APPLIED STATISTICS

WEEK 8

Confidence Interval Estimation on Two Samples – PART II

Inference about Two Populations

- Confidence Intervals for the difference between two means.
- Confidence Intervals for the difference between two proportions
- Confidence Intervals for the ratio of two variances.

Estimating the Difference between Two Means

CASE 1: VARIANCES KNOWN

CASE 2: VARIANCES UNKNOWN BUT EQUAL

CASE 3: VARIANCES UNKNOWN AND UNEQUAL

Estimating the Difference between Two Proportions

Estimating the Ratio of Two Variances

Estimating the Difference between Two Proportions

Confidence Intervals for the difference between two populations proportions $P_1 - P_2$

- For qualitative data, we compare the population proportions of the occurence of a certain event.

Examples:

- Comparing defective rates between two machines.
- Comparing the effectiveness of new drug versus older one
- Comparing market share before and after advertising compaign.

Parameter and Statistic

* Parameter:

→ When the data are qualitative, we can only count the occurences of a certain event in two populations, and calculate proportions.

→ The parameter we want to estimate is

$$P_1 - P_2$$

* Statistic

→ An estimator of $P_1 - P_2$ is

$\hat{P}_1 - \hat{P}_2$ (the difference between the sample proportions)

$\hat{P}$ → sample proportion

$Pop1 \rightarrow S1$ $\quad$ $Pop2 \rightarrow S2$

General Formula for CI:

Point Estimate $\mp$ ME

Point Estimator:

— Two random samples are drawn from two populations $\rightarrow$ $\hat{p}_1 - \hat{p}_2$

— The number of successes in each sample is recorded.

— The sample proportions are computed.

Sample #1

Sample size, n_1

Number of successes, X_1

Sample Proportion:

$$\hat{p}_1 = \frac{X_1}{n_1}$$

Sample #2

Sample size, n_2

Number of successes, X_2

Sample Proportion:

$$\hat{p}_2 = \frac{X_2}{n_2}$$

Confidence Interval on the Difference in the Population Proportions

If $\hat{p}_1$ and $\hat{p}_2$ are the sample proportions of observations in two independent random samples of sizes n_1 and n_2 that belong to a class of interest, an **approximate two-sided 100(1 – α)% confidence interval on the difference in the true proportions $p_1 - p_2$ is**

(actual) $P_1 - P_2$ $\hat{q}_1$

$$\hat{P}_1 - \hat{P}_2 - z_{\alpha/2}\sqrt{\frac{\hat{P}_1(1-\hat{P}_1)}{n_1} + \frac{\hat{P}_2(1-\hat{P}_2)}{n_2}}$$

Point Estimator

$$\leq P_1 - P_2 \leq \hat{P}_1 - \hat{P}_2 + z_{\alpha/2}\sqrt{\frac{\hat{P}_1(1-\hat{P}_1)}{n_1} + \frac{\hat{P}_2(1-\hat{P}_2)}{n_2}}$$

Point Estimator

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution

Question 1:

Consider the process of manufacturing crankshaft bearings

Suppose that a modification is made in the surface finishing process and that, subsequently, a second random sample of 85 bearings is obtained. The number of defective bearings in this second sample is 8. Therefore, because $n_1 = 85$, $\hat{p}_1 = 10/85 = 0.1176$, $n_2 = 85$, and $\hat{p}_2 = 8/85 = 0.0941$. Obtain an approximate 95% confidence interval on the difference in the proportion of detective bearings produced under the two processes.

Sample #1

$n_1 = 85$

$\hat{P}_1 = \frac{10}{85} = \boxed{0.1176}$

Sample #2

$n_2 = 85$

$\hat{P}_2 = \frac{8}{85} = \boxed{0.0941}$

Question 1:

$1-\alpha = 0.95 \Rightarrow \alpha = 0.05$

$\alpha/2 = 0.025$

$$\hat{P}_1 - \hat{P}_2 \mp Z_{0.025} \sqrt{\frac{\hat{P}_1(1-\hat{P}_1)}{n_1} + \frac{\hat{P}_2(1-\hat{P}_2)}{n_2}}$$

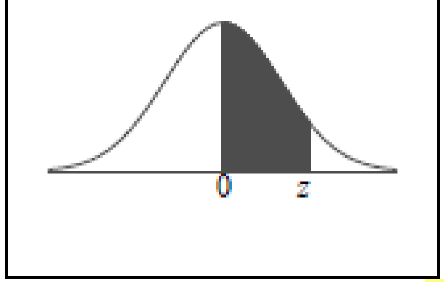
$$(0.1176 - 0.0941) \mp 1.96 \sqrt{\frac{0.1176(1-0.1176)}{85} + \frac{0.0941(1-0.0941)}{85}}$$

This simplifies to

$$\underbrace{-0.0685}_{\text{Lower Limit}} \leq P_1 - P_2 \leq \underbrace{0.1155}_{\text{Upper Limit}}$$

Interpretation:

This confidence Interval includes zero. Based on the sample data, it seems unlikely that the changes made in the surface finish process have reduced the proportion of defective bearings being produced.



$\alpha/2 = 0.025$

$\downarrow$

$z_{0.025} = \boxed{1.96}$

STANDARD NORMAL DISTRIBUTION TABLE

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952

Question 2:

A manufacturer turns out a product item that is labeled either "defective" or "not defective". In order to estimate the proportion defective, a random sample of 100 items is taken from production, and 10 are found to be defective. Following implementation of a quality improvement program, the experiment is conducted again. A new sample of 100 is taken, and this time only 6 are found to be defective.

$P_1 - P_2$ P_1

a. Give a **95%** confidence interval on $p_1 - p_2$, where p_1 is the population proportion defective before improvement and p_2 is the proportion defective after improvement.

b. Is there information in the confidence interval found in part (a) that would suggest that $p_1 > p_2$? Explain.

$P_1 > P_2$

P_2

Question 2- Solution:

Sample #1 ~~Quality~~
(Before Improvement)
Case

Samples #2 ~~Quality~~
(After Improvement)
Case

$$\hat{P}_1 = \frac{10}{100} = 0.1 \qquad \hat{P}_2 = \frac{6}{100} = 0.06$$

$$z_{\alpha/2} = z_{0.025} = 1.96$$

$$\underbrace{(0.1 - 0.06)} \mp 1.96 \sqrt{\frac{(0.1)(0.9)}{100} + \frac{(0.06)(0.94)}{100}}$$

a) $\boxed{-0.035 < P_1 - P_2 < 0.115}$

Question 2- Solution:

b. Is there information in the confidence interval found in part (a) that would suggest that $p_1 > p_2$? Explain.

$$-0.035 < P_1 - P_2 < 0.115$$

Interpretation:

Since the interval contains the value of 0 (zero), there is no reason to believe that the new process produces a significant decrease in the proportion of defectives over the existing method.

Estimating the Ratio of Two Variances

Confidence Interval on the Ratio of Two Variances

Ratio of Two Standard Dev.?

If s_1^2 and s_2^2 are the sample variances of random samples of sizes n_1 and n_2, respectively, from two independent normal populations with unknown variances σ_1^2 and σ_2^2, then a 100(1 – α)% confidence interval on the ratio σ_1^2/σ_2^2 is

$$\frac{s_1^2}{s_2^2} f_{1-\alpha/2,\, n_2-1,\, n_1-1} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{s_1^2}{s_2^2} f_{\alpha/2,\, n_2-1,\, n_1-1}$$

where $f_{\alpha/2,\, n_2-1,\, n_1-1}$ and $f_{1-\alpha/2,\, n_2-1,\, n_1-1}$ are the upper and lower $\alpha/2$ percentage points of the F-distribution with n_2-1 numerator and n_1-1 denominator degrees of freedom, respectively.

Note that: A confidence interval on the ratio of the standard deviations can be obtained by taking square roots in the given equation above ↑.

F-distribution: Critical Values

$$P[f_{1-\alpha/2}(v_1, v_2) < F < f_{\alpha/2}(v_1, v_2)] = 1 - \alpha,$$

where $f_{1-\alpha/2}(v_1, v_2)$ and $f_{\alpha/2}(v_1, v_2)$ are the values of the F-distribution with v_1 and v_2 degrees of freedom, leaving areas of $1 - \alpha/2$ and $\alpha/2$, respectively, to the right.

$\frac{\sigma_1^2}{\sigma_2^2}$ or $\frac{\sigma_1}{\sigma_2}$

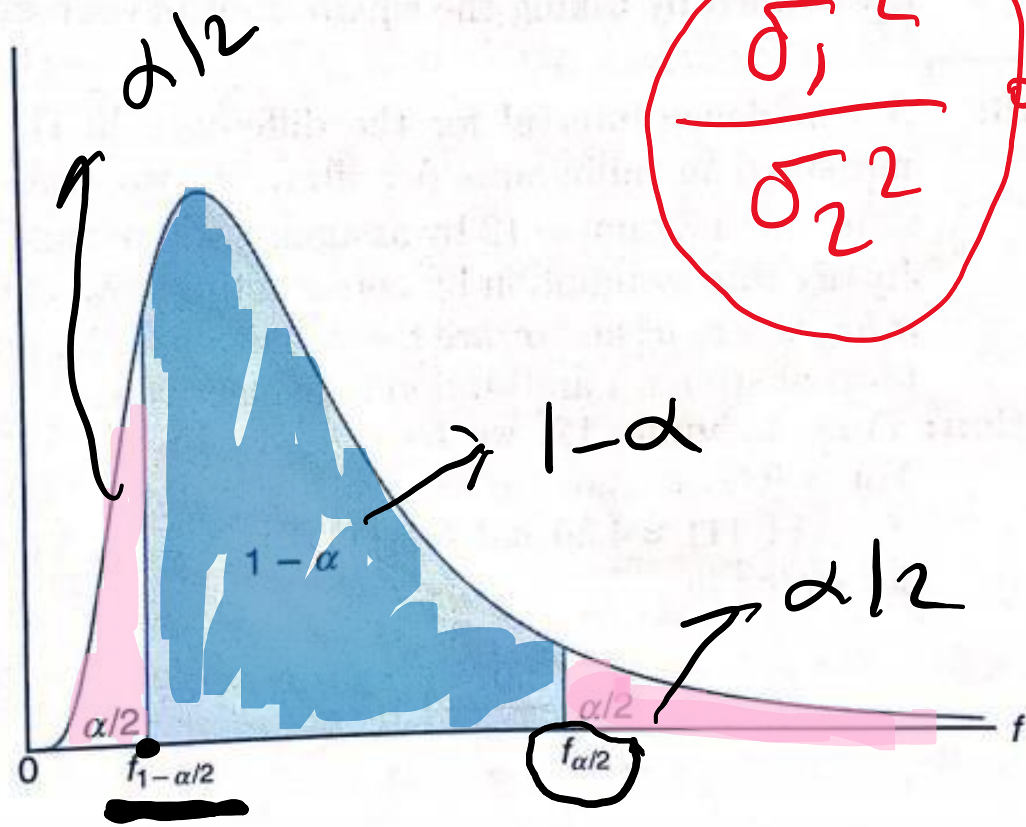


Figure 8: $P[f_{1-\alpha/2}(v_1, v_2) < F < f_{\alpha/2}(v_1, v_2)] = 1 - \alpha.$

Question 3:

A manufacturer of electric irons produces these items in two plants. Both plants have the same suppliers of small parts. A saving can be made by purchasing thermostats for plant B from a local supplier. A single lot was purchased from the local supplier, and a test was conducted to see whether or not these new thermostats were as accurate as the old. The thermostats were tested on tile irons on the 550^{0}F setting, and the actual temperature were read to the nearest 0.1^{0}F with a thermocouple. The data are as follows:

$n_{new} = 23$

New Supplier (^{0}F)					
530.3	559.3	549.4	544.0	551.7	566.3
549.9	556.9	536.7	558.8	538.8	543.3
559.1	555.0	538.6	551.1	565.4	554.9
550.0	554.9	554.7	536.1	569.1	
Old Supplier (^{0}F)					
559.7	534.7	554.8	545.0	544.6	538.0
550.7	563.1	551.1	553.8	538.8	564.6
554.5	553.0	538.4	548.3	552.9	535.1
555.0	544.8	558.4	548.7	560.3	

$n_{old} = 23$

$1-\alpha = 0.95 \quad \alpha = 0.05 \quad \alpha/2 = 0.025$

Find a 95% confidence intervals for σ_1^2/σ_2^2 and σ_1/σ_2 where σ_1^2 and σ_2^2 are the population variances of the thermostat readings for the new and old suppliers, respectively.

Question 3-Solution:

$n_N = n_{old} = 23$

$S_N^2 = 105.9271, \quad S_0^2 = 77.4138$

Use Sample Variance Formula !

$$\frac{S_1^2}{S_2^2} f_{1-\alpha/2, n_2-1, n_1-1} < \frac{\sigma_1}{\sigma_2} \le \frac{S_1^2}{S_2^2} f_{\alpha/2, n_2-1, n_1-1}$$

$$\frac{105.9271}{77.4138} f_{1-0.025, 22, 22} < \frac{\sigma_N^2}{\sigma_0^2} < \frac{105.9271}{77.4138} f_{0.025, 22, 22}$$

Note that:

$$f_{1-\alpha/2}(\vartheta_1, \vartheta_2) = \frac{1}{f_{\alpha/2}(\vartheta_2, \vartheta_1)}$$

$$\frac{105.9271}{77.4138} \cdot \frac{1}{2.358} < \frac{\sigma_N^2}{\sigma_0^2} < \frac{105.9271}{77.4138}(2.358)$$

$$0.58 \le \frac{\sigma_N^2}{\sigma_0^2} \le 3.23$$

$$0.76 \le \frac{\sigma_N}{\sigma_0} \le 1.80$$

$$0.58 \le \frac{\sigma_N^2}{\sigma_0^2} \le 3.23$$

$$0.76 \le \frac{\sigma_N}{\sigma_0} \le 1.80$$

Interpretation:

Since the intervals contain 1, we will assume that the variability did not change with the local supplier.

Remember that:
Formula for Sample Variance

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

s^2 = sample variance

x_i = value of i^{th} element (observation)

$\bar{x}$ = sample mean

n = sample size

The F Distribution: Values of F (alpha = 0.025)

ϑ_2

0.025 $f_{0.025, 22, 22}$ ϑ_1

Degees of Freedom of the numerator df1

df2	21	22	23	24	25	30	40	60	120	10000
1	994.3033	995.351	996.3405	997.2719	998.0868	1001.405	1005.596	1009.787	1014.036	1018.227
2	39.45024	39.45206	39.45479	39.45661	39.45752	39.46479	39.47298	39.48117	39.48935	39.49754
3	14.15515	14.14378	14.13355	14.12423	14.11536	14.08057	14.03646	13.99212	13.94733	13.90254
4	8.545953	8.53322	8.521624	8.510824	8.501047	8.461257	8.411121	8.360416	8.309144	8.257985
5	6.314167	6.301093	6.289042	6.278015	6.26784	6.226855	6.175071	6.122548	6.069286	6.015966
6	5.153822	5.140578	5.128385	5.117215	5.10687	5.065203	5.012453	4.958906	4.90445	4.849767
7	4.452033	4.438618	4.426312	4.415	4.404541	4.362391	4.308873	4.254389	4.19891	4.143033
8	3.98461	3.971081	3.958661	3.947207	3.936663	3.894002	3.839773	3.784436	3.727934	3.670891
9	3.651962	3.638291	3.625743	3.61419	3.603532	3.560416	3.505477	3.449301	3.39179	3.333568
10	3.403471	3.389687	3.377025	3.365358	3.354586	3.311015	3.255394	3.198409	3.139917	3.080515
11	3.210943	3.197044	3.184283	3.172516	3.161645	3.11762	3.061331	3.003535	2.944077	2.883539
12	3.057465	3.043453	3.030578	3.018712	3.007742	2.963276	2.906347	2.84777	2.78736	2.725685
13	2.932239	2.918128	2.905153	2.893188	2.882132	2.837254	2.7797	2.720355	2.659021	2.596238
14	2.828159	2.813948	2.800874	2.788809	2.777654	2.732378	2.674227	2.614158	2.551928	2.488036
15	2.740265	2.725955	2.712781	2.700645	2.68939	2.643731	2.585011	2.524231	2.461121	2.396149
16	2.665061	2.650651	2.637393	2.625171	2.613845	2.567816	2.508528	2.447067	2.383111	2.317094
17	2.599975	2.58548	2.572136	2.55983	2.548418	2.502041	2.442228	2.380105	2.315325	2.248271
18	2.543089	2.528509	2.515094	2.502702	2.49122	2.444509	2.384184	2.321421	2.255838	2.187775
19	2.492953	2.478288	2.464787	2.452325	2.440771	2.393733	2.332925	2.269552	2.203194	2.134129
20	2.448417	2.433673	2.420101	2.40756	2.395936	2.3486	2.287322	2.223359	2.156241	2.08621
21	2.408591	2.393776	2.380133	2.367528	2.35584	2.308212	2.24648	2.181949	2.114092	2.043116
22	2.372765	2.357879	2.344173	2.331504	2.319752	2.27184	2.209674	2.144596	2.076028	2.004121
23	2.340364	2.325415	2.311637	2.298904	2.287095	2.23892	2.176343	2.110724	2.041475	1.968651
24	2.310919	2.295906	2.282071	2.269275	2.257408	2.208978	2.146002	2.079872	2.009948	1.936236
25	2.28404	2.268962	2.255071	2.242224	2.230301	2.181622	2.118263	2.051635	1.981057	1.906471
26	2.259405	2.244271	2.230323	2.217419	2.205446	2.156526	2.092804	2.0257	1.954483	1.879037
27	2.236746	2.221554	2.20755	2.194597	2.182574	2.133426	2.069342	2.001784	1.929948	1.853657

Question 4:

Sample standard deviation values are given

A company manufactures impellers for use in jet-turbine engines. One of the operations involves grinding a particular surface finish on a titanium alloy component. Two different grinding processes can be used, and both processes can produce parts at identical mean surface roughness. The manufacturing engineer would like to select the process having the least variability in surface roughness. A random sample of $n_1 = 11$ parts from the first process results in a sample standard deviations $s_1 = 5.1$ microinches, and a random sample of $n_2 = 16$ parts from the second process results in a sample standard deviation of $s_2 = 4.7$ microinches. Find a 90% confidence interval on the ratio of the two standard deviations, σ_1 / σ_2.

Assuming that the two processes are independent and that surface roughness is normally distributed.

$1-\alpha = 0.90 \quad \alpha = 0.10 \quad \alpha/2 = 0.05$

$1-\alpha/2 = 0.95$

$$\frac{S_1^2}{S_2^2} f_{1-\alpha/2,\, n_2-1,\, n_1-1} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} f_{\alpha/2,\, n_2-1,\, n_1-1}$$

$n_1 = 11$

$n_2 = 16$

$$\frac{(5.1)^2}{(4.7)^2} \underbrace{f_{0.95,\, 15,\, 10}} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{(5.1)^2}{(4.7)^2} \underbrace{f_{0.05,\, 15,\, 10}}$$

$$\boxed{\frac{(5.1)^2}{(4.7)^2} (0.39) \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{(5.1)^2}{(4.7)^2} (2.85)}$$

$$f_{0.95,\, 15,\, 10} = \frac{1}{f_{0.05,\, 10,\, 15}} = \frac{1}{2.54}$$

$$= 0.39$$

$$\boxed{0.678 \le \frac{\sigma_1}{\sigma_2} \le 1.832}$$

Interpretation:

Since the confidence interval includes unity $\left(\frac{\sigma_1}{\sigma_2} = 1\right)$, we cannot claim that the standard deviations of surface roughness for the two process are different at the 90% level of confidence.

0.95 quantiles for F distributions ($f_{0.05,d_1,d_2}$ values)

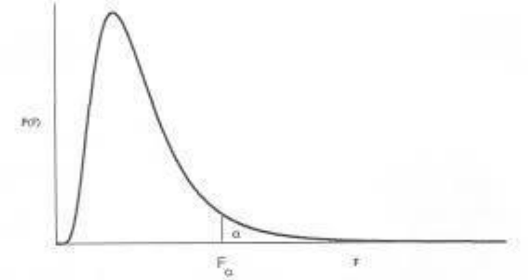


Figure K.1: The F distribution

This table gives $f_{0.05,d_1,d_2}$ values for different (d_1, d_2)'s, where f_{a,d_1,d_2} is defined such that $P(F(d_1, d_2) > f_{a,d_1,d_2}) = a$ and $F(d_1, d_2)$ is the F distribution with (d_1, d_2) degrees of freed

d_1 (degrees of freedom for the numerator)

d_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99

$$0.58 \leq \frac{\sigma_N^2}{\sigma_0^2} \leq 3.23$$

$$0.76 \leq \frac{\sigma_N}{\sigma_0} \leq 1.80$$

Interpretation:

Since the intervals contain 1, we will assume that the variability did not change with the local supplier.

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