Question 1 (Q1) (12 points)
Limits

(a) (5 points) Evaluate the following limit or explain why it does not exist:

$$\lim_{x\to 1} \frac{x^2-5x+4}{|x-1|}$$

left limit: $x<1$, so $|x-1| = 1-x$

$$\lim_{x\to 1^-} \frac{x^2-5x+4}{1-x} = \lim_{x\to 1^-} \frac{(x-1)(x-4)}{1-x} = 3$$

right limit: $x>1$, so $|x-1| = x-1$

$$\lim_{x\to 1^+} \frac{x^2-5x+4}{x-1} = \lim_{x\to 1^+} \frac{(x-1)(x-4)}{x-1} = -3$$

Since left and right limit are different, $\lim_{x\to 1} \frac{x^2-5x+4}{|x-1|}$ DOES NOT EXIST.

(b) (7 points) For

$$f(x) = \frac{\sin(2x)}{x}, x \neq 0,$$

define a continuous extension $F(x)$ of the function $f(x)$, that has domain $\mathbb{R}$. Show whether $F(x)$ is differentiable.

* A continuous extension $F(x)$ is a function for which $F(x) = f(x)$ on the domain of f. (and defined such that F is continuous ~~m~~ also at the points where f is not defined.)

↳ we need to define $F(0)$ such that F is continuous at 0.

$$F(0) = \lim_{x\to 0} \frac{\sin(2x)}{x} = 2 \quad (f \text{ is continuous on } \mathbb{R}\setminus\{0\})$$

→ for $x \neq 0$, $f = F$ is differentiable

→ at $x = 0$, $F'(0) = \lim_{h\to 0} \frac{F(0+h)-F(0)}{h}$

$$= \lim_{h\to 0} \frac{f(h)-2}{h} = \lim_{h\to 0} \frac{\frac{\sin(2h)}{h} - 2}{h}$$

$$= \lim_{h\to 0} \frac{\sin(2h)-2h}{h^2} \overset{H}{=} \lim_{h\to 0} \frac{2\cos(2h)-2}{2h}$$

$$\overset{H}{=} \lim_{h\to 0} \frac{-4\sin(2h)}{2} = 0$$

↳ left and right limit are the same, (~~and~~ the limit exists, so F is differentiable at 0.

Question 2 (Q2) (13 points)
Derivatives

(a) (5 points) Find the derivative of $f(x)$:

$$f(x) = \sqrt{x\sqrt{x}}$$

$f(x) = x^{\frac{3}{4}} \Rightarrow f'(x) = \frac{3}{4} x^{-1/4} = \frac{3}{4} \frac{1}{\sqrt[4]{x}}$

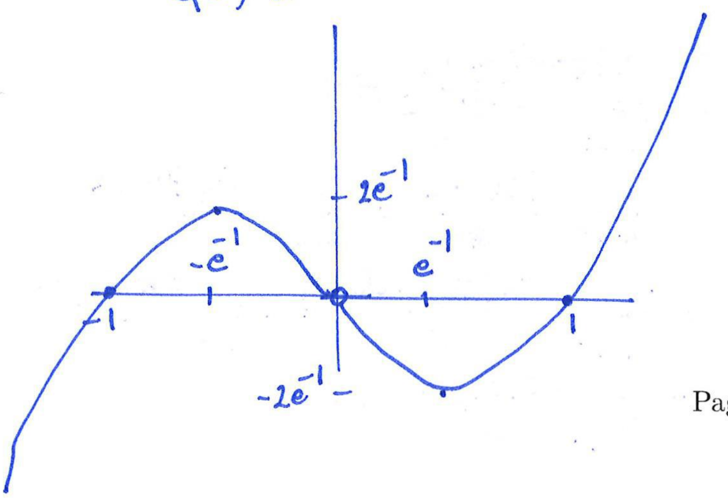
(b) (8 points) Find at what point(s) x the tangent to $f(x)$ is parallel to $y = -2x + 2$, for:

$$f(x) = x^2 + 8x + 2$$

- 2 parallel lines have the same slope.
  $y = -2x + 2$ has slope $m = -2$
- $f'(x) = 2x + 8$
  ↳ we need to find x_0, such that $f'(x_0) = -2$

$-2 = 2x_0 + 8 \Rightarrow x_0 = -5$

Q3) 6.



$f(1) = f(-1) = 0$

Question 3 (Q3) (15 points)

Sketching the graph of a function

(a) (15 points) Let :

$$f(x) = x\ln(x^2)$$

1. Determine the domain of f. Is f continuous on its domain?
2. Compute the first derivative of f. Determine from this derivative for what values of x the function f is increasing or decreasing. Does it have local minima or maxima ?If yes, at which values of x?
3. Compute the second derivative of f. Determine from this derivative for what values of x the function f is convex (concave up) or concave (concave down). Does it have inflection points? If yes, at which values of x?
4. Find all the asymptotes of f.
5. Is f even? Is f odd? Why?
6. Sketch the graph of f based on your previous answers, and show the properties found in your previous answers in the graph.

1) domain : $\mathbb{R}\setminus\{0\}$ f is continuous on its domain

2) $f'(x) = \ln(x^2) + \frac{2x^2}{x^2} = \ln(x^2) + 2$

$f'(x) = 0 \Leftrightarrow \ln(x^2) = -2 \Leftrightarrow \ln|x| = -1 \Leftrightarrow x = \pm e^{-1}$

$f'(x) < 0$ for $|x| < e^{-1}$ $(-e^{-1} < x < e^{-1})$

$f'(x) > 0$ for $|x| > e^{-1}$ $(x < -e^{-1} \vee x > e^{-1})$

↳ $-e^{-1}$ is a local maximum, e^{-1} is a local minimum

3) $f''(x) = \frac{2x}{x^2} = \frac{2}{x}$

↳ $f''(x) > 0$ if $x > 0$, $f''(x) < 0$ if $x < 0$

no inflection points (since $f''(x) = 0$ has no solutions).

4) $f(x)$ does not have asymptotes.

$\lim_{x\to 0^+} x\ln(x^2) = \lim_{x\to 0^+} 2x\ln(x) = \lim_{x\to 0^+} \frac{2\ln(x)}{\frac{1}{x}} \overset{H}{=}$

$\lim_{x\to 0^+} \frac{\frac{2}{x}}{-\frac{1}{x^2}} = 0$ (since f is odd, $\lim_{x\to 0^+} f(x) = -\lim_{x\to 0^-} f(x)$)

$\lim_{x\to +\infty} f(x) = +\infty$ (no HA), $\lim_{x\to +\infty} \frac{f(x)}{x} = +\infty$ (no OA)

(no need to check $\lim_{x\to -\infty} f(x)$, since $f(x)$ is odd.)

5) $f(x)$ is odd, since $f(-x) = (-x)\ln((-x)^2) = -x\ln(x^2)$
$= -f(x)$

6)

x		$-e^{-1}$		0		e^{-1}	
$f(x)$		$2e^{-1}$				$-2e^{-1}$	
$f'(x)$	↗		↘		↘		↗
$f''(x)$		$\cap$				$\cup$	

(sketch: see pg. 2)

Question 4 (Q4) (12 points)
Integrals

(a) (5 points) Evaluate the following integral:

$$\int_0^{\pi/2} \sin(2x)e^{-x}dx$$

Integration by parts : $dV = e^{-x}dx$ $V = -e^{-x}$
$U = \sin(2x)$ $dU = 2\cos(2x)dx$

$$\int_0^{\pi/2} \sin(2x)e^{-x}dx = \cancel{[-e^{-x}\sin(2x)]_0^{\pi/2}} + 2\int_0^{\pi/2} e^{-x}\cos(2x)dx$$

$\hookrightarrow dV = e^{-x}dx$ $V = -e^{-x}$
$U = \cos(2x)$ $dU = -2\sin(2x)dx$

$$= 2[-e^{-x}\cos(2x)]_0^{\pi/2} - 4\int_0^{\pi/2} e^{-x}\sin(2x)dx$$

$$\Rightarrow 5\int_0^{\pi/2}\sin(2x)e^{-x}dx = 2(+e^{-\pi/2}+1) \Rightarrow \int_0^{\pi/2}\sin(2x)e^{-x}dx = \frac{2}{5}(e^{-\pi/2}+1)$$

(b) (7 points) Evaluate the following integral, or show that it diverges:

$$\int_0^{+\infty} \frac{x}{\sqrt{x^2+3}}dx$$

$$\int_0^{+\infty}\frac{x\,dx}{\sqrt{x^2+3}} = \lim_{R\to+\infty}\int_0^R \frac{x\,dx}{\sqrt{x^2+3}} = \lim_{y\to+\infty}\int_3^y \frac{du}{2\sqrt{u}} = \lim_{y\to+\infty}(\sqrt{y}-\sqrt{3}) = +\infty$$

$u = x^2+3$; $x=0 \to u=3$
$du = 2x\,dx$ $x = R \to u = R^2+3 = y$
$\frac{1}{2}du = x\,dx$ $\to u \to \infty$ as $x \to \infty$

$\Rightarrow$ this integral diverges to $+\infty$

Question 5 (Q5) (14 points)
Sequences, Series

(a) (6 points) Determine whether the given series converges absolutely, converges conditionally or diverges using an appropriate test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n!}}$$

1) test for absolute convergence : ratio test.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n!}}{\sqrt{(n+1)!}} = \lim_{n\to\infty} \frac{1}{\sqrt{(n+1)}} = 0 < 1$

↳ according to the ratio test, this series is absolutely convergent.

(b) (8 points) Determine whether the given series converges absolutely, converges conditionally or diverges using an appropriate test.

$$\sum_{n=1}^{\infty} \frac{\ln(n+3)}{n^3}$$

* this is a positive series, so if it converges, it converges absolutely.

* we test for absolute convergence with the limit comparison test, and compare to $\Sigma \frac{1}{n^2}$ (which we know is a converging series)

$\Rightarrow \lim_{n\to\infty} \frac{\frac{\ln(n+3)}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln(n+3)}{n}$

↳ we calculate this limit by calculating the limit of the function $\frac{\ln(x+3)}{x}$, $x \in (0, +\infty)$

$\lim_{x\to+\infty} \frac{\ln(x+3)}{x} \overset{H}{=} \lim_{x\to+\infty} \frac{\frac{1}{x+3}}{1} = 0$

$\Rightarrow \lim_{n\to+\infty} \frac{\ln(n+3)}{n} = 0$. Since $\Sigma \frac{1}{n^2}$ converges, $\Sigma \frac{\ln(n+3)}{n^3}$ also converges absolutely.

Question 6 (Q6) (14 points)
Differential Equations

(a) (7 points) Find the solution $y = y(x)$ to the given differential equation.

$$e^x \frac{dy}{dx} + 2e^x y = 1$$

this is a linear nonhomogeneous equation.

1) Homogeneous solution

$\cancel{e^x} \frac{dy}{dx} + 2\cancel{e^x} y = 0 \Leftrightarrow \frac{dy}{dx} = -2y \Rightarrow \int \frac{dy}{y} = -2\int dx \Rightarrow \ln|y| = -2x + C$

$\Rightarrow y_H(x) = K e^{-2x}$

2) Parameter variation $y(x) = K(x)e^{-2x}$, $y'(x) = K'(x)e^{-2x} - 2K(x)e^{-2x}$

$\hookrightarrow e^x(K'(x)e^{-2x} - \cancel{2K(x)e^{-2x}}) + \cancel{2e^x K(x)e^{-2x}} = 1$

$\Rightarrow K'(x) = e^x$

$\Rightarrow K(x) = e^x + C$

$\Rightarrow y(x) = K(x)e^{-2x} = (e^x + C)e^{-2x} = e^{-x} + Ce^{-2x}$

(b) (7 points) Find the solution $y = y(x)$ to the given initial value problem.

$$\begin{cases} xy' + y = \cos(x) \\ y(\pi/2) = 4/\pi \end{cases}$$

another linear, non-homogeneous ODE.

1) Homogeneous solution.

$x y' = -y \Leftrightarrow \int \frac{dy}{y} = -\int \frac{dx}{x} \Rightarrow \ln|y| = -\ln|x| + C$

$\Rightarrow y_H(x) = \frac{K}{x}$

2) Parameter variation $y(x) = \frac{K(x)}{x}$, $y'(x) = \frac{K'(x)}{x} - \frac{K(x)}{x^2}$

$\cancel{x}\left(\frac{K'(x)}{\cancel{x}} - \cancel{\frac{K(x)}{x^2}}\right) + \cancel{\frac{K(x)}{x}} = \cos(x) \Rightarrow K'(x) = \cos(x) \Rightarrow K(x) = \sin(x) + C$

$\Rightarrow y(x) = \frac{K(x)}{x} = \frac{\sin(x)}{x} + \frac{C}{x}$

3.) Initial condition $y(\frac{\pi}{2}) = \frac{\sin(\pi/2)}{\frac{\pi}{2}} + \frac{C}{\frac{\pi}{2}} = \frac{2}{\pi} + C \cdot \frac{2}{\pi} = \frac{4}{\pi} \Rightarrow C = 1$

$\hookrightarrow y(x) = \frac{\sin(x)}{x} + \frac{1}{x}$

Question 7 (Q7) (12 points)

Multivariate Calculus, Partial Derivatives

(a) (6 points) Let $f(x, y) = e^{xy}\cos(x^2 + y^2)$. Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{xy}\cos(x^2+y^2)\right) = y\,e^{xy}\cos(x^2+y^2) \pm e^{xy}\cdot 2x\cdot \sin(x^2+y^2)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^{xy}\cos(x^2+y^2)\right) = x\,e^{xy}\cos(x^2+y^2) - e^{-xy}\,2y\,\sin(x^2+y^2)$$

(b) (6 points) Find $\frac{\partial f}{\partial s}$ for $f(x, y)$ with $x = t\cos s,\ y = t\sin s$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = -\frac{\partial f}{\partial x}\,t\,\sin(s) + \frac{\partial f}{\partial y}\cdot t\cdot\cos(s)$$

Question 8 (Q8) (8 points)
Double Integrals

Evaluate the double integral:

$$\int_0^1 \int_0^y xye^{x^2} dxdy$$

$$\int_0^1 \int_0^y xye^{x^2} dx\,dy = \int_0^1 y \int_0^y xe^{x^2} dx\,dy = \frac{1}{2} \int_0^1 y \int_0^{y^2} e^{u} du\,dy = \frac{1}{2} \int_0^1 y \left(e^{y^2}-1\right) dy$$

$u = x^2$
$\frac{1}{2} du = xdx$
$x = y \rightarrow u = y^2$
$x = 0 \rightarrow u = 0$

$$\frac{1}{2} \int_0^1 y\left(e^{y^2}-1\right) dy = \frac{1}{2} \int_0^1 y e^{y^2} dy - \frac{1}{2} \int_0^1 y\,dy$$

$$= \frac{1}{4} \left[e^{u}\right]_0^1 - \frac{1}{4} \left[y^2\right]_0^1 = \frac{1}{4}(e-1-1+0) = \frac{e-2}{4}$$

$u = y^2$
$\frac{1}{2} du = y\,dy$
$u = 0 \leftrightarrow y = 0$
$u = 1 \leftrightarrow y = 1$