Statistics: Tutorial sheet 5

Mandatory Exercises

Exercise 1. Suppose we have a random sample $X_1, \ldots, X_n$ from Normal$(0, \theta)$.

a. Calculate the MME $\hat{\theta}_{MM}$ of θ. Is this an unbiased estimator?

b. Calculate the MLE $\hat{\theta}_{ML}$ of θ. Is this an unbiased estimator?

c. Are $\hat{\theta}_{MM}$ and $\hat{\theta}_{ML}$ UMVU estimators of θ? Hint: $\mathbb{E}(X_1^4) = 3\theta^2$.

d. Find the MME and MLE of $\eta = \sqrt{\theta}$. Are these unbiased estimators of η?

Exercise 2. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$g(x \mid \theta) = \frac{1}{2\theta\sqrt{x}} e^{-\sqrt{x}/\theta} \quad \text{where } x > 0, \theta > 0.$$

It is known that

$$\mathbb{E}(X_1^k) = \theta^{2k}\Gamma(1 + 2k),$$

where $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

a. Find sufficient statistics for θ.

b. Derive the method of moment estimator $\hat{\theta}_{MM}$ based on the lowest (integer) moment possible. Is this estimator unbiased? Is it based on a sufficient statistic?

c. Derive the maximum likelihood estimator $\hat{\theta}_{ML}$. Is this estimator unbiased? Is it based on a sufficient statistic?

d. Based on the information derived thus far, can you already rule out one of the estimators as being a UMVUE?

e. Argue whether any of the above estimators is a UMVUE based on the Cramer-Rao lower bound.

Practice Exercises

Exercise 1. We are given the statistical model $\{\text{Bernoulli}(p) \mid p \in [0, 1]\}$, that is

$$g(x \mid p) = p^x(1-p)^{1-x}.$$

Both the moment and maximum likelihood estimator are given by $\overline{X}$.

a. Show that the Bernoulli statistical model is an exponential family.

b. Show that $\overline{X}$ is the UMVU estimator for p_0.

Exercise 2. In this exercise we study an iid random sample $X_1, \dots, X_n$ from a population in the statistical model $\{g(x \mid \theta) \mid \theta \in \Theta\}$.

a. Prove that the set of order statistics $T = (X_{(1)}, \dots, X_{(n)})$ is sufficient for θ_0.

b. We are interested in finding an unbiased estimator for $\tau(\theta) = \mathbb{P}_\theta(X_1 \leq 2)$. Show that $W(\boldsymbol{X}) = \mathbb{1}_{\{X_1 \leq 2\}}$ is unbiased.

c. Use Rao-Blackwellisation to find a better unbiased estimator for $\tau(\theta)$.

d. Show that the obtained estimator converges to $\tau(\theta_0)$ as $n \to \infty$.