Calculus

Revision

Alexia Briassouli, Otti D'Huys,

Overview

- Limits and continuity
- Differentiation
- Integration
- Sequences and Series
- Differential Equations
- Multivariate calculus:
  - partial derivatives
  - double integrals

This overview does only contains the main outlines (many things are omitted). All material in the lecture slides and notes is examinable!

Limits

The **limit** $\lim_{x\to a} f(x) = L$ if $\forall\epsilon > 0\ \exists\delta > 0 : |x-a| < \delta \Leftrightarrow |f(x) - L| < \epsilon$.

- Left (right) limits: $x < a$ ($x > a$). The limit only exists if left and right limit are equal.
- $\lim_{x\to a} f(x) = +\infty$ if $\forall M > 0\ \exists\delta > 0 : |x-a| < \delta \Rightarrow f(x) > M$.
- $\lim_{x\to+\infty} f(x) = L$ if $\forall\epsilon > 0\ \exists N > 0 : x > N \Rightarrow |f(x) - L| < \epsilon$.
- A function is continuous at a if $\lim_{x\to a} f(x) = f(a)$. A function is continuous **on its domain** if it is continuous at every point **of its domain**.
- Asymptotes: a function $f(x)$
    - has a horizontal asymptote $y = a, a \in \mathbb{R}$ if $\lim_{x\to\pm\infty} f(x) = a$.
    - has a vertical asymptote $x = b, b \in \mathbb{R}$ if $\lim_{x\to b^\pm} f(x) = \pm\infty$.
    - has an oblique asymptote $y = ax + b, a \neq 0$ if $\lim_{x\to\pm\infty}(f(x) - ax) = b$, with $\lim_{x\to\pm\infty} \frac{f(x)}{x} = a$.

Derivatives

The **derivative** of $y = f(x)$ with respect to x is given by

$$y'(x) = f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists and is finite.

- The derivative $f'(x_0)$ is the slope of the tangent line to the function at x_0
- Left/right limits lead to left/right derivatives - $f(x)$ is differentiable at a if left and right derivative are equal. $f(x)$ is differentiable on its domain if it is differentiable at every point at its domain.
- 2 important rules to calculate derivatives: chain rule, product rule.
- Sign of $f'(x) \to$ increasing/decreasing intervals/possible extrema.
- Sign of $f''(x) \to$ convex/concave intervals/possible inflection points.
- l'Hopital rules: using derivatives to calculate indeterminate limits of the form $\left[\frac{0}{0}\right]$ and $\left[\frac{\infty}{\infty}\right]$:

Integration

- An indefinite integral can be seen as antiderivative:

$$F'(x) = f(x) \Leftrightarrow F(x) + C = \int f(x)dx$$

- The definite integral $\int_a^b f(x)dx = F(b) - F(a)$ is the area under the curve between a and b (Riemann sums)
- Fundamental theorem of Calculus: $\frac{d}{dx}\int_a^x f(t)dt = f(x)$
- Methods to calculate integrals: substitution (inverse chain rule), integration by parts (inverse product rule), partial fraction decompostion,...
- Improper integrals (can converge, diverge, or diverge to $\pm\infty$):
  - $\int_a^\infty f(x)dx) = \lim_{t\to\infty}\int_a^t f(x)dx$
  - if $\lim_{x\to a^+} f(x) = \pm\infty$, $\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$

Sequences

- A sequence $\{a_n\}$ is an ordered list of numbers $a_1, a_2, \dots, a_n, \dots$
- Main question: does the sequence converge? $\lim_{n\to\infty} a_n = A$?
- How to calculate the limit of a sequence? Calculate the limit of the function!
  If $f(x)$ is defined for all $x \geq n_0$ and $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$, then:
  $$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$$
  .
- If $a_n \to A$, then $f(a_n) \to f(A)$ for a continuous function f.

Series

- A series is a formal sum of infinitely many terms:
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$
- A series is a sequence of **partial sums**:
$$s_n = s_{n-1} + a_n = \sum_{j=1}^{n} a_j$$
- Main question: does a series converge?
- Two important series:
  - The **geometric** series $\sum_{n=0} ar^n = \frac{a}{1-r}$. The geometric series converges absolutely for $|r| < 1$ (by direct calculation of the limit).
  - The **p-series** $\sum_{n=1} n^{-p}$. The p-series converges for $p > 1$ (by integral test).
- Absolute convergence: convergence in absolute value. Absolute convergence implies convergence.
- Conditional convergence: convergence, but no absolute convergence.

Convergence tests

For divergence (any series)

- n-th term divergence test: if the sequence a_n does not converge to 0, the series $\sum a_n$ diverges.

For positive series/absolute convergence

- Integral test
- Comparison test
- Limit comparison test
- Ratio test

For alternating series

- Alternating series test

Convergence - possible solution strategy

To determine whether a series $\sum a_n$ converges:

1. Does the sequence $a_n \to 0$? If not, the series $\sum a_n$ diverges.
2. Is the series positive, i.e $a_n > 0$ for all n? If not, first test for absolute convergence with the series $\sum |a_n|$.
3. Optional, if the terms look complicated: Replace by a positive series $\sum b_n$, for which $\lim_{n\to\infty} \frac{|a_n|}{b_n} = L$, with $0 < L < \infty$ (limit comparison test).
4. Do you see terms a^n and factorials $n!$? Try the ratio test.
5. If the ratio test is inconclusive, and/or you see denominators that are polynomials in n, try the limit comparison test. Compare with a suitable p-series.
6. The p-series themselves are solved with the integral test (but you are allowed to know which p-series converge, no need to calculate the integral).
7. If you have an alternating series that is not absolutely convergent, check for conditional convergence with the alternating series test.

Differential equations

- A **Differential Equation (DE)** $f(y', y, x) = 0$ is an equation involving one or more derivatives of an unknown function $y(x)$
- A **solution of a differential equation** $y(x)$ on an interval is any function satisiying the differential equation.
- An IVP (initial value problem) also provides a value for $y(x_0)$, so that the equation has a unique solution.
- A homogenous differential equation has the form $f(y', y) = 0$.
- A linear differential equation is linear in $y(x)$, $y'(x)$, ... (not in x!). You can add up solutions of homogeneous linear differential equations.

Differential equations

Possible check list for first order differential equation solving:

1. Is the DE separable? If so, you can put all terms involving x on the left hand side and all terms involving y on the right hand side, and integrate both sides.
2. Is the DE linear and homogeneous and first order? Then it is separable.
3. Is the DE linear, but not homogeneous? First solve the homogenous equation. Then solve the non-homogeneous equation by parameter variation $y(x) = K(x)y_H(x)$.
4. Always explain your method and reasoning! *Any method that results in a correct solution is allowed. But, if you make a mistake, you only get intermediate marks if we can follow your reasoning.*

Partial derivatives

- The **partial derivatives** of $f(x, y)$ with respect to x, y are given by:
$$\frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$\frac{\partial}{\partial y} f(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$
if these limits exist.
- The **normal line** to $z = f(x, y)$ at $(a, b, f(a, b))$:
$$f(a,b) - z = \frac{x-a}{f_x(a,b)} = \frac{y-b}{f_y(a,b)}$$
- The chain rule in two dimensions: for $z(x, y)$, $x(s, t)$ and $y(s, t)$,
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
- For higher order partial derivatives, the order of derivation does not matter (if all partials are continuous):
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Double integrals

- We consider double integrals $\iint_R f(x,y)dA$ for (piecewise) continuous functions $f(x,y)$ on a bounded domain R.
- Just like for one dimension, the integral is a limit of a Riemann sum.
- The double integral is calculated as an iterated (inner + outer) integral

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy\,.$$

- Fubini's theorem: it does not matter how you set up your integral (which variable you choose for your outer integral), as long as your integration limits describe the region R correctly.