Cumulative Distribution Function

For any discrete r.v. with possible values $x_1, x_2, \ldots, x_n$, the events $\{X = x_1\}, \{X = x_2\}, \ldots, \{X = x_n\}$ are mutually exclusive.

Therefore, $P\{X \le x\} = \sum_{x_i \le x} f(x_i)$

The cumulative distribution function can be denoted by $F(x) = P\{X \le x\}$ for any number x, it means it is the prob. that X will be at most x.

Properties of $F(x)$

1. $F(x) = P(X \le x) = \sum_{x_i \le x} f(x_i)$
2. $0 \le F(x) \le 1$ it means $F(-\infty) = 0$, $F(+\infty) = 1$
3. $F(x)$ is nondecreasing and right continuous

   If $x \le y$, then $F(x) \le F(y)$

Ex:

y	1	2	3	4
$p(y)$	0.4	0.3	0.2	0.1

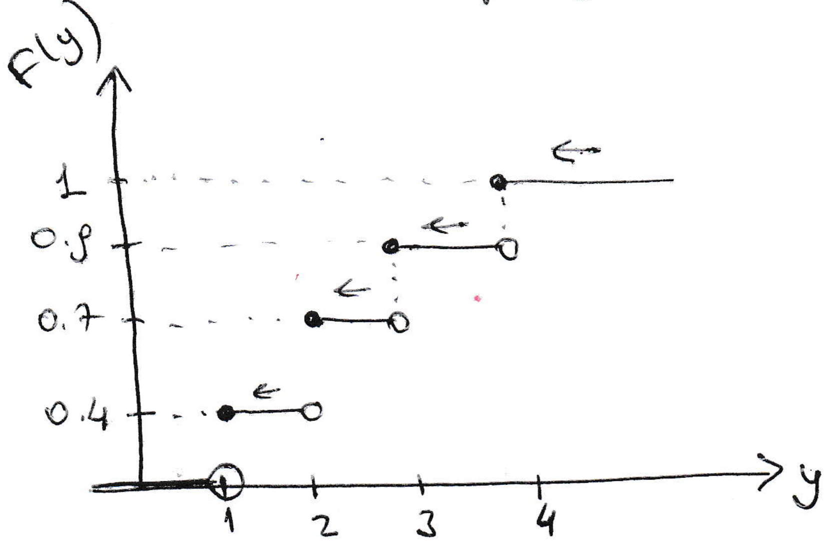
$F(1) = P(Y \le 1) = P(Y = 1) = 0.4$

$F(2) = P(Y \le 2) = P(Y = 1) + P(Y = 2) = 0.4 + 0.3 = 0.7$

$F(3) = P(Y \le 3) = P(Y=1) + P(Y=2) + P(Y=3) = 0.9$

$F(4) = P(Y \le 4) = P(Y=1) + P(Y=2) + P(Y=3) + P(Y=4) = 1$

$$F(y) = \begin{cases} 0 & \text{if } y<1 \\ 0.4 & \text{if } 1 \le y < 2 \\ 0.7 & \text{if } 2 \le y < 3 \\ 0.9 & \text{if } 3 \le y < 4 \\ 1 & \text{if } y \ge 4 \end{cases}$$

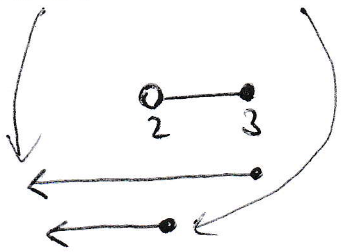


$F(1.1) = P(Y \le 1.1) = P(Y \le 1) = 0.4$

$P(2 < Y \le 3) = P(Y \le 3) - P(Y \le 2) = 0.9 - 0.7 = 0.2$

$\downarrow$

$P(Y=3) = 0.2$

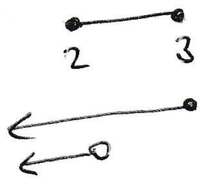


$P(2 \le Y \le 3) = P(Y \le 3) - P(Y < 2)$

$= P(Y \le 3) - P(Y \le 1)$

$= F(3) - F(1)$

$= 0.9 - 0.4 = 0.5$



for any two numbers a and b and $a<b$,

$P(a \le X \le b) = F(b) - F(a^-)$

a^- $\rightarrow$ represents the largest possible X value that is strictly less than a.

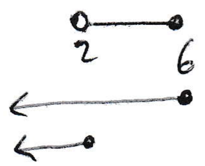
If a and b are integers

$$P(a \le X \le b) = P(X = a \text{ or } a+1 \text{ or} \dots \text{or } b)$$

$$= F(b) - F(a-1)$$

Ex: $F(x) = \begin{cases} 0 & x < 1 \\ 1/3 & 1 \le x < 4 \\ 1/2 & 4 \le x < 6 \\ 5/6 & 6 \le x < 10 \\ 1 & x \ge 10 \end{cases}$

Find $P(2 < X \le 6) = P(X \le 6) - \underbrace{P(X \le 2)}_{F(2) = F(1)} = \frac{5}{6} - \frac{1}{3} = \frac{3}{6} = \frac{1}{2}$



$P(X=3) + P(X=4) + P(X=5) + P(X=6)$

Find $P(X=4) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Find $P(X=6) = F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = 2/6$

Find $P(X=10) = F(10) - F(6) = 1 - 5/6 = 1/6$

$f(x) = \begin{cases} 1/3 & x = 1 \\ 1/6 & x = 4 \\ 2/6 & x = 6 \\ 1/6 & x = 10 \\ 0 & \text{o.w.} \end{cases}$

Homework Questions

H.W.1 $f(x) = \frac{x}{15}$, $x = 1, 2, 3, 4, 5 \Rightarrow F(x) = ?$

H.W.2 $F(x) = \begin{cases} 0 , & x < 1 \\ 0.5 , & 1 \leq x < 3 \\ 1 , & 3 \leq x \end{cases}$ Find $f(x)$ and graph $f(x)$.

H.W.3 $F(x) = \begin{cases} 0 & x < -10 \\ 0.25 & -10 \leq x < 30 \\ 0.75 & 30 \leq x < 50 \\ 1 & x \geq 50 \end{cases}$

a) $P(X \leq 50) = ?$

b) $P(X \leq 40) = ?$

c) $P(40 \leq X \leq 60) = ?$

d) $P(X < 0) = ?$

e) $P(0 \leq X < 10) = ?$

f) $P(-10 < X < 10) = ?$

g) Find $f(x)$

Some Distributions for Discrete Random Variables

Discrete Uniform Distribution :

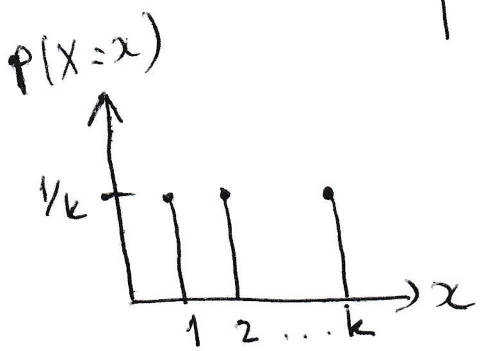
The simplest discrete r.v. is one that assumes only a finite number of possible values, each with equal probability.

A r.v. X has a discrete uniform distribution if each of the n values in its range, say $x_1, x_2, \ldots, x_n$, has equal probability.

$$f(x_i) = \frac{1}{n}$$

Suppose that the value of a r.v. X is equally likely to be any one of the k integers $1, 2, \ldots, k$.

$$P(X=x) = \begin{cases} \frac{1}{k}, & x = 1, 2, \ldots, k \\ 0, & \text{o.w.} \end{cases}$$



Ex: We have 10 balls with numbers $0, 1, \ldots, 9$ in a bag. If we pick one ball from the bag, the prob. of selecting any of them is $\frac{1}{10}$.

Bernoulli Distribution

A random experiment consists of a trial with only 2 possible outcomes.

It is assumed that the prob. of a success in trials is constant. (considering different trials of the random experiment)

If $X \sim \text{Bernoulli}(p)$

$$P(X=x) = p^x (1-p)^{1-x} \quad x = 0, 1 \quad 0 < p < 1$$

failure, success

$$= 0, \quad \text{o.w.}$$

p : prob. of success

$1-p$: " " failure

Ex: We roll a dice. What is the prob. of having the outcome '6'?

This is a Bernoulli trial with prob. of success $p = 1/6$ and prob. of failure $1-p = 5/6$

$$P(X=1) = \frac{1}{6}^1 \cdot \frac{5}{6}^{1-1} = \frac{1}{6}$$

Binomial Distribution

A random experiment consists of n Bernoulli trials such that

* Trials are independent
* Each trial results in only two possible outcomes, "success" and "failure".
* The prob. of success in each trial, p, remains constant.

The r.v. X that equals the number of trials that result in a success has a binomial r.v. with parameters $0<p<1$, and $n=1,2,\ldots$

The p.m.f of X is

$$f(x)=P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}, \quad x=0,1,\ldots,n$$

$$=0, \quad \text{o.w.}$$

or we can say that we have a random experiment repeated n times.

X: # of occurences of event A.

p: prob. of event A occurs.

$1-p$: " " " A does not occur.

Ex: We flip a fair coin 10 times. What is the prob. that at least 1 of them is Head?

$X \sim Bin(n=10, p=\frac{1}{2})$ p: prob. of having the outcome Head.

$$P(X=x)=\binom{10}{x}\frac{1}{2}^x\left(1-\frac{1}{2}\right)^{10-x}, \quad x=0,1,\ldots,10$$

$$P(X\geq 1)=1-P(X<1)=1-\underbrace{P(X=0)}_{\text{none of them is head, all are tail.}}=1-\binom{10}{0}\frac{1}{2}^{10}=1-\frac{1}{2^{10}}=0.999.$$

$\binom{n}{x} \to$ is the total number of different sequences of trials that contain x successes and $n-x$ failures.

Binomial expansion formula;

For constants a and b, the binomial expansion is;

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

When $a=p$ and $b=1-p$, we observe that sum of the prob. for a binomial r.v. is 1.

Binomial coefficients;

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{k} = \binom{n}{n-k}, \quad \binom{n}{1} = \binom{n}{n-1} = n$$

Ex: Let us flip a coin 2 times and let X is the number of times we get head.

$$X \sim Bin(n=2, p=\tfrac{1}{2})$$

$$f(x) = P(X=x) = \binom{2}{x} p^x (1-p)^{2-x}, \quad x = 0, 1, \ldots, n$$

Sample space	X	$P(X=x)$
TT	0	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad (\frac{1}{2})^2 \to P(X=0) = \binom{2}{0} \frac{1}{2}^0 \frac{1}{2}^2 = 1 \cdot (\frac{1}{2})^2$
TH	1	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ $\Big\}$ $2 \cdot (\frac{1}{2})^2 \to P(X=1) = \binom{2}{1} \frac{1}{2}^1 \cdot \frac{1}{2}^1 = 2 \cdot (\frac{1}{2})^2$
HT	1	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
HH	2	$\frac{1}{2} \cdot \frac{1}{2} = (\frac{1}{2})^2 \to P(X=2) = \binom{2}{2} \frac{1}{2}^2 \cdot \frac{1}{2}^0 = 1 \cdot (\frac{1}{2})^2$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$\left(\frac{1}{2}+\frac{1}{2}\right)^2 = \sum_{k=0}^{2} \binom{2}{k} \frac{1}{2}^k \frac{1}{2}^{2-k}$$

$$\left(\frac{1}{2}\right)^2 + 2\cdot\frac{1}{2}\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2 = 1\cdot\left(\frac{1}{2}\right)^2 + 2\cdot\left(\frac{1}{2}\right)^2 + 1\cdot\left(\frac{1}{2}\right)^2$$

Ex: When we flip the coin 3 times. X: # of heads we obtain

$$\underbrace{TTT}_{X=0} \quad \underbrace{TTH, HTT, THT}_{X=1}, \underbrace{THH, HHT, HTH}_{X=2}, \underbrace{HHH}_{X=3}$$

$$\left(\frac{1}{2}+\frac{1}{2}\right)^3 = \sum_{k=0}^{3} \binom{3}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{3-k}$$

$$(a+b)^3 = 1\cdot a^3 + 3a^2b + 3ab^2 + 1b^3 \rightarrow \binom{3}{3}\frac{1}{2}^3\cdot\frac{1}{2}^0$$

$$\binom{3}{0}\frac{1}{2}^0\frac{1}{2}^3 \quad \binom{3}{1}\frac{1}{2}^1\frac{1}{2}^2 \quad \binom{3}{2}\frac{1}{2}^2\frac{1}{2}^1$$

Poisson Distribution

Let X be the number of events occur in a period (unit) of time interval with mean λ, then

$$P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!}, & x=0,1,\ldots \\ & \lambda>0 \\ 0, & \text{o.w.} \end{cases}$$

Note: $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda} \Rightarrow \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1$

Ex: In a toyshop, in a period of 1 hour, 2 toys are sold on the average ($\lambda=2$). If the number of toys sold in the toy shop is distributed as Poisson, what is the prob. that at least 2 toys are sold in the toyshop in a period of 1 hour?

$$P(X \geq 2) = 1 - P(X<2) = 1 - P(X \leq 1) = 1 - \{P(X=0) + P(X=1)\}$$

$$= 1 - P(X=0) - P(X=1) \Rightarrow 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!}$$

$$= 1 - e^{-2} - 2e^{-2} = 0.594$$

Note: As the number of trials in a binomial experiment increases to infinity and probability of getting success everges to zero, then the mean of the dist. remains constant and binomial dist. emerges to poisson dist. with $\lambda = n.p$.