Stochastic Simulation (MIE1613H) - Homework 2

Due: February 22, 2023

- Submit your homework on Quercus as a single PDF file by the deadline. Late submissions are penalized 10% each day the homework is late.
- At the top of your homework include your **name, student number, department, and program**.
- You may discuss the assignment with other students in general terms, but each student must solve the problems, write the code and the solutions individually.
- The simulation models must be programmed in Python. You must include both the source code (including comments to make it easy to follow) and the output of the simulation in your submission.
- **It is very important to explain your answers and solution approach clearly and not just provide the code/results**. Full marks will only be given to correct solutions that are fully and clearly explained.

Problem 1. (20 Pts.) Consider the $AR(1)$ model described in Section 3.3. of the textbook. Derive an expression for the asymptotic MSE of the sample mean $\bar{Y}(m)$. **HINT**: Start with the definition of asymptotic bias from Eq. (5.4) and asymptotic variance from Eq. (5.7).

Problem 2. (15 Pts.) Consider the $M(t)/M/\infty$ model of the parking lot example from the lecture.

Using simulation, you have generated n iid sample paths of the process $\{N(t); 0 \leq t \leq T = 24\}$, i.e., the number of cars in the parking lot at time t during a 24-h period, and starting with 0 cars in the parking lot, i.e., $N(0) = 0$. Denote the simulated sample paths by $N_1(t), N_2(t), ..., N_n(t)$.

For each of the performance measures of interest below, propose an unbiased (if possible) estimator using the simulation output.

(a) Expected average number of cars in the parking lot during a day

(b) Expected maximum number of cars in the parking lot during a day

(c) Probability of having more than 5 cars left in the parking lot at the end of the day

(d) 0.95th quantile of the maximum number of cars in the parking lot during the day

Problem 3. (15 Pts.) (Down-and-out call option) Another variation of European options are barrier options. Denote the stock price at time t by $X(t)$ and assume that it is modelled as a Geometric Brownian Motion (GBM). A down-and-out call option with barrier B and strike price K has payoff

$$I\left\{\min_{0\leq t\leq T} X(t) > B\right\}(X(T) - K)^+,$$

where $I\{A\}$ is the indicator function of event A. This means that if the value of the asset falls below \$$B$ before the option matures then the option is worthless. Hence, the value of the option is

$$E\left[e^{-rT} I\left\{\min_{0\le t\le T} X(t) > B\right\}(X(T)-K)^+\right].$$

Using the same parameters as in the Asian option example, i.e., $T = 1$, $X(0) = \$50$; $K = \$55$; $r = 0.05$ and $\sigma^2 = (0.3)^2$, estimate the value of this option for barriers $B = 35, 40, 45$ and provide an intuitive reason for the effect of increasing the barrier on the value of the option. Use $m = 64$ steps when discretizing the GBM and use $n = 40,000$ replications. Report a 95% confidence interval for your estimates.

Problem 4. (15 Pts.) (Chapter 4, Exercise 4) Beginning with the PythonSim event-based $M/G/1$ simulation, implement the changes necessary to make it an $M/G/s$ simulation (a single queue with s servers). Keep the arrival rate at $\lambda = 1$ and use average service time $\tau = 0.8 \times s$, and simulate the system for $s = 1, 2, 3$.

(a) Report the estimated expected number of customers in the system (including customers in the queue and service), expected system time, and expected number of busy servers in each case.

(b) Compare the results and state clearly what you observe. What you're doing is comparing queues with the same service capacity, but with 1 fast server as compared to 2 or more slow servers.

HINT: You need to modify the logic, not just set the number of available servers to s. The attribute "NumberOfUnits" of the Resource object returns the number of available units for any instance of the object.

Problem 5. (15 Pts.) (Chapter 4, Exercise 5) Modify the PythonSim event-based simulation of the $M/G/1$ queue to simulate a $M/G/1/c$ retrial queue. This means that customers who arrive to find $c < \infty$ customers in the system (including the customer in service) leave immediately, but arrive again after an exponentially distributed amount of time with mean MeanTR. Do we need the arrival rate λ to be smaller than service rate $1/\tau$ for the system to reach steady-state? Explain your answer using numerical evidence from the simulation model.

HINT: The existence of retrial customers should not affect the arrival process for first-time arrivals.

Problem 6. (20 Pts.) A long-term care home in Toronto is planning to test all of its 170 staff (nurses and caregivers) before they start their shift using a new COVID-19 Rapid Test. Consider the 7AM shift and assume that staff arrive between 6:30AM and 7AM for their shift with arrival times uniformly distributed during the 30-minute period prior to the shift. Test times take on average 15 minutes and can be modelled using an Erlang distribution with 4 phases.

(a) Provide a 95% Confidence Interval for the expected average waiting time (excluding the test time) of staff when there are 10 parallel servers (testing stations) available. Assume that staff wait in a single First-Come, First-Serverd queue before being tested. Use 100 replications.

(b) Approximately how many servers are required to keep the average waiting time below 15 minutes?

HINT: You may want to schedule the arrival events for all 170 staff to the calendar at the beginning of each replication.