Exercise Sheet 9

June 15th 2023

Submission of the homework assignments until June 22nd, 11:30 am online in TUM-moodle in groups of two. Please put the *full names* and *student IDs* of you *and* your partner on all parts of your submission. The solution will be discussed in the classes and published on moodle after some time.

Homework

Problem H 36 - Moment Generating Functions (Continuous Case) [5 pts.]

a) Let X be exponentially distributed with the parameter λ. Find the corresponding moment generating function $M_X(t)$. What is special about $M_X(t)$ in this case? Additionally verify your found $M_X(t)$ by calculating the expected value and variance of X.

b) Assume that Y and Z are independent normal random variables with expected value and variance (μ_1, σ_1^2) and (μ_2, σ_2^2), respectively. Show that $Y + Z$ is normal with $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Solution:

a) We start from the probability density of a exponentially distributed random variable being

$$f_X(x) = \lambda e^{-\lambda x}.$$

We simply compute the moment generating function by its definition in the continuous case

$$\begin{aligned}
M(t) &= \mathbb{E}(e^{tX}) \\
&= \int_0^\infty e^{tX} \lambda e^{-\lambda x} dx \\
&= \lambda \int_0^\infty e^{-(\lambda - t)x} dx \\
&= \frac{\lambda}{t - \lambda} \left[e^{-(\lambda - t)x} \right]_0^\infty \\
&= \frac{\lambda}{\lambda - t} \quad \text{if } t < \lambda. \checkmark
\end{aligned}$$

Importantly, this is restricted to $t < \lambda$ because if $t > \lambda$ the integral does not converge ($\lim\limits_{x\to\infty} e^{-(\lambda-t)x} = \infty$ in the anti-derivative in this case) and if $t = \lambda$ the fraction $\frac{\lambda}{\lambda-t}$ is not well-defined. ✓
To obtain the expected value and the variance of X we need to calculate the first and second derivative of $M(t)$:

$$M'(t) = \frac{\lambda}{(\lambda-t)^2}$$
$$M''(t) = \frac{2\lambda}{(\lambda-t)^3}.$$

By this we find

$$\mathbb{E}(X) = M'(0) = \frac{1}{\lambda}$$

and

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = M''(0) - (M'(0))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

These two results reproduce the known formulas for $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ of an exponentially distributed random variable. ✓

b) From the lecture we know that the moment generating function of a sum of two random variables Y and Z equals to the product of the moment generating functions:

$$M_{Y+Z}(t) = M_y(t) \cdot M_Z(t).$$

Additionally, it was shown that the moment generating function of the normal distribution with mean μ and variance σ^2 is

$$M_{\mathcal{N}(\mu,\sigma^2)}(t) = e^{t\mu+(t\sigma)^2/2}.$$

Considering now $Y \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Z \sim \mathcal{N}(\mu_2, \sigma_2^2)$, we find

$$\begin{aligned} M_{Y+Z}(t) &= e^{t\mu_1+(t\sigma_1)^2/2} \cdot e^{t\mu_2+(t\sigma_2)^2/2} \\ &= e^{t(\mu_1+\mu_2)+(t^2(\sigma_1^2+\sigma_2^2)/2}. \end{aligned}$$

This resembles the moment generating function of a normally distributed random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. By the one-to-one correspondence between moment generating functions and the probability distribution we hence conclude that $Y + Z$ is indeed distributed by $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. ✓✓

Problem H 37 - Central Limit Theorem [7 pts.]

Let $X_1, \ldots, X_n$, $n \in \mathbb{N}$ be independent and identically distributed (*iid*) random variables with exponential distribution with parameter $\lambda = 1$ each. Define

$$S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad Z_n = \frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}}.$$

a) Calculate the probability density functions f_{S_n} and f_{Z_n}, $n \in \mathbb{N}$, explicitly.

b) Illustrate f_{S_n} and f_{Z_n} graphically for $n = 1, 2, 3, 4, 5$. Do this by help of your computer in two different plots, one for S_n and one for Z_n. Include the standard normal distribution to your plot for Z_n.

c) Prove that f_{Z_n} converges to the probability density function of the standard normal distribution for $n \to \infty$.
Hint: your strategy should be to calculate the Taylor expansion of $\ln(f_{Z_n})$ *at* 0 *up to second order. Further, use Stirling's approximation:*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \textit{as } n \to \infty \quad \textit{or} \quad \ln(n!) = n\ln(n) - n + \mathcal{O}(\ln(n)).$$

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Solution:

a) A single one of the random variables X_i has the probability density

$$f_{X_i}(x_i) = e^{-x_i}, \quad (x \geq 0)$$

according to the instruction that says that $\lambda = 1$. The probability density function of the summed variable S_n can be found by means of a proof by induction. For $n = 1$ we obviously have $f_{S_1}(x) = f_{X_1}(x_1)$. For $n = 2$, the density of S_2 is obtained by the convolution of f_{X_1} and f_{X_2} (as all X_i are independent):

$$\begin{aligned} f_{S_2}(x) &= \int_{-\infty}^{\infty} f_{X_1}(y) \cdot f_{X_2}(x-y)dy \\ &= \int_{-\infty}^{\infty} e^{-y} \cdot e^{-x+y}dy \\ &= e^{-x}\int_0^x 1dy = e^{-x} \cdot x = f_{(S_1+X_2)}(x) \end{aligned}$$

for $x \geq 0$. By continuing this in mind, we see that for each next step $n \to n+1$ the exponentials e^y and e^{-y} will cancel out and the integration yields just the anti-derivative of the polynomial expression x^{n-1} which is x^n/n. Following this scheme, we come to the conjecture that

$$f_{S_{n+1}}(x) = \frac{e^{-x}x^n}{n!} \quad \text{or} \quad f_{S_n}(x) = \frac{e^{-x}x^{(n-1)}}{(n-1)!}. \checkmark$$

To prove this, we perform the induction step $n \to n+1$:

$$\begin{aligned} f_{S_{n+1}}(x) &= \int\limits_{-\infty}^{\infty} f_{S_n}(y) \cdot f_{X_{n+1}}(x-y)dy \\ &= \int\limits_{-\infty}^{\infty} \frac{e^{-y}y^{(n-1)}}{(n-1)!} \cdot e^{-x+y}dy \\ &= e^{-x}\int\limits_{0}^{x} \frac{y^{(n-1)}}{(n-1)!}dy = e^{-x} \cdot \frac{x^n}{n!}, \end{aligned}$$

matching the conjectured formula. As the base case has been given already, the proof is complete and we have

$$f_{S_n}(x) = \frac{e^{-x}x^{(n-1)}}{(n-1)!} \quad \text{for } x \geq 0 \text{ and } n \in \mathbb{N}. \checkmark$$

(Another way to find this is to use the moment generating function from H 36

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1-t}.$$

By this we can determine the moment generating function of the summed random variable S_n,

$$M_{S_n}(t) = \prod_{i=1}^{n} \frac{1}{1-t} = \left(\frac{1}{1-t}\right)^n,$$

being well-defined for $t < 1$. Although this is a bit cheating, a look-up in a table of generating functions yields that this is the moment generating function of a *Gamma* distribution with parameters n and $\lambda = 1$, so the corresponding probability distribution is

$$f_{S_n}(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{\Gamma(n)} = \frac{e^{-x}x^{n-1}}{(n-1)!}, \quad (x \geq 0)$$

where the Gamma function $\Gamma(n)$ equals to $(n-1)!$ in the case of integers.)
Next, we need to find the expected value and the variance of S_n. The first can be calculated by

$$\mathbb{E}(S_n) = \int\limits_0^\infty s f_{S_n}(s)ds = \int\limits_0^\infty \frac{e^{-s}s^n}{(n-1)!}ds = \frac{\Gamma(n+1)}{(n-1)!} = \frac{n!}{(n-1)!} = n$$

where we used that $\int\limits_0^\infty e^{-x}x^n dx = \Gamma(n+1)$ with the Gamma function $\Gamma(z)$ being equal to $(z-1)!$ for $z \in \mathbb{N}$. For the second, we need

$$\mathbb{E}(S_n^2) = \int\limits_0^\infty s^2 f_{S_n}(s)ds = \int\limits_0^\infty \frac{e^{-s}s^{n+1}}{(n-1)!}ds = \frac{\Gamma(n+2)}{(n-1)!} = \frac{(n+1)!}{(n-1)!} = (n+1)n,$$

again using the Gamma function. By this, the variance is

$$\text{Var}(S_n) = \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2 = (n+1)n - n^2 = n^2 + n - n^2 = n.$$

This means that the random variable Z_n takes the form

$$Z_n = \frac{S_n - n}{\sqrt{n}}$$

if $S_n \geq 0$, i.e. $Z_n \geq -\sqrt{n}$. ✓ To find its probability distribution we detour to the cumulative distribution

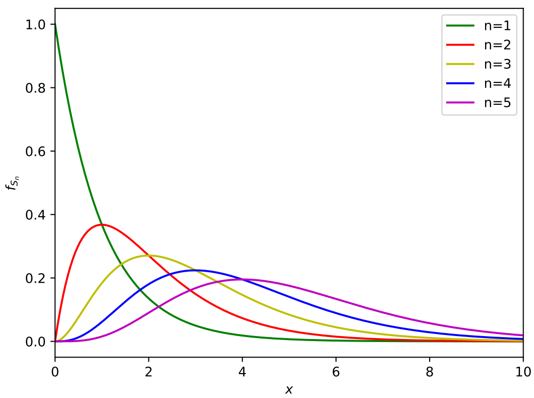
$$F_{Z_n}(x) = Pr(Z_n \leq x) = Pr\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) = Pr\left(S_n \leq \sqrt{n}x + n\right) = F_{S_n}(\sqrt{n}x+n).$$

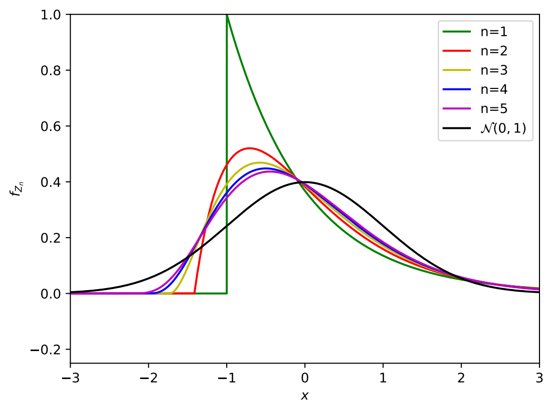
Taking its derivative finally yields

$$\begin{aligned} f_{Z_n}(x) = \frac{d}{dx}F_{Z_n}(x) = \frac{d}{dx}\left(F_{S_n}(\sqrt{n}x+n)\right) \\ = \sqrt{n}\frac{d}{dx}F_{S_n}(\sqrt{n}x+n) = \sqrt{n}f_{S_n}(\sqrt{n}x+n) \\ = \sqrt{n}\frac{e^{-\sqrt{n}x-n}(\sqrt{n}x+n)^{(n-1)}}{(n-1)!} \end{aligned}$$

being no-zero for $x \geq \sqrt{n}$. ✓

b) One should obtain the following plots. ✓





c) (Note that the hint was misleading in the original version. One should first apply Stirling's approximation and then Taylor-expand the logarithm.) We have shown already in the previous that the density of Z_n is

$$f_{Z_n}(x) = \sqrt{n} \cdot f_{S_n}(n + \sqrt{n}x) = \sqrt{n}\frac{e^{-\sqrt{n}x-n}(\sqrt{n}x+n)^{(n-1)}}{(n-1)!}.$$

As we are interested in the limit $n \to \infty$ we may use Stirling's approximation for the factorial, $k! \sim \sqrt{2\pi k} k^k e^{-k}$. This yields the estimations

$$\sqrt{n}\frac{e^{-\sqrt{n}x-n}(\sqrt{n}x+n)^{(n-1)}}{(n-1)!} \approx \sqrt{n} \cdot \frac{e^{-\sqrt{n}x-n}(\sqrt{n}x+n)^{(n-1)}}{\sqrt{2\pi(n-1)}(n-1)^{n-1}e^{-n+1}}$$

$$\approx \sqrt{n} \cdot \frac{e^{-\sqrt{n}x-n} \cdot n^{(n-1)} \left(\frac{x}{\sqrt{n}}+1\right)^{(n-1)}}{\sqrt{2\pi(n-1)}(n-1)^{n-1}e^{-n+1}} \approx \sqrt{n} \cdot \frac{e^{-\sqrt{n}x-n} \cdot n^{(n-1)} \left(\frac{x}{\sqrt{n}}+1\right)^{(n-1)}}{\sqrt{2\pi n}n^{(n-1)}e^{-n}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\sqrt{n}x}\left(\frac{x}{\sqrt{n}}+1\right)^{(n-1)} \approx \frac{1}{\sqrt{2\pi}} \cdot e^{-\sqrt{n}x}\left(\frac{x}{\sqrt{n}}+1\right)^{n} \equiv \frac{1}{\sqrt{2\pi}} \cdot g(x). \checkmark$$

Applying now the logarithm to the term $g(x)$ brings us to

$$\ln(g(x)) = -\sqrt{n}x + n \cdot \ln\left(\frac{x}{\sqrt{n}}+1\right).$$

By means of the Taylor expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \mathcal{O}(x^3)$$

we get

$$\ln(g(x)) \approx -\sqrt{n}x + n \cdot \left(\frac{x}{\sqrt{n}} - \frac{x^2}{2 \cdot n}\right) = -\frac{x^2}{2}.$$

So, we can further estimate

$$f_{Z_n}(x) = \sqrt{n} \cdot f_{S_n}(n+\sqrt{n}x) \approx \frac{1}{\sqrt{2\pi}} \cdot e^{\ln(g(x))} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

As this equals to the density of the standard normal distribution we have shown that $f_{Z_n}(x)$ in fact converges to the density of $\mathcal{N}(0,1)$. This is exactly what we observe in the right one of the above plots. ✓

Problem H 38 - Bullheads' Estimator **[4 pts.]**

On a trip to the Neckar in the south of Heilbronn, the students Roman and Renis find European bullheads in the water (a sort of freshwater fish). Suppose that they find $n \in \mathbb{N}$ different animals of independent and identically distributed sizes $X_1, \ldots, X_n$, respectively. A quick look-up on their phones yields that the X_i are uniformly distributed on some interval $[0, L]$, $L \in \mathbb{R}^+$, but unfortunately they cannot find the actual value of L.

a) Define a coefficient C such that the weighted sample mean

$$C \cdot \frac{\sum_{i=1}^n X_i}{n}$$

is an unbiased estimator for L.

b) In comparison to that construct a maximum likelihood estimator for L. Is this estimator unbiased?

Solution:

a) We define the sample mean from X_1 to X_n by

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n}.$$

To solve the first part of the exercise, we determine its expected value in dependence of L:

$$\mathbb{E}(\bar{X}) = \mathbb{E}\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{1}{n} \cdot \sum_{i=1}^n \frac{L}{2} = \frac{L}{2}. \checkmark$$

(We used the expected value of a uniformly distributed random variable.) If we choose $C = 2$, doubling the sample mean, we have

$$\mathbb{E}(2 \cdot \bar{X}) = 2 \cdot \mathbb{E}(\bar{X}) = L.$$

So, this is an unbiased estimator for L. ✓

b) Now, we are interested in finding a maximum likelihood estimator. Denote by $x_1, \dots, x_n$ the actual samples observed by the students. Since all sample variables X_i are uniformly distributed on $[0, L]$, their density function is

$$f_{X_i}(x_i) = \begin{cases} \frac{1}{L} & \text{if } x \in [0, L] \\ 0 & \text{else.} \end{cases}$$

Accordingly, the product $\Pi_{i=1}^n f_{X_i}(x_i)$ equals to zero if there exists at least one $x_i \notin [0, L]$. Otherwise, all factors are equal to $1/L$. Define $\alpha = \min_{i \leq i \leq n} x_i$ and $\beta = \max_{i \leq i \leq n} x_i$. Using these abbreviations, the likelihood function can be expressed as

$$\mathcal{L}(x_1, \dots, x_n; L) = \begin{cases} \frac{1}{L^n} & \text{if } 0 \leq \alpha \leq \beta \leq L \\ 0 & \text{else.} \end{cases}$$

Hence, if we assume that $x_i \in [0, L]$ holds for all $1 \leq i \leq n$, the likelihood function is $\frac{1}{L^n}$, strictly monotonously decreasing with respect to L. So, the maximum of $\mathcal{L}(x_1, \dots, x_n; L)$ is at $L = \beta$. The maximum likelihood estimator, denoted by Y, hence is

$$Y = \max\{X_1, \dots, X_n\}. \checkmark$$

By means of a counterexample, we can prove that Y is not unbiased. For this, consider $n = 1$. Since X_1 is uniformly distributed on $[0, L]$ we know that $\mathbb{E}(X_1) = L/2$ and thus

$$\mathbb{E}(Y) = \mathbb{E}(X_1) = \frac{L}{2} \neq L.$$

This already implies that Y is not unbiased with respect to L. ✓

Problem H 39 - A MLE for the Log-Normal Distribution **[5 pts.]**

A random variable X is called log-normally (logarithmically normally) distributed with parameters μ and $\sigma > 0$, $\mu, \sigma \in \mathbb{R}$ if it has the probability density function

$$f_X(x) \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} \cdot \exp\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right) & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}$$

In this case $\mathbb{E}(X)$ is $\exp(\mu + \sigma^2/2)$. Further, it holds that the random variable $Y = \ln(X)$ is normally distributed with parameters μ and σ.

a) Find a maximum likelihood estimator (*MLE*) for μ.

b) Prove or disprove that your constructed estimator is unbiased.

Solution:

a) Denote by $x_1, \dots, x_n \in \mathbb{R}$ the actual samples. The likelihood function is, according to its definition,

$$\begin{aligned} L(x_1, \dots, x_n; \mu, \sigma) &= \prod_{i=1}^n f_X(x_i) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma x_i} \cdot \exp\left(-\frac{(\ln(x_i) - \mu)^2}{2\sigma^2}\right) \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^n} \prod_{i=1}^n \frac{1}{x_i} \cdot \exp\left(-\frac{(\ln(x_i) - \mu)^2}{2\sigma^2}\right). \checkmark \end{aligned}$$

We aim at a value for μ that maximizes L. Like in the previous exercises, here it is sufficient to find a value of μ that maximizes the log-likelihood function. By the properties for the logarithm we find

$$\begin{aligned} \ln\left(L(x_1, \dots, x_n; \mu, \sigma)\right) &= \ln\left(\frac{1}{(\sqrt{2\pi}\sigma)^n} \prod_{i=1}^n \frac{1}{x_i} \cdot \exp\left(-\frac{(\ln(x_i) - \mu)^2}{2\sigma^2}\right)\right) \\ &= -n\ln(\sqrt{2\pi}\sigma) - \sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n \frac{(\ln(x_i) - \mu)^2}{2\sigma^2} \equiv f(\mu). \checkmark \end{aligned}$$

Upon differentiation of $f(\mu)$ with respect to μ we obtain

$$f'(\mu) = -\sum_{i=1}^n \frac{2(\ln(x_i) - \mu) \cdot (-1)}{2\sigma^2} = \frac{\sum_{i=1}^n (\ln(x_i) - \mu)}{\sigma^2}.$$

It follows:

$$f'(\mu) = 0 \quad \Leftrightarrow \mu = \frac{\sum_{i=1}^n \ln(x_i)}{n}.$$

To prove that this indeed is a maximum, we calculate the second derivative with respect to μ:

$$f''(\mu) = \frac{\sum_{i=1}^{n}(-1)}{\sigma^2} = -\frac{n}{\sigma^2}.$$

Obviously, this is negative, regardless of μ. So, $\hat{\mu} = \frac{\sum_{i=1}^{n}\ln(X_i)}{n}$ is the desired maximum likelihood estimator of the parameter μ of the logarithmic standard normal distribution. ✓

b) To check whether or not the estimator $\hat{\mu}$ is biased, we need to examine if $\mathbb{E}(\hat{\mu}) = \mu$ holds or not. By the linearity of the expected value we find

$$\mathbb{E}(\hat{\mu}) = \frac{\sum_{i=1}^{n}\mathbb{E}(\ln(X_i))}{n}. ✓$$

Using the information of the instruction that $\ln(X_i)$ is normally distributed with parameters μ and σ. By this, we have

$$\mathbb{E}(\hat{\mu}) = \frac{\sum_{i=1}^{n}\mu}{n} = \mu,$$

implying that the maximum likelihood estimator $\hat{\mu}$ is unbiased for the parameter μ of the logarithmic normal distribution. ✓