Retreat 2016, questions and answers

Discrete Probability Theory (IN0018) (Technische Universität München)

Technische Universit¨at Mu¨nchen Faculty of Computer Science
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Sif the written exam
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Discrete probability theory

Name	First name	Matriculation number	
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General notes on the exam

- The specification of the exam includes six tasks and a table of standard nor- mal distribution.
- Please do not write in pencil, red or green ink.
- Apart from your writing utensils and a handwritten DIN A4 sheet of paper, no other aids are allowed.
- The processing time is 120 minutes.

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Special remarks:

	A1	A2	A3	A4	A5	A6	Σ	Proofreader
Total	6	7	5	7	8	7	40	
Initial correction								
Second correction								

Task 1 (1+1+2+2 points)

To prepare for the DWT exam, the students Anna, Bea and Ca- rola, together with their fellow students David, Emil and Felix, randomly divide into 3 disjunctive study groups of 2 persons each. All splits are equally probable.

1. Show that there are 15 ways to par- tition the 6 people into groups of two.
2. Determine the probability that all learning groups are mixed, that is, each learning group consists of one student.
3. What is the probability that all learning groups are mixed if Anna and Da- vid form a common learning group.
4. Show or disprove the following statement: The union of the two event- nesses „Anna and David form a learning group" resp. „Bea and Carola form a Learning group" is independent of the event that all learning groups are mixed.

Proposed

1. Als Ergebnismenge Ω wählen wir alle Partitionen der 6 Studenten in Mengen der Kardinalität 2. Dafür nehmen wir vorerst an, dass die Lerngruppen unterscheidbar sind. Folglich gibt es $\binom{6}{2}$ Kombinationen aus 2 Studenten für die erste Lerngruppe, $\binom{4}{2}$ for the second learning group and $\binom{2}{2}$ for the third learning group. To set the order among the learning groups, we divide by 3! and get
$$|\Omega| = \frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = \frac{90}{6} = 15.$$
2. *Let E* be the event that all learning groups are mixed. We are looking for the probability $\Pr[E]$. Since we consider a Laplace probability space, $\Pr[E] = |E|/|\Omega|$ holds. Note that each element in *E of* a bijective mapping from boys to girls. Thus $|E| = 3!$ and it follows that
$$\Pr[E] = \frac{3!}{15} = \frac{2}{5} = 0{,}4.$$
3. *Let F* denote the event that Anna and David are in a study group. We are looking for the conditional probability $\Pr[E \mid F]$. If Anna and David are already in a learning group, there are still
$$|F| = \frac{\binom{4}{2}\binom{2}{2}}{2!} = \frac{}{2} = 3$$
possible distributions for the remaining students. Moreover, for 2 of the 3 elements of F all learning groups are mixed, namely for $\{\{A, D\}, \{B, E\}, \{C, F\}\}$ and $\{\{A, D\}, \{B, F\}, \{C, E\}\}$. Therefore $|E \cap F| = 2$ and we get
$$\Pr[E \mid F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{|E \cap F|}{|F|} = \frac{2}{3} \approx 0.66667.$$

4. Let G be the event that Bea and Carola are in a study group. We to know whether the events E and $F \cup G$ are independent. For reasons of symmetry $\Pr[F] = \Pr[G] = 3/15$. Furthermore, the third learning group is uniquely determined if Anna and David as well as Bea and Carola are in a learning group.
have come together. Therefore $\Pr[F \cap G] = 1/15$. According to the sieve formula the probability of $F \cup G$ *is* given by

$$\Pr[F \cup G] = \Pr[F] + \Pr[G] - \Pr[F \cap G] = \frac{3}{15} + \frac{3}{15} - \frac{1}{15} = \frac{1}{3} \approx 0.33333.$$

Since the events E and G cannot occur simultaneously, the following also applies

$$\Pr[E \cap (F \cup G)] = \Pr[(E \cap F) \cup (E \cap G)] = \Pr[(E \cap F) \cup \emptyset] = \Pr[E \cap F] = \frac{2}{15} \approx 0.13333.$$

Overall it follows that

$$\Pr[E \cap (F \cup G)] = \frac{2}{15} = \frac{2}{5} - \frac{1}{3} = \Pr[E] - \Pr[F \cup G],$$

with which we have shown the independence of E and $F \cup G$.

Task 2 (1+3+3 points)

In a marathon, teams consisting of 2 or 4 runners may compete. However, each runner, regardless of the other participants, has a probability of reaching
$0 \leq p \leq 1$ does not reach the finish line. If less than half of the runners of a team reach the finish, the team is disqualified.

1. Determine, as a function of p, the probability that a team of two or four will be disqualified.

2. For which values of p is the probability of disqualification for a team of two really smaller than for a team of four.

3. After the run, the sports reporter Anja assumes that $p \leq 2/100$. In order to check this statement, she determines how many of the total 10000 runners have the have reached the target. Perform a suitable hypothesis test and determine the largest possible rejection range with which Anja's claim can be rejected at a significance level of 0.05. Use the table of the standard normal distribution for this purpose. Use also the table of the standard normal distribution on the last sheet of the specification.

Proposed

1. Let X and Y be random variables that count how many runners in a team of two and four reach the goal, respectively. Due to the independence between the two teams
   La¨ufern, X and Y are binomially distributed with parameters p and 2 and p and 4, respectively. Therefore

   $$\Pr[X < 2/2] = \Pr[X = 0] = \binom{2}{0} \cdot p^2 \cdot (1-p)^0 = p^2$$

   and

   $$\Pr[Y < 4/2] = \Pr[Y \leq 1] = \binom{4}{0} \cdot p^4 \cdot (1-p)^0 + \binom{4}{1} \cdot p^3 \cdot (1-p) = -3p^4 + 4p^3 .$$

2. We are looking for all values for p such that $\Pr[X < 2/2] < \Pr[Y < 4/2]$. By rearranging and substituting we get

   $$0 < \Pr[Y < 4/2] - \Pr[X < 2/2] = -3p^4 + 4p^3 - p^2 .$$

   So we have to determine the values of p between 0 and 1 for which the polynomial $-3p^4 + 4p^3 - p^2$ is positive. Obviously, $-3p^4 + 4p^3 - p^2$ has a double zero for $p = 0$. Factoring accordingly, we obtain the term $p^2 \cdot (3p^2 - 4p + 1)$.
   The polynomial $3p^2 - 4p + 1$ has again zeros at the points

   $$p = \frac{4 + \sqrt{16 - 4 \cdot 3 \cdot 1}}{6} = 1$$

   and

   $$p = \frac{4 - \sqrt{16 - 4 \cdot 3 \cdot 1}}{6} = \frac{1}{3}.$$

   Since $3p^4 + 4p^3 - p^2$ is positive for $p \to \infty$ $3p^4 + 4p^3$ p^2 can only be positive in the interval $(1/3, 1)$. Thus, the probability of a disqualification for teams of two is exactly smaller than for teams of four, if $1/3 < p < 1$ holds.

3. To test Anja's null hypothesis, namely $H_0 : p_0 \leq 0.02$, we perform an ap-proximate binomial test. This is possible, since the number of runners that are
is binomially distributed and with 10000 runners we can assume that the number of samples is sufficiently large to approximate with the normal distribution. The test size is

$$Z = \frac{h - 10000 \cdot 0{,}02}{\sqrt{10000 \cdot 0{,}02 \cdot (1 - 0{,}02)}} = \frac{h - 200}{14},$$

where h *is* the number of runners who did not make it to the finish line. The null hypothesis can be rejected with a significance level of 0.05 if $Z > z_{1-0{,}05}$ holds. So we have to choose h such that

$$h > 200 + 14 z_{0{,}95} \approx 200 + 14 \cdot 1.65 = 223.1.$$

We have taken the 0.95 quantile of the standard normal distribution from the table in the appendix. At a significance level of 0.05, we obtain the largest possible rejection range if we reject Anja's hypothesis for $h \geq 224$.

Task 3 (2+3 points)

The washing bear Andreas prefers to eat cherries. Each day Andreas captures a random number X of the eastern fruits. The probability generating function of X is defined as:

$$G_X(s) = \exp\left(5 \cdot (s^2 - 1)\right).$$

1. Show that the expected value and variance of X are given by $\mathsf{E}[X] = 10$ and $\mathrm{Var}[X] = 20$.

2. Use the Chebyshev inequality to show that the probability of Andreas finding at least 20 cherries in a day is at most *1/5*.

Proposed solution

1. The expected value of X is given by

$$\mathsf{E}[X] = G'_X(1) = \exp\left(5 \cdot (1^2 - 1)\right) \cdot (10 \cdot 1) = 10.$$

Furthermore, the second derivative of the function G_X at the point 1 is calculated to be

$$G''_X(1) = \exp\left(5 \cdot (1^2 - 1)\right) \cdot (10 \cdot 1)^2 + \exp\left(5 \cdot (1^2 - 1)\right) \cdot 10 = 110.$$

The variance of X is therefore

$$\mathrm{Var}[X] = G''_X(1) + G'_X(1) - \left(G'_X(1)\right)^2 = 110 + 10 - 10^2 = 20.$$

2. The probability that Andreas finds at least 20 cherries in one day can be estimated with

$$\Pr[X \geq 20] = \Pr[X - \mathsf{E}[X] \geq 10] \leq \Pr[|X - \mathsf{E}[X]| \geq 10].$$

The upper inequality is valid since $X - \mathsf{E}[X] \geq 10$ implies the event $|X - \mathsf{E}[X]| \geq 10$. According to the Chebyshev inequality we can further estimate to

$$\Pr[|X - \mathsf{E}[X]| \geq 10] \leq \frac{\mathrm{Var}[X]}{10^2} = \frac{20}{100} = \frac{1}{5}.$$

Overall, we have thus shown that $\Pr[X \geq 20] \leq 1/5$ holds.

Task 4 (3+3+1 points)

The unpu¨nktliche student Adam has arranged to meet the even more unpu¨nktliche student Ben at the university. Let X be a random variable indicating Adam's lateness in hours. Furthermore, let Y be Ben's tardiness in hours. The joint density of the two random variables is described by:

$$f_{X,Y}(x, y) = \begin{cases} (e^{-x+y})/(e-c) & \text{if } x, y \in [0, 1] \text{ and } x \le y \\ 0 & \text{otherwise} \end{cases}.$$

1. Find a value for c such that $f_{X,Y}$ is a permissible density function.
2. Set up the edge densities f_X and f_Y.
3. Examine the random variables X and Y for independence.

Proposed solution

1. In order for the given function to be an admissible density, the following must be true. $f_{X,Y}(x, y) \ge 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(s, t)\, ds dt = 1$. The first condition is for $c < e$ is trivially satisfied. For the second condition we consider the integral
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(s, t)\, ds dt = \int_0^1\int_0^t \frac{e^{-s+t}}{e-c}\, ds dt = \frac{1}{e-c}\int_0^1\int_0^t e^{-s+t}\, ds dt.$$
The primitive function of e^{-s+t} with respect to s is $-e^{-s+t}$. We can thus resolve the inner integral to
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(s, t)\, ds dt = \frac{1}{e-c}\int_0^1 \left[-e^{-s+t}\right]_0^t dt = \frac{1}{e-c}\int_0^1 -1 + e^t\, dt.$$
The primitive function of $e^t - 1$ with respect to t is again given by $e^t - t$. Finally we obtain
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(s, t)\, ds dt = \frac{1}{e-c}\left[e^t - t\right]_0^1 = \frac{(e-1)-(1-0)}{e-c} = \frac{e-2}{e-c}.$$
For this term to be 1, we choose $c = 2$. Furthermore, since $2 < e$ holds, $f_{X,Y}$ for $c = 2$ an admissible density.

2. In the case $x < 0$ or $x > 1$, the marginal density of X is trivially 0. For all other values of x, that is, $0 \le x \le 1$, f_X is given by
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, t)\, dt = \int_x^1 \frac{e^{-x+t}}{e-2}\, dt = \frac{1}{e-2}\int_x^1 e^{-x+t}\, dt.$$
The primitive function of e^{-x+t} with respect to t is e^{-x+t} and we can further to
$$f_X(x) = \frac{1}{e-2}\left[e^{-x+t}\right]_x^1 = \frac{e^{-x+1} - e^{-x+x}}{e-2} = \frac{e^{1-x} - 1}{e-2}.$$

Similarly, the marginal density of Y in the interval $0 \leq y \leq 1$ is calculated to be

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(s, y)\, ds = \int_0^y \frac{e^{-s+y}}{e-2}\, ds = \frac{1}{e-2} \cdot \int_0^y e^{-s+y}\, ds.$$

The primitive function of e^{-s+y} with respect to s *is* already known from the first task. It is $-e^{-s+y}$ and it follows

$$f_Y(y) = \frac{1}{e-2} \left[-e^{-s+y}\right]_0^y = \frac{-e^{-y+y} + e^{-0+y}}{e-2} = \frac{e^y - 1}{e-2}.$$

In total we obtain the edge densities

$$f_X(x) = \begin{cases} (e^{1-x} - 1)/(e-2) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} (e^y - 1)/(e-2) & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

3. For two random variables to be independent, the joint density must be the product of the marginal densities. For X and Y this is obviously not the case, because e.g. $f_X(3/4) > 0$ and $f_Y(1/4) > 0$ although $f_{X,Y}(3/4, 1/4) = 0$. The joint density in this case is 0 because $1/4 > 3/4$.

Task 5 (4+2+2 points)

Ada breaks a chocolate bar of length ℓ at a random location X into two pieces, where X is equally distributed on the interval $[0, \ell]$. Ada keeps the larger for itself. The smaller piece, whose length is described by the random variable Y, she gives to her sister Bianca.

1. Show that the distribution function of Y is given by:

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 2y/\ell & \text{if } 0 \le y \le \ell/2 \\ 1 & \text{if } \ell/2 < y \end{cases}.$$

2. Determine the density function and the expected value of Y.

3. Bianca wants estimate the unknown length the chocolate bar by the of her piece. Construct with the help of Y an expectation-guided estimator and a maximum likelihood estimator for ℓ.

Proposed

1. To determine *the* distribution function of Y we first consider the probability $\Pr[Y \le y]$. In the cases $y \le 0$ or $\ell/2 \le y$ trivially $\Pr[Y \le y] = 0$ or $\Pr[Y \le y] = 1$. Therefore, let $0 \le y \le \ell/2$.

   We now make a case distinction with respect to X. For $X \le \ell/2$, $Y = X$. Otherwise, for $X > \ell/2$, $Y = \ell - X$. Since both cases are disjoint we can transform the probability $\Pr[Y \le y]$ to

$$\begin{aligned}\Pr[Y \le y] &= \Pr[Y \le y, X \le \ell/2] + \Pr[Y \le y, X > \ell/2] \\ &= \Pr[X \le y, X \le \ell/2] + \Pr[n - X \le y, X > \ell/2].\end{aligned}$$

   From $0 \le y \le \ell/2$ we conclude, on the one hand, that

$$\Pr[X \le y, X \le \ell/2] = \Pr[X \le y] = \frac{y}{\ell},$$

   and on the other hand that

$$\Pr[\ell - X \le y, X > \ell/2] = \Pr[X > \ell - y, X > \ell/2] = \Pr[X > \ell - y] = 1 - \frac{\ell - y}{\ell}.$$

   Overall, we therefore obtain a probability of

$$\Pr[Y \le y] = \frac{y}{\ell} + 1 - \frac{\ell - y}{\ell} = \frac{2y}{\ell}$$

   and the distribution function of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 2y/\ell & \text{if } 0 \le y \le \ell/2 \\ 1 & \text{if } \ell/2 < y \end{cases}.$$

2. To determine the density function of Y, we derive F_Y *with respect to* y and continuously continue the function at the non-differentiable points $y = 0$ and $y = \ell/2$ on the left and right sides, respectively

$$f_Y(y) = F_Y'(y) = \begin{cases} 2/\ell & \text{if } 0 \leq y \leq \ell/2 \\ 0 & \text{otherwise} \end{cases}.$$

We can see from the density function that Y *is* uniformly distributed on the interval $[0, \ell/2]$. The expected value of Y is therefore

$$\mathrm{E}[Y] = \frac{0 + \ell/2}{2} = \frac{\ell}{4}.$$

3. From the expectation value of Y we can directly determine an expectation-true estimator for ℓ. As estimator we choose $4Y$,

$$\mathrm{E}[4Y] = 4 \cdot \mathrm{E}[Y] = 4 \cdot \frac{\ell}{4} = \ell.$$

For the maximum likelihood estimator we assume that $y > 0$ is the concrete length of Bianca's piece of the chocolate bar. According to the density function from the first task, the likelihood function of ℓ is given by

$$L(y; n) = \begin{cases} 2/\ell & \text{if } \ell \geq 2y \\ 0 & \text{otherwise} \end{cases}.$$

Since $2/\ell$ *is* strictly monotonically decreasing on the interval $(0, \infty)$, the likelihood function for $\ell = 2y$ is maximum. Thus, $2Y$ is a maximum likelihood estimator for ℓ.

Task 6 (2+3+2 points)

The weatherman Alice has determined that a rainy day is followed with probability 1/2 by a sunny day and otherwise by a rainy day. A sunny day, on the other hand, is followed with probability 3/4 by a sunny day and otherwise by a rainy day. Let X_t be a random variable such that $X_t = 1$ if the *tth* day is sunny. If, on the other hand, the *t-th day is* rainy, then $X_t = 0$.

1. Model the Markov chain $\{X_t\}_{t\geq 0}$. In particular, sketch the transition graph and specify the transition matrix.
2. Assume that on the 0-th day it is sunny. Prove by induction over t that

$$q_t = \left(\frac{1 - 4^{-t}}{3}, \frac{2 + 4^{-t}}{3}\right).$$

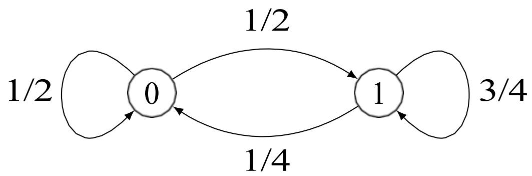
3. Determine the stationary distributions of the Markov chain $\{X_t\}_{t\geq 0}$.

Proposed solution

1. Since the probabilities $\Pr[X_t = 0]$ or $\Pr[X_t = 1]$ depend only on the previous day, we can model the weather on the t-th day by a Markov chain $\{X_t\}_{t\geq 0}$ over the state set $S = \{0, 1\}$. With the transition probabilities aus der Angabe erhalten wir als Übergangsmatrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}.$$

Graphically, the Markov chain can be represented by the following transition graph:



2. As required we prove the identity from the specification by means of an induction over t. We start with the induction base $t = 0$. According to the specification today is sunny. The initial distribution q_0 is therefore given by

$$q_0 = (0, 1) = \left(\frac{1 - 4^{-0}}{3}, \frac{2 + 4^{-0}}{3}\right).$$

Für den Induktionsschritt sei $t > 0$ und wir nehmen an, dass

$$q_{t-1} = \left(\frac{1 - 4^{-(t-1)}}{3}, \frac{2 + 4^{-(t-1)}}{3}\right)$$

has already been proved. We now want to show that this identity is also true for t holds. From the definition of q_t follows

$$q_t = q_{t-1} \cdot P = q_{t-1} \cdot \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}.$$

If we now use the induction hypothesis, we get

$$q_t = \left(\frac{1 - 4^{-(t-1)}}{3}, \frac{2 + 4^{-(t-1)}}{3}\right) \cdot \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}.$$

For $(q_t)_0$ this results in a value of

$$(q_t)_0 = \frac{1 - 4^{-(t-1)}}{3} \cdot \frac{1}{2} + \frac{2 + 4^{-(t-1)}}{3} \cdot \frac{1}{4} = \frac{4 - 4^{-(t-1)}}{12} = \frac{1 - 4^{-t}}{3}.$$

Similarly, $(q_t)_1$ is calculated to be

$$(q_t)_1 = \frac{1 - 4^{-(t-1)}}{3} \cdot \frac{1}{2} + \frac{2 + 4^{-(t-1)}}{3} \cdot \frac{3}{4} = \frac{8 + 4^{-(t-1)}}{12} = \frac{2 + 4^{-t}}{3}.$$

Overall, we have thus shown that

$$q_t = \left(\frac{1 - 4^{-t}}{3}, \frac{2 + 4^{-t}}{3}\right).$$

3. Since the transition diagram of the Markov chain is strongly coherent, it is an irreducible Markov chain. It is also aperiodic, since each state has a loop on itself. Overall, the Markov chain is ergodic and the Fundamental Theorem for ergodic Markov chains can be applied. It states that independent of the initial distribution $\lim_{t\to\infty} q_t = \pi$ holds, where π is the unique stationary distribution of the Markov chain. From the second subtask
schlussfolgern wir, dass $\{X_t\}_{t\geq 0}$ genau eine stationäre Verteilung besitzt, nämlich

$$\lim_{t\to\infty} \left(\frac{1 - 4^{-t}}{3}, \frac{2 + 4^{-t}}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}\right).$$

This table of the standard normal distribution contains the values of $\Phi(x)$ for $0 \leq x \leq 2.99$. For example, $\Phi(1.55) \approx 0.939$.

$\Phi(x)$	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,500	0,504	0,508	0,512	0,516	0,520	0,524	0,528	0,532	0,536
0,1	0,540	0,544	0,548	0,552	0,556	0,560	0,564	0,567	0,571	0,575
0,2	0,579	0,583	0,587	0,591	0,595	0,599	0,603	0,606	0,610	0,614
0,3	0,618	0,622	0,626	0,629	0,633	0,637	0,641	0,644	0,648	0,652
0,4	0,655	0,659	0,663	0,666	0,670	0,674	0,677	0,681	0,684	0,688
0,5	0,691	0,695	0,698	0,702	0,705	0,709	0,712	0,716	0,719	0,722
0,6	0,726	0,729	0,732	0,736	0,739	0,742	0,745	0,749	0,752	0,755
0,7	0,758	0,761	0,764	0,767	0,770	0,773	0,776	0,779	0,782	0,785
0,8	0,788	0,791	0,794	0,797	0,800	0,802	0,805	0,808	0,811	0,813
0,9	0,816	0,819	0,821	0,824	0,826	0,829	0,831	0,834	0,836	0,839
1,0	0,841	0,844	0,846	0,848	0,851	0,853	0,855	0,858	0,860	0,862
1,1	0,864	0,867	0,869	0,871	0,873	0,875	0,877	0,879	0,881	0,883
1,2	0,885	0,887	0,889	0,891	0,893	0,894	0,896	0,898	0,900	0,901
1,3	0,903	0,905	0,907	0,908	0,910	0,911	0,913	0,915	0,916	0,918
1,4	0,919	0,921	0,922	0,924	0,925	0,926	0,928	0,929	0,931	0,932
1,5	0,933	0,934	0,936	0,937	0,938	0,939	0,941	0,942	0,943	0,944
1,6	0,945	0,946	0,947	0,948	0,949	0,951	0,952	0,953	0,954	0,954
1,7	0,955	0,956	0,957	0,958	0,959	0,960	0,961	0,962	0,962	0,963
1,8	0,964	0,965	0,966	0,966	0,967	0,968	0,969	0,969	0,970	0,971
1,9	0,971	0,972	0,973	0,973	0,974	0,974	0,975	0,976	0,976	0,977
2,0	0,977	0,978	0,978	0,979	0,979	0,980	0,980	0,981	0,981	0,982
2,1	0,982	0,983	0,983	0,983	0,984	0,984	0,985	0,985	0,985	0,986
2,2	0,986	0,986	0,987	0,987	0,987	0,988	0,988	0,988	0,989	0,989
2,3	0,989	0,990	0,990	0,990	0,990	0,991	0,991	0,991	0,991	0,992
2,4	0,992	0,992	0,992	0,992	0,993	0,993	0,993	0,993	0,993	0,994
2,5	0,994	0,994	0,994	0,994	0,994	0,995	0,995	0,995	0,995	0,995
2,6	0,995	0,995	0,996	0,996	0,996	0,996	0,996	0,996	0,996	0,996
2,7	0,997	0,997	0,997	0,997	0,997	0,997	0,997	0,997	0,997	0,997
2,8	0,997	0,998	0,998	0,998	0,998	0,998	0,998	0,998	0,998	0,998
2,9	0,998	0,998	0,998	0,998	0,998	0,998	0,998	0,999	0,999	0,999