Exercise Sheet 3

May 3rd 2023

Submission of the homework assignments until May 11th, 11:30 am online in TUM-moodle in groups of two. Please put the *full names* and *student IDs* of you *and* your partner on all parts of your submission. The solution will be discussed in the classes one week after.

Homework

Problem H 9 - Chuck-a-luck [3 pts.]

In the game *chuck-a-luck*, a player pays a stake of one dollar to the casino, chooses one of the six numbers $1, 2, \ldots, 6$ and then throws three standard dices. If none of the three dices shows the chosen number, the player looses and the casino keeps the stake. Elsewise, he or she receives a payout of 1:1 if the number appears once, a 2:1 payout if it appears twice and a 3:1 payout if it appears on all of the dices. It is said that the casino has an advantage of 7.9 % in this game. Is this true? Additionally, determine the variance of the win.

Solution:
Denote by X the win ($X \geq 0$) or loss ($X < 0$) of the player. Each of the fair dices shows the chosen number by probability $\frac{1}{6}$. From direct considerations or via the binomial distribution we see that the different outcomes of X have the probabilities $Pr(X = -1) = \left(\frac{5}{6}\right)^3$, $Pr(X = 1) = 3 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2$, $Pr(X = 2) = 3 \cdot \left(\frac{1}{6}\right)^2 \cdot \frac{5}{6}$ and $Pr(X = 3) = \left(\frac{1}{6}\right)^3$ ✓.
This yields the expected value of $\mathbb{E}(X) = -\frac{17}{216} \approx -0.079$, i.e. the casino indeed has an advantage of 7.9 %. ✓
The variance is easily calculated via $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \approx 1.245 - 0.006 = 1.239$ ✓.

Problem H 10 - A bug's race [5 pts.]

The two ants Atta and Flik are competing in a race. The start is at 0 cm of an infinite folding rule. In each round of the game a fair coin is thrown to decide which of the ants will move forward by 1 cm. If the coin falls heads, Atta moves forward, if it falls tails, it is Flik's turn. The ant that first has a lead of 4 cm over the other will win the race. Define the discrete random variable X as the number of centimeters the winning ant has moved forward until the end of the race.

a) Denote by E the event that Atta and Flik are on par after the first two rounds. Further define the two events F and G that Atta or Flik, respectively, has a lead of 2 cm after the first two rounds. Determine $\mathbb{E}(X|E)$, $\mathbb{E}(X|F)$ and $\mathbb{E}(X|G)$ in terms of $\mathbb{E}(X)$.

b) Use your results from (a) to calculate $\mathbb{E}(X)$.

Hint: you may use the partitioning theorem for the expectation value from the lecture.

Solution:
Clearly, the sample space of the first two rounds is $\Omega = \{HH, HT, TH, TT\}$. The event E is equivalent to the subset $\{HT, TH\}$ which has the probability $Pr(E) = \frac{1}{2}$. The events F and G are formed by $\{HH\}$ and $\{TT\}$, respectively, with the probabilities $Pr(F) = Pr(G) = \frac{1}{4}$.

a) If both Atta and Flik took a step of 1 cm in the first two rounds the situation resembles the start. The only difference is that X is raised by 1. From the linearity of expectation we know

$$\mathbb{E}(X|E) = \mathbb{E}(X+1) = \mathbb{E}(X) + 1. \checkmark$$

Consider now the case of F that Atta has a lead of 2 cm after the first two rounds. After the two following rounds three things may have happened: Atta's lead keeps constant at 2 cm (if HT and TH was thrown), increases to 4 cm (HH) or the two ants are again on par with each other (TT). Let us denote these events by E', F' and G'. Obviously, we have $Pr(E') = Pr(E) = \frac{1}{2}$, $Pr(F') = Pr(F) = \frac{1}{4}$ and $Pr(G') = Pr(G) = \frac{1}{4}$.
In the first case, the situation is the same as two rounds before except for the fact that the conditional random variable $X|F$ is raised by 1, i.e.

$$\mathbb{E}\left((X|F)\,|E'\right) = \mathbb{E}\left((X|F) + 1\right) = \mathbb{E}\left(X|F\right) + 1.$$

In the second case Atta's lead increases by two more cm to 4 cm and she wins. Hence we have

$$\mathbb{E}\left((X|F)\,|F'\right) = 4.$$

In the third case, however, Flik is able to catch up and we are back to the situation at the start of the race - just with X having raised by two. So, here it holds that

$$\mathbb{E}\left((X|F)\,|G'\right) = \mathbb{E}\left(X+2\right) = \mathbb{E}\left(X\right) + 2. \checkmark$$

Now, it is time to apply the partitioning theorem for the expectation value of the conditional random variable $X|F$:

$$\begin{aligned}
\mathbb{E}\left(X|F\right) &= \mathbb{E}\left((X|F)\,|E'\right)\cdot Pr(E') + \mathbb{E}\left((X|F)\,|F'\right)\cdot Pr(F') \\
&\quad + \mathbb{E}\left((X|F)\,|G'\right)\cdot Pr(G') \\
&= \left(\mathbb{E}\left(X|F\right)+1\right)\cdot\frac{1}{2} + 4\cdot\frac{1}{4} + \left(\mathbb{E}\left(X\right)+2\right)\cdot\frac{1}{4} = \\
&= \frac{1}{2}\cdot\mathbb{E}\left(X|F\right) + 2 + \frac{1}{4}\cdot\mathbb{E}\left(X\right). \checkmark
\end{aligned}$$

This can be brought to the form

$$\mathbb{E}(X|F) = 2\cdot\left(2 + \frac{1}{4}\cdot\mathbb{E}\left(X\right)\right) = 4 + \frac{1}{2}\cdot\mathbb{E}\left(X\right).$$

From the symmetry between the conditional random variables $X|F$ and $X|G$ we obtain the same expected value for $X|G$:

$$\mathbb{E}(X|G) = 4 + \frac{1}{2}\cdot\mathbb{E}\left(X\right). \checkmark$$

b) Using the results from the first part of the problem, we now apply the partitioning theorem for the expected value to X which yields

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(X|E)\cdot Pr(E) + \mathbb{E}(X|F)\cdot Pr(F) + \mathbb{E}(X|G)\cdot Pr(G)\\ &= (\mathbb{E}(X)+1)\cdot\frac{1}{2} + (4+\frac{1}{2}\cdot\mathbb{E}(X))\cdot\frac{1}{4} + (4+\frac{1}{2}\cdot\mathbb{E}(X))\cdot\frac{1}{4}\\ &= \frac{3}{4}\cdot\mathbb{E}(X) + \frac{5}{2}.\end{aligned}$$

This equation is solved by

$$\mathbb{E}(X) = 4\cdot\frac{5}{2} = 10. \checkmark$$

Problem H 11 - Joint Probability Distributions **[6 pts.]**

An urn contains 4 red, 2 white and 4 green balls. We randomly draw 3 balls without replacement. Let X and Y be random variables counting the number of red and white balls drawn, respectively.

a) Determine the joint probability distribution $f_{X,Y}$, the two marginal distributions f_X and f_Y as well as $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

b) Calculate $Pr(X \geq Y)$ and $Pr(X = 2|X \geq Y)$.

Solution:

a) In order to have a simple probability space we assume that all balls (also those of equal color) are distinguishable. We model a draw from the urn as an unordered set and thus have $\binom{10}{3}$ different elementary events that all have equal probability to occur. By doing so, we have to determine the relative frequencies of the different events.
(Alternatively, the draw of three balls could be also modeled as ordered triple which would yield $\binom{10}{3}\cdot 3! = 10\cdot 9\cdot 8$ elementary events. In this view, the number of elementary events of some specific type would raise equally by the factor 3! because there is 3! ordered triples for each unordered set of three elements. Apparently, these two different modelings are equivalent in regard of the resulting probabilities.)
The joint probability of the events X and Y is defined as $f_{X,Y}(x,y) := Pr(X = x, Y = y)$. Since there is just two white balls the value range of Y is $W_Y = \{0,1,2\}$. $W_X = \{0,1,2,3\}$ since there is more red balls available than balls drawn. Obviously, $f_{X,Y}(x,y) = 0$ holds if $x+y > 3$, so we assume $x+y \leq 3$ for the following.
What is the number of draws with exactly x red and y white balls? To construct these we can choose x of the four red balls for which we have $\binom{4}{x}$ possibilities. Additionally, we can choose y of the two white balls which can be done in $\binom{2}{y}$ different ways. The remaining $3-x-y$ balls have to be green. For these to be chosen there are $\binom{4}{3-x-y}$ different possibilities. In total, we have $\binom{4}{x}\cdot\binom{2}{y}\cdot\binom{4}{3-x-y}$ different draws with exactly x red and y white balls. ✓

As all elementary event have the same probability the joint probability distribution can be expressed as

$$f_{X,Y}(x,y) \begin{cases} \frac{\binom{4}{x}\cdot\binom{2}{y}\cdot\binom{4}{3-x-y}}{\binom{10}{3}} & \text{if } x \in W_X, y \in W_Y, x+y \leq 3 \\ 0 & \text{else.} \checkmark \end{cases}$$

We now simply put in the actual values of x and y and arrange the resulting values of $f_{X,Y}$ in a table.

$y\backslash x$	0	1	2	3
0	1/30	1/5	1/5	1/30
1	1/10	4/15	1/10	0
2	1/30	1/30	0	0

Now, the marginal densities $f_X(x) := \Pr(X = x)$ and $f_Y(y) := \Pr(Y = y)$ are found by summing over the rows or columns, respectively. ✓

$$f_X(x) \begin{cases} \frac{1}{6} & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{3}{10} & \text{if } x = 2 \\ \frac{1}{30} & \text{if } x = 3 \\ 0 & \text{else.} \end{cases}$$

$$f_Y(y) \begin{cases} \frac{7}{15} & \text{if } y = 0 \\ \frac{7}{15} & \text{if } y = 1 \\ \frac{1}{15} & \text{if } y = 2 \\ 0 & \text{else.} \end{cases}$$

This yields the expected values

$$\mathbb{E}(X) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{30} = \frac{6}{5},$$

$$\mathbb{E}(Y) = 0 \cdot \frac{7}{15} + 1 \cdot \frac{7}{15} + 2 \cdot \frac{1}{15} = \frac{3}{5}. \checkmark$$

b) To calculate $Pr(X \geq Y)$ one has to sum over all cells in the table in which $x \geq y$ holds.

$y\backslash x$	0	1	2	3
0	1/30	1/5	1/5	1/30
1	1/10	4/15	1/10	0
2	1/30	1/30	0	0

By this we find

$$Pr(X \geq Y) = \frac{1+6+6+1+8+3}{30} = \frac{5}{6}. \checkmark$$

For $Pr(X = 2|X \geq Y)$ we need the probability $Pr(X = 2 \cap X \geq Y)$. Since there is just three balls drawn, "$X = 2$" implies $X \geq Y$. Thus, it holds $Pr(X = 2 \cap X \geq Y) = Pr(X = 2) = f_X(2) = \frac{3}{10}$.

$$Pr(X = 2|X \geq Y) = \frac{Pr(X = 2 \cap X \geq Y)}{Pr(X \geq Y)} = \frac{3/10}{5/6} = \frac{9}{25}. \checkmark$$

Problem H 12 - Discrete Probability Distributions **[6 pts.]**

Determine the probability distribution function of the following discrete random variables $X_1, \ldots, X_6$.

a) A single fair dice is thrown. X_1 is the thrown number.

b) Ten fair dices are thrown at the same time. X_2 is defined as the number of dices that show the number "1".

c) A single fair dice is thrown. $X_3 = 1$ if the thrown number is "1" and $X_3 = 0$ else.

d) A single fair dice is thrown repeatedly until it shows "1" first. X_4 counts the total number of throws in the trial.

e) A single fair dice is thrown repeatedly until "1" has been thrown 10 times. X_5 is the number of throws that are needed to achieve that.

f) Ten fair dices are thrown at the same time, and this is repeated 10^{10} times. X_6 counts how often all ten dices show the number "1" at the same time.

Solution:

a) Uniform distribution, $X_1 \sim Uni(1, 6)$. $\checkmark$

b) Binomial distribution, $X_2 \sim Bin(10, \frac{1}{6})$. $\checkmark$

c) Bernoulli distribution, $X_3 \sim Ber(\frac{1}{6})$ (or the trivial case of a binomial distribution, $X_3 \sim Bin(1, \frac{1}{6})$). $\checkmark$

d) Geometric distribution, $X_4 \sim Geo(\frac{1}{6})$ (or the trivial case of a negative binomial distribution, $X_4 \sim NegBin(1, \frac{1}{6})$). $\checkmark$

e) Negative binomial distribution, $X_5 \sim NegBin(10, \frac{1}{6})$. $\checkmark$

f) Binomial distribution, $X_6 \sim Bin(10^{10}, \frac{1}{6^{10}})$ which can be approximated by the Poisson distribution, $X_6 \sim Poi(\lambda)$ with $\lambda = \left(\frac{10}{6}\right)^{10}$. $\checkmark$

Problem H 13 - Counting variables and probability distributions [3 pts.]

Prove the following theorem:
Let (Ω, Pr) be a probability space and $A_1, \dots, A_n$ independent events in (Ω, Pr) that occur with the same probability p. Then the *counting variable* Z defined as

$$Z := \sum_{j=1}^{n} \mathbf{1}_{A_j}$$

has the binomial distribution $Bin(n, p)$.
(Note that $\mathbf{1}_{A_j}$ is the indicator function of the set A_j, i.e. $\mathbf{1}_{A_j}(\omega) = 1$ if $\omega \in A_j$ and $\mathbf{1}_{A_j}(\omega) = 0$ else.)

Solution:
The random variable Z takes the value k if and only if k arbitrary events of $A_1, \dots, A_n$ happen and the remaining $n-k$ do not happen. More formal, we may consider all sets $T = \{i_1, \dots, i_k\}$ of k distinct indices within $[1, n]$. In this notation, the occurring events are $\{A_i | i \in T\}$ and the non-occurring $\{A_j | j \notin T\}$. That way we write for the event $Z = k$

$$\{Z = k\} = \sum_{T} \left(\bigcap_{i \in T} A_i \cap \bigcap_{j \notin T} A_j^c \right). \checkmark$$

In total, the sum has $\binom{n}{k}$ summands according to all different index sets T from $[1, n]$ with k elements. ✓ For a specific T we may split the probability

$$Pr\left(\bigcap_{i \in T} A_i \cap \bigcap_{j \notin T} A_j^c \right) = \Pi_{i \in T} Pr(A_i) \cdot \Pi_{j \notin T} A_j^c = p^k \cdot (1-p)^{n-k}$$

which is guaranteed for $A_1, \dots, A_n$ being (mutually) independent implies for any choice of $\{A_i | i \in T\}$ and $\{A_j | j \notin T\}$ (lemma on independent events from the lecture). Adding up all these probabilities immediately yields the statement to show. ✓