Question 1 [25 points] - True/False Questions

For each of the following questions, argue whether you agree or disagree. Motivate your answer (points are only awarded for motivated answers). In all of the following, $X_1, \dots, X_n$ is a random sample from some distribution $f(\boldsymbol{x} \mid \theta)$.

(a) **[5 points]** Let $\hat{\theta}$ be a best unbiased estimator (UMVUE) of θ and let τ be some arbitrary function. Then, $\hat{\eta} = \tau\left(\hat{\theta}\right)$ is a best unbiased estimator of $\eta = \tau(\theta)$

(b) **[5 points]** Let $\hat{\theta}_1$ be a consistent estimator of θ and $\hat{\theta}_2$ be an unbiased estimator of θ. Then, $\hat{\theta}_1$ is also unbiased, but $\hat{\theta}_2$ is not necessarily consistent.

(c) **[5 points]** The family of distributions $\left\{\frac{1}{2\sigma}e^{-\frac{x-\mu}{2\sigma}} \mid \mu \in \mathbb{R}, \sigma > 0\right\}$ forms a location-scale family.

(d) **[5 points]** Let $T(\boldsymbol{X}) = \prod_{i=1}^n X_i$ be a sufficient statistic for θ. The estimator $\hat{\theta} = \frac{1}{n}\sum_{i=1}^n \log(X_i)$ can a best unbiased estimator.

(e) **[5 points]** We have two estimators, $W_1(\boldsymbol{X})$ with $\mathbb{E}_\theta(W_1(\boldsymbol{X})) = \tau(\theta)$ and $W_2(\boldsymbol{X})$ with $\mathbb{E}_\theta(W_2(\boldsymbol{X})) = 0$ for all $\theta \in \Theta$. In addition, $\mathbb{E}_\theta(W_1(\boldsymbol{X})W_2(\boldsymbol{X})) = 0$ for all $\theta \in \Theta$. Then, $W_1(\boldsymbol{X})$ is the best unbiased estimator of $\tau(\theta)$.

Question 2 [75 points] - Carnaval in Maastricht

Your course lecturer is looking forward to celebrate carnaval in Maastricht. He spent weeks to create his colourful home-made costume. While he is happy with the result, he is afraid the costume will not survive heavy rainfall. Therefore, he only wants to wear his costume when the probability of rainfall is low. Since you don't trust the weather forecasts, you decide to help your lecturer predict the amount of rainfall (in mm) in Maastricht around carnaval.

After doing some research into weather modelling, you learn that the amount of daily rainfall can be modelled by the Gamma distribution with pdf

$$g(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 \leq x < \infty, \quad \alpha, \beta > 0,$$

with $\mathbb{E}_{\boldsymbol{\theta}}(X) = \alpha\beta$ and $\mathbb{V}\mathrm{ar}_{\boldsymbol{\theta}}(X) = \alpha\beta^2$, where $\theta = (\alpha, \beta)$. Let $X_1, \ldots, X_n$ be a random sample from $g(x \mid \alpha, \beta)$ where X_i represent the amount of rainfall on day i. You have collected historical rainfalls for 15 historical carnaval days, say $x_1, \ldots, x_{15}$, and recorded the following information:

$$n = 15, \qquad \sum_{i=1}^{n} x_i = 120, \qquad \sum_{i=1}^{n} x_i^2 = 960.$$

(a) **(10pts)** Find sufficient statistics for (α, β). What would be the sufficient statistic if α was known?

For questions (b)-(e), assume that α is a known constant!

(b) **(10pts)** Show that the maximum likelihood estimator is equal to $\hat{\beta}_{ML} = \frac{\bar{X}}{\alpha}$.

(c) **(10pts)** Show that the method of moments estimators based on the first and second moment are given by

$$\hat{\beta}_{MM1} = \frac{\bar{X}}{\alpha} \quad \text{and} \quad \hat{\beta}_{MM2} = \sqrt{\frac{1}{n\alpha(1+\alpha)} \sum_{i=1}^{n} X_i^2},$$

respectively.

(d) **(15pts)** Are the estimators found in (b) and (c) unbiased estimators of β? Is any of them a UMVUE of β? Motivate your answer.

(e) **(10pts)** Show that $\hat{\beta}_{MM2}$ is a consistent estimator of β.

(f) **(10pts)** If the rainfall is more than 20mm, your lecturer's costume is unlikely to survive. Therefore, if the probability of more than 20mm of rainfall is larger than 5%, you will advise your lecturer to wear a different costume. Assume that $\alpha = 1$ and compute the probability of more than 20mm of rainfall *by maximum likelihood.* State your advice to your lecturer.

Hint: $\mathbb{P}_\beta(X > x)$ is just a function of β. Also, $\Gamma(1) = 1$.

(g) **(10pts)** Derive *either* the MME *or* the MLE of α in the case where both α and β are unknown. **Note**: it may not be possible to derive both of these estimators analytically, so choose wisely which of these two estimators you are going to try to derive.