$$f(x:\theta) = \underbrace{c(\alpha)}_{h(x)} e^{-(|x-\theta|^{\alpha})}$$ $c(\alpha)$ = normalize constant

$x, \theta \in \mathbb{R}$

$\theta x \quad -x^{\alpha}$ $\quad 0 < \alpha \leq 2$

$\underbrace{c(\alpha) e}_{g(\theta)} \quad \underbrace{1 \cdot e}_{h(x)}$ $\quad (1, 2)$

$$f(x:\theta) = h(x)\, g(\theta) \exp(\gamma(\theta)\, T(x))$$

$$c(\alpha) \; \exp(\alpha \log(x-\theta)) \qquad e^{\cdots} = \exp(\ldots)$$

$$\Rightarrow c(\alpha) \frac{1}{e^{|x-\theta|^{\alpha}}}$$

$\alpha = 1$ için

$$\Rightarrow \underbrace{c(1)}_{h(x)} \frac{1}{e^{x-\theta}} \Rightarrow c(1) \underbrace{\frac{1^{x}}{e^{x-\theta}}}_{\gamma(\theta)} \quad \exp(\gamma(\theta) T(x))$$

$c = 1, 2$

$\frac{1}{e^{x}} = T(x) \qquad \frac{1}{e^{\theta}} = -\log \theta \qquad c(\alpha) = 1, 2$

~~$g(\theta) = \theta$~~

~~$h(x) = c(\alpha) + \theta$~~

~~$g(\theta) = c(1) e$~~ $\log x = (T(x))$ $\qquad c(1), c(2)$

~~$h(x) = 1^{x}$~~ $\qquad h(x) = 1^{x} \cdot \theta$

$\hat{T} = \sum x_i = E(T) = \theta$ ise unbiased $\quad g(\theta) = c(1) e$

$\alpha = 1$ için üstel ailede sayılmaz

$\alpha = 2$

$$f(x:\theta) = c(2)\, e^{-(|x-\theta|)^2} = \underbrace{c(2)}_{h(x)} \underbrace{e^{-x^2+2x\theta-\theta^2}}_{h(x)} \cdot e$$

$$\underbrace{c(2) e^{-\theta^2}}_{g(\theta)} \quad \underbrace{e^{-x^2}}_{h(x)} \quad \underbrace{e^{2x\theta}}_{\gamma(\theta) T(x)}$$

Only $\alpha = 2$ equalize to exponential family for negative binomial

$$f(x;\theta) = \begin{cases} \frac{1}{2} e^{-|x-\theta|} & x \geq 0 \\ & x < 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$E(x) = \frac{1}{2} \int_{-\infty}^{0} x \cdot e^{\theta - x} \, dx + \int_{0}^{\infty} x \, e^{\theta - x} \, dx$$

$x = u \qquad dx = du$

$$\Rightarrow \int_{-\infty}^{\theta} u e^{-\theta - u} \, du \Rightarrow \frac{1}{\cancel{2}} \cancel{2} u e^{-\theta - u} \Big|_{-\infty}^{\theta} = u e^{-u\theta} \Big|_{-\infty} = \theta - 1$$

$$\int_{\theta}^{\infty} u e^{\theta - u} \, du = \frac{1}{\cancel{2}} \cancel{2} u e^{-u} \Big|_{\theta}^{\infty} = 1 + \theta$$

$$E(x) = \frac{1}{2}(\theta \cancel{-1} \cancel{+1} + \theta) = \frac{2\theta}{2} = \theta$$

Varyans $= \operatorname{Var}(X) = E(x_i^2) - [E(x)]^2$

$\theta - x = u$

$du = dx$

$$E(x^2) = \frac{1}{2} \int_{-\infty}^{\infty} x^2 f(x;\theta) \, dx$$

$$\Rightarrow \frac{1}{2}\left(3 \cdot u^3 e^{u} \Big|_{-\infty}^{\theta} + 3u^3 e^{u} \Big|_{\theta}^{\infty}\right)$$

$$\Rightarrow \left(\frac{3}{2} \theta^3 e^{-\infty} - \frac{3}{2}\right) + \frac{3}{2} - \frac{3\theta^3 e^{\theta}}{2}$$

$$= e^{2\theta} \Rightarrow e^2 e^{\theta}$$

$$= \theta^2 + 2$$

$E(x^2) - (E(x))^2$

$\operatorname{Var}(x) = \cancel{\theta^2} + 2 - \cancel{\theta^2}$

$= 2$

MLE

$$I(\theta) = E\left[\frac{\partial^2 f(x:\theta)}{\partial\theta^2}\right]$$

Asymtotic Yakınsama

$$\hat{\theta} \sim \left(\theta, \frac{1}{I(\theta)}\right) \qquad I(\theta) = n$$

Cramer-Rao $Var(\hat{\theta}) \geq \frac{1}{nI(\theta)}$

$$l(\theta : x) = -n\log 2 - \sum^{n} |x_i - \theta|$$

→ 2. türevi MLE'dir Eğer negatifse

$$\frac{\partial^2 (l(\theta : x)}{\partial^2 \theta} = \sum x_i \frac{1}{\theta} = \frac{\sum x_i}{n}$$

$$\frac{\partial}{\partial\theta}|x-\theta| = \begin{cases} -1 & x<\theta \\ 0 & x=\theta \\ 1 & x>\theta \end{cases}$$

$$X_n = \frac{\sum x_i}{n}$$

$$n = \frac{\sum x_i}{x_n}$$

$$E(T(x)) = \frac{\sum x_i}{n} \qquad E(T(\theta)) = \frac{\sum X_i}{\theta}$$

$$I(\theta) = xn$$

$$Var(\hat{\theta}) \geq I(\theta)^{-1}n^{-1} = \frac{1}{I(\theta)n} \qquad Var(\hat{\theta}) = \frac{1}{n}$$

Mean

f) $MSE = Var(\bar{x}) + Bias^2$ $E(\bar{x}) = \theta = E(x_i)$ unbiased

$$Bias(\bar{x}) = E(\bar{X}) - \theta = \frac{\sum E(x_i)}{n} - \theta = 0$$

$$= \theta - \theta = 0$$

$$MSE = Var(\bar{x}) - Bias^2 = Var\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n^2}\sum Var(x_i)$$

$$= \frac{1}{n^2} n 2 = \frac{2}{n}$$

two tails one tail

9) 2 tables => 1.96 0.95 1.645

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\hat{\theta} - \theta}{\sqrt{Var(\theta)}} = \frac{\hat{\theta} - \theta}{\sqrt{Var(\theta)}}$$

Confidence Interval

Standart Deviation $= \sqrt{\frac{1}{n}} = \frac{1}{\sqrt{n}}$

2

$$\hat{\theta} - z_{1-\alpha/2} \sqrt{Var(\theta)} < \theta < \hat{\theta} + 1.96\sqrt{Var(\theta)}$$

0.975

$$\hat{\theta} \pm 1.96 \frac{1}{\sqrt{n}}$$

$\mu = 10$

$\sigma = 0.03$

$E(x) + Var(x) = 10 + 0.03^2$

f) $E(\bar{x}) = E\left(\frac{1}{n}\sum x_i\right)$ $E(x_i) = \theta$ $Var(x_i) = 2$

parameter

$Bias = E(\bar{x}) - \theta = \frac{1}{n} n.\theta - \theta = \theta - \theta = 0$

$Var(\bar{x}) = Var\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n^2}\sum Var(x_i)$

$= \frac{1}{n^2} n.2$

$\theta_1 = \frac{1}{n} \sum x_i$

$\theta_2 = \frac{2}{n} \sum x_i$

$\theta_3 = 0.5$

$Q_4 = \frac{(\sum x_i + 1)}{n+2}$

$Bias = E(\pi_1) - \pi_1$ $\quad \bar{x}_n = \frac{\sum x_i}{n}$

$= \frac{1}{n} n\pi = \pi$ unbiased $\quad n = \frac{\sum x_i}{\bar{x}}$

$MSE(Q_1, \pi) = Var(\theta_1)' \cdot \frac{\pi(1-\pi)}{n}$

Binomial $= \binom{n}{x} p(1-p)^{x-n}$

$Bin(1, \pi) = f(\pi(1-\pi)) = \pi(1-\pi)$

$Var(\bar{x}) = \frac{1}{n^2} \sum^{n} x_i$ $\quad MSE = \frac{1}{n^2} n \underbrace{\boxed{\pi(1-\pi)}}^{Var(\bar{x})\ Bin(1,\pi)\ MLE} = \frac{\pi(1-\pi)}{n}$

$= \frac{n}{n^2}$ $\quad \Rightarrow Var(\bar{X}) + Bias^2$

$= \frac{1}{n}$ $\quad \Rightarrow \frac{\pi(1-\pi)}{n}$

2) $\frac{2}{n} \sum_{i=1}^{n/2} E(x_i) = \frac{2}{\cancel{n}} \cdot \frac{\cancel{n}}{2} \cdot \pi = \pi$

unbiased

$Var(\bar{x}) = \frac{4}{n^2} \cdot \frac{n}{2} \pi(1-\pi) = \frac{2n(1-\pi)}{\pi^2}$

3. $0.5 \neq 0 \Rightarrow$ Bias

UMVUE

If $E(T)$ has a bias we don't have a variance

$MSE = (Bias^2)$

Maximum likelihood for the Negative Binomial Exercise Sheet 4

Negative Binomial $(r_0, p)$

$$f(x;p) = \binom{x+r_0-1}{x} p^{r_0} (1-p)^x \quad x \in \mathbb{N}_0$$

Exponential Family Theorem

$$f_X(x;\theta) = h(x) g(\theta) e^{\gamma(\theta) T(x)} \quad \text{or } h(x) g(\theta) \exp(\gamma(\theta) T(x))$$

$$f(x;p) = \underbrace{\binom{x+r_0-1}{x}}_{h(x)} \underbrace{p^{r_0}}_{g(p)} e^{\underbrace{x}_{T(x)} \underbrace{(\log(1-p))}_{\gamma(p)}}$$

$$\underbrace{T(x) = \sum_{i=1}^{n} T(x) = \sum_{i=1}^{n} x_i}$$

sufficient and complete

---

Likelihood Function

$$L(r,p \mid x) = \prod_{i=1}^{n} f(x;p)$$

$$= \prod_{i=1}^{n} \binom{x+r-1}{x} p^r (1-p)^x$$

$$= \prod_{i=1}^{n} \binom{x+r-1}{x} p^{rn} (1-p)^{\sum_{i=1}^{n} x}$$

$$\ln(f(x;p)) = \log L(f(x;p))$$

$$= \log\left[\prod_{i=1}^{n} \binom{x+r-1}{x} \times p^{rn} (1-p)^{\sum_{i=1}^{n} x}\right]$$

$\prod$

* In Çarpım sembolü içine girerken toplam $\Sigma$ sembolü oluyor.

$$= \log\left[\prod_{i=1}^{n} \binom{x+r-1}{x}\right] + \log\left[(1-p)^{\Sigma x_i}\right] + \log(p^{rn})$$

$$= \sum_{i=1}^{n} \log\binom{x+r-1}{x} + \log(1-p) \sum_{i=1}^{n} x_i + rn \log(p)$$

Maximum likelihood for the Negative Binomial Exercise Sheet 4

Negative Binomial $(r_0, p)$

$$f(x;p) = \binom{x+r_0-1}{x} p^{r_0} (1-p)^x \qquad x \in \mathbb{N}_0$$

Exponential Family Theorem

$$f_X(x;\theta) = h(x)\, g(\theta)\, e^{\gamma(\theta) T(x)} \quad \text{or} \quad h(x)\, g(\theta) \exp(\gamma(\theta) T(x))$$

$$f(x:p) = \underbrace{\binom{x+r_0-1}{x}}_{h(x)} \underbrace{p^{r_0}}_{g(p)} e^{\underbrace{x}_{T(x)} \underbrace{(\log(1-p))}_{\gamma(p)}}$$

$T(x) = \sum_{i=1}^{n} T(x) = \sum_{i=1}^{n} x_i$ sufficient and complete

---

Likelihood Function

$$L(r,p \mid x) = \prod_{i=1}^{n} f(x;p)$$

$$= \prod_{i=1}^{n} \binom{x+r-1}{x} p^{r} (1-p)^{x}$$

$$= \prod_{i=1}^{n} \binom{x+r-1}{x} p^{rn} (1-p)^{\sum_{i=1}^{n} x}$$

$$\ln(f(x:p)) = \log L(f(x:p))$$

$$= \log\left[\prod_{i=1}^{n} \binom{x+r-1}{x} \times p^{rn} (1-p)^{\sum_{i=1}^{n} x}\right]$$

$\Pi$ In Carpim sembolü icine girerken toplam $\Sigma$ sembolu oluyor.

$$= \log\left[\prod_{i=1}^{n} \binom{x+r-1}{x}\right] + \log\left[(1-p)^{\Sigma x_i}\right] + \log(p^{rn})$$

$$= \sum_{i=1}^{n} \log\binom{x+r-1}{x} + \log(1-p) \sum_{i=1}^{n} x_i + rn \log(p)$$

$$\frac{\partial \ln(r,p|:x)}{\partial p} = \frac{1}{1-p}\sum_{i=1}^{n} x_i + \frac{rn}{p}$$

$$= -(1-p)^{-1} n\bar{x} + rnp^{-1}$$

$\bar{x}_n = \frac{1}{n}\sum x_i$

$n\bar{x}_n = \sum x_i$

$$\frac{\partial^2 \ln(r,p:x)}{\partial p^2} = (-1.-1\cdot 1)(1-p)^{-1-1} n\bar{x}_n - rnp^{-2}$$

$$= -\frac{n\bar{x}_n}{(1-p)^2} - \frac{rn}{p^2}$$

$$\left.\frac{\partial \ln(r,p:x)}{\partial p}\right|_{p=\hat{p}} = 0 \Rightarrow -\frac{1}{1-\hat{p}}\sum x_i + \frac{rn}{p} = 0$$

$$\frac{rn}{\hat{p}} = \frac{\sum x_i}{1-\hat{p}}$$

$$\frac{1-\hat{p}}{\hat{p}} = \frac{\sum x_i}{rn}$$

$$\frac{1}{\hat{p}} - 1 = \frac{\bar{x}_n}{r}$$

$$\frac{1}{\hat{p}} = \frac{\bar{x}_n + r}{r}$$

$$\hat{p} = \frac{r}{r+\bar{x}_n}$$

$$\frac{\partial^2 \ln(r,p:x)}{\partial p^2} = -\left(\frac{n\bar{x}_n}{(1-p)^2} + \frac{rn}{p^2}\right) < 0$$

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta}x}$$

$h(x) = 1^x$ $\quad \gamma(\theta) = \frac{-1}{\theta} \Rightarrow \log\left(\frac{-1}{\theta}\right)$

$g(\theta) = \frac{1}{\theta}$ $\quad T(x) = x \Rightarrow \log(x) = nx = \theta$

$E(T(x)) = \theta$ ise unbiased

$E(x) = $ mean $\quad$ Variance $= E(x^2)$

Gamma $(\alpha, \beta) \rightarrow \alpha/\beta \quad \alpha/\beta^2$

$(n, \theta) \rightarrow n\theta \quad n\theta^2$

first condition

Gamma icin $n\theta \neq \theta \Rightarrow$ bias

2.condition $\gamma(\theta) = \frac{1}{n} E(T(x)) = \frac{1}{\cancel{n}} \cancel{n}\theta = \theta \Rightarrow$ unbiased

Sufficient

$T = \sum T(x) = \theta \Rightarrow \sum x_i \Rightarrow$ ise complete of sufficien

Likelihood Function

$$L(\theta) = L[\theta/x.] = f(x:\theta) \Rightarrow \prod_{i=1}^{n} f(x:\theta)$$

$$\hat{\theta}_{MLE} = \max L(f(x:\theta))$$

$$L(\theta) = \prod^{n} f(x:\theta) = \prod \frac{1}{\theta} e^{-\frac{1}{\theta}x} = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_i}$$

Log-likelihood

$$l(\theta) = \log(L(f(x:\theta)) = \ln\left(\frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_i}\right)$$

$$= \log\theta^{-n} + \log e^{-\frac{1}{\theta}\sum x_i}$$

$$= -n\log\theta - \frac{1}{\theta}\sum x_i \quad \ln e$$

$$\frac{\partial l(f(x:\theta))}{\partial\theta} = \frac{\partial}{\partial\theta}\left(-n\log\theta - \frac{1}{\theta}\Sigma x_i\right) = 0$$

$$\boxed{= \frac{-n}{\theta} + \frac{1}{\theta^2}\Sigma x_i}$$

$$= \frac{1}{\theta^2}\Sigma x_i = \frac{n}{\theta} \quad \theta$$

mean

$$\hat{\theta}_{mean} \Rightarrow \quad \Sigma x_i = \frac{n\cancel{\theta^2}}{\cancel{\theta}} \quad n\theta \Rightarrow E(x)$$

$$\frac{\partial^2 l(f(x:\theta))}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(\frac{\partial l(f(x:\theta))}{\partial\theta}\right) = \frac{\partial}{\partial\theta}\left(\frac{-n}{\theta} + \frac{1}{\theta^2}\Sigma x_i\right)$$

$$\theta = \frac{1}{2}\left(\frac{1}{x}\right)\Sigma x$$

$$= \frac{n}{\theta^2} - \frac{2}{\theta^3}\Sigma x_i < 0 \qquad \theta = \frac{1}{n}\Sigma x_i \text{ Diyelim ki}$$

Direk ispat buraya kadar

Zaten (-) li sonuca elde ettik.

$$= \frac{n}{\left(\frac{1}{n}\Sigma x_i\right)^2} - \frac{2}{\left(\frac{1}{n}\Sigma x_i\right)^3}\Sigma x_i$$

$$\frac{n^3 - 2n^3}{\Sigma x_i^2} = \frac{-n^3}{\Sigma x_i^2} < 0$$

Bize parametreli olarak MLE isterse

$3\theta$

$$\hat{\theta}_{MLE} = \frac{\Sigma x_i}{n} = \bar{x}$$ ise her dağılıma uyum sağlamaz.

Üstel dağılımın minimum-variance unbiased estimator (UMVUE)