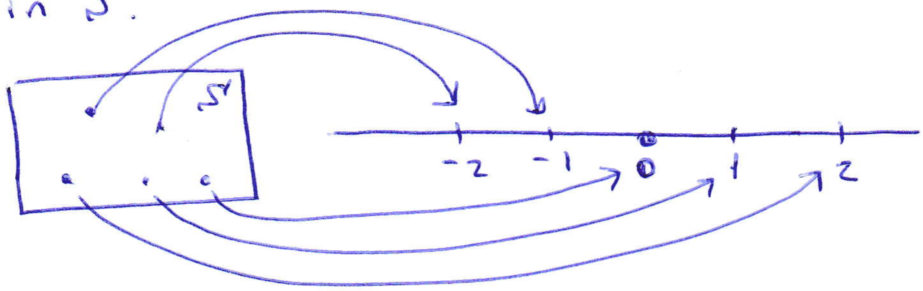
Random Variables and Probability Distributions

Random variables

We often summarize the outcome from a random experiment by a simple number. The variable that associates a number with the outcome of a random experiment is referred to as a random variable.

A random variable is a function that assigns real number to each outcome in the sample space of a random experiment.

For a given sample space S' of some experiment, a r.v. (random variable) is any rule that associates a number with each outcome in S'.



A r.v. is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the r.v. is denoted by lowercase letter such as $x = 70$ milliamperes.

A discrete r.v is a r.v. with a finite (or countably infinite) range.

Ex: number of scratches on a surface, number of transmitted bits received in error, proportion of defective parts among 1000 tested

A continuous r.v is a r.v. with an interval (either finite or infinite) of real numbers for its range.

Ex: electrical current, length, pressure, temperature, time, voltage, weight

Discrete Random Variable and Its Probability Distributions

A set is discrete either if it consists of a finite number of elements or if its elements can be listed so that we can start counting them even if they are infinite.

A random variable is said to be discrete if its set of possible values is a discrete set. Thus a discrete r.v. is a r.v. with a finite (or countably infinite) range.

A prob. distribution or prob. mass function of a discrete r.v. is

$$f(x_i) = P(X = x_i) \quad i = 1, 2, \ldots, n$$

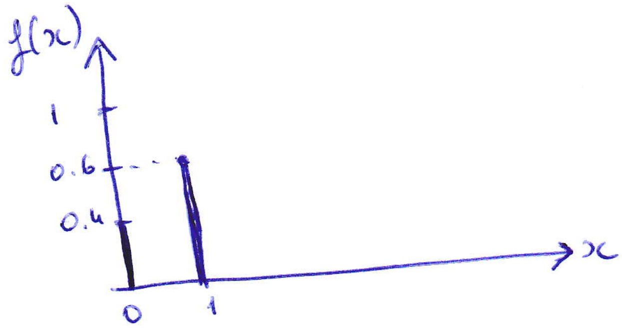
Conditions of a prob. mass function

① $f(x_i) \geq 0$, $\sum_{\text{all } x_i} P(X = x_i) = 1$ or $\sum_{i=1}^{n} f(x_i) = 1$

<u>Ex</u>: $X = \begin{cases} 1, & \text{if customer purchases a radial tire} \\ 0, & \text{if customer purchases a bias-ply tire} \end{cases}$

If we know that 60% of the customer select radial tires, then

$$P(X = x) = \begin{cases} 0.6, & \text{if } x = 1 \\ 0.4, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$



<u>Ex</u>: When we flip a coin 3 times. Let X denote the number of trials in which we observe heads. Determine the prob. mass function of X.

Sample Space	$X = x$	$P(X = x)$
TTT	0	$(1/2)^3$
HTT	1	$\frac{1}{2} \cdot \left(\frac{1}{2}\right)^2$
THT	1	$\frac{1}{2}\left(\frac{1}{2}\right)^2$
TTH	1	$\frac{1}{2}\left(\frac{1}{2}\right)^2$

Sample space	$X = x$	$P(X = x)$
HHT	2	$\frac{1}{2} \cdot \left(\frac{1}{2}\right)^2$
HTH	2	$\frac{1}{2}\left(\frac{1}{2}\right)^2$
THH	2	$\frac{1}{2}\left(\frac{1}{2}\right)^2$
HHH	3	$\left(\frac{1}{2}\right)^3$

$$P(X=x)=\begin{cases}\left(\frac{1}{2}\right)^3, & x=0\\ 3\cdot\left(\frac{1}{2}\right)^3, & x=1\\ 3\left(\frac{1}{2}\right)^3, & x=2\\ \left(\frac{1}{2}\right)^3, & x=3\end{cases}$$

$$\sum_{x=0}^{3} P(X=x)=\frac{1}{8}+\frac{3}{8}+\frac{3}{8}+\frac{1}{8}=\frac{8}{8}=1 \checkmark$$

Ex: There is a chance that a bit transmitted through a digital transmission channel is received in error. Let X equal the number of bits in error in the next 4 bits transmitted. The possible values for X are {0,1,2,3,4}. Based on a model for the errors that is presented later (binomial distr.) prob. dist. is determined (or prob. of the values) as follows:

$P(X=0)=0.6561$

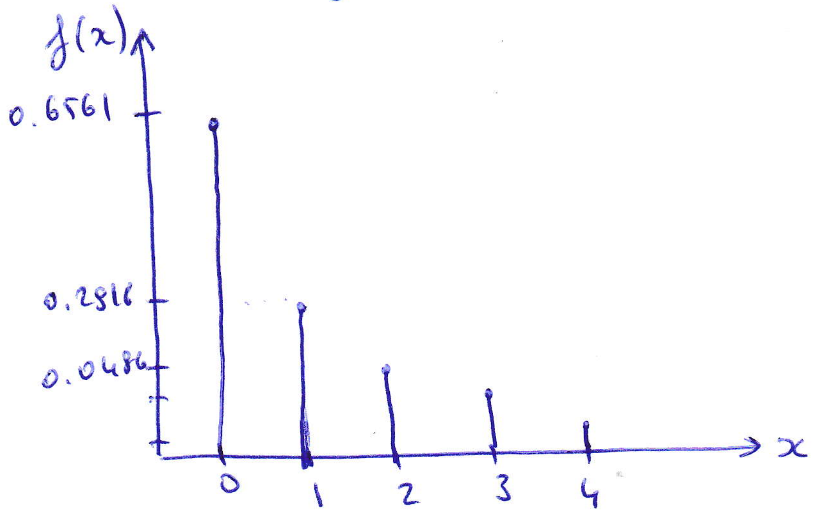
$P(X=1)=0.2916$

$P(X=2)=0.0486$

$P(X=3)=0.0036$

$P(X=4)=0.0001$

The prob. dist of X is specified by the possible values along with the prob. of each.



Parameter of a Prob. Distribution

If we don't know what percent of the customers select radials, we let the probability of such a selection to be α. Then,

$$P(X=x)=\begin{cases}\alpha, & \text{if } x=1\\ 1-\alpha, & \text{if } x=0\\ 0, & \text{otherwise}\end{cases}$$

Then α is the parameter of this p.m.f. (prob. mass function)

Ex:

x	-2	-1	0	1	2
$f(x)$	$1/8$	$2/8$	$2/8$	$2/8$	$1/8$

a) Find $P(X\leq 2)$

$P(X\leq 2)=P(X=-2)+P(X=-1)+P(X=0)+P(X=1)+P(X=2)=1$

b) Find $P(X>-2)$

$P(X>-2)=1-P(X\leq -2)=1-\frac{1}{8}=7/8$

c) Find $P(-1\leq X\leq 1)$

$P(-1\leq X\leq 1)=P(X=-1)+P(X=0)+P(X=1)$

$=2/8+2/8+2/8=6/8$

d) $P(X\leq -1 \text{ or } X=2)=?$

$P(X\leq -1 \text{ or } X=2)=P(X\leq -1)+P(X=2)$

$=P(X=-2)+P(X=-1)+P(X=2)$

$=1/8+2/8+1/8=4/8$ //

Ex: $f(x)=8/7\cdot\left(\frac{1}{2}\right)^x \quad x=1,2,3$

a) Find $P(X\leq 1)$

$P(X\leq 1)=P(X=1)=\frac{8}{7}\cdot\frac{1}{2}=4/7$

b) Find $P(X>1)$

$P(X>1)=1-P(X\leq 1)=1-\frac{4}{7}=\frac{3}{7}$

c) Find $P(2<X<6)$

$P(2<X<6)=P(X=3)=\frac{8}{7}\cdot\frac{1}{2^3}=\frac{1}{7}$ //

Ex: $f(x) = \dfrac{2x+1}{25}$, $x = 0,1,2,3,4$

a) Find $P(X=4)$

$$P(X=4) = \frac{2.4+1}{25} = \frac{9}{25}$$

b) Find $P(X>-10)$

$$P(X>-10) = 1$$

c) Find $P(X \leq 1)$

$$P(X \leq 1) = P(X=0) + P(X=1)$$
$$= \frac{1}{25} + \frac{3}{25} = \frac{4}{25}$$

Ex: $f(x) = c.\left(\frac{1}{4}\right)^x$ for $x = 1,2,3,\ldots$, then $c = ?$ ($f(x)$ is a p.m.f)

$$\sum_x f(x) = 1 \Rightarrow c\Big[\underset{a}{\frac{1}{4}} + \underset{a.r}{\frac{1}{4}^2} + \underset{a.r^2}{\frac{1}{4}^3} + \underset{a.r^3}{\frac{1}{4}^4} + \ldots\Big]$$

$$c\left[\frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\right)\right]$$

It is a geometric series with $a = \frac{1}{4}$ $r = \frac{1}{4}$

Note:

$$(*) \quad \sum_{k=0}^{\infty} a.r^k = \frac{a}{1-r}$$

$$a + a.r + a.r^2 + a.r^3 + \ldots$$

when $a = 1$ then

$$1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$$

$$c.\frac{a}{1-r} = 1$$

$$c.\frac{\frac{1}{4}}{1-\frac{1}{4}} = 1 \Rightarrow c = 3$$

H.w. Under which condition does $f(x) = (1-k).k^x$ $x = 0,1,2,\ldots$ is a p.m.f of X. Try to detemine it.