2x} & 0 \le x < 18 \\ \frac{x-6}{2} & x \geq 18. \end{cases}$$

(b) $g(0) = 4$
$g(1) = 3$
$g(2) = 2$
$g(3) = 1$
$g(4) = 0.$

answer!

So take $f(0) = 0$
$f(1) = 2$
$f(2) = 4$
$f(3) = 1$

(Check: $(g\circ f)(0) = g(f(0)) = g(0) = 4$ ✓
$(g\circ f)(1) = g(f(1)) = g(2) = 2$ ✓
$(g\circ f)(2) = g(f(2)) = g(4) = 0$ ✓
$(g\circ f)(3) = g(f(3)) = g(1) = 3$ ✓)