Exercise Sheet 2

April 26th 2023

Submission of the homework assignments until May 4th, 11:30 am online in TUM-moodle in groups of two. Please put the *full names* and *student IDs* of you *and* your partner on all parts of your submission. The solution will be discussed in the classes one week after.

Homework

Problem H 5 - Muffins à la Bayes [4 pts.]

The pastry cook Francois bakes the most delicious blueberry muffins in town. In order to prepare them, he orders four boxes of blueberries at the fruit merchant every week. Unfortunately, the fruit merchant sometimes makes mistakes and delivers boxes with raspberries instead. Based on his long experience with the fruit merchant, Francois knows that he erroneously will receive one, two, three or four boxes of raspberries in his weekly order of four boxes with a probability of $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{6}$ or $\frac{1}{4}$, respectively.

a) Determine the probability that more than half of the four boxes in a weekly delivery contain raspberries!

b) To check the delivery, Francois randomly opens one of the (uniformly distributed) four boxes and, what a bad luck, finds a box of raspberries. Given this, what is the probability to find more than half of the boxes filled by raspberries now?

Solution:
We denote the event that i of the four boxes contain raspberries by E_i. By this definition, the sets E_i and E_j are disjoint if $i \neq j$.

a) The probability of interest simply is $Pr(E_3 \cup E_4) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$. ✓

b) Denote by F the event that Francois's random sample contains raspberries. We need to find $Pr(E_3 \cup E_4|F) = Pr(E_3|F) + Pr(E_4|F)$. Since the events E_i are pairwise disjoint and $F \subseteq \cup_{i=1}^{4} E_i$, we may apply Bayes' theorem:

$$Pr(E_j|F) = \frac{Pr(F|E_j) \cdot Pr(E_j)}{\sum_{i=1}^{4} Pr(F|E_i) \cdot Pr(E_i)}. \checkmark$$

If there are exactly i boxes with raspberries in the four boxes of the delivery, the probability to draw one of them in the check is $Pr(F|E_i) = i/4$, and the probabilities $Pr(E_i)$ are given. We have

$$\sum_{i=1}^{4} Pr(F|E_i) \cdot Pr(E_i) = \frac{1}{4} \cdot \frac{1}{8} + \frac{2}{4} \cdot \frac{1}{16} + \frac{3}{4} \cdot \frac{1}{6} + \frac{4}{4} \cdot \frac{1}{4} = \frac{7}{16}. \checkmark$$

This yields

$$Pr(E_3|F) = \frac{\frac{3}{4} \cdot \frac{1}{6}}{\frac{7}{16}} = \frac{2}{7} \text{ as well as } Pr(E_4|F) = \frac{\frac{4}{4} \cdot \frac{1}{4}}{\frac{7}{16}} = \frac{4}{7},$$

so in total $Pr(E_3 \cup E_4|F) = \frac{6}{7}$. ✓ It is interesting to observe that the conditional probability based on the information gained by the check is significantly greater than the *a priori* probability from subtask (a).

Problem H 6 - Balls 'n' Boxes [4 pts.]

We play a game where we draw 40 numbered balls from three different boxes. The balls are numbered by $1, 2, \ldots, 40$ and are distributed to the three boxes as follows:

- All balls which show a number divisible by 4, i.e. 4, 8, ..., 40 are put to box 1.
- From the remaining balls all those that show a prime number are put to box 2.
- The rest of the balls is put to box 3.

First, we randomly choose one of the boxes where each of them is equally likely to be chosen. Second, we randomly draw a ball from the chosen box where similarly all balls from that box are drawn with the same probability.

a) Denote by E the event that the number on the drawn ball is less than 10. Calculate $Pr(E)$.

b) Let be F the event that the number on the drawn ball is prime. Prove or disprove the independence of the two events E and F.

Solution:

a) It helps to determine how many balls and which of them are in the boxes first. We have $40/4 = 10$, so there are 10 balls in the first box (those with the numbers 4, 8, ..., 40). Since these are all numbers divisible by 4 none of them is prime. Therefor the second box contains *all* prime numbers until 40. We count them to be 12 prime numbers. The third box contains the remaining $40 - 10 - 12 = 18$ balls. ✓
Let B_i be the event that box i was chosen. It is easy to see that the events B_1, B_2, B_3 are pairwise disjoint and thus form a partition of the probability space. For each box we have $Pr(B_i) = 1/3$ by the assumption of this task.
Two of the balls in box 1 have a number less than 10 (numbers 4 and 8), and because all of the ten balls in box 1 have equal probability to be drawn, we find the conditional probability $Pr(E|B_1) = 2/10 = 1/5$. By similar reasoning we obtain $Pr(E|B_2) = 4/12 = 1/3$ and $Pr(E|B_3) = 3/18 = 1/6$.

Now we are able to calculate the probability of interest by means of the theorem of the total probability

$$\begin{aligned}
Pr(E) &= \sum_{i=1}^{3} Pr(E|B_i) \cdot Pr(B_i) \\
&= \frac{1}{5} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} \\
&= \frac{1}{15} + \frac{1}{9} + \frac{1}{18} \\
&= \frac{6}{90} + \frac{10}{90} + \frac{5}{90} = \frac{7}{30}. \checkmark
\end{aligned}$$

b) Independence of E and F is fulfilled if $Pr(E) \cdot Pr(F) = Pr(E \cap F)$ holds. From above we know $Pr(E) = 7/30$. For F we repeat the fact that box 2 consists of the prime numbers between 1 and 40. Thus, F is equivalent to the event B_2, to choose box 2, at the start of the trial. So, $Pr(F) = 1/3$. ✓
The intersection $E \cap F$ is the event that the drawn number is prime and less than 10 at the same time. Among the 12 prime numbers in box 2 there are exactly 4 less than 10, so $Pr(E \cap F|B_2) = 4/12 = 1/3$. For the other two boxes we have $Pr(E \cap F|B_1) = Pr(E \cap F|B_3) = 0$ because they do not contain any prime number. We again apply the theorem of the total probability to calculate the probability of $E \cap F$:

$$\begin{aligned}
Pr(E \cap F) &= \sum_{i=1}^{3} Pr(E \cap F|B_i) \cdot Pr(B_i) \\
&= 0 + \frac{1}{3} \cdot \frac{1}{3} + 0 \\
&= \frac{1}{9}.
\end{aligned}$$

However, we now see that

$$Pr(E) \cdot Pr(F) = \frac{7}{30} \cdot \frac{1}{3} = \frac{7}{90} \neq \frac{10}{90} = \frac{1}{9} = Pr(E \cap F),$$

i.e. E and F are not independent. ✓

Problem H 7 - Independent events [4 pts.]

Consider a discrete probability space to the sample space Ω.

a) Let $A, B \subseteq \Omega$ be two disjoint subsets of Ω. Show that A and B are independent if and only if one of the events has zero probability.

b) Let $C, D \subseteq \Omega$ be events with $C \subset D$. When are C and D independent? Consider all possible cases.

c) Let $E, F \subseteq \Omega$ be two events with $Pr(E) > 0$ and $Pr(E|F) > Pr(E)$. Prove that $Pr(E|F^c) < Pr(E)$ holds in this case.

d) We call a collection of events $\{E_i\}_{i\in\mathbb{N}}$ pairwise independent if for any two indices $i \neq j$ the events E_i and E_j are independent. Show that this does not imply that the total collection is independent as well.

Solution:

a) The two sets $A, B \subseteq \Omega$ are given to be disjoint, i.e. we have $A \cap B = \emptyset$. Formally, we need to show that $Pr(A \cap B) = Pr(A) \cdot Pr(B) \Leftrightarrow Pr(A) = 0 \vee Pr(B) = 0$.
Let us start with the implication "$\Rightarrow$". Assume that A and B are independent. In this case it holds that

$$0 = Pr(\emptyset) = Pr(A \cap B) = Pr(A) \cdot Pr(B)$$

which directly yields that $Pr(A) = 0$ or $Pr(B) = 0$.
Vice versa, for "$\Leftarrow$ we assume $Pr(A) = 0$ or $Pr(B) = 0$ is true. It directly follows that $Pr(A) \cdot Pr(B) = 0$. By the definition of the sets we have $Pr(A \cap B) = Pr(\emptyset) = 0$. Hence, A and B are stochastically independent. ✓

b) For events $C, D \subseteq \Omega$ with the relation $C \subset D$ it holds that $Pr(C \cap D) = Pr(C)$. By definition, C and D are independent if and only if $Pr(C) \cdot Pr(D) = Pr(C \cap D) = Pr(C)$. This equation is satisfied if $Pr(C) = 0$ or $Pr(D) = 1$ holds. In other words, D needs to be the sure event or C needs to be impossible. ✓

c) The statement is intuitive: the assumption is that the event E has higher probability to occur if F already has happened (compared to the unconditional case). Hence, E should have smaller probability to occur if the complementary event F^c has happened.
A formal proof starts by applying the theorem of total probability:

$$Pr(E) = Pr(E|F) \cdot Pr(F) + Pr(E|F^c) \cdot Pr(F^c).$$

By assumption, we have $Pr(E|F) > Pr(E)$. Together with the above, this inequality yields

$$Pr(E) > Pr(E) \cdot Pr(F) + Pr(E|F^c) \cdot Pr(F^c),$$

which is equivalent to

$$Pr(E) \cdot (1 - Pr(F)) > Pr(E|F^c) \cdot Pr(F^c).$$

We see that the two factors $(1 - Pr(F))$ and $Pr(F^c)$ are equal, and further that they are greater than 0 since

$$1 - Pr(F) = 1 - \frac{Pr(E \cap F)}{Pr(E|F)} > 1 - \frac{Pr(E \cap F)}{Pr(E)} \geq 1 - \frac{Pr(E)}{Pr(E)} = 0.$$

Thus, we may divide both sides of the inequality by the factor in common which results in the relation to show. ✓

Alternatively:
$Pr(E) > 0$ implies $E \neq \emptyset$. Now exclude the two cases $F = \emptyset$ and $F = \Omega$: if $F = \emptyset$, $Pr(E|\emptyset) = 0$ which contradicts $Pr(E|F) > Pr(E) > 0$. If $F = \Omega$, $Pr(E|\Omega) = Pr(E)$ which contradicts $Pr(E|F) > Pr(E)$. So, $\emptyset \subset F \subset \Omega$ and in particular $0 < Pr(F) < 1$ and hence also $0 < Pr(F^c) < 1$.
We now derive from the assumption

$$\begin{aligned} Pr(E) &< Pr(E|F) \\ Pr(E) \cdot Pr(F) &< Pr(E|F) \cdot Pr(F) = Pr(E \cap F) \quad | \ Pr(E) - \cdots \\ Pr(E) - Pr(E) \cdot Pr(F) &> Pr(E) - Pr(E \cap F) \\ Pr(E) \cdot (1 - Pr(F)) = Pr(E) \cdot Pr(F^c) &> Pr(E \cap F^c) \\ Pr(E) &> \frac{Pr(E \cap F^c)}{Pr(F^c)} = Pr(E|F^c). \end{aligned}$$

d) Although mutual independence trivially implies pairwise independence the opposite implication does not hold in general. This can be seen from the following counterexample:
Consider a probability space to the samples $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ where each elementary event ω_i has equal probability. Additionally, we define the events E_1, E_2, E_3 by

$$E_1 = \{\omega_1, \omega_4\}, E_2 = \{\omega_2, \omega_4\} \text{ and } E_3 = \{\omega_3, \omega_4\}$$

and some arbitrary E_i for $i \geq 4$. Obviously, we have $Pr(E_1) = Pr(E_2) = Pr(E_3) = \frac{1}{2}$, and, further, all pairs of the events E_1, E_2, E_3 have one element in common (which is ω_4). Thus, the probability of all pairwise intersections of E_1, E_2, E_3 is equal to $\frac{1}{4}$. Hence, the events E_1, E_2, E_3 are pairwise independent. However, it holds that

$$Pr(E_1 \cap E_2 \cap E_3) = Pr(\{\omega_4\}) = \frac{1}{4} \neq \frac{1}{8} = Pr(E_1) \cdot Pr(E_2) \cdot Pr(E_3),$$

i.e. the events are not mutually independent. ✓

Problem H 8 - Expectation value and variance [6 pts.]

Consider a gamble where two fair dices are thrown independently. The number of points a player gets is determined by multiplying the square of the number on the first dice by the number shown on the second one. Denote this random variable by X.

a) Calculate $\mathbb{E}(X)$.

b) Calculate $Var(X)$.

c) Write down an ansatz to calculate the third central and non-central moment of X.

d) Draw a graph of both the probability mass function and the cumulative distribution function. Please do this as a programming task or make use of some adequate software.

Solution:

a) Define the random variable Y as the square of the number on the first dice and Z the number on the second dice, then $X = Y \cdot Z$. The probability distribution of Z is $Pr(Z = i) = \frac{1}{6}$ for $i = 1, \ldots, 6$, for Y we have $Pr(1) = Pr(4) = Pr(9) = Pr(16) = Pr(25) = Pr(36) = \frac{1}{6}$. As the two dices are thrown independently, we may write

$$\mathbb{E}(X) = \mathbb{E}(Y) \cdot \mathbb{E}(Z).$$

It is easy to calculate $\mathbb{E}(Z) = \frac{1}{6} \cdot \sum_{i=1}^{6} i = \frac{21}{6}$ and $\mathbb{E}(Y) = \frac{1}{6} \cdot \sum_{i=1}^{6} i^2 = \frac{91}{6}$, so $\mathbb{E}(X) = \frac{637}{12} = 53\frac{1}{12}$. ✓
One might try to further decompose $\mathbb{E}(Y) = \mathbb{E}(Z' \cdot Z')$ (Z' denoting the number on the first dice). This erroneously would yield $\mathbb{E}(Y) = \frac{7}{2} \cdot \frac{7}{2} = \frac{49}{4} \neq \frac{91}{6}$. The mistake done here is that the variable Z' is not independent from itself and hence the product formula of the expectation value cannot be applied to it.

b) We can use the formula

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2.$$

To determine the first summand we split it to two independent random variables

$$\mathbb{E}(X^2) = \mathbb{E}((Y \cdot Z)^2) = \mathbb{E}(Y^2 \cdot Z^2) = \mathbb{E}(Y^2) \cdot \mathbb{E}(Z^2) = \mathbb{E}(Y^2) \cdot \mathbb{E}(Y) \checkmark$$

because $Z^2 = Y$. $\mathbb{E}(Y^2)$ has to be computed explicitly via $\frac{1}{6} \cdot \sum_{i=1}^{6} i^4 = \frac{2275}{6}$. Together, this yields

$$\mathbb{E}(X^2) = \frac{2275}{6} \cdot \frac{91}{6} = \frac{207025}{36},$$

so

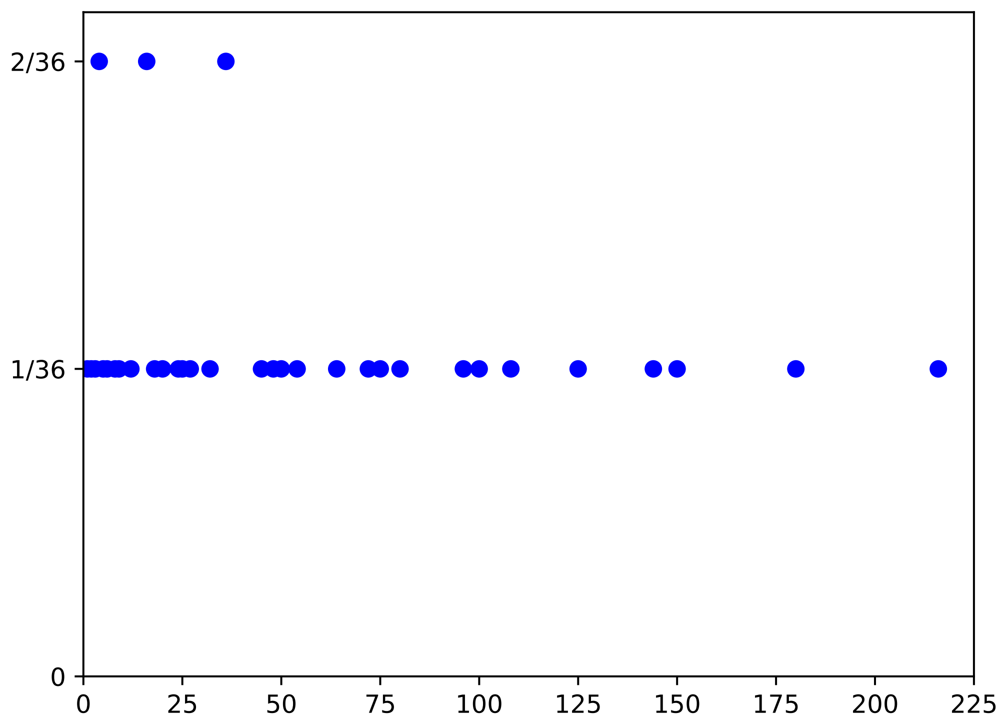
$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{207025}{36} - (\frac{637}{12})^2 \approx 2932.9. \checkmark$$

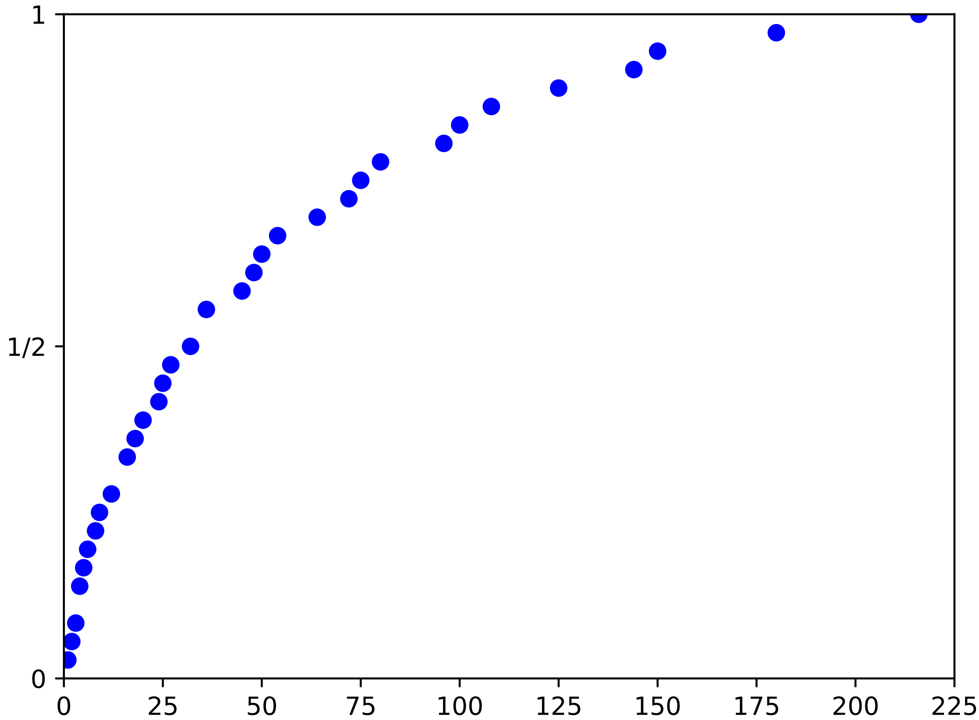
c) The third central and non-central moments of X are defined as

$$\mathbb{E}((X - \mathbb{E}(X))^3) \text{ and } \mathbb{E}(X^3)\ , \quad \text{respectively.} \checkmark$$

d) Using python (including some libraries) yields the plots below. ✓✓

```python
#!/usr/bin/python3

import numpy as np
import matplotlib.pyplot as plt

X=[(i+1)**2*(j+1) for i in range(6) for j in range(6)]

unique_values = list(set(X))
rf = [ np.sum( [k == u  for k in X] )/36. for u in unique_values]

plt.plot(unique_values,rf,"bo")
plt.xlim([0,225])
plt.ylim([0,0.06])
plt.yticks([0.,1./36,2./36.],["0","1/36","2/36"])
plt.show()

crf = [ np.sum( [k <= u  for k in X] )/36. for u in unique_values]
plt.plot(unique_values,crf,"bo")
plt.xlim([0,225])
plt.ylim([0,0.06])
plt.yticks([0.,0.5,1.],["0","1/2","1"])
plt.show()
```





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