Statistics: Tutorial sheet 3

Mandatory Exercises

Exercise 1. Find sufficient statistics for random samples from the following distributions. Use both the Factorization Theorem and the formula for exponential families (when applicable).

a. Beta(α,β)

b. Poisson(λ)

c. Uniform(a,b)

Exercise 2. Consider a random sample $X_1, \ldots, X_n$ from the pdf

$$f(x \mid \mu, \sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \quad \mu < x < \infty,\ 0 < \sigma < \infty.$$

Find the sufficient statistics for (μ,σ).

Exercise 3. Find the MME, based on the lowest moment possible, of θ based on a random sample of the following pdfs. Hint: recognizing the pdfs as corresponding to familiar distributions will help in calculating the required moment(s).

a. $f(x \mid \theta) = \frac{1}{x!} e^{-\theta} \theta^x$, where $x = 0, 1, \ldots$ and $\theta > 0$.

b. $f(x \mid \theta) = \theta x^{\theta - 1}$, where $0 < x < 1$ and $\theta > 0$.

c. $f(x \mid \theta) = \theta^2 x e^{-\theta x}$, where $x > 0$ and $\theta > 0$.

Practice Exercises

Exercise 1. We have the statistical model $\{\text{Poisson}(\lambda) \mid \lambda > 0\}$.

a. Show that $T(\boldsymbol{X}) = \sum_{i=1}^n X_i$ is sufficient by calculating the conditional distribution of $\boldsymbol{X}$ given $T(\boldsymbol{X})$.

b. Show that $T(\boldsymbol{X}) = \sum_{i=1}^n X_i$ is sufficient by using the factorization theorem.

Exercise 2. Let $X_1, \ldots, X_n$ be iid distributed with cdf

$$P(X \leq x) = \begin{cases} 0 & x < 0 \\ (x/\beta)^\alpha & 0 \leq x \leq \beta \\ 1 & x > \beta \end{cases}$$

a. Find the pdf of X_1.

b. Find a two-dimensional sufficient statistic for (α, β).

c. Is this statistical model an exponential family?

d. What system of equations do you have to solve to find the method of moment estimators for α and β? Note, don't actually solve them!

Exercise 3. Let $X_1, \ldots, X_n$ be a random sample from a population belonging to the exponential family

$$g(x \mid \theta) = h(x)c(\theta)e^{\sum_{j=1}^{m} w_j(\theta)t_j(x)}.$$

Prove that $T(\boldsymbol{X}) = (\sum_{i=1}^{n} t_1(X_i), \ldots, \sum_{i=1}^{n} t_m(X_i)$ is a sufficient statistic for θ_0 by using the factorization theorem.

Exercise 4. Suppose we have the statistical model $\{g(x \mid \theta) \mid \theta > 0\}$, where

$$g\left(x \mid \theta\right) = \theta x^{\theta - 1} \qquad \text{if } 0 \leq x \leq 1.$$

a. Find a sufficient statistic for θ.

b. Find the moment estimator for θ_0.

c. Is the moment estimator based on a sufficient statistic? What does this tell us?