UYGULAMA.1. (DEVAMI) [8]

$\hat{\theta}_{MLE} = \dfrac{\sum X_j}{n}$ TAHMİNLEYİCİSİ, θ PARAMETRESİ İÇİN EN KÜÇÜK VARYANSLI SAPMASIZ TAHMİNLEYİCİ MİDİR? ($\hat{\theta}_{MLE}$ MVUE midir?)

Çözüm : (a) $E(\hat{\theta}_{MLE}) - \theta = 0$ OLDUĞU GÖSTERİLMİŞTİ.

(b) $CRLB_\theta(Var(\hat{\theta}_{MLE})) = \dfrac{\theta^2}{n}$ OLARAK DAHA ÖNCE BULUNMUŞTU.

$Var(X_j) = \sigma_{X_j}^2 = \theta^2$ OLARAK DAHA ÖNCE BULUNMUŞTU.

$$\Rightarrow VAR(\hat{\theta}_{MLE}) = VAR\left(\frac{\sum X_j}{n}\right) = \frac{1}{n^2} Var\left(\sum X_j\right) = \frac{1}{n^2}\sum Var(X_j)$$

$$= \frac{1}{n^2}\sum_{j=1}^{n}\theta^2 = \frac{1}{n^2} n\theta^2 \Rightarrow Var(\hat{\theta}_{MLE}) = \frac{\theta^2}{n}$$

BÖYLECE, $CRLB_\theta(Var(\hat{\theta}_{MLE})) = Var(\hat{\theta}_{MLE}) = \dfrac{\theta^2}{n}$

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(a) ve (b) ÇÖZÜMLERİ GEREĞİ, $\hat{\theta}_{MLE} = \dfrac{\Sigma X_j}{n}$,
θ PARAMETRESİ İÇİN EN KÜÇÜK VARYANSLI SAPMASIZ TAHMİNLEYİCİ DİR (MVUE DİR).

UYGULAMA. 2. $X_1, X_2, \ldots, X_n$ RANDOM ÖRNEĞİ, θ_1 ve θ_2 PARAMETRELERİNE SAHİP NORMAL POPULASYON-DAN ALINMIŞ OLSUN. θ_2 PARAMETRESİNİN BİLİNDİĞİ VARSAYILSIN. (a) θ_1 PARAMETRESİNİN E.Y.O. TAHMİNLEYİCİSİNİ BULUNUZ. (b) θ_1 HAKKINDAKİ ÖRNEK FISHER BİLGİSİNİ ELDE EDİNİZ.
(c) BULDUĞUNUZ $\hat{\theta}_{1_{MLE}}$, θ_1 İÇİN SAPMASIZ TAHMİNLEYİCİ MİDİR? (d) $\hat{\theta}_{1_{MLE}}$ SAPMASIZ TAHMİNLEYİ

Cİ İSE, $\hat{\theta}_{1_{MLE}}$ NİN VARYANSI İÇİN CRAMÉR–RAO ALT SINIRINI ELDE EDİNİZ. (e) $\hat{\theta}_{1_{MLE}}$ NİN VARYANSINI BULUNUZ. (f) $\hat{\theta}_{1_{MLE}}$, θ PARAMETRESİ İÇİN EN KÜÇÜK VARYANSLI SAPMASIZ TAHMİNLEYİCİ MİDİR?

ÇÖZÜM: (a) $X_j \sim N(\theta_1, \theta_2)$

$$f(x_j;\theta_1,\theta_2) = \frac{1}{\sqrt{\theta_2}\sqrt{2\pi}} e^{-(x_j-\theta_1)^2/2\theta_2}, \quad \theta_2 > 0$$

$$-\infty < x_j < \infty$$

$$= (\theta_2 2\pi)^{-\frac{1}{2}} e^{-(x_j-\theta_1)^2/2\theta_2}$$

$X_1, X_2, \ldots, X_n$ RANDOM ÖRNEĞİ İÇİN OLABİLİRLİK FONKSİYONU

$$L_{X_1,\ldots,X_n}(\theta_1, \theta_2 | x_1, \ldots, x_n) = \prod_{j=1}^{n} (2\pi\theta_2)^{-\frac{1}{2}} e^{-(x_j-\theta_1)^2/2\theta_2}$$

$$= (2\pi\theta_2)^{-\frac{n}{2}} e^{-\frac{1}{2\theta_2}\sum_{j=1}^{n}(x_j-\theta_1)^2}, \quad \begin{array}{c} -\infty < x_j < \infty \\ \theta_2 > 0 \end{array}$$

LOG-OLABİLİRLİK FONKSİYONU,

$$\log L(\theta_1, \theta_2 | x_1, \ldots, x_n) = -\frac{n}{2}\log(2\pi\theta_2) - \frac{1}{2\theta_2}\sum_{j=1}^{n}(x_j-\theta_1)^2$$

$$\Rightarrow \log L(\theta_1, \theta_2 | x_1, \ldots, x_n) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\theta_2 - \frac{1}{2\theta_2}\sum_{j=1}^{n}(x_j-\theta_1)^2$$

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$$S(\theta_1) = \frac{\partial \log L(\theta_1, \theta_2 | x_1, \ldots, x_n)}{\partial \theta_1} = -\frac{(2\theta_2)\left[-2\sum_{j=1}^{n}(x_j - \theta_1)\right]}{4\theta_2^2}$$

$$\Rightarrow S(\theta_1) = \frac{4\theta_2 \sum (x_j - \theta_1)}{4\theta_2^2} = \frac{\sum (x_j - \theta_1)}{\theta_2}$$

$$\Rightarrow S(\theta_1) = \frac{\sum x_j - n\theta_1}{\theta_2}, \quad \theta_2 > 0.$$ VE θ_2 NİN BİLİNDİĞİ VARSAYILIYOR.

LOG_OLABİLİRLİK FONKSİYONUNU MAXIMUM YAPMAK İÇİN,

$$S(\theta_1) = 0 \Rightarrow \frac{\sum x_j - n\hat{\theta}_{1_{MLE}}}{\theta_2} = 0$$

$$\Rightarrow \sum x_j - n\hat{\theta}_{1_{MLE}} = 0 \Rightarrow \hat{\theta}_{1_{MLE}} = \frac{\sum X_j}{n}, \quad \theta_2 > 0$$

$$\frac{\partial S(\theta_1)}{\partial \theta_1} = \frac{\partial^2 \log L}{\partial \theta_1^2} = -\frac{n\theta_2}{\theta_2^2} = -\frac{n}{\theta_2}$$

LOG-OLABİLİRLİK FONKSİYONUNUN İKİNCİ TÜREVİ

$$I\!I(\theta_1) = -\frac{\partial S(\theta_1)}{\partial \theta_1} = -\frac{\partial^2 \log L}{\partial \theta_1^2} = -\left(-\frac{n}{\theta_2}\right) = \frac{n}{\theta_2}$$

θ_1 HAKKINDAKİ ÖRNEK FISHER BİLGİSİ

$$E\left(I\!I(\theta_1)\right) = \mathcal{I}(\theta_1) = E\left(\frac{n}{\theta_2}\right) = \frac{n}{\theta_2}$$

θ_1 HAKKINDAKİ TEORİK FISHER BİLGİSİ

(c) $$E(e^{tx}) = M_x(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x-\theta_1)^2/2\theta_2}\, dx$$

$$= e^{\theta_1 t + \frac{\theta_2 t^2}{2}}$$

O HALDE, $M_{x_j}(t) = e^{\theta_1 t + \frac{\theta_2 t^2}{2}}$, $j = 1, 2, \ldots, n$

$N(\theta_1, \theta_2)$ NİN MTF SİNİN t YE GÖRE TÜREVİ,

$$\frac{dM_{x_j}(t)}{dt} = M'_{x_j}(t) = (\theta_1 + \theta_2 t)\, e^{\theta_1 t + \frac{\theta_2 t^2}{2}}$$

$$E(X_j) = \mu_1 = M'_{x_j}(t=0) = \theta_1$$ (X_j NİN ANAKÜTLE ORTALAMASI)

$M_{x_j}(t)$ NİN t YE GÖRE İKİNCİ TÜREVİ,

$$M''_{x_j}(t) = \theta_2\, e^{\theta_1 t + \frac{\theta_2 t^2}{2}} + (\theta_1 + \theta_2 t)^2\, e^{\theta_1 t + \frac{\theta_2 t^2}{2}}$$

$$M''_{x_j}(t) = \left[\theta_2 + (\theta_1 + \theta_2 t)^2\right] e^{\theta_1 t + \frac{\theta_2 t^2}{2}}$$

$$E(X_j^2) = \mu_2 = M''_{x_j}(t=0) = \theta_1^2 + \theta_2$$

$$\mu_2 = \theta_1^2 + \theta_2$$

$VAR(X_j) = \sigma_{X_j}^2 = \theta_1^2 + \theta_2 - \theta_1^2 = \theta_2$

$\hat{\theta}_{1_{MLE}}$, θ_1 İÇİN SAPMASIZ MIDIR ?

$$E(\hat{\theta}_{1_{MLE}}) = E\left(\frac{\sum X_j}{n}\right) = \frac{1}{n} \sum E(X_j) = \frac{1}{n} n\theta_1 = \theta_1$$

$\Rightarrow E(\hat{\theta}_{1_{MLE}}) - \theta_1 = 0 \Rightarrow \hat{\theta}_{1_{MLE}}$ SAPMASIZ DIR.

(d) $CRLB_{\theta_1}(Var(\hat{\theta}_{1_{MLE}})) = (\mathbb{I}(\theta_1))^{-1} = \left(\frac{n}{\theta_2}\right)^{-1} = \frac{\theta_2}{n}$

(e) $\hat{\theta}_{1_{MLE}}$ NİN VARYANSINI DAHA ÖNCE BULMUŞTUK.

(f) $\hat{\theta}_{1_{MLE}}$ SAPMASIZ DIR VE ,

$$CRLB_{\theta_1}(Var(\hat{\theta}_{1_{MLE}})) = VAR(\hat{\theta}_{1_{MLE}})$$

O HALDE; $\hat{\theta}_{1_{MLE}}$, θ_1 İÇİN EN KÜÇÜK VARYANSLI SAPMA-SIZ TAHMİNLEYİCİDİR (MINIMUM VARIANCE UNBIASED ESTIMATOR)

EV ÖDEVİ: UYGULAMA.2. DE ÇÖZÜMLERİ YAPILAN TÜM SORULARI BU KEZ θ_1 PARAMETRESİ BİLİNİYOR, θ_2 PARAMETRESİ BİLİNMİYOR DURUMU İÇİN ÇÖZÜNÜZ.