APPLIED STATISTICS

WEEK 6

Hypothesis Testing (Single Sample)

Tests of the Population Proportion

- Involves categorical variables.
- Two possible outcomes
  - "Success" (a certain characteristic is present)
  - "Failure" (the characteristic is not present)
- Fraction or proportion of the population in the "success" category is denoted by p.

Proportions

- Sample proportion in the success category is denoted by $\hat{p}$.

$$\hat{p} = \frac{X}{n} = \frac{\text{number of successes in sample}}{\text{sample size}}$$

- When "$np(1-p) > 9$", $\hat{p}$ can be approximated by a normal distribution with mean and standard deviation:

$$\boxed{\mu_{\hat{p}} = p}, \quad \boxed{\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}}$$

check this requirement!

Hypothesis Tests for Proportions

* The sampling distribution of $\hat{p}$ is approximately normal, so the test statistic is a z-value

$$\boxed{z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}}$$

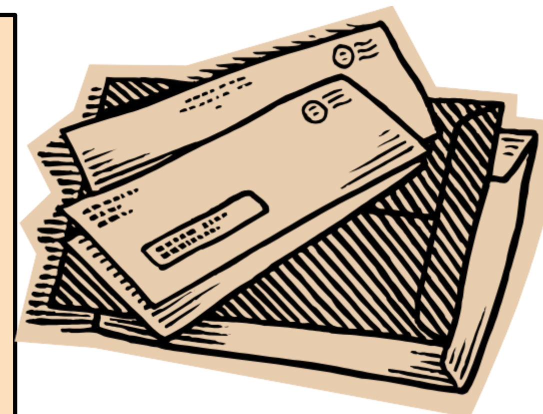
z-test statistic value

Example: Z Test for Proportion

Question 5: A marketing company claims that it receives 8% responses from its mailing.

To test this claim, a random sample of 500 were surveyed with 25 responses.

Test at the $\alpha = .05$ significance level.



$$\hat{p} = \frac{25}{500} = \boxed{0.05}$$

$$\hat{p} =$$

Null hypothesis

Alternative hypothesis

Z Test for Proportion: Solution

Two-tailed test

H_0: P = .08

H_1: P ≠ .08

α = .05

n = 500, $\hat{p}$ = .05

Test Statistic:

$$z = \frac{\hat{p} - P_0}{\sqrt{\frac{P_0(1-P_0)}{n}}} = \frac{.05 - .08}{\sqrt{\frac{.08(1-.08)}{500}}} = -2.47$$

Z-stat

Critical Values: ± 1.96

critical values

NP(1-P) > 9

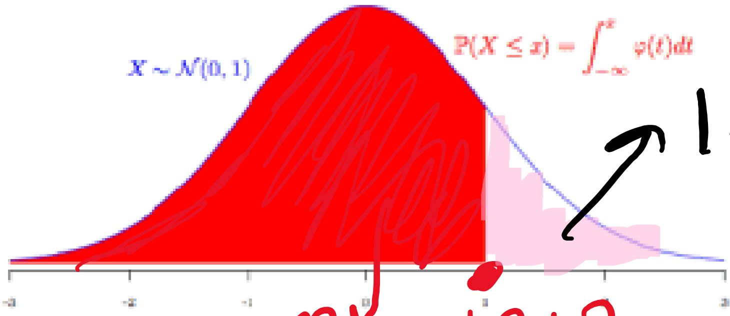
Decision:

Critical value App. Solution

Reject H_0 at α = .05

Conclusion:

There is sufficient evidence to reject the company's claim of 8% response rate.



0.9932 +2,47

1 − 0.9932 = 0.0068

$\alpha/2 = 0.025$

$z = -2.47$

$+2.47$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

$$N P(1-P) > 9$$

$$\underbrace{500(0.05)(0.95)}_{23.75} > 9 \checkmark$$

Let's utilize normal approximation

Testing Procedure

I, Critical value Approach

II. P_ value Approach

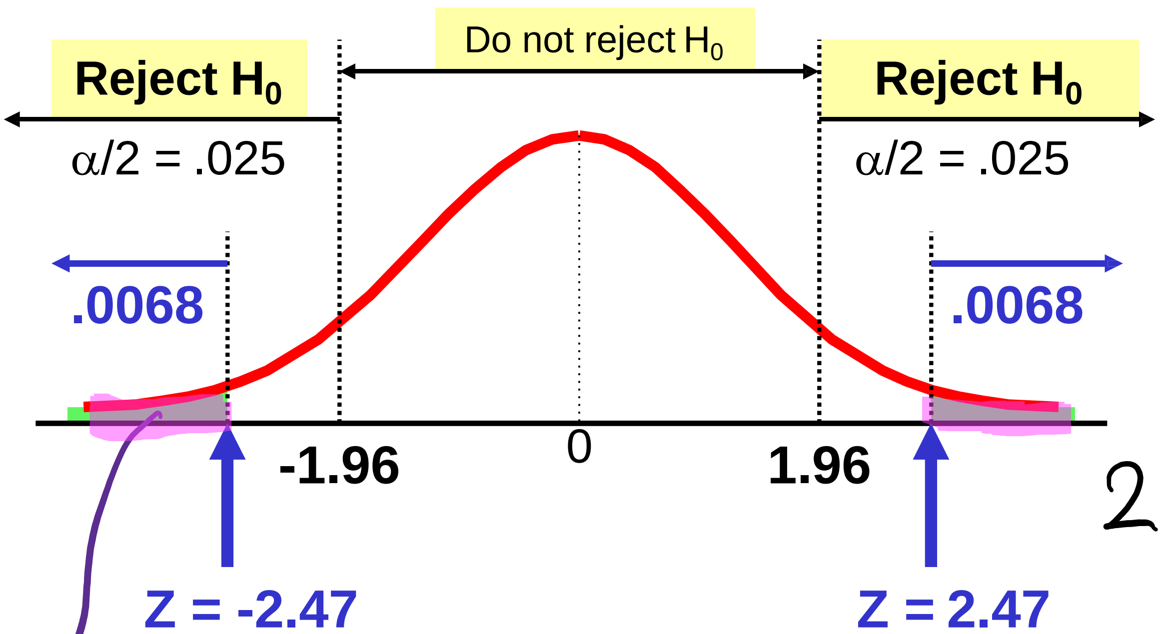
p-Value Solution

level of significance
$\alpha = 0.05$

(continued)

Calculate the p-value and compare to α

(For a two sided test the p-value is always two sided)



p-value = .0136:

$$P(Z \le -2.47) + P(Z \ge 2.47)$$

$$= 2(.0068) = 0.0136$$

$2p = 2(0.0068)$

$\Rightarrow \boxed{0.0136}$

$1 - 0.9932$
$= 0.068$

Reject H_0 since p-value = .0136 < α = .05 *

Major Components of a Hypothesis Testing Procedure

- Null and Alternative Hypotheses $\rightarrow H_0: ?$ $\rightarrow H_1: ?$
- Level of Significance, α = ?
- Test statistic (this is typically the name of the test as well, z-test, t-test, etc.)
- Distribution of the test statistic, if null is true.
- Acceptance/rejection regions and conclude

  OR

  ↓ Critical Value approach
- Compare p-value with α and conclude

  ↓ p-value approach

Chi-Square (χ^2) Test for Variance

- Tests a population variance or standard deviation
- Assume that population is approximately normally distributed
- Test Statistic : Chi-Square

Chi-Square Test for Variance

Test statistic

$$\chi^2 = \frac{(n-1) \cdot S^2}{\sigma_0^2}$$

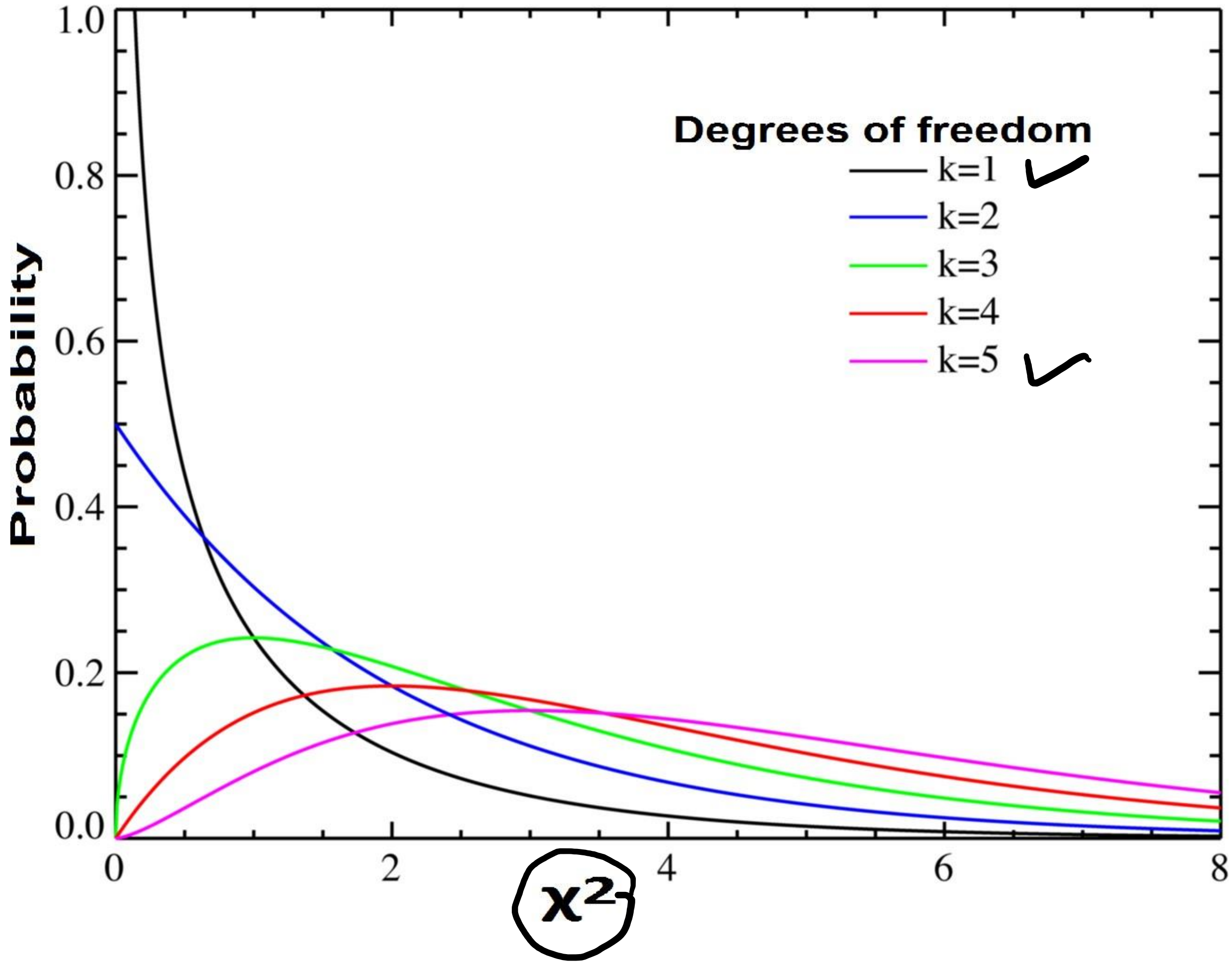
Sample variance

Hypothesized population variance

- If null hypothesis is true, this quantity follows a chi-square distribution with (n-1) degrees of freedom.

- If the observed test statistic is on the extremes of this distribution, we reject the null hypothesis.

Chi-Square Probability Density Function



Chi-Square (χ^2) Test Example

Question 6: Is the variation in boxes of cereal, measured by the **standard deviation**, equal to **15** grams?

A random sample of $\boxed{\textbf{25}}$ boxes had a standard deviation of **17.7** grams. Test at the **.05** level of significance.

→ Chi-Square Test

$H_0: \sigma = 15$

$H_1: \sigma \neq 15$

Chi-Square (χ^2) Test Solution

Sample variance

Testing for pop standard deviation

H0: $\sigma = 15$

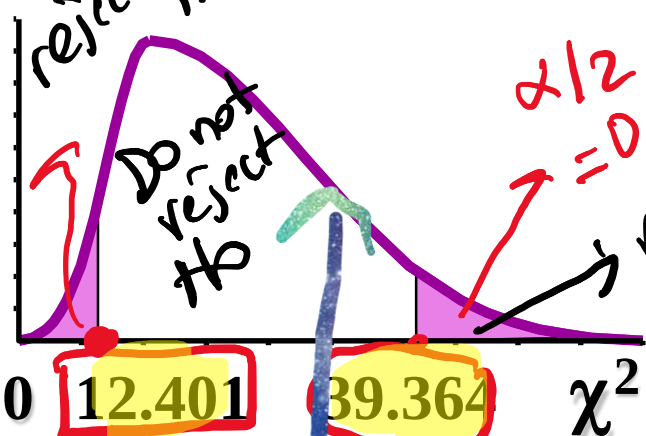
Ha: $\sigma \neq 15$

$\alpha = .05$ $n = 25$

df = 25 - 1 = 24

Critical Value(s):

Two tailed test



1-α/2 33.42 critical value

Test Statistic:

$$\chi^2 = \frac{(n-1) S^2}{\sigma_0^2} = \frac{(25-1) 17.7^2}{15^2}$$

$$\boxed{= 33.42}$$

Fail to reject Ho.

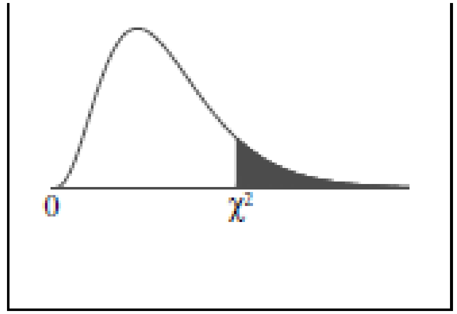
Decision:

Do not reject Ho at $\alpha = .05$

Conclusion:

There is no evidence σ is not 15.

$\alpha/2 = 0.025$



$1-\alpha/2 = 0.975$

$\alpha/2 = 0.025$

$1-\alpha/2 = 1-0.025 = 0.975$

The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928

critical value

Note about Chi-Square Test

* For a two-tailed test at the α-level of significance, the critical region is

critical value

computed stat value

$\chi^2 < \chi^2_{1-\alpha/2}$ or $\chi^2 > \chi^2_{\alpha/2}$

* For the one-sided alternative $\sigma^2 < \sigma_0^2$, the critical region is

$\chi^2 > \chi^2_{\alpha}$

critical value

Upper Tail Test or a lower Tail Test