Exercise Sheet 11

June 29nd 2023

Submission of the homework assignments until July 6th, 11:30 am online in TUM-moodle in groups of two. Please put the *full names* and *student IDs* of you *and* your partner on all parts of your submission. The solution will be discussed in the classes and published on moodle after some time.

Homework

Problem H 45 - Hypotheses on Blueberries **[6 pts.]**

Welcome back to the pastry factory! A box of blueberries needed to bake the most delicious blueberry muffins should have a nominal weight of 100 g. From a weekly delivery of a large number of boxes the following random sample of size n =10 was drawn:

1	2	3	4	5	6	7	8	9	10
98 g	101 g	100 g	96 g	102 g	98 g	96 g	101 g	97 g	100 g

.

a) Define a null-hypothesis $H_{0,t}$ that is adequate for a two-sided Gaussian test of the assumption that the observed mean is the true one.

b) Find the two-sided confidence interval of the level 95% if the true variance is known to be $\sigma^2 = 4$ (in g^2). Based on this, should one be suspicious that the true mean differs from the nominal value?

c) Repeat this in the case of an unknown variance.

d) Perform a t-test for the one-sided hypothesis $H_{0,o}$: "$\mu \geq 100$" on the significance level of $\alpha = 0.05$.

Solution:

a) The null hypothesis is simply $H_{0,t} : \mu = \mu_0$ where μ_0 is the nominal value 100 g. ✓ Accordingly, the alternative hypothesis is $H_{1,t} : \mu \neq \mu_0$.

b) The level of confidence is $95\% = 1 - \alpha$, so $\alpha = 0.05$ or $1 - \frac{\alpha}{2} = 0.975$. The sample size is $n = 10$ and the sample mean is $\bar{X} = 98.9$. The standard deviation is known to be $\sigma = 2$ g. A look in the table yields the required quantile of the standard normal distribution: $z_{0.975} = 1.96$. By this, we obtain the confidence interval

$$\left[98.9 - 1.96 \cdot \frac{2}{\sqrt{10}}, 98.9 + 1.96 \cdot \frac{2}{\sqrt{10}}\right] = [97.7, 100.1]\,,$$

which contains with a probability of 95% the true mean value. The nominal value of 100 is in this interval included, so there is no reason to be suspicious. ✓✓

c) In case of unknown variance, we need the sample variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \approx 4.77 \qquad \Rightarrow s = 2.183$$

We again have $1 - \frac{\alpha}{2} = 0.975$ and further have $n - 1 = 9$ degrees of freedom. The required quantile of the t-distribution is $t_{0.975;9} = 2.262$ as found from a table. The resulting interval of confidence is

$$\left[98.9 - 2.262 \cdot \frac{2.183}{\sqrt{10}}, 98.9 + 2.262 \cdot \frac{2.183}{\sqrt{10}}\right] = [97.3, 100.5]\,,$$

being larger than the previous one. ✓✓

d) We already have the sample mean and variance. The required test value is

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{98.9 - 100}{0.690} = -1.593,$$

referring to test statistic that is t-distributed with $n-1$ degrees of freedom. For this one-sided test we need the quantile $t_{1-\alpha;n-1} = t_{0.95;9} = 1.833$. The one-sided null-hypothesis $H_{0,o}$ will be neglected if $t < -1.833$, i.e. here we will keep $H_{0,o}$. ✓

Problem H 46 - Testing a Vaccine **[6 pts.]**

A new vaccine against some new virus is tested at $n = 100$ persons. It is speculated that each of these persons independently will be immune against that virus by the probability p. The pharma company claims that $p \geq 0.9$.

a) Perform an appropriate hypothesis test on the significance level of 5% to determine the largest possible critical region K for the company's hypothesis.

b) A neutral researcher is skeptical - he claims that $p < 0.8$. Consider now the null-hypothesis of the pharma company in contrast to the researcher's assumption as alternative hypothesis. Determine the type-II error with respect to the critical region found before. Use the approximation by the standard normal distribution.
Hint: The function

$$f(x) = \frac{a - bx}{\sqrt{bx(1-x)}}, \; 0 < a < b$$

is strictly monotonously decreasing on the interval $x \in [0, 1]$.

Solution:

a) Denote by X the number of immune persons. It is clear from the instruction that X is binomially distributed with parameters $n = 100$ and p. We perform an approximate binomial test with the null hypothesis $H_0 : p \geq p_0$ where $p_0 = 0.9$. The test statistic is given as

$$Z = \frac{X - np_0}{\sqrt{np_0(1-p_0)}} = \frac{X - 100 \cdot 0.9}{\sqrt{100 \cdot 0.9 \cdot 0.1}} = \frac{X - 90}{\sqrt{9}} = \frac{X - 90}{3}. ✓$$

On the significance level of α we will neglect the null hypothesis if $Z < z_\alpha$, being equivalent to $X < 3 \cdot z_\alpha + 90$ in this exercise. The α-quantile with $\alpha = 0.05$ might not be tabulated but by the identity $\Phi(-x) = 1 - \Phi(x)$ we can use that $z_{1-\alpha} = -z_\alpha$. So, we obtain $z_{0.05} = -z_{0.95}$, and get from the table $z_{0.05} = -z_{0.95} = -1.65$. By this, the testing statistic will be in the region of rejection K if $X < 90 - 3 \cdot 1.65 = 85.05$ holds. The greatest possible critical region / region of rejection thus is $K = \{0, \dots, 85\}$. ✓

b) The type-II error means that H_0 is false while $H_1 : p < p_1 = 0.8$ - the hypothesis of the neutral researcher - is true, but the test statistic is not in the region of rejection of H_0. So, we consider

$$\sup_{p<p_1} Pr_p(X \notin K) = \sup_{p<p_1} Pr_p(X \geq k+1). \checkmark$$

Following the hint, we approximate by the standard normal distribution. From a theorem of the lecture we know that

$$Z' = \frac{X - np}{\sqrt{np(1-p)}}$$

can be assumed to be normally distributed. By this, we find

$$\begin{aligned} Pr_p(X \geq k+1) &= Pr_p(Z' \cdot \sqrt{np(1-p)} + np \geq k+1) \\ &= Pr_p\left(Z' \geq \frac{k+1-np}{\sqrt{np(1-p)}}\right) \\ &= 1 - Pr_p\left(Z' < \frac{k+1-np}{\sqrt{np(1-p)}}\right) \\ &\approx 1 - \Phi\left(\frac{k+1-np}{\sqrt{np(1-p)}}\right). \checkmark \end{aligned}$$

The cumulative normal distribution Φ is strictly monotonously increasing. We further use the hint with $a = k+1$, $b = n$ and $x = p$ to see that the argument of Φ is strictly monotonously decreasing with respect to p. ✓

$$\begin{aligned} \sup_{p<p_1} Pr_p(X \geq k+1) &= 1 - \inf_{p<p_1} \Phi\left(\frac{k+1-np}{\sqrt{np(1-p)}}\right) = 1 - \Phi\left(\frac{k+1-np_1}{\sqrt{np_1(1-p_1)}}\right) \\ &= 1 - \Phi\left(\frac{85+1-80}{\sqrt{16}}\right) = 1 - \Phi\left(\frac{6}{4}\right) = 1 - \Phi(1.5) = 1 - 0.933 = 0.067. \checkmark \end{aligned}$$

Problem H 47 - Helping an Economy Student **[4 pts.]**

Laura, an economy student, hypothesizes that a certain advertisement for a mobile dating app is more attractive to women than to men. The advertisement was rated on some continuous scale by a sample of $N_1 = 8$ women and $N_2 = 10$ men. This yielded the sample means $\hat{\mu}_1 = 7$ and $\hat{\mu}_2 = 5.5$ as well as the sample variances $s_1^2 = 1$ and $s_2^2 = 1.7$. Using the confidence level $\alpha = 0.01$, provide Laura a test for whether the female mean is greater than the male mean. Assume two normal distributions with $\sigma_1^2 = \sigma_2^2 = \sigma$.

Solution:
We need to test the null-hypothesis $H_0 : \mu_1 > \mu_2$ against the alternative hypothesis $H_1 : \mu_1 \leq \mu_2$, referring to the true mean values μ_1 and μ_2. In the case of unknown but equal variances we need the test statistic

$$T = \frac{\hat{\mu}_1 - \hat{\mu}_2}{\sqrt{(N_1 - 1)s_1^2 + (N_2 - 1)s_2^2}} \sqrt{\frac{N_1 + N_2 - 2}{\frac{1}{N_1} + \frac{1}{N_2}}}. \checkmark$$

Using the values given in the instruction, we get

$$T = \frac{7 - 5.5}{\sqrt{7 \cdot 1 + 9 \cdot 1.7}} \sqrt{\frac{16}{\frac{1}{8} + \frac{1}{10}}} = \frac{1.5}{\sqrt{22.3}} \sqrt{\frac{640}{9}} = \frac{1.5}{0.5600} \approx 2.679. \checkmark$$

The table of the t-distribution yields that $t_{1-\alpha;N} = t_{.99;16} = 2.583$ where we used $\alpha = 0.01$ and $N = N_1 + N_2 - 2 = 16$, the degrees of freedom. ✓ The null hypothesis will be rejected if $T < t_{1-\alpha;N}$. As we have $2.679 > 2.583$, the samples do not give rise to reject the null-hypothesis - the data supports Laura's hypothesis. ✓

Problem H 48 - A Dice to Determine Exam Grades **[5 pts.]**

Linsen buys a biased dice at the toy store. The vendor assures that the dice falls on the number "one" with the probability of $\frac{1}{3}$ while all other numbers occur with the same probability. In order to check whether this is true or wrong, Linsen throws the dice for $n = 100$ times and observes the following frequencies:

Number	1	2	3	4	5	6
Frequency	30	15	18	14	10	13

.

Should Linsen believe the vendor's claim on a significance level of 0.01? Perform an appropriate hypothesis test.
Hint: You may approximate $\chi^2_{k,\alpha}$, i.e. the α quantile of the χ^2 distribution with k degrees of freedom, by

$$k \cdot \left(1 - \frac{2}{9k} + z_\alpha \cdot \sqrt{\frac{2}{9k}}\right)^3$$

where z_α denotes the α-quantile of the standard normal distribution.

Solution:
The total number of throws $30 + 15 + 18 + 14 + 10 + 13$ indeed is equal to $n = 100$. Denote by X_j the discrete random variable for the number on the dice of the jth throw ($1 \leq j \leq n$). The vendor claims that the probability for a "1" is $p_1 = 1/3$ while any other number $i \in \{2, \ldots 6\}$ occurs by the same probability, namely $p_i = (1 - (1/3))/5 = 2/15$. We perform a χ^2 goodness-of-fit test to the null-hypothesis

$$H_0 : Pr(X = i) = p_i \quad \text{for } i \in \{1, \ldots 6\}, \checkmark$$

where X is just the result in some throw. For $1 \leq i \leq 6$, let h_i be the frequency of the respective number i on the dice. The test statistic T is defined as

$$T = \sum_{i=1}^{6} \frac{(h_i - n \cdot p_i)^2}{n \cdot p_i}. \checkmark$$

Using the reported values we calculate

$$\begin{aligned}
T &= \frac{(30 - 100/3)^2}{100/3} + \frac{(15 - 200/15)^2}{200/15} + \frac{(18 - 200/15)^2}{200/15} \\
&\quad + \frac{(14 - 200/15)^2}{200/15} + \frac{(10 - 200/15)^2}{200/15} + \frac{(13 - 200/15)^2}{200/15} \\
&= \frac{1}{3} + \frac{5}{24} + \frac{49}{30} + \frac{1}{30} + \frac{5}{6} + \frac{1}{120} \\
&= \frac{366}{120} = \frac{61}{20} = 3.05. \checkmark
\end{aligned}$$

If this value is larger than $\chi^2_{M-1;1-\alpha} = \chi^2_{6-1;1-0.01} = \chi^2_{5;0.99}$ ($M - 1 = 5$ being the number of degrees of freedom in the case of a dice with 6 sides), Linsen should reject the null-hypothesis. ✓ We use the approximation of the hint and find

$$\chi_{5;0.99} \approx 5 \cdot \left(1 - \frac{2}{9 \cdot 5} + z_{0.99} \cdot \sqrt{\frac{2}{9 \cdot 5}}\right)^3 \approx 5 \cdot \left(1 - \frac{2}{45} + 2.33 \cdot \sqrt{\frac{2}{45}}\right)^3 \approx 15.1.$$

Since $3.05 < 15.1$, Linsen cannot reject the claim of the vendor based on her sample. ✓