Advanced Mathematics and Statistics
Module 2 - Advanced Statistical Methods

Exercise 1. If the joint probability density function of (X, Y) is defined by

$$f(x,y) = \begin{cases} x(y-x)\mathrm{e}^{-y} & 0 < x < y < +\infty \\ 0 & \text{elsewhere} \end{cases}$$

(a) Determine the marginal distributions of X and Y.

(b) How can you simulate realizations of the random variable Y?

(c) Identify the probability distribution of $U = X/Y$.

Solution.

(a)

$$\begin{aligned} f_X(x) &= x \int_x^{+\infty} (y-x)\,\mathrm{e}^{-y}\,\mathrm{d}y\, \mathbb{1}_{(0,+\infty)}(x) = x\,\mathrm{e}^{-x} \int_0^{+\infty} w\,\mathrm{e}^{-w}\,\mathrm{d}w\, \mathbb{1}_{(0,+\infty)}(x) \\ &= x\,\mathrm{e}^{-x}\, \mathbb{1}_{(0,+\infty)}(x) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \mathrm{e}^{-y} \int_0^{y} x(y-x)\,\mathrm{d}x\, \mathbb{1}_{(0,+\infty)}(y) = \mathrm{e}^{-y} \int_0^{1} y^3\, w(1-w)\,\mathrm{d}w\, \mathbb{1}_{(0,+\infty)}(y) \\ &= \frac{1}{\Gamma(4)}\, y^3\,\mathrm{e}^{-y}\, \mathbb{1}_{(0,+\infty)}(y) \end{aligned}$$

(b) Note that $Y \sim \mathrm{Ga}(4,1)$. Hence, Y equals in distribution the sum of 4 independent and identically distributed random variables having a negative-exponential distribution with parameter 1, i.e.

$$Y \stackrel{\mathrm{d}}{=} X_1 + X_2 + X_3 + X_4$$

where $X_i \stackrel{\mathrm{iid}}{\sim} \mathrm{E}(1)$. If $U_1, \ldots, U_4$ are iid from a Unif$(0,1)$, then $X_i = -\log(1-U_i)$ for each $i = 1, 2, 3, 4$. Hence to simulate Y, one may proceed as follows:

(1) For $i = 1, 2, 3, 4$

  (1.1) generate $U_i \sim \mathrm{Unif}(0,1)$

  (1.1) set $X_i = -\log(1-U_i)$

(2) Set $Y = X_1 + X_2 + X_3 + X_4$

(c) Set $U = g_1(X,Y) = X/Y$ and $V = g_2(X,Y) = Y$ and note that

$$g_1^{-1}(u,v) = uv, \qquad g_2^{-1}(u,v) = v.$$

The Jacobian of the transformation is

$$J = \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}$$

and $|\det(J)| = |v|$. Hence

$$\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(uv,v)\,|v| = uv(v-uv)|v|\,\mathrm{e}^{-v}\,\mathbb{1}_{(0,+\infty)}(v)\,\mathbb{1}_{(0,v)}(uv) \\
&= v^3\,\mathrm{e}^{-v}\,u(1-u)\,\mathbb{1}_{(0,+\infty)}(v)\,\mathbb{1}_{(0,1)}(u) \\
&= \frac{1}{\Gamma(4)} v^3\,\mathrm{e}^{-v}\,\mathbb{1}_{(0,+\infty)}(v)\,\frac{\Gamma(4)}{\Gamma(2)\Gamma(2)}\,u(1-u)\,\mathbb{1}_{(0,1)}(u) \\
&= f_V(v)\,f_U(u)
\end{aligned}$$

which entails that $V \perp U$ and $U \sim \mathrm{Beta}(2,2)$.

Exercise 2. Let X and Y be independent random variables such that

$$f_X(x) = 2\,\mathrm{e}^{-2x}\,\mathbb{1}_{(0,+\infty)}(x), \quad f_Y(y) = 4y\,\mathrm{e}^{-2y}\,\mathbb{1}_{(0,+\infty)}(y).$$

(a) Determine $\mathbb{P}[X < Y]$

(b) Show that the moment generating function of Y exists and determine it.

(c) Using the result in (b) evaluate the first two moments of Y, i.e. $\mathbb{E}Y$ and $\mathbb{E}Y^2$

Solution.

(a)

$$\begin{aligned}
\mathbb{P}[X < Y] &= \int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{y} \mathrm{d}x\, f_{X,Y}(x,y) = \int_{-\infty}^{+\infty} f_Y(y)\Big\{\int_0^y f_X(x)\,\mathrm{d}x\Big\}\,\mathrm{d}y \\
&= 4\int_0^{+\infty} y\,\mathrm{e}^{-2y}\,\Big\{\int_0^y 2\,\mathrm{e}^{-2x}\,\mathrm{d}x\Big\}\,\mathrm{d}y \\
&= 4\int_0^{+\infty} y\,\mathrm{e}^{-2y}\left(1 - \mathrm{e}^{-2y}\right)\mathrm{d}y \\
&= 4\left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{4}.
\end{aligned}$$

(b)

$$\begin{aligned}
m_Y(t) &= \int_{-\infty}^{+\infty} \mathrm{e}^{ty}\, f_Y(y)\,\mathrm{d}y = \int_0^{+\infty} \mathrm{e}^{ty}\, 4y\,\mathrm{e}^{-2y}\,\mathrm{d}y \\
&= 4\int_0^{+\infty} y\,\mathrm{e}^{-(2-t)y}\,\mathrm{d}y
\end{aligned}$$

and it is apparent that $m_Y(t) < +\infty$ for any $t < 2$. Hence, there exists a neighbourhood of the origin where m_Y is defined. We can conclude that, for any $t < 2$,

$$m_Y(t) = \frac{4}{(2-t)^2}.$$

(c) Note that since $m_Y'(t) = 8(2-t)^{-3}$, one has

$$\mathbb{E}Y = m_Y'(0) = 1.$$

On the other hand, $m_Y''(t) = 24(2-t)^{-4}$ and

$$\mathbb{E}Y^2 = m_Y''(0) = \frac{3}{2}.$$

Exercise 3. Let X be a random variable whose distribution is negative exponential with parameter 1, i.e. $f_X(x) = \mathrm{e}^{-x}\mathbb{1}_{(0,+\infty)}(x)$.

(a) What is the distribution of $Y = 1-\mathrm{e}^{-X}$? Provide an answer and motivate it, without actually determining f_Y through the change of variable formula.

(b) If $X_1, X_2, \ldots$ are random variables that are independent and identically distributed, with the same law as X, let $X_{(n)} = \max\{X_1, \ldots, X_n\}$ be the n–th order statistics. Evaluate the cumulative distribution function of $W_n = X_{(n)} - \log n$. (*Optional question*: Does it admit a limit as $n \to \infty$?)

(c) If $X_1, X_2, \ldots$ are random variables that are independent and identically distributed, with the same law as X, show that $\bar{X}_n/2$ converges in quadratic mean to $1/2$? Does the convergence hold true also almost surely?

Solution.

(a) Since $F_X(x) = (1 - \mathrm{e}^{-x})\, \mathbb{1}_{(0,+\infty)}(x)$, then

$$Y = 1 - \mathrm{e}^{-X} = F_X(X)$$

and $Y \sim \mathrm{Unif}(0, 1)$.

(b) Since $F(x) = (1 - \mathrm{e}^{-x})\, \mathbb{1}_{(0,+\infty)}(x)$, for any $x > 0$ one has

$$F_{X_{(n)}}(x) = (F(x))^n = (1 - \mathrm{e}^{-x})^n.$$

From this

$$F_{W_n}(x) = \mathbb{P}[X_{(n)} \leq x + \log n] = (1 - \mathrm{e}^{-x-\log n})^n = \left(1 - \frac{\mathrm{e}^{-x}}{n}\right)^n$$

for any $x > 0$. It can be seen that as $n \to \infty$.

$$F_{W_n}(x) \longrightarrow \mathrm{e}^{-\mathrm{e}^{-x}}$$

(c) Since $\mathbb{E}(\bar{X}_n/2) = 1/2$, one has

$$\mathbb{E}\left|\frac{1}{2}\,\bar{X}_n - \frac{1}{2}\right|^2 = \mathrm{Var}(\bar{X}_n/2) = \frac{1}{4n} \to 0$$

s $n \to \infty$. If we let $X'_n = X_n/2$, the sequence of random variables $(X'_n)_{n\geq 1}$ satisfies the assumption of the SLLN, which states the following

Let $(X'_n)_{n\geq 1}$ be a sequence of iid random variables. Then $\bar{X}'_n \xrightarrow{a.s.} \mu$ if and only if $\mathbb{E}X'_i = \mu < \infty$

According to this, one also has a stronger convergence result and $\bar{X}_n/2 \xrightarrow{\text{a.s}} 1/2$, as $n \to \infty$.