Exercise Sheet 1

April 19th 2023

Submission of the homework assignments until April 27th, 9:45 am online in TUM-moodle in groups of two. Please put the *full names* and *student IDs* of you *and* your partner on all parts of your submission. The solution will be discussed in the classes one week after.

Homework

Problem H 1 - Sample spaces [4 pts.]

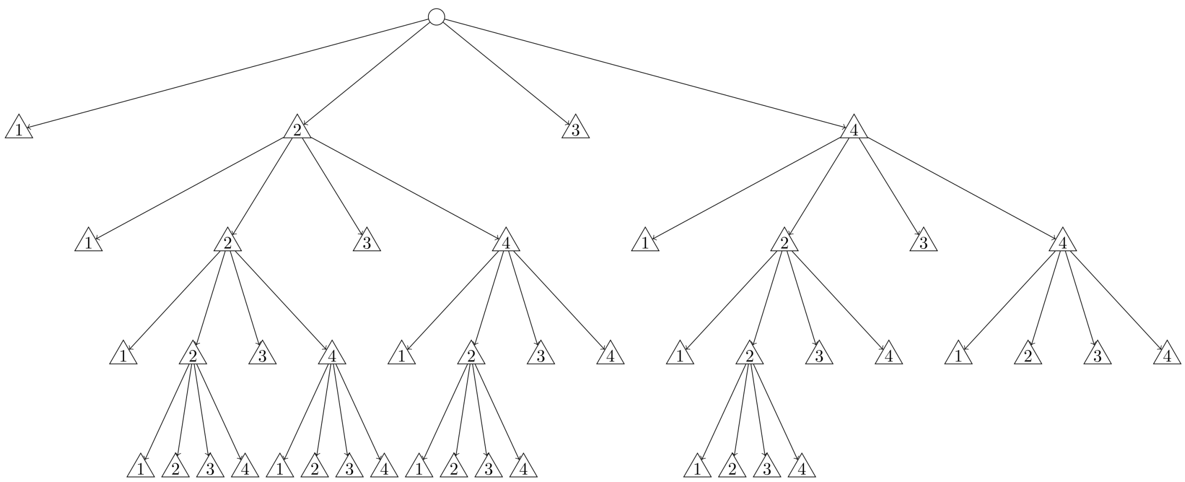
Consider throwing a balanced tetrahedron with the numbers 1, 2, 3 and 4 on its sides. We play a game where we throw until we get an odd number or the sum of the thrown numbers is greater than or equal to 9 or when the tetrahedron has been thrown four times.

a) Sketch the sample space by a tree.

b) Assume a trial with the maximum number of throws. What is the probability that the sum of the numbers is less than 8 in that case?

Solution:

a) For the sketch ✓✓.



b) We consider only trials with the maximum number of 4 throws (condition), i.e. the 16 outcomes 2221, 2222, ..., 2224 on level 4 in the tree. 2221 is the only one with sum less than 8, so the probability for this event is 1/16. This is not to be confused with the unconditioned probability to get a sum less than 8 regardless of the number of throws. ✓✓

Problem H 2 - A fair game? **[4 pts.]**

Alex, Brenda and Claudia play a game where they toss a coin one after each other. The probability to obtain "head" is given by $0 < p < 1$. The order is always alphabetically, i.e. Alex $\rightarrow$ Brenda $\rightarrow$ Claudia, and then again Alex etc. Whoever throws "tail" first wins the round of the game. Do Alex, Brenda and Claudia have equal probabilities to win?

Solution:
Note that a round of the game does not end before "tail" is thrown, i.e. the sample space is (countably) infinite. We see that the winner depends only on the number of throws until the end of the round. Denote by n the number of heads thrown in the round and further define $j = n \mod 3$. For example, Alex wins the game only if "head" is thrown 0, 3, 6, ... times, followed by tails which is exactly in the case $j = 0$. Analogously, Brenda wins if $j = 1$ and Claudia if $j = 2$. Let be $Pr\left[E_n\right]$ be the probability that the game ends after exactly n throws of heads. This is the probability to successively throw n times heads and after that get a "tail",

$$Pr\left[E_n\right] = p^n \cdot (1-p). \checkmark$$

From above we see that Alex's, Brenda's and Claudia's chance to win are

$$Pr\left[A\right] = \sum_{i=0}^{\infty} Pr\left[E_{3i+0}\right], \; Pr\left[B\right] = \sum_{i=0}^{\infty} Pr\left[E_{3i+1}\right], \; Pr\left[C\right] = \sum_{i=0}^{\infty} Pr\left[E_{3i+2}\right],$$

respectively. For general $j \in \{0, 1, 2\}$ we can calculate these probabilities by

$$\sum_{i=0}^{\infty} Pr\left[E_{3i+j}\right] = \sum_{i=0}^{\infty} p^{3i+j} \cdot (1-p) = p^j \cdot (1-p) \cdot \sum_{i=0}^{\infty} (p^3)^i = p^j \cdot \frac{1-p}{1-p^3}. \checkmark$$

Note, that $\sum_{i=0}^{\infty} (p^3)^i$ is a geometric series which converges to $1/(1-p^3)$ since $|p^3| < 1$. Hence, we obtain

$$\sum_{i=0}^{\infty} Pr\left[A\right] = \frac{1-p}{1-p^3}, \; \sum_{i=0}^{\infty} Pr\left[B\right] = \frac{p \cdot (1-p)}{1-p^3}, \; \sum_{i=0}^{\infty} Pr\left[C\right] = \frac{p^2 \cdot (1-p)}{1-p^3}. \checkmark$$

We can see easily that $Pr\left[A\right] > Pr\left[B\right] > Pr\left[C\right]$ for any $0 < p < 1$, i.e. Alex is most likely to win the game in any case - the game is not fair! ✓

Problem H 3 - Inclusion-exclusion identity **[5 pts.]**

In the lecture you have learned the identity

$$Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le n} Pr\left(\bigcap_{j=1}^{r} A_{i_j}\right).$$

This means to consider the intersection of all singlets, then of all pairs, then of all triplets etc. of events $A_1, \dots, A_n$ and add or subtract the referring probabilities in an alternating manner. Prove this statement by mathematical induction with respect to n, the number of events.

Solution:
The trivial case $n = 1$ reduces the formula to

$$Pr\left(\bigcup_{i=1}^{1} A_i\right) = Pr\left(\bigcap_{j=1}^{1} A_j\right) \Leftrightarrow Pr(A_1) = Pr(A_1)$$

which is true but does not suffice for the following. We need to consider $n = 2$ as the base step. We recall from the lecture that for two events A_1, A_2 we have

$$Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2 \setminus A_1).$$

We can decompose A_2 by $A_2 = (A_2 \setminus A_1) \cup (A_1 \cap A_2)$ where $A_2 \setminus A_1$ and $A_1 \cap A_2$ are disjoint sets. Thus, $Pr(A_2) = Pr(A_2 \setminus A_1) + Pr(A_1 \cap A_2)$, or equivalently

$$Pr(A_2 \setminus A_1) = Pr(A_2) - Pr(A_1 \cap A_2).$$

Taken together, we obtain

$$Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2) - Pr(A_1 \cap A_2). \checkmark$$

Now, we need to take the induction step $n \to n + 1$, $n \geq 2$ where we assume that the formula holds for n.

$$\begin{aligned}
Pr\left(\bigcup_{i=1}^{n+1} A_i\right) &= Pr\left((\bigcup_{i=1}^{n} A_i) \cup A_{n+1}\right) \\
&= Pr\left(\bigcup_{i=1}^{n} A_i\right) + Pr(A_{n+1}) - Pr\left((\bigcup_{i=1}^{n} A_i) \cap A_{n+1}\right) \checkmark \\
&= Pr\left(\bigcup_{i=1}^{n} A_i\right) + Pr(A_{n+1}) - Pr\left(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right) \\
&\overset{*}{=} \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} Pr\left(\bigcap_{j=1}^{r} A_{i_j}\right) + Pr(A_{n+1}) \\
&\quad - \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} Pr\left((\bigcap_{j=1}^{r} A_{i_j}) \cap A_{n+1}\right) \checkmark \\
&\overset{**}{=} \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r < (n+1)} Pr\left(\bigcap_{j=1}^{r} A_{i_j}\right) + Pr(A_{n+1}) \\
&\quad + \sum_{r=2}^{n+1} (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_{r-1} < i_r = (n+1)} Pr\left((\bigcap_{j=1}^{r} A_{i_j}) \cap A_{n+1}\right) \checkmark \\
&\overset{***}{=} \sum_{r=1}^{n+1} (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq (n+1)} Pr\left(\bigcap_{j=1}^{r} A_{i_j}\right). \checkmark
\end{aligned}$$

At (*) the induction hypothesis was used.
At (**), we changed from $\leq n$ to $< (n+1)$ in the first summation and changed the index in the second. The latter was done by considering A_{n+1} in the iterated intersection which raised r by 1 and sets the last index to $i_r = (n+1)$.
At (***) the two summations were combined to a single one. The first summation considers all summands with $i_r < (n+1)$, the second one all with $i_r = (n+1)$. However, this misses the corresponding summand of $r = 1$ as it starts at $r = 2$. This is

$$(-1)^{r+1} \sum_{1 \leq i_1 < \cdots < i_{r-1} < i_r = (n+1)} Pr\left(\bigcap_{j=1}^{r} A_{i_j}\right)\Bigg|_{k=1} = Pr(A_{n+1}),$$

i.e. it refers to the middle summand $Pr(A_{n+1})$.

Problem H 4 - A special case of the inclusion-exclusion identity [6 pts.]

Roberto, the waiter in an Italian restaurant, is oblivious - he mixes up what the guests order from him.

a) Consider that $n = 4$ guests order four different dishes from Roberto but, thanks to Roberto's bad memory, it is completely random whom of the four guests he will serve to each of the four dishes. Calculate the probability P that nobody gets the dish that he or she ordered.

b) What is the behavior of P for the case $n \to \infty$?

Solution:

a) We define the sample space Ω by all $4! = 24$ distributions of dishes to guests (i.e. all permutations of 4 different objects), and let be A_i, $i = 1, 2, 3, 4$ the event that the guest i gets the right dish. So, $A_1 \cup A_2 \cup A_3 \cup A_4$ is the event that at least one of the guests gets what he ordered. This is exactly the opposite of the event of interest, i.e.

$$Pr\left((A_1 \cup A_2 \cup A_3 \cup A_4)^c\right) = 1 - Pr\left(A_1 \cup A_2 \cup A_3 \cup A_4\right). \checkmark$$

For the last part we can use the inclusion-exclusion principle (the theorem of Poincaré-Sylvester). For the special case that the probability of the intersection of r events depends only on the number r of considered event but not on the events themselves this theorem takes the form

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \cdot \binom{n}{r} \cdot Pr\left(\bigcap_{i=1}^{r} A_i\right). \checkmark$$

$\binom{n}{r}$ is the number of subsets with r elements of a set with n distinct elements. Instead of considering all different intersections of r sets it is sufficient to consider one of them in this case.

In the context of the exercise we determine the probabilities of these intersections to be

$$
\begin{aligned}
Pr(A_1) &= \frac{3!}{4!} = \frac{1}{4}, \\
Pr(A_1 \cap A_2) &= \frac{2!}{4!} = \frac{1}{12}, \\
Pr(A_1 \cap A_2 \cap A_3) &= \frac{1!}{4!} = \frac{1}{24}, \\
Pr(A_1 \cap A_2 \cap A_3 \cap A_4) &= \frac{0!}{4!} = \frac{1}{24}. \checkmark
\end{aligned}
$$

This yields

$$
\begin{aligned}
Pr\left(A_1 \cup A_2 \cup A_3 \cup A_4\right) &= \binom{4}{1} \cdot \frac{1}{4} - \binom{4}{2} \cdot \frac{1}{12} + \binom{4}{3} \cdot \frac{1}{24} - \binom{4}{4} \cdot \frac{1}{24} \\
&= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} = \frac{5}{8}.
\end{aligned}
$$

This means that at least one guest gets the right dish by $5/8$ and hence by $1-5/8 = 3/8$ nobody gets the right dish. ✓

b) For general n we conclude from above that

$$
Pr\left(\bigcap_{i=1}^{r} A_i\right) = \frac{(n-r)!}{n!}.
$$

Hence, we have

$$
\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \cdot \binom{n}{r} \cdot \frac{(n-r)!}{n!} = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r!} = 1 - \sum_{r=0}^{n} \frac{(-1)^r}{r!}. \checkmark
$$

The event of interest is the complementary event which thus has the probability $\sum_{r=0}^{n} \frac{(-1)^r}{r!}$. In the limit of $n \to \infty$ this converges to $1/e$ by the series representation of the exponential function, $\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} = \exp(-1)$. ✓