Statistics: Tutorial sheet 4

Mandatory Exercises

Exercise 1. Find the MLE of θ based on a random sample from the following pdfs and show whether they are unbiased estimators.

a. $f(x \mid \theta) = \frac{1}{2\theta} e^{-|x|/\theta}$, where $\theta > 0$.

b. $f(x \mid \theta) = (1-\theta)\theta^x$, where $x = 0, 1, 2, \ldots$ and $0 < \theta < 1$.

c. $f(x \mid \theta) = \theta^2 x e^{-\theta x}$, where $x > 0$ and $\theta > 0$.

Exercise 2. Let $X_1, \ldots, X_n$ be a random sample from the population

$$f(x \mid \theta) = \theta x^{\theta - 1} \quad \text{where } 0 < x < 1,\ \theta > 0.$$

a. Find a sufficient statistic for θ.

b. Derive the MLE of θ.

c. Compare the MLE to the MME of Mandatory Exercise 3(b) in Tutorial 3. Which of these is unbiased? Which of these is based on a sufficient statistic? What does that intuitively tell you?

Exercise 3. Let $X_1, \ldots, X_n$ be a random sample from Uniform$(0, \theta)$.

a. Show that the MLE of θ is given by $\hat{\theta}_{ML} = \max_i X_i$.

b. Show that the cdf of $\hat{\theta}_{ML}$ is equal to

$$G_{ML}(x \mid \theta) = \left(\frac{x}{\theta}\right)^n, \quad 0 \leq x \leq \theta.$$

and that the pdf is given by

$$g_{ML}(x \mid \theta) = \frac{n x^{n-1}}{\theta^n}, \quad 0 \leq x \leq \theta.$$

Hint: recall Practice Exercise 1 from Tutorial 1.

c. Derive the expectation and variance of $\hat{\theta}_{ML}$.

d. Derive the MME of θ based on the lowest moment possible and find it expectation and variance.

e. Calculate the MSE of the MLE and MME. Which estimator would you prefer?

f. You should have found that the MLE is a biased estimator. Can you come up with a transformation that would make the MLE unbiased?

Practice Exercises

Exercise 1. Find the ML estimator for p_0 in the statistical model $\{\text{Bernoulli}(p) \mid p \in [0,1]\}$. In this case we have

$$g(x \mid p) = p^x(1-p)^{1-x} \qquad \text{if } x \in \{0,1\}.$$

Exercise 2. Let $L(\theta \mid \boldsymbol{x})$ be the likelihood corresponding to a statistical model. Prove that $L(\theta \mid \boldsymbol{x})$ has a *unique* maximum at $\tilde{\theta} \in \Theta$ if and only if $\ell(\theta \mid x)$ has a *unique* maximum at $\tilde{\theta}$. Hint: Use that the logarithm is a monotone increasing function, that is $\log y_1 > \log y_2$ if and only if $y_1 > y_2$.

Exercise 3. Prove Lemma 4.14, which says the following: Let $\Theta \in \mathbb{R}$ be an interval and let $h : \Theta \to \mathbb{R}$ be a function that is twice differentiable. Suppose that there exists a unique stationary point $\tilde{\theta} \in \Theta$, that is $h'(\tilde{\theta}) = 0$, and that it satisfies the second derivative test $h''(\tilde{\theta}) < 0$. Then $\tilde{\theta}$ is the unique point at which the global maximum is attained. Hint: use the mean value theorem.

Exercise 4. The goal of this exercise is to combine most of what we have learned until now to tackle a complete statistics problem. An insurance company has asked us for help to predict the expected amount of money that they will have to pay next month. We have the following data:

- The number of monthly claims that had to be paid in the previous n months. We will denote this n_i for $1 \le i \le n$.
- The value of each individual claim. We write $c_{i,j}$, where $1 \le j \le n_i$, for the value of claim j in month i.

We use the following model strategy. The number of claims can be modelled nicely as a random sample from a Poisson population:

$$N_i \sim \text{Poisson}(\mu).$$

This follows by a similar intuition as for the milk store in week one, we have many independent clients with small probability to enter a claim. We model the claim size as all independent with an exponential distribution:

$$C_{ij} \sim \text{Exponential}(\lambda).$$

a. Write down a formula for the total amount that has to be paid in month i.

b. Find the expected value for the total amount that has to be paid in month i.

c. Let $\theta = (\mu, \lambda)$. Show that the joint density

$$f(n_i, c_{i1}, c_{i2}, \dots, c_{in_i} \mid \theta)$$

of $(N_i, C_{i1}, C_{i2}, \dots, C_{iN_i})$ is equal to

$$\left(\prod_{j=1}^{n_i} \lambda e^{-\lambda c_{ij}}\right) e^{-\mu} \frac{\mu^{n_i}}{n_i!}.$$

d. Derive the maximum likelihood estimator for the expected value of the amount that has to be paid next month.