MIE 1613 HW1

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MEng - MIE
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1. Assume X is continuous & uniformly distributed in $[2, 10]$. We are intrested in $\theta = E[(X-5)^+]$. Note $a^+ = Max(a, 0)$

a) Compute θ exactly using the definition of expected value. Hint $f(x) = \frac{1}{b-a}$, $x \in [a,b]$ & 0 otherwise

$$x \in [2, 10] \rightarrow f(x) = \frac{1}{10-2} = \frac{1}{8}$$

$$\theta = E[(x-5)^+] = \int_2^{10} (x-5)^+ \cdot f(x) dx$$

$$(x-5)^+ = \begin{cases} 0 & \text{if } 2 \leq x \leq 5 \\ a^+ & \text{if } 5 \leq x \leq 10 \end{cases}$$

$$\theta = \int_2^{10} (x-5)^+ \cdot f(x)\, dx = \int_2^{10} (x-5)^+ \cdot \frac{1}{8} dx$$

$$= \frac{1}{8}\left[\int_2^5 (x-5)^+ dx + \int_5^{10} (x-5)^+ dx\right]$$

$$= \frac{1}{8}\left[\int_2^5 0\, dx + \int_5^{10} (x-5)^+ dx\right]$$

$$= \frac{1}{8}\left[0 + \int_5^{10} (x-5)^+ dx\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}x^2\Big|_5^{10} + 5x\Big|_5^{10}\right]$$

$$= \frac{1}{8}\left[\left(\frac{75}{2}\right) - (25)\right] = \frac{25}{16} \approx 1.5625$$

$$\theta = \frac{25}{16}$$ //

4. What is the standard error of the estimator $\bar{X}_n$ where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ & X_i's are i.i.d

$E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{\cancel{n}}\cdot \cancel{n}\,E[X]$

↳ linearity of expectation property

$E[\bar{X}_n] = E[X]$

$Var(\bar{X}_n) = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2} Var\left[\sum_{i=1}^{n} X_i\right]$

↳ Constant Multiplication Property of Variance

$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$ → Sum of the Variance of i.i.d variables is equal to sum of each var

$= \frac{1}{n^{\cancel{2}}}\cdot \cancel{n}\, Var(X)$

↳ Since X_i's are i.i.d

$Var(\bar{X}_n) = \frac{1}{n^2} Var(X)$

let $\sigma^2 = Var(X)$

Thus the standard error (SE) of the Sample Mean is

$$SE = \sqrt{\frac{Var(X)}{n}} = \frac{\sqrt{Var(X)}}{\sqrt{n}} = \frac{\sigma}{\sqrt{n}}$$

$$SE = \frac{\sigma}{\sqrt{n}} \quad //$$

5. Prove the following

Let $\mu_x = E[x]$ & $\sigma^2 = Var(x)$

a) $Var(aX+b) = a^2 Var(x)$

$Var(ax+b) = E[((aX+b) - E[aX+b])^2]$

↳ Property of Variance (i)

$V(x) = E[(x-\mu_x)^2]$

$= E[((aX+\cancel{b}) - aE[X] - \cancel{b})^2]$

↳ (ii) expected value of a constant b is equal to b

↳ (iii) linearity of expectation

$= E[(aX - a\mu_x)^2]$

$= E[a^2(x-\mu_x)^2]$

$= a^2 E[(x-\mu_x)^2]$ ← (iii)

$= a^2 Var(X)$ ← (i)

$\therefore$ Thus $Var(aX+b) = a^2 Var(X)$ //

b) $Var(X+Y) = Var(X) + Var(Y) = 2Cov(X,Y)$

Let $\mu_X = E[X]$, $\mu_Y = E[Y]$, $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$$Var(X+Y) = E\left[\left((X+Y) - E(X+Y)\right)^2\right] \quad (i)$$

$$= E\left[(X+Y-\mu_X-\mu_Y)^2\right]$$

$$= E\left[\left((X-\mu_X)+(Y-\mu_Y)\right)^2\right]$$

$$= E\left[(X-\mu_X)^2 + (Y-\mu_Y)^2 + 2(X-\mu_X)(Y-\mu_Y)\right]$$

$$= E\left[(X-\mu_X)^2\right] + E\left[(Y-\mu_Y)^2\right] + 2E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

From (i) $\Rightarrow Var(X) + Var(Y) + 2Cov(X,Y)$

$\therefore$ Thus, $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$ //

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- Program: M.Eng.

In [5]:
```
import numpy as np
import matplotlib.pyplot as plt

# fix random number seed
np.random.seed(1)
```

1. ¶

1b)

In [6]:
```
# X is cont. U[2,10]
# theta = E[(X-5)^+]
# a^+ = Max(a,0) -> a^+ = 0 if a < 0
#                -> a^+ = a if a >= 0

ab_value_list = []  # initalize list
a = 2
b = 10
n = 100000  # 100k samples

for i in range(n):
    random_value = np.random.random()  # create random float between 0.0 & 1.0
    # scale by difference between a & b and shift by a
    ab_value = (b-a) * random_value + a
    # output is uniformed list of values between 2 & 10
    ab_value_list.append(ab_value)

theta = []
for x in ab_value_list:
    if x < 5:   # if x is less than 5 then a^+ = 0
        theta.append(0)
    else:       # if x >= 5 then a^+ = a
        theta.append((x-5))
```

In [7]:
```
average_theta_list = []
n = 0
for value in theta:  # loop through the values in theta
    n += 1
    if n == 1:
        average_theta_list.append(value)  # append the first value of theta
    else:
        average_theta_list.append(
            (1/n) * ((n-1)*average_theta_list[-1] + value))
        # running average = adding the new number to the old average and divid
ing by the total number of samples

theta_mean = np.mean(theta)  # mu of theta list
theta_std = np.std(theta)  # std of theta list


print("Average theta =", theta_mean)
# 95% confidence interval: alpha = mu +- 1.96 std/sqrt(n)
print('95% Confidence interval theta:', np.mean(theta),
      "+/-", 1.96*np.std(theta, ddof=1)/np.sqrt(n))

# 95% confidence interval: alpha = mu +- 1.96 std/sqrt(n)
alpha_neg = theta_mean - (1.96*theta_std)/np.sqrt(n)
alpha_plus = theta_mean + (1.96*theta_std)/np.sqrt(n)

print(" negative bound =", alpha_neg)
print(" Positive bound =", alpha_plus)
```

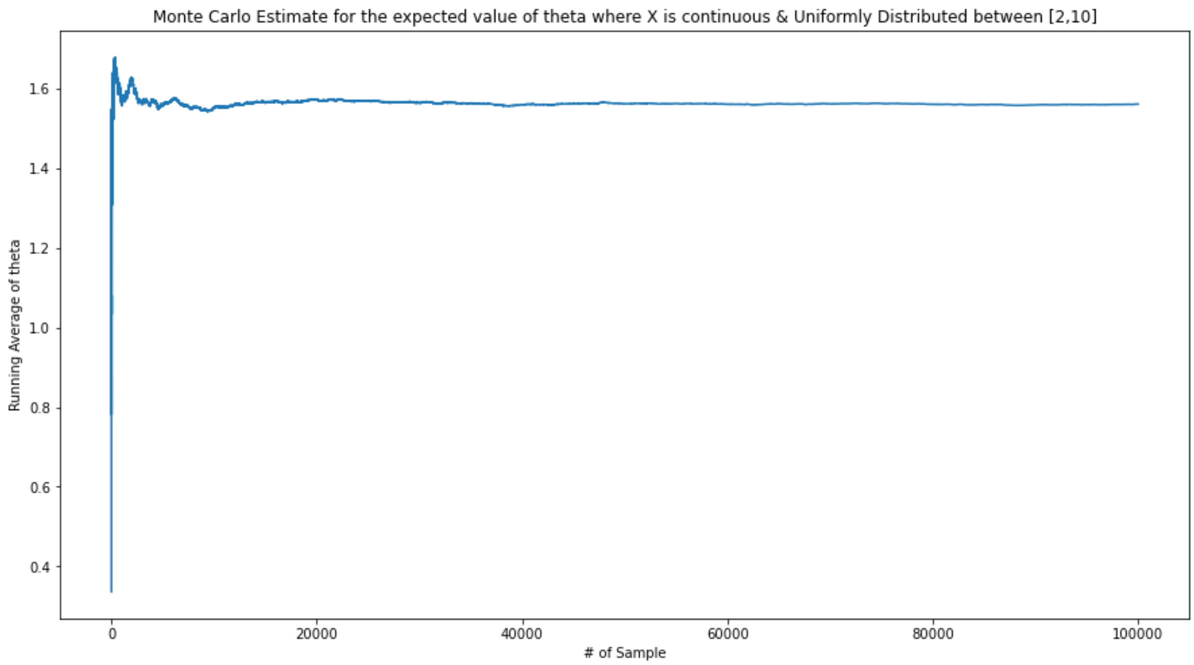
```
Average theta = 1.5613895874744266
95% Confidence interval theta: 1.5613895874744266 +/- 0.010318273591054833
 negative bound = 1.5510713654748687
 Positive bound = 1.5717078094739845
```

- 1a) theta = 1.5625

1c)

In [8]:
```
plt.figure(figsize=(15, 8))
plt.plot(average_theta_list)
plt.title(
    "Monte Carlo Estimate for the expected value of theta where X is continuou
s & Uniformly Distributed between [2,10]")
plt.xlabel("# of Sample")
plt.ylabel("Running Average of theta")
```

Out[8]:
```
Text(0, 0.5, 'Running Average of theta')
```



2.

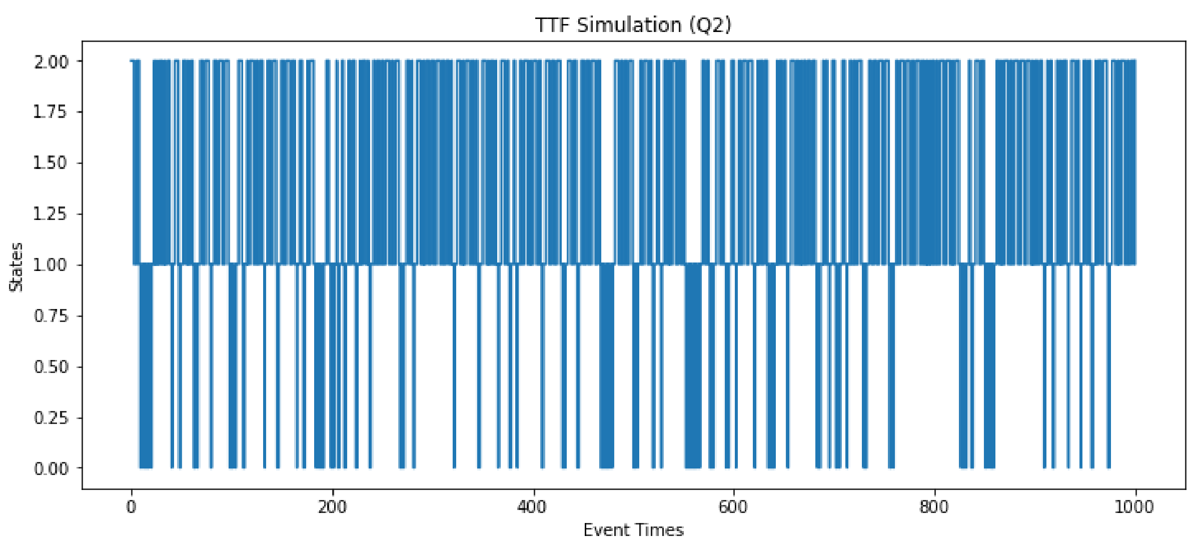
In [99]:
```
def Q2_TTF(T):
    # pesudo random
    np.random.seed(1)

    # start with 2 functioning components at time 0
    clock = 0
    S = 2
    # initialize the time of events
    NextRepair = float('inf')
    NextFailure = np.ceil(6*np.random.random())
    # lists to keep the event times and the states
    EventTimes = [0]
    States = [S]

    A_t = {}

    '''
    Logic: While the runtime is less than the value of T, clock is stored as t
he min value of either the repair or failure time.
        If the repair value is less than the failure, we initalize the NextRep
air to inf, however if S = 1 then we reset and schedule the next repair and fa
ilure time.
        Similarly if the repair value is greater than the failure, we have the
same condition but initalize NextFailure to inf.
    '''

    while clock < T:  # changed while loop from S>0 to clock<T
        # advance the time till next event

        clock = min(NextRepair, NextFailure, T)  # Stops when clock = T

        if NextRepair < NextFailure:  # next event is completion of a repair
            S = S + 1
            if S == 2:
                # Can only be 1 or 2 thus NextRepair = inf if S!=1
                NextRepair = float('inf')
            elif S == 1:  # When S = 1, schedule next repair and failure
                NextRepair = clock + 2.5
                NextFailure = clock + np.ceil(6*np.random.random())

        else:  # next event is a failure
            S = S - 1
            if S == 0:
                # Can only be 1 or 0 thus NextFailure = inf if S!=1
                NextFailure = float('inf')
            elif S == 1:
                NextRepair = clock + 2.5
                NextFailure = clock + np.ceil(6*np.random.random())

        # save the time and state
        EventTimes.append(clock)
        States.append(S)
        last_clock = clock

        # Stores value of 1 when State = 1 at time t
        for idx, state in enumerate(States):
```

```
            if state == 2:
                A_t[EventTimes[idx]] = 1
            else:
                A_t[EventTimes[idx]] = 0

    # plot the sample path
    print('T =', int(T))
    print('Clock =', last_clock)
    print('Event Time length =', len(EventTimes))
    print('A_t =', sum(A_t.values())/len(EventTimes))
    plt.figure(figsize=(12, 5))
    plt.plot(EventTimes, States, drawstyle='steps-post')
    plt.title("TTF Simulation (Q2)")
    plt.xlabel("Event Times")
    plt.ylabel("States")
    plt.show()
```

2a)

In [100]:
```
Q2_TTF(1000)
```

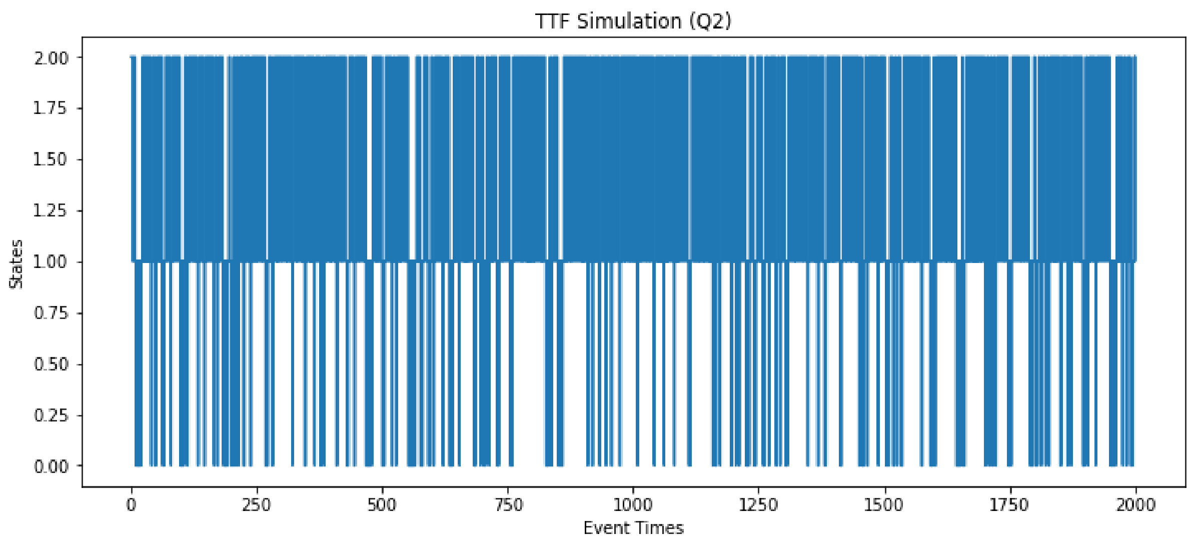
```
T = 1000
Clock = 1000
Event Time length = 513
A_t = 0.32748538011695905
```



2b)

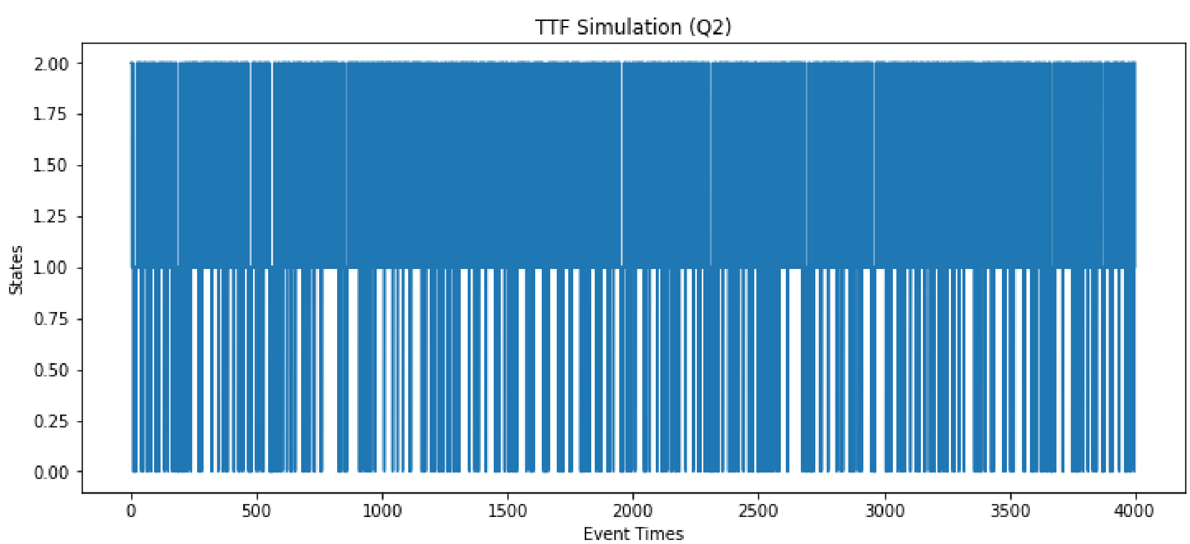
In [101]:
```
Q2_TTF(2000)
```

```
T = 2000
Clock = 2000
Event Time length = 1031
A_t = 0.3326867119301649
```



In [102]:
```
Q2_TTF(4000)
```

```
T = 4000
Clock = 4000.0
Event Time length = 2083
A_t = 0.33461353816610656
```



- T=1k: 0.32748538011695905
- T=3K: 0.3326867119301649
- T=4k: 0.33461353816610656

Observations

- The fraction of the system being fully functional increases with time.

3.

In [117]:
```
def Q3_TTF(N):
    # Set number of replications
    Rep = 1000
    # Define lists to keep samples of the outputs across replications
    TTF_list = []

    # fix random number seed
    np.random.seed(1)

    for rep in range(0, Rep):
        clock = 0
        S = N
        Spare = N - 1  # number of spare components

        # initialize the time of events
        NextRepair = float('inf')
        NextFailure = np.ceil(6*np.random.random())
        EventTimes = [0]
        States = [S]

        '''
        Logic: For each replication, while S > 0, if the repair value is less
than the failure, if S = N then there is no need to schedule a repair.
            However if not then the NextRepair will be scheduled if there is a
spare avaliable.
            If NextRepair >= NextFailure then the next failure is scheduled in
itally unless S = 0.
            If S!=0 then we schedule the next repair [The reason why the NextF
ailure is outside is because scheduling nextfailure does not depend on the spa
re components]
            If the repair value is greater than the failure, we have the same
condition but initalize NextFailure to inf.
        '''

        while S > 0:
            # advance the time till next event
            clock = min(NextRepair, NextFailure)

            if NextRepair < NextFailure:  # next event is completion of a repa
ir
                S = S + 1
                Spare = Spare + 1 # Spare components increases with S

                if Spare == N: # Max spare components = N-1
                    Spare = N-1

                if S == N: # Max components -> no repairs
                    NextRepair = float('inf')

                else:
                    if Spare > 0:  # If there are spare components available
                        Spare = Spare
                        NextRepair = clock + 2.5

            else:  # next event is a failure
                S = S - 1
```

```
                NextFailure = clock + np.ceil(6*np.random.random())

                if S == 0:
                    NextFailure = float('inf')
                else:
                    if Spare > 0: # check if there are spare components availa
ble
                        Spare = Spare - 1
                        if S == (N-1):
                            NextRepair = clock + 2.5

            # save the time and state
            EventTimes.append(clock)
            States.append(S)

        # save the TTF and average # of func. components
        TTF_list.append(clock)

        #From Q1
        average_TTF_list = []
        n = 0
        for value in TTF_list:
            n += 1
            if n == 1:
                average_TTF_list.append(value)
            else:
                average_TTF_list.append((1/n) * ((n-1)*average_TTF_list[-1] +
value))

    print("For N = ", N)
    print('TTF List Length =', len(TTF_list))
    print('Estimated expected TTF:', np.mean(TTF_list))
    print('95% CI for TTF:', np.mean(TTF_list), "+/-", 1.96*np.std(TTF_list, d
dof = 1)/np.sqrt(Rep))
```

In [118]:
```
Q3_TTF(2)
```

```
For N =  2
TTF List Length = 1000
Estimated expected TTF: 14.193
95% CI for TTF: 14.193 +/- 0.7212458114694061
```

In [119]:
```
Q3_TTF(3)
```

```
For N =  3
TTF List Length = 1000
Estimated expected TTF: 87.648
95% CI for TTF: 87.648 +/- 5.523633399450503
```

In [121]:
```
Q3_TTF(4)
```

```
For N =  4
TTF List Length = 1000
Estimated expected TTF: 648.805
95% CI for TTF: 648.805 +/- 42.26686864295534
```

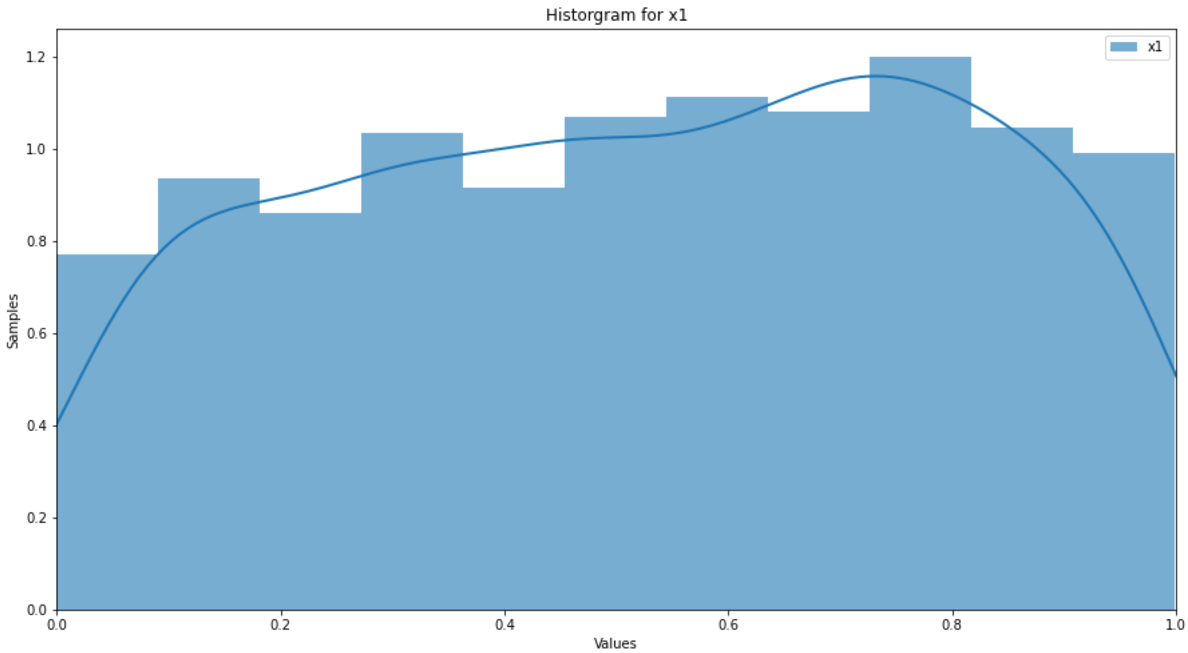
6.

In [124]:
```
# X_n = 1/n sum(X_i)
def X_n(n):  # create a X_n function
    sum_xn = 0
    for i in range(n):
        # Random uniformed values between 0 & 1
        X = np.random.uniform(0, 1, 1000)
        sum_xn += X  # sum
    X_n = (1/n)*sum_xn
    return X_n


x1 = X_n(1)
x2 = X_n(2)
x30 = X_n(30)
x500 = X_n(500)

x_s = {"x1": x1, "x2": x2, "x30": x30, "x500": x500}

for x in (x_s):
    print("Length of " + x + ":", len(x_s[x]))
```

```
Length of x1: 1000
Length of x2: 1000
Length of x30: 1000
Length of x500: 1000
```

In [130]:
```
import seaborn as sns
kwargs = dict(hist_kws={'alpha': .6}, kde_kws={'linewidth': 2})

for key, value in x_s.items():
    plt.figure(figsize=(15, 8))
    sns.distplot(value, **kwargs, label=key)
    plt.xlim([0, 1])
    plt.title(f"Historgram for {key}")
    plt.xlabel('Values')
    plt.ylabel('Samples')
    plt.legend()
    plt.show()
```

```
c:\Users\William Hazen\anaconda3\envs\venv\lib\site-packages\seaborn\distribu
tions.py:2619: FutureWarning: `distplot` is a deprecated function and will be
removed in a future version. Please adapt your code to use either `displot`
(a figure-level function with similar flexibility) or `histplot` (an axes-lev
el function for histograms).
  warnings.warn(msg, FutureWarning)
```




```
c:\Users\William Hazen\anaconda3\envs\venv\lib\site-packages\seaborn\distribu
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removed in a future version. Please adapt your code to use either `displot`
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  warnings.warn(msg, FutureWarning)
```
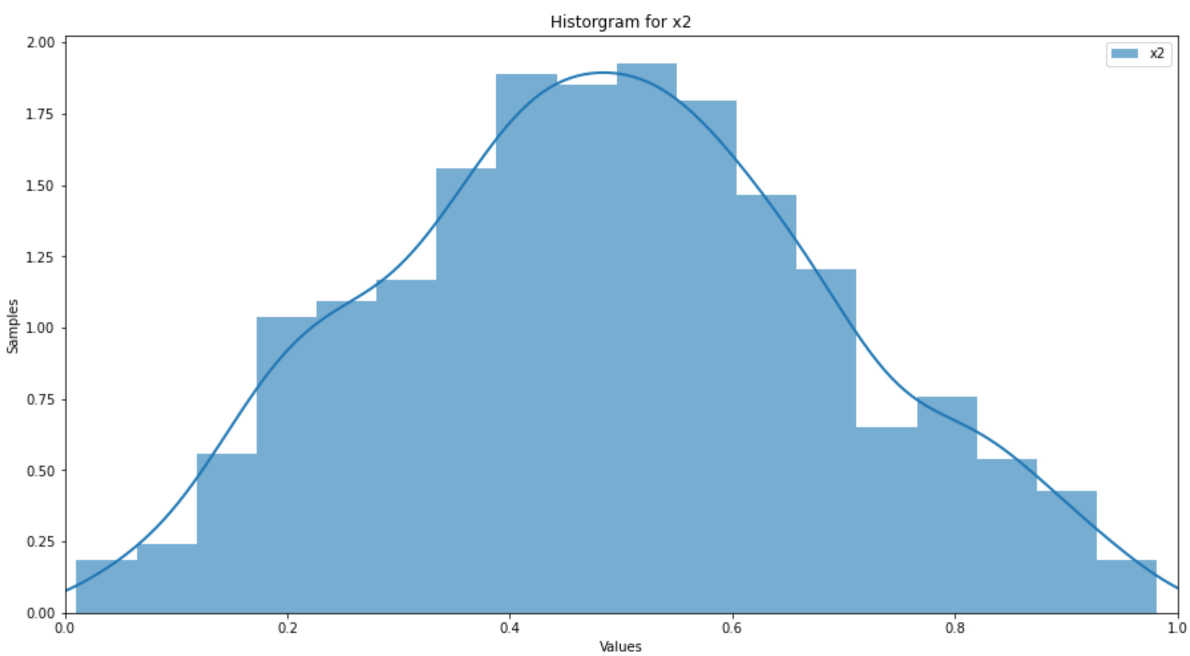
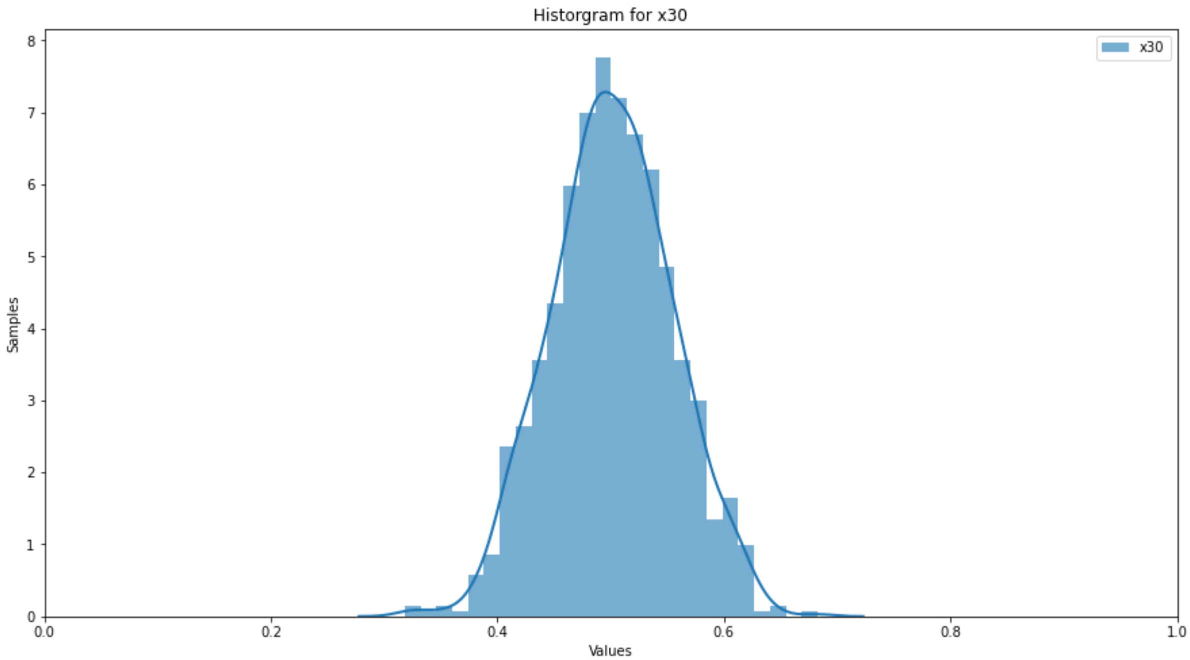
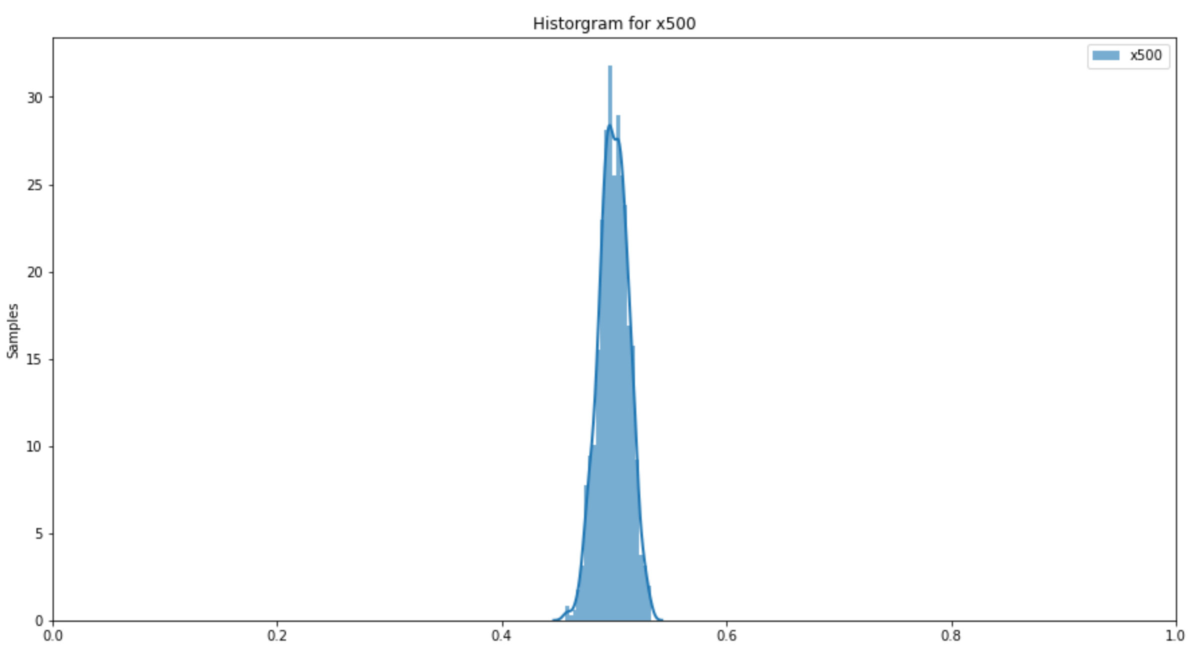



```
c:\Users\William Hazen\anaconda3\envs\venv\lib\site-packages\seaborn\distributions.py:2619: FutureWarning: `distplot` is a deprecated function and will be removed in a future version. Please adapt your code to use either `displot` (a figure-level function with similar flexibility) or `histplot` (an axes-level function for histograms).
  warnings.warn(msg, FutureWarning)
```




```
c:\Users\William Hazen\anaconda3\envs\venv\lib\site-packages\seaborn\distributions.py:2619: FutureWarning: `distplot` is a deprecated function and will be removed in a future version. Please adapt your code to use either `displot` (a figure-level function with similar flexibility) or `histplot` (an axes-level function for histograms).
  warnings.warn(msg, FutureWarning)
```



In [126]:
```
plt.figure(figsize=(15, 8))
for key, value in x_s.items():
    sns.distplot(value, **kwargs)
plt.title("Historgram for All")
plt.xlabel('Values')
plt.ylabel('Samples')
plt.legend()
plt.show()
```

```
c:\Users\William Hazen\anaconda3\envs\venv\lib\site-packages\seaborn\distribu
tions.py:2619: FutureWarning: `distplot` is a deprecated function and will be
removed in a future version. Please adapt your code to use either `displot`
(a figure-level function with similar flexibility) or `histplot` (an axes-lev
el function for histograms).
  warnings.warn(msg, FutureWarning)
c:\Users\William Hazen\anaconda3\envs\venv\lib\site-packages\seaborn\distribu
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  warnings.warn(msg, FutureWarning)
c:\Users\William Hazen\anaconda3\envs\venv\lib\site-packages\seaborn\distribu
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removed in a future version. Please adapt your code to use either `displot`
(a figure-level function with similar flexibility) or `histplot` (an axes-lev
el function for histograms).
  warnings.warn(msg, FutureWarning)
No handles with labels found to put in legend.
```

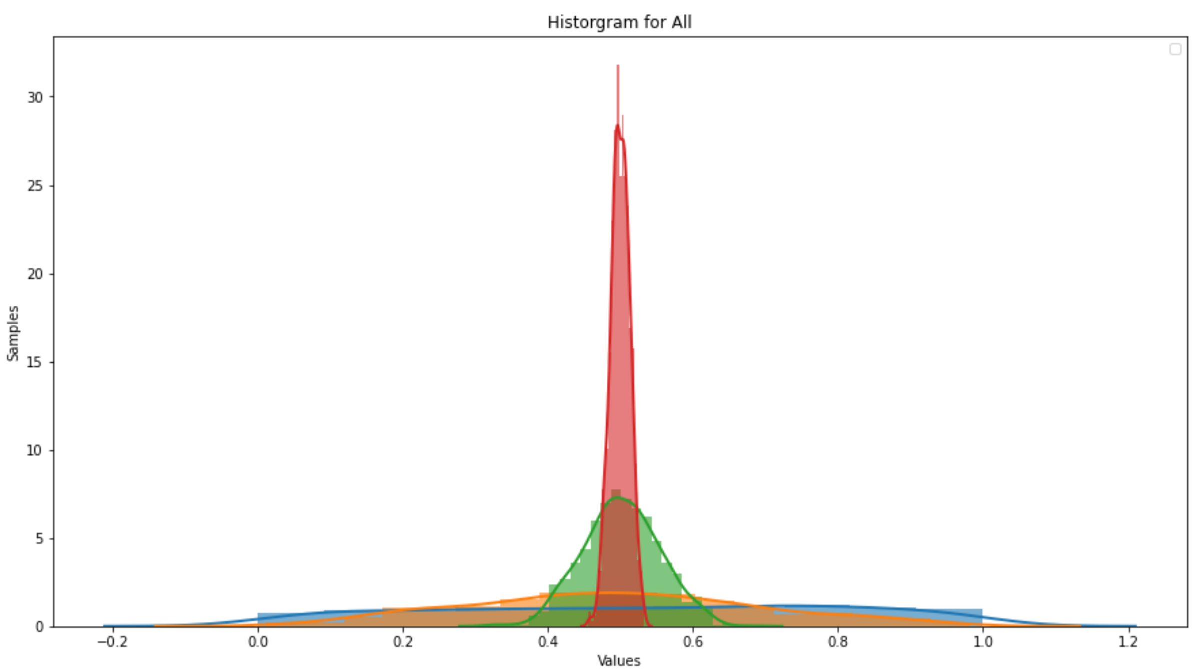

What happens to the mean and variability of the sample means as n increases? Relate your observations to the Central Limit Theorem discussed in the class.

- As n increases the mean and variability converge to a central point. In X_1, we can see high variability and as n increases we can see the variability decrease and the mean hovers around 0.5 at X_500. Thus as the sample size increases the distributation closely resembles a normal distribution and becomes tightly clustered around the mean, which is a behaviour that agrees with the Central Limit Theorem.

In [131]:
```
%%shell
jupyter nbconvert --to html C:\Users\William Hazen\OneDrive - University of To
ronto\Term 2 (W)\MIE 1613\Assignments\A1\HW1.ipynb
```

```
UsageError: Cell magic `%%shell` not found.
```