Discrete Probability
Theory (INHN0020)
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Summer Term 2023

Exercise Sheet 8

June 8th 2023

Submission of the homework assignments until June 15th, 11:30 am online in TUM-moodle in groups of two. Please put the *full names* and *student IDs* of you *and* your partner on all parts of your submission. The solution will be discussed in the classes and published on moodle after some time.

Homework

Problem H 32 - Radioactive Decay Chain [5 pts.]

A radioactive decay chain is a series of decays resulting in a sequence of different decay products, each with a specific half-life τ_i. Consider the following decay chain:

$$
\begin{tikzcd}
 & A \arrow[dl, "\tau_1"] \arrow[dr, "\tau_2"] & \\
B \arrow[d, "\tau_3"] & & Y \\
X & &
\end{tikzcd}
$$

where material A decays either to material B or Y, and material B decays to material X.

a) Given an amount of $M_A(0)$ of atoms of type A at time $t = 0$, write an expression for $M_A(t)$ for $t > 0$.

b) Additionally given an amount of $M_B(0)$ of atoms of type B at time $t = 0$, find the amount of material B for $t > 0$. Explain your steps. *(Hint: it may help to repeat your knowledge on differential equations.)*

Solution:

a) The half-life τ_i translates to the rate λ_i by $\lambda_i = \frac{\ln(2)}{\tau_i}$. ✓ For simplicity of notation, we assume this transformation to be done for the three different types of decay in this task and write λ_1, λ_2 and λ_3 for the following. Material A decays by two so-called *channels* that are independent and exclusive. The probability of a single atom to have not decayed to B is $\exp(-\lambda_1 t)$, to have not decayed to Y $\exp(-\lambda_2 t)$. Hence, the probability that it "survives" until time T, i.e. to neither have decayed to B nor to Y, is $Pr(t > T) = \exp(-\lambda_1 t) \cdot \exp(-\lambda_2 t) = \exp(-(\lambda_1 + \lambda_2)t)$ (we may multiply the probabilities of the independent events). The total amount of A is a multiple of such independent atoms, so we obtain by multiplication with $M_A(0)$:

$$M_A(t) = M_A(0) \cdot e^{-(\lambda_1+\lambda_2)t}. \checkmark$$

b) One way to solve this problem is in terms of a stochastic process that leads to a differential equation. Consider the interval in time $[t, t+dt[$ where dt is meant to be infinitesimally small. We now focus on the *change* of the amount of material B in this time: this is determined by the number of atoms of type A right now decaying to B and the number of atoms of type B right now decaying to X. For very small dt these two contributions are proportional to $\lambda_i \cdot dt$ in best approximation, so we get

$$dM_B(t) = \lambda_1 M_A(t)dt - \lambda_3 M_B(t)dt. \checkmark$$

This rate equation yields the first-order inhomogeneous linear differential equation

$$\frac{dM_B(t)}{dt} = \lambda_1 M_A(t) - \lambda_3 M_B(t) = \lambda_1 M_A(0)e^{-(\lambda_1+\lambda_2)t} - \lambda_3 M_B(t)$$

where we used the result for $M_A(t)$ from the first part. The strategy to solve this differential equation has two steps:

- The general solution of the connected homogeneous differential equation

  $$\frac{dM_B(t)}{dt} = -\lambda_3 M_B(t)$$

  is $M_B(t) = C_B \exp(-\lambda_3 t)$ with some positive constant C_B. $\checkmark$
- A particular solution of the inhomogeneous differential equation is found by variation of parameters. Using the ansatz $M_B(t) = C_B(t)\exp(-\lambda_3 t)$ in the equation leads to

  $$\frac{dM_B(t)}{dt} = \frac{dC_B(t)}{dt}e^{-\lambda_3 t} - \lambda_3 C_B(t)e^{-\lambda_3 t} = \lambda_1 M_A(0)e^{-(\lambda_1+\lambda_2)t} - \lambda_3 M_B(t)e^{-\lambda_3 t}.$$

  After a bit of algebra we get by integration of the derivative $\frac{dC_B(t)}{dt}$

  $$C_B(t) = \frac{\lambda_1}{\lambda_3 - \lambda_1 - \lambda_2} M_A(0)e^{(\lambda_3-\lambda_1-\lambda_2)t} + D_B$$

  where D_B is constant. This yields

  $$M_B(t) = \frac{\lambda_1}{\lambda_3 - \lambda_1 - \lambda_2} M_A(0)e^{-(\lambda_1+\lambda_2)t} + D_B e^{-\lambda_3 t}.$$

We may determine the constant D_B by the condition on $M_B(t)$ at $t = 0$:

$$M_B(0) = \frac{\lambda_1}{\lambda_3 - \lambda_1 - \lambda_2} M_A(0) + D_B.$$

In conclusion we get

$$M_B(t) = M_B(0)e^{-\lambda_3 t} + \frac{\lambda_1}{\lambda_3 - \lambda_1 - \lambda_2} M_A(0)(e^{-(\lambda_1+\lambda_2)t} - e^{-\lambda_3 t}). \checkmark$$

For intuition, we see that the amount of material B depends (i) on the amount of B at $t = 0$ that decays exponentially, (ii) on the income from material A at t, depending on the current amount of A a t, (iii) and, in turn, the decay of these incoming atoms.

Problem H 33 - Poisson's Mall **[5 pts.]**

a) A shopping center has three entrances that we label by A, B and C. The customers arrive at the entrances by three independent Poisson processes with the rates per hour of $\lambda_A = 80$, $\lambda_B = 120$ and $\lambda_C = 200$, respectively. Find the distribution of the number of clients which arrive in the first hour.

b) A store of the shopping center is neighbored to entrance A with the lowest rate. To increase the frequency at A the store establishes the following prize draw: each second customer entering at A gets a coupon of the store as a present. Determine a general expression for the probability that at a point in time $t > 0$ an entering customer does or does not win a coupon. For this, you may assume that the rate is still constant at λ_A.

Solution:

a) At each of the entrances we have a Poisson process, so the number of customers entered within the first hour is Poisson-distributed with the parameters $t \cdot \lambda_i$ ($t = 1\text{h}$). More formally, denote by X_A, X_B and X_C the number of clients having entered at A, B and C within the first hour then $X_i \sim Po(\lambda_i t)$, $i = A, B, C$. Since X_A, X_B and X_C are independent the random variable $X \equiv X_A + X_B + X_C$ is also Poisson-distributed with parameter $(\lambda_A + \lambda_B + \lambda_C)t$ (Note, that the statement was presented in the lecture for two summands only but generalizes to N summands. A proof for two summands can be done by a convolution of random variables.). So, the total number X of clients arriving within the first hour is Poisson-distributed with $\lambda t = 400$, $X \sim Po(400)$. ✓✓

b) The number of customers having entered at A until $t > 0$ is Poisson distributed with parameter $\lambda_A t$, i.e.

$$Pr(X_A(t) = k) = \frac{1}{k!}(\lambda_A t)^k e^{-\lambda_A t}, \; k = 0, 1, 2, \ldots. \checkmark$$

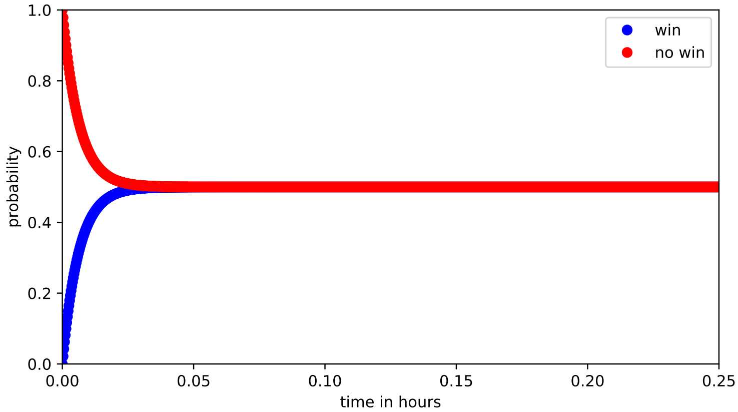
A customer entering at t will only win if the number of costumers having entered before is odd such that he or she is the 2nd, 4th, 6th etc. costumer. We assume that the probability of this event - to have an odd number of successes in the interval $[0, t[$ - is the same as for the interval $[0, t]$. By this, we find the probability that a costumer entering at t may win:

$$\begin{aligned} Pr(X_A \, \text{odd}) &= \sum_{j=0}^{\infty} Pr(X_A = 2j + 1) \\ &= \sum_{j=0}^{\infty} \frac{1}{(2j+1)!}(\lambda_A t)^{2j+1} e^{-\lambda_A t} \\ &= \sinh(\lambda_A t) e^{-\lambda_A t}. \checkmark \end{aligned}$$

Here, the series representation of sinh was used in the last step.
Analogously we get for the probability that an entering client does not win (which happens for an even number of costumers in the time right before t)

$$\begin{aligned} Pr(X_A\,\text{even}) &= \sum_{j=0}^{\infty} Pr(X_A = 2j) \\ &= \sum_{j=0}^{\infty} \frac{1}{(2j)!}(\lambda_A t)^{2j} e^{-\lambda_A t} \\ &= \sum_{j=0}^{\infty} \frac{1}{(2j)!}(\lambda_A t)^{2j} e^{-\lambda_A t} \\ &= \cosh(\lambda_A t) e^{-\lambda_A t}, \checkmark \end{aligned}$$

here with the function cosh.
It is interesting to visualize these two probabilities with the value of λ_A given in the instruction.



We see, that while in the first phase it is more likely to have no win the situation more and more converges to a 50:50 chance. This is intuitive since in the beginning it is very likely to be the first customer - that does not win - while later on the number of costumers entered X_A is likely to be high and then almost equally likely to be even or odd. Additionally, notice that the two probabilities add up to 1 at all points in time t.

Problem H 34 - Assessing Blueberries in 2D **[4 pts.]**

The quality of the blueberries grown in a farm is evaluated. The quality depends both on the size and the color. Let us assume that X and Y are two continuous random variables that measure the deviation of the size and color from the optimal values x_0 and y_0, respectively, on some certain scales. The joint probability density is given by

$$f_{X,Y}(x,y) = \frac{\exp(-2/3 \cdot (x^2 - x \cdot y + y^2))}{2\pi \cdot \sqrt{3/4}}.$$

a) Show that X and Y are standard normally distributed.

b) Are the random variables X and Y independent?

Solution:

a) In order to prove that X and Y are standard normally distributed the marginal densities $f_X(x)$ and $f_Y(y)$ are required. For this we assume that x or y, respectively, has a fixed value and integrate over the other variable. For example, the marginal density of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} \frac{\exp(-2/3 \cdot (x^2 - x \cdot t + t^2))}{2\pi \cdot \sqrt{3/4}} dt. \checkmark$$

This expression should equal to $\exp(-1/2 \cdot x^2)/\sqrt{2\pi}$ if X has the standard normal distribution. To verify this we first transform the exponent:

$$-\frac{2}{3} \cdot \left(x^2 - x \cdot t + t^2\right) = -\frac{1}{2} \cdot x^2 - \frac{2}{3} \cdot \left(\frac{1}{4} \cdot x^2 - x \cdot t + t^2\right) = -\frac{1}{2} \cdot x^2 - \frac{2}{3} \cdot \left(t - \frac{1}{2} \cdot x\right)^2.$$

If we use this in the integral above we obtain

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \frac{\exp(-1/2 \cdot x^2)}{\sqrt{2\pi}} \cdot \frac{\exp(-2/3 \cdot (t - 1/2 \cdot x)^2)}{\sqrt{2\pi} \cdot \sqrt{3/4}} dt \\ &= \frac{\exp(-1/2 \cdot x^2)}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \frac{\exp(-1/2 \cdot (t - 1/2 \cdot x)^2)/(3/4))}{\sqrt{2\pi} \cdot \sqrt{3/4}} dt. \checkmark \end{aligned}$$

The term in front of the integral sign is the desired probability density function of the standard normal distribution. Further one observes that the integrand corresponds to the probability density function of a normally distributed random variable with the expected value $\frac{1}{2} \cdot x$ and standard deviation $\sqrt{3/4}$. Hence, the integral takes the value 1 by which we have shown that indeed

$$f_X(x) = \frac{\exp(-1/2 \cdot x^2)}{\sqrt{2\pi}}.$$

(Alternatively, one may use the Gaussian integral.) From the symmetry between Y and X it is clear that Y has the standard normal distribution as well. ✓

b) Assume that X and Y are independent. Then $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ should hold. By a simple counter example we see, however, that this condition is not met for the blueberries. For instance, for the choice $x = 0$ and $y = 0$ we have

$$f_{X,Y}(0, 0) = \frac{1}{2\pi \cdot \sqrt{3/4}} \neq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} = f_X(0) \cdot f_Y(0).$$

So, X and Y are not independent. ✓

Problem H 35 - Unlikely Bacterial Mutations **[5 pts.]**

In a pharmaceutical research lab a bacterial population of size $N = 2000$ is exposed to a certain chemical substance that may induce a genetic mutation. This may happen for one bacterium independently from other effects with the probability of 0.5%. Determine the probability to generate at most six mutated bacteria

a) exactly,

b) in approximation using the Poisson distribution,

c) and in approximation based on the central limit theorem.

Solution:

Denote by X the total number of mutated bacteria. From the instruction we see that X has the binomial distribution with the parameters $N = 2000$ and $p = 0.005$. Further use the notation $q = 1 - p$.

a) For the exact calculation we use the probability mass function of the binomial distribution $Pr(X \leq i) = \sum_{k=0}^{i} \binom{N}{k} p^k q^{N-k}$. In this specific example we have

$$Pr(X \leq 6) = \sum_{k=0}^{6} \binom{2000}{k} \cdot \left(\frac{1}{200}\right)^k \cdot \left(\frac{199}{200}\right)^{2000-k} \approx 0.1295. \checkmark$$

b) By the law of rare events we approximate X by a Poisson-distributed random variable Y with the parameter $\lambda = Np = 10$. It holds

$$Pr(Y \leq 6) = \sum_{i=0}^{6} \frac{e^{-\lambda}\lambda^i}{i!} = \frac{1}{e^{10}} \cdot \sum_{i=0}^{6} \frac{10^i}{i!} = \frac{1}{e^{10}} \cdot \frac{25799}{9} \approx 0.1301. \checkmark$$

c) Let $\mu = \mathbb{E}(X) = Np = 10$ and $\sigma^2 = \text{Var}(X) = Npq = 9.95$ be the expected value and the variance. In order to approximate by the normal distribution we may directly apply De Moivre's limit theorem, a corollary to the central limit theorem, because X is binomially distributed. By this theorem the random variable $H = \frac{X-Np}{\sqrt{Npq}}$ is approximately standard normally distributed. Hence, it holds

$$Pr(X \leq 6) = Pr\left(\frac{X - Np}{\sqrt{Npq}} \leq \frac{6 - Np}{\sqrt{Npq}}\right) = Pr\left(H \leq \frac{6 - Np}{\sqrt{Npq}}\right) \approx \Phi\left(\frac{6 - Np}{\sqrt{Npq}}\right)$$

where Φ is the cumulative distribution of the standard normal distribution. $\checkmark$ We use the property $\Phi(-x) = 1 - \Phi(x)$ as well as a tabulated value of Φ (e.g. from the table presented in the lecture) to find

$$\Phi\left(\frac{6 - Np}{\sqrt{Npq}}\right) = \Phi\left(-\frac{4}{\sqrt{Npq}}\right) = 1-\Phi\left(\frac{4}{\sqrt{Npq}}\right) \approx 1-\Phi\left(1.27\right) \approx 1-0.898 = 0.102. \checkmark$$

Since N is relatively small in this case it is reasonable to apply the continuity correction term that was presented in the lecture:

$$Pr(X \leq 6) \approx \Phi\left(\frac{6 + 0.5 - Np}{\sqrt{Npq}}\right).$$

Using the specific values we get

$$\Phi\left(-\frac{3.5}{\sqrt{9.95}}\right) = 1 - \Phi\left(\frac{3.5}{\sqrt{9.95}}\right) \approx 1 - \Phi(1.11) \approx 1 - 0.864 = 0.136.$$

By comparing this with the exact result, we see that this term improves the approximation. ✓