MIE 1613 A2

William Hazen
1009231225
02-22-2023
MIE MEng.

1. Consider the AR(1) Model in Section 3.3. Derive an expression for the asymptotic MSE of the sample mean $\bar{Y}(m)$

AR(1) Model for the ith simulation output (within a single replication

$$Y_i = \mu + \varphi(Y_{i-1} - \mu) + X_i \quad , \quad i = 1, 2 \ldots \qquad (3.7)$$

- X_i are i.i.d RV with mean 0 & σ^2 variance
- $|\varphi| < 1$
- μ is constant
- Y_0 distribution is given & independent of X's

Properties of AR(1):

$$E(Y_i) = \mu + \varphi^i(E(Y_0) - \mu) \xrightarrow{i \to \infty} \mu \qquad (3.9)$$

$$Var(Y_i) = \varphi^{2i} var(Y_0) + \sigma^2 \sum_{j=0}^{i-1} \varphi^{2j} \xrightarrow{i \to \infty} \frac{\sigma^2}{1-\varphi^2} \qquad (3.10)$$

$$Corr(Y_i, Y_{i+j}) \xrightarrow{i \to \infty} \varphi^j \qquad (3.10.1)$$

Asymptotic bias:

$$\beta = \lim_{m \to \infty} m \cdot (E(\bar{Y}(m)) - \mu)$$

$$= \lim_{m \to \infty} m \cdot \left(E\left(\frac{1}{m} \sum_{i=1}^{m} Y_i\right) - \mu\right)$$

$$= \lim_{m \to \infty} m \left(\frac{1}{m} E\left(\sum_{i=1}^{m} Y_i\right) - \mu\right)$$

$$= \lim_{m \to \infty} E\left(\sum_{i=1}^{m} Y_i - m\mu\right)$$

constant

$$= \lim_{m \to \infty} \sum_{i=1}^{m} (E(Y_i) - \mu)$$

$$\beta = \sum_{i=1}^{\infty} (E(Y_i) - \mu)$$

Sub (3.9) into $E(Y_i)$

$$\beta = \sum_{i=1}^{\infty} (\cancel{\mu} + \varphi^i(E(Y_0) - \mu) - \cancel{\mu})$$

Thus $\beta = \sum_{i=1}^{\infty} \varphi^i \left[E(Y_0) - \mu \right]$ //

$$Var(\bar{Y}(m)) = \frac{\sigma^2}{m}\left(1 + 2\sum_{k=1}^{m-1}\left(1 - \frac{k}{m}\right)\rho_k\right) \quad (5.6)$$

Asymptotic Variance

$$\gamma^2 = \lim_{m \to \infty} m\,Var(\bar{Y}(m))$$

$$= \lim_{m \to \infty} \sigma^2\left(1 + 2\sum_{k=1}^{m-1}\left(1 - \frac{k}{m}\right)\rho_k\right)$$

$$\gamma^2 = \sigma^2\left(1 + 2\sum_{k=1}^{\infty}\rho_k\right) \quad (5.7)$$

When $m \to \infty$, $\sigma^2 = Var(Y_m)$, $\rho_k = Corr(Y_m, Y_{m+k})$ are no longer functions of m

When $\gamma^2 \exists \Rightarrow Var(\bar{Y}(m)) \approx \frac{\gamma^2}{m}$ for large m

Thus the $MSE(\bar{Y}(m)) = E\left[(\bar{Y}(m) - \mu)^2\right] = Bias^2(\bar{Y}(m)) + Var(\bar{Y}(m))$

$$MSE(\bar{Y}(m)) \approx \frac{\beta^2}{m^2} + \frac{\gamma^2}{m} \quad (5.8)$$ //

To simplify (5.8) we have

$$\beta = \sum_{i=1}^{\infty} \varphi^i \left[E(Y_0) - \mu \right] \Rightarrow |\varphi| < 1 \ \& \ \frac{E[Y_0] - \mu}{1 - \varphi}$$

$$\gamma^2 = \sigma^2\left(1 + 2\sum_{k=1}^{\infty}\rho_k\right)$$

for large m : $\sigma^2 = \frac{\sigma^2}{1 - \varphi^2}$ ← 3.10 & $\rho_k = Corr(Y_m, Y_{m+k})$

thus $\gamma^2 = \frac{\sigma^2}{1 - \varphi^2}\left(1 + 2\sum_{k=1}^{\infty} Corr(Y_m, Y_{m+k})\right)$

now applying (3.10.1) $\text{Corr}(Y_m, Y_{m+k}) \xrightarrow{m \to \infty} \varphi^k$

therefore $\gamma^2 = \frac{\sigma^2}{1-\varphi^2}\left(1 + 2\underbrace{\sum_{k=1}^{\infty} \varphi^k}_{|\varphi|<1 \to \frac{1}{1-\varphi}}\right)$

Hence, $MSE(\bar{Y}(m)) = \frac{\beta^2}{m^2} + \frac{\gamma^2}{m}$

$$= \frac{\left(\frac{E[Y_0]-\mu}{1-\varphi}\right)^2}{m^2} + \frac{\frac{\sigma^2}{1-\varphi^2}\left(1+2\left(\frac{1}{1-\varphi}\right)\right)}{m}$$

//

2. $M(t)/M/\infty$. n iid sample paths of the process $\{N(t);\ 0 \le t \le T = 24\}$, $N(0)=0$, simulated sample paths by $N_1(t), N_2(t), \ldots, N_n(t)$.

a) Expected average number of cars in a day.

$$\frac{1}{n}\sum_{i=1}^{n} \overline{N_i(t)}$$, where $\overline{N_i(t)}$ is the average number of cars in a sample path

$$\overline{N_i(t)} = E\left[\sum_{i=1}^{n} N_i(t)\right] = \frac{1}{24-0}\int_0^{24} N_i(t)\,dt \rightarrow n\,E[N(t)]$$

$N_i(t)$: iid RV

thus,

$$\frac{1}{n}\sum_{i=1}^{n} \overline{N_i(t)} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{24}\int_0^{24} N_i(t)\,dt$$

Expected average number of cars in a day is,

$$\frac{1}{n}\cdot\frac{1}{24}\sum_{i=1}^{n}\int_0^{24} N(t)\,dt$$ //

b) Expected max number of cars in a day.

$$\frac{1}{n}\sum_{i=1}^{n} \max(N_i(t))$$, where $t \in [0, 24]$ //

c) Probability of having more than 5 cars left at the end of the day

$$P\{N_i(24) > 5\} = \frac{1}{n}\sum_{i=1}^{n} I\{N_i(24) > 5\}$$

I is an indicator function

thus by taking the sample average where $I\{N_i(24) > 5\} = 1$ and $I\{N_i(24) \le 5\} = 0$ //

d) 0.95th quantile of the Max number of cars in the Parking lot during the day

$SN_I = \text{Sort}\left(\max\left(N_i(t)\right)\right)$ $i \in [0, n]$ & $t \in [0, 24]$

$I = \lceil 0.95 \cdot n \rceil$ Ceil

thus $SN_I \rightarrow$ 0.95th quantile of max number of cars