MIE 1613 HW1

William Hazen
MEng - MIE
1009231225
2023-02-01

1. Assume X is continuous & Uniformly distributed in [2, 10]
We are intrested in $\theta = E[(x-5)^+]$. Note $a^+ = Max(a, 0)$

a) Compute θ exactly using the definition of Expected Value. Hint $f(x) = \frac{1}{5-a}$, $x \in [a,b]$ & 0 otherwise

$x \in [2,10] \rightarrow f(x) = \frac{1}{10-2} = \frac{1}{8}$

$$\theta = E[(x-5)^+] = \int_2^{10} (x-5)^+ \cdot f(x)dx$$

$$(x-5)^+ = \begin{cases} 0 \text{ if } 2 \le x \le 5 \\ a^+ \text{ if } 5 \le x \le 10 \end{cases}$$

$$\theta = \int_2^{10} (x-5)^+ \cdot f(x)\,dx = \int_2^{10} (x-5)^+ \cdot \frac{1}{8}\,dx$$

$$= \frac{1}{8}\left[\int_2^5 (x-5)^+ dx + \int_5^{10} (x-5)^+ dx\right]$$

$$= \frac{1}{8}\left[\int_2^5 0\,dx + \int_5^{10} (x-5)^+ dx\right]$$

$$= \frac{1}{8}\left[0 + \int_5^{10} (x-5)^+ dx\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}x^2\Big|_5^{10} + 5x\Big|_5^{10}\right]$$

$$= \frac{1}{8}\left[\left(\frac{75}{2}\right) - (25)\right] = \frac{25}{16} \approx 1.5625$$

$\theta = \frac{25}{16}$ //

4. What is the standard error of the estimator $\bar{X}_n$ where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ & X_i's are i.i.d

$E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n} \cdot n\, E[X]$

↳ linearity of expectation property

$E[\bar{X}_n] = E[X]$

$Var(\bar{X}_n) = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2} Var\left[\sum_{i=1}^{n} X_i\right]$

↳ constant multiplication property of variance

$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$

↳ sum of the variance of i.i.d variables is equal to sum of each var

$= \frac{1}{n^2} \cdot n\, Var(X)$

↳ since X_i's are iid

$Var(\bar{X}_n) = \frac{1}{n} Var(X)$

let $\sigma^2 = Var(X)$ ⇒ $Var(\bar{X}_n) = \frac{\sigma^2}{n}$

Thus the standard error (SE) of the sample mean is

$$SE = \sqrt{\frac{Var(X)}{n}} = \frac{\sqrt{\sigma^2}}{\sqrt{n}} = \frac{\sigma}{\sqrt{n}}$$

$$SE = \frac{\sigma}{\sqrt{n}} //$$

5. Prove the following

Let $\mu_x = E[X]$ & $\sigma^2 = Var(X)$

a) $Var(aX+b) = a^2 Var(X)$

$Var(ax+b) = E\left[\left((aX+b) - E[aX+b]\right)^2\right]$

↳ Property of Variance (i)
$V(X) = E[(X-\mu_x)^2]$

$= E\left[\left((aX+b) - aE[X] - b\right)^2\right]$

↳ (ii) expected value of a constant b is equal to b

↳ (iii) linearity of expectation

$= E\left[(aX - a\mu_x)^2\right]$

$= E\left[a^2 (X-\mu_x)^2\right]$

$= a^2 E\left[(X-\mu_x)^2\right]$ ← (iii)

$= a^2 Var(X)$ ← (i)

∴ Thus $Var(aX+b) = a^2 Var(X)$ //

b) $Var(X+Y) = Var(X) + Var(Y) = 2Cov(X,Y)$

Let $\mu_X = E[X]$, $\mu_Y = E[Y]$, $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$Var(X+Y) = E[((X+Y) - E(X+Y))^2]$ ← (i)

$= E[(X+Y-\mu_X-\mu_Y)^2]$

$= E[((X-\mu_X)+(Y-\mu_Y))^2]$

$= E[(X-\mu_X)^2 + (Y-\mu_Y)^2 + 2(X-\mu_X)(Y-\mu_Y)]$

$= E[(X-\mu_X)^2] + E[(Y-\mu_Y)^2] + 2E[(X-\mu_X)(Y-\mu_Y)]$

From (i) $\Rightarrow Var(X) + Var(Y) + 2Cov(X,Y)$

$\therefore$ Thus, $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$ //