EN YÜKSEK OLABİLİRLİK TAHMİNLEME YÖNTEMİ VE EN KÜÇÜK VARYANSLI SAPMASIZ TAHMİNLEYİCİLER KONULARI İLE İLGİLİ UYGULAMALAR : [1]

UYGULAMA. 1. (θ PARAMETRELİ ÜSTEL POPULASYON)

$X_1, X_2, \ldots, X_n$ RANDOM ÖRNEĞİ, ORTALAMASI θ OLAN ÜSTEL POPULASYONDAN ALINMIŞTIR. (a) θ PARAMETRESİNİN EN YÜKSEK OLABİLİRLİK TAHMİNLEYİCİSİNİ BULUNUZ. (b) θ PARAMETRESİ HAKKINDAKİ ÖRNEK FISHER BİLGİSİNİ VE TEORİK FISHER BİLGİSİNİ BULUNUZ. (c) $\hat{\theta}_{MLE}$, θ İÇİN SAPMASIZ BİR TAHMİNLEYİCİ MİDİR? (Orijin Etrafındaki Birinci ve İkinci Momentleri Bulmak İçin MTF KULLANINIZ.

UYGULAMA-1. İN ÇÖZÜMÜ:

$$X_j \sim \text{ÜSTEL}(\theta), \quad E(X_j) = \theta$$

$$f(x_j;\theta) = \frac{1}{\theta} e^{-x_j/\theta}, \quad x_j > 0, \quad \theta > 0$$

VERİLEN n HACİMLİK RANDOM ÖRNEĞİN OLABİLİRLİK FONKSİYONU,

$$L_{X_1,\dots,X_n}(\theta \mid x_1,\dots,x_n) = \prod_{j=1}^{n} f(x_j;\theta) = \prod_{j=1}^{n} \frac{1}{\theta} e^{-x_j/\theta}$$

$$= \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{j=1}^{n} x_j}, \quad x_j > 0, \quad \theta > 0$$

OLABİLİRLİK FONKSİYONU (θ DEĞİŞKENDİR!)

LOG-OLABİLİRLİK FONKSİYONU,

$$\log L_{X_1,\dots,X_n}(\theta \mid x_1,\dots,x_n) = \log\left(\frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_j}\right)$$

[3]

$$\log L_{X_1,\dots,X_n}(\theta|x_1,\dots,x_n) = \log 1 - \log \theta^n - \frac{1}{\theta}\sum_{j=1}^{n} x_j$$

$$= -n\log\theta - \frac{1}{\theta}\sum_{j=1}^{n} x_j$$ (LOG-OLABİLİRLİK FONKSİYONU)

SKOR FONKSİYONU,

$$S(\theta) = \frac{\partial \log L(\theta|x_1,\dots,x_n)}{\partial\theta} = -\frac{n}{\theta} + \frac{\sum x_j}{\theta^2}$$

LOG-OLABİLİRLİK FONKSİYONUNUN θ YA GÖRE BİRİNCİ TÜREVİ

LOG-OLABİLİRLİK FONKSİYONUNU MAXIMUM YAPMAK İÇİN,

$$S(\theta) = \frac{\partial \log L(\theta|x_1,\dots,x_n)}{\partial\theta} = 0 \Rightarrow -\frac{n}{\hat{\theta}_{MLE}} + \frac{\sum x_j}{\hat{\theta}^2_{MLE}} = 0$$

$$\Rightarrow \frac{\sum x_j - n\hat{\theta}_{MLE}}{\hat{\theta}^2_{MLE}} = 0 \Rightarrow \sum x_j - n\hat{\theta}_{MLE} = 0,$$

ÇÜNKÜ $\hat{\theta}_{MLE} > 0$.

$$\Rightarrow \hat{\theta}_{MLE} = \frac{\sum X_j}{n} = \bar{X}$$ YANİ, θ NIN E.Y.O. TAHMİNLEYİCİSİ ÖRNEK ORTALAMASIDIR. [4]

(b) θ HAKKINDAKİ ÖRNEK FISHER BİLGİSİ,

$$\frac{\partial S(\theta)}{\partial \theta} = \frac{\partial^2 \log L(\theta|x_1,\dots,x_n)}{\partial \theta^2} = \frac{n}{\theta^2} - \frac{2\theta(\sum x_j)}{\theta^4}$$

$$\frac{\partial S(\theta)}{\partial \theta} = \frac{n}{\theta^2} - \frac{2\sum x_j}{\theta^3}$$ LOG-OLABİLİRLİK FONKSİYONUNUN İKİNCİ TÜREVİ

ÖRNEK FISHER BİLGİSİ,

$$\mathbb{I}(\theta) = -\frac{\partial S(\theta)}{\partial \theta} = -\left(\frac{n}{\theta^2} - \frac{2\sum x_j}{\theta^3}\right) = -\frac{n}{\theta^2} + \frac{2\sum x_j}{\theta^3}$$ (ÖRNEK FISHER BİLGİSİ)

TEORİK FISHER BİLGİSİ,

$$\mathcal{I}(\theta) = E(\mathbb{I}(\theta)) = E\left(-\frac{n}{\theta^2} + \frac{2\sum x_j}{\theta^3}\right) = -\frac{n}{\theta^2} + \frac{2E(\sum X_j)}{\theta^3}$$

[5]

$$\Rightarrow \mathcal{I}(\theta) = -\frac{n}{\theta^2} + \frac{2\sum E(X_j)}{\theta^3} = -\frac{n}{\theta^2} + \frac{2\sum_{j=1}^{n}\theta}{\theta^3}$$

$$= -\frac{n}{\theta^2} + \frac{2n\theta}{\theta^3} = -\frac{n}{\theta^2} + \frac{2n}{\theta^2} \Rightarrow \mathcal{I}(\theta) = \frac{n}{\theta^2}$$

θ HAKKINDAKİ TEORİK FISHER BİLGİSİ

(c) θ PARAMETRELİ ÜSTEL İÇİN MOMENT TÜRETEN FONKSİYON,

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx}\,\frac{1}{\theta}\,e^{-x/\theta}\,dx = \frac{1}{\theta}\int_0^\infty e^{tx}e^{-x/\theta}\,dx$$

$$= \frac{1}{\theta}\int_0^\infty e^{-\left(\frac{1}{\theta}-t\right)x}\,dx = \frac{1}{\theta}\cdot\frac{1}{\frac{1}{\theta}-t} = \frac{1}{1-\theta t}$$

Böylece, $E(X_j)=\theta$ olan ÜSTEL DAĞILIM İÇİN,

$$M_{x_j}(t) = \frac{1}{1-\theta t} = (1-\theta t)^{-1}$$ ELDE EDİLMİŞ OLDU.

$$M'_{x_j}(t) = \frac{dM_{x_j}(t)}{dt} = \frac{\theta}{(1-\theta t)^2} \Rightarrow M'_{x_j}(t=0) = E(X_j) = \mu_1 = \theta$$

$$M''_{x_j}(t) = \frac{d^2 M_{x_j}(t)}{dt^2} = \frac{2\theta^2(1-\theta t)}{(1-\theta t)^4} = \frac{2\theta^2}{(1-\theta t)^3}$$

$$\mu_2 = E(X_j^2) = M''_{x_j}(t=0) = 2\theta^2.$$ ORİJİN ETRAFINDAKİ İKİNCİ MOMENT.

$$Var(X_j) = \sigma_{X_j}^2 = \mu_2 - (\mu_1)^2 = 2\theta^2 - \theta^2 = \theta^2$$

ŞİMDİ, $\hat{\theta}_{MLE}$ NİN YANİ E.Y.O. TAHMİNLEYİCİSİNİN θ PARAMETRESİ İÇİN SAPMASIZ OLUP OLMADIĞINA BAKALIM:

$$E(\hat{\theta}_{MLE}) = E\left(\frac{\Sigma X_j}{n}\right) = \frac{1}{n} E(\Sigma X_j) = \frac{1}{n} \sum_{j=1}^{n} E(X_j)$$

$$E(\hat{\Theta}_{MLE}) = \frac{1}{n}\sum_{j=1}^{n}\Theta = \frac{1}{n}n\Theta = \Theta.$$

O HALDE,

$$E(\hat{\Theta}_{MLE}) - \Theta = \Theta - \Theta = 0.$$

YANİ, $\hat{\Theta}_{MLE}$ Θ İÇİN SAPMASIZ TAHMİNLEYİCİDİR.

$\hat{\Theta}_{MLE}$, Θ İÇİN SAPMASIZ İDİ. $\hat{\Theta}_{MLE}$ NİN VARYANSI İÇİN CRAMÉR-RAO ALT SINIRI,

$$CRLB_{\Theta}(Var(\hat{\Theta}_{MLE})) = \frac{1}{\mathbb{I}(\Theta)} = \left[\mathbb{I}(\Theta)\right]^{-1}$$

$$= \frac{1}{\frac{n}{\Theta^2}} = \frac{\Theta^2}{n}.$$

$\hat{\Theta}_{MLE}$ NİN VARYANSI İÇİN CRAMÉR-RAO ALT SINIRI.