# Applied Statistics

for Computer Science BSc, Exam

# Probability Theory and Mathematical Statistics

for Computer Science Engineering BSc, Term grade

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## Main topics

1. Probability theory
2. Statistics

Mathematical tools: combinatorics, calculus

Computer tool: Matlab

Book:

Yates, Goodman:

Probability and Stochastic Processes: A Friendly Introduction for Electrical and Computer Engineers

# Lecture 7

# Expectation and variance

## Definition of the expectation

The expectation of a random variable is its mean value or its average.

In the discrete case it was

$$\mathbb{E}X = \sum\nolimits_i x_i P(X = x_i) \tag{1}$$

We can not apply it for absolutely continuous distributions because in that case

$P(X = x_i) = 0,\ \forall x_i \in \mathbb{R}$.

However, if we divide the real line into short intervals, then we obtain a similar formula

$$\mathbb{E}X \approx \sum\nolimits_i x_i P(x_i \le X < x_{i+1}) = \sum\nolimits_i x_i [F_X(x_{i+1}) - F_X(x_i)], \tag{2}$$

where $F_X$ is the CDF. If $X$ has PDF $f_X$, then (2) gives

$$\mathbb{E}X \approx \sum_i x_i \int_{x_i}^{x_{i+1}} f_X(x)dx \approx \int_{-\infty}^{+\infty} x f_X(x)dx.$$

# Definition of the expectation...

Let the random variable $X$ have PDF $f$.
If $\int_{-\infty}^{+\infty} |x| f(x) dx$ is finite, then we say that $X$ has a finite expectation.
In this case the quantity

$$\boxed{\mathbb{E}X = \int_{-\infty}^{+\infty} x f(x) dx} \tag{3}$$

exists, it is finite, and it is unique. Then the number $\mathbb{E}X$ is called the expectation of $X$.

# Definition of the expectation...

**Remark.** Both the discrete and the absolute continuous cases the expectations are particular cases of the following general notion of the expectation.
The general notion of the expectation is

$$\mathbb{E}X = \int_{\Omega} X(\omega) dP(\omega) = \int_{-\infty}^{+\infty} x \, dF_X(x). \tag{4}$$

where the first integral is a Lebesgue integral and the second one is a Lebesgue–Stieltjes integral. We do not go into the details.

# Expectation of the uniform distribution

Let $X$ be uniformly distributed on the interval $[a, b]$. Then

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \left[ \frac{x^2}{2} \right]_a^b = \frac{a+b}{2}.$$

So the expectation is the middle of the interval $[a, b]$.

# The Cauchy distribution has no expectation

Let $X$ have Cauchy distribution. Then

$$\int_0^\infty x f_X(x) dx = \int_0^\infty \frac{x}{1+x^2} dx = \left[\frac{1}{2}\ln(1+x^2)\right]_0^\infty = \infty.$$

Similarly,

$$\int_{-\infty}^0 x f_X(x) dx = -\infty$$

So

$$\int_{-\infty}^{+\infty} x f_X(x) dx$$

is not defined.

## The expectation is linear

**Theorem.**
Assume that the expectation of $X$ is finite.
Let $c \in \mathbb{R}$.
Then the expectation of $cX$ is finite and

$$\boxed{\mathbb{E}(cX) = c\mathbb{E}X}.$$

If the expectations of $X$ and $Y$ are finite,
then the expectation of $X + Y$ is also finite and

$$\boxed{\mathbb{E}(X + Y) = \mathbb{E}X + \mathbb{E}Y}.$$

# The properties of the expectation

**Theorem.**

a) If $X \geq 0$, then $\mathbb{E}X \geq 0$.

b) If $X \geq Y$, then $\mathbb{E}X \geq \mathbb{E}Y$.

c) If $X \geq 0$ and $\mathbb{E}X = 0$, then $P(X = 0) = 1$.

## Calculation of the expectation

**Theorem.**
Let $f$ be the PDF of the r.v. $X$.
Let $g$ be a Borel measurable function.
If $\int |g(x)|f(x)dx < \infty$, then

$$\boxed{\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x)f(x)dx}.$$

# Moments

Let $k \geq 0$. The $k$th moment of $X$ is

$$\mathbb{E}X^k.$$

It can be calculated as

$$\mathbb{E}X^k = \int_{-\infty}^{\infty} x^k f(x)dx,$$

where $f$ is the PDF $X$.

## Moments...

**Exercise.** Calculate of the moments of the exponential distribution.
**Solution.**

$$\mathbb{E}X^k = \int_{-\infty}^{\infty} x^k f(x)dx = \int_0^{\infty} x^k \lambda e^{-\lambda x} dx = \left[-x^k e^{-\lambda x}\right]_0^{\infty} -$$

$$-\int_0^{\infty} (-k\, x^{k-1} e^{-\lambda x})dx = \frac{k}{\lambda} \int_0^{\infty} x^{k-1} \lambda e^{-\lambda x} dx = \frac{k}{\lambda} \mathbb{E}X^{k-1}.$$

As $\mathbb{E}X^0 = \int\limits_{-\infty}^{\infty} f(x)dx = 1$,

so using the above recursion

$$\mathbb{E}X^k = k!/\lambda^k.$$

In particular

$$\mathbb{E}X = 1/\lambda.$$

## The variance

**Definition.** The variance of $X$ is

$$\boxed{\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - \mathbb{E}^2 X}\ .$$

**Proposition.**

$$\boxed{\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)}\ .$$

If $X$ and $Y$ are independent, then

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

## Exercise

Let

$$f(x) = \begin{cases} 3x^2, & \text{if} \qquad x \in [0, c], \\ 0, & \text{if} \qquad x \notin [0, c] \end{cases}$$

be the PDF of the r.v. $X$.
Find the value of $c$.
Find the corresponding CDF.
Calculate the value of $P(X > 0.5)$
Find $\mathbb{E}X$ and $\mathrm{Var}(X)$

## Exercise...

**Solution.**

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{0}^{c} 3x^2 dx = 3\left(\frac{c^3}{3} - \frac{0^3}{3}\right) = c^3$$

So $c = 1$.
Using $F(x) = \int_{-\infty}^{x} f(t)dt$, we get that the CDF of $X$ is

$$F(x) = \begin{cases} 0, & \text{if} \quad x \leq 0, \\ x^3, & \text{if} \quad 0 < x \leq 1, \\ 1, & \text{if} \quad 1 < x. \end{cases}$$

# Exercise...

$$P(X > 0.5) = 1 - F(0.5) = 1 - 0.5^3 = 1 - 0.125 = 0.875$$

$$\mathbb{E}X = \int_0^1 x \cdot 3x^2 dx = \frac{3}{4}$$

$$\mathbb{E}X^2 = \int_0^1 x^2 \cdot 3x^2 dx = \frac{3}{5}$$

$$\mathrm{Var}(X) = \mathbb{E}X^2 - \mathbb{E}^2 X = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

# The uniform distribution

Choose a point randomly from the interval $[a, b]$.
Denote by $X$ the position of the point.
We say that $X$ has uniform distribution on the interval $[a, b]$.
The CDF of $X$ is

$$F(t) = \begin{cases} 0, & \text{if} \quad t \leq a, \\ \frac{t-a}{b-a}, & \text{if} \quad a < t \leq b, \\ 1, & \text{if} \quad b < t. \end{cases}$$

# The CDF of the uniform distribution

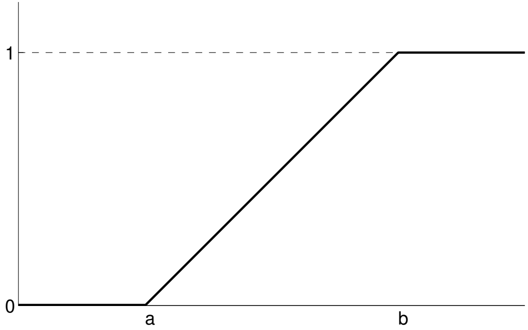


Figure: The CDF of the uniform distribution

# The PDF of the uniform distribution

$$f(t) = \begin{cases} \frac{1}{b-a}, & \text{if} \qquad t \in [a, b], \\ 0, & \text{if} \qquad t \notin [a, b]. \end{cases}$$

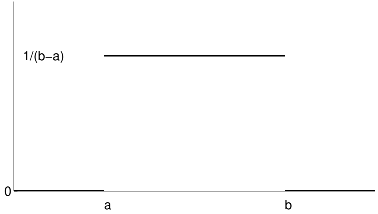


Figure: The PDF of the uniform distribution

## The expectation and the variance of the uniform distribution

The expectation is

$$\mathbb{E}X = \frac{a+b}{2}.$$

Te second moment is

$$\mathbb{E}X^2 = \int_a^b x^2/(b-a)dx = (b^3 - a^3)/(3(b-a)).$$

So the variance is

$$\mathrm{Var}(X) = \mathbb{E}X^2 - \mathbb{E}^2 X = (b^2 + ab + a^2)/3 - (a+b)^2/4 =$$

$$= (b-a)^2/12.$$

# The CDF of the exponential distribution

$$F(x) = \begin{cases} 0, & x \leq 0, \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}$$

is the CDF, where $\lambda$ is a positive parameter.

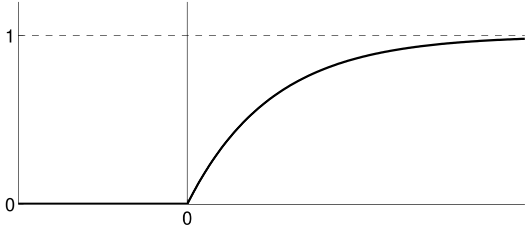


Figure: The CDF of the exponential distribution

# The PDF of the exponential distribution

$$f(x) = \begin{cases} 0, & x \leq 0, \\ \lambda e^{-\lambda x}, & x > 0 \end{cases}$$

is the PDF.

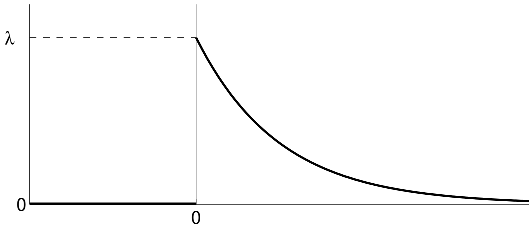


Figure: The PDF of the exponential distribution

## Properties of the exponential distribution

The $k$th moment of the exponential distribution is

$$\mathbb{E}X^k = k!/\lambda^k$$

for $k = 0, 1, 2, \ldots$.
Its expectation is

$$\mathbb{E}X = 1/\lambda.$$

Its variance is

$$\mathrm{Var}(X) = \mathbb{E}X^2 - \mathbb{E}^2 X = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

The exponential distribution is memoryless, that is

$$P(X < t + s | X \geq t) = P(X < s), \qquad t > 0, \; s > 0.$$

# The Erlang distribution

If $X_1, X_2, \ldots, X_n$ are independent and all of them have exponential distribution with parameter $\lambda$, then

$$Y_n = X_1 + X_2 + \cdots + X_n$$

has $\Gamma$ distribution with rank $n$ and parameter $\lambda$, that is its PDF is

$$f_n(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad \text{if} \quad t > 0.$$

It is also called Erlang distribution.