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MIE 1613 HW1

1. Assume X is continuous & uniformly distributed in $[2, 10]$. We are intrested in $\theta = E[(X-5)^+]$. Note $a^+ = \text{Max}(a, 0)$

a) Compute θ exactly using the definition of expected value. Hint $f(x) = \frac{1}{b-a}$, $x \in [a,b]$ & 0 otherwise

$$x \in [2, 10] \rightarrow f(x) = \frac{1}{10-2} = \frac{1}{8}$$

$$\theta = E[(x-5)^+] = \int_2^{10} (x-5)^+ \cdot f(x)dx$$

$$(x-5)^+ = \begin{cases} 0 \text{ if } 2 \leq x \leq 5 \\ a^+ \text{ if } 5 \leq x \leq 10 \end{cases}$$

$$\theta = \int_2^{10} (x-5)^+ \cdot f(x)\, dx = \int_2^{10} (x-5)^+ \cdot \frac{1}{8} dx$$

$$= \frac{1}{8}\left[\int_2^5 (x-5)^+ dx + \int_5^{10} (x-5)^+ dx\right]$$

$$= \frac{1}{8}\left[\int_2^5 0\, dx + \int_5^{10} (x-5)^+ dx\right]$$

$$= \frac{1}{8}\left[0 + \int_5^{10} (x-5)^+ dx\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}x^2\Big|_5^{10} + 5x\Big|_5^{10}\right]$$

$$= \frac{1}{8}\left[\left(\frac{75}{2}\right) - (25)\right] = \frac{25}{16} \approx 1.5625$$

$$\theta = \frac{25}{16}$$

4. What is the standard error of the estimator $\bar{X}_n$ where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ & X_i's are i.i.d

$E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{\cancel{n}} \cdot \cancel{n} E[X]$

↳ linearity of expectation property

$E[\bar{X}_n] = E[X]$

$Var(\bar{X}_n) = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2} Var\left[\sum_{i=1}^{n} X_i\right]$

↳ constant multiplication property of variance

$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$ → sum of the variance of i.i.d variables is equal to sum of each var

$= \frac{1}{n^{\cancel{2}}} \cdot \cancel{n}\, Var(X)$

↳ since X_i's are iid

$Var(\bar{X}_n) = \frac{1}{n^2} Var(X)$

let $\sigma^2 = Var(X)$

Thus the standard error (SE) of the sample mean is

$$SE = \sqrt{\frac{Var(X)}{n}} = \frac{\sqrt{Var(X)}}{\sqrt{n}} = \frac{\sigma}{\sqrt{n}}$$

$$SE = \frac{\sigma}{\sqrt{n}}$$ //

5. Prove the following

Let $\mu_x = E[X]$ & $\sigma^2 = Var(X)$

a) $Var(aX+b) = a^2 Var(X)$

$$Var(aX+b) = E[((aX+b) - E[aX+b])^2]$$

Property of Variance (i)
$V(X) = E[(X-\mu_x)^2]$

$$= E[((aX+\cancel{b}) - aE[X] - \cancel{b})^2]$$

(ii) expected value of a constant b is equal to b

(iii) linearity of expectation

$$= E[(aX - a\mu_x)^2]$$

$$= E[a^2(X-\mu_x)^2]$$

$$= a^2 E[(X-\mu_x)^2] \quad \Leftarrow (iii)$$

$$= a^2 Var(X) \quad \Leftarrow (i)$$

$\therefore$ Thus $Var(aX+b) = a^2 Var(X)$ //

b) $Var(X+Y) = Var(X) + Var(Y) = 2Cov(X,Y)$

Let $\mu_X = E[X]$, $\mu_Y = E[Y]$, $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$$Var(X+Y) = E[((X+Y) - E(X+Y))^2] \leftarrow (i)$$

$$= E[(X+Y-\mu_X-\mu_Y)^2]$$

$$= E[((X-\mu_X)+(Y-\mu_Y))^2]$$

$$= E[(X-\mu_X)^2 + (Y-\mu_Y)^2 + 2(X-\mu_X)(Y-\mu_Y)]$$

$$= E[(X-\mu_X)^2] + E[(Y-\mu_Y)^2] + 2E[(X-\mu_X)(Y-\mu_Y)]$$

From (i) $\Rightarrow Var(X) + Var(Y) + 2Cov(X,Y)$

$\therefore$ Thus, $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$ //