Assignment I1

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21-02-2019

1.

We wish to show that $L_1 := \{a^i b^j a^j \mid 0 \le i, 0 \le j\}$ is non regular, we proceed by applying the pumping lemma.
Let $k \ge 0$ be given, choose $z = a^p b^q a^q$ for some $p, q \ge 0$ such that $|z| > k$ i.e. $p + 2q > k$, let $u, v, w \in \Sigma^*$ be given such that $z = uvw$ and $|uv| < k$ and $|v| > 0$ we then get a few cases of how u, v, w could look:
Case 1: $u = a^l$, $v = a^j$, $w = a^{p-j-l} b^q a^q$ for $l \ge 0$ and $p \ge j > 0$ such that $p - l - j \ge 0$.
Case 2: $u = a^{p-l}$, $v = a^l b^j$, $w = b^{q-j} a^q$ for $p \ge l \ge 0$ and $j > 0$ since if $j = 0$ we are in case 1.
Case 3: $u = a^p b^j$, $v = b^l$, $w = b^{q-j-l} a^q$ for $l > 0$ and $j \ge 0$ such that $q - l - j \ge 0$.
Case 4: $u = a^p b^{q-j}$, $v = b^j a^l$, $w = a^{q-l}$ for $l > 0$ and $j \ge 0$ and $|uv| = p + q + l < k$.
Is it overkill to go through all these cases? absolutely.
Case 1: Consider

$$\begin{aligned} uv^i w &= a^l a^{ij} a^{p-j-l} b^q a^q = a^{l+ij+p-j-l} b^q a^q \\ &= a^{(i-1)j+p} b^q a^q \in L_1 \iff (i-1)j + p \in \mathbb{N} \end{aligned}$$

Now we want to find an $i \ge 0$ such that $uv^i w \notin L_1$, but as shown above we cannot choose such an i since j is positive and smaller than p, and as such we are forced to choose $p = 0$ to avoid case 1 altogether.
After forcing $p = 0$ case 2 becomes absorbed into case 3, and becomes:
Case 3': $u = b^j$, $v = b^l$, $w = b^{q-j-l} a^q$ for $l > 0$ and $j \ge 0$ such that $q - l - j \ge 0$. Consider

$$\begin{aligned} uv^i w &= b^j b^{il} b^{q-j-l} a^q = b^{q+(i-1)l} a^q \\ uv^i w &\in L_1 \iff q + (i-1)l = q \iff (i-1)l = 0 \end{aligned}$$

As in case 1, we wish $uv^i w \notin L_1$ so we can choose $i \ne 1$ since $l > 0$.
Case 4: After forcing $p = 0$ case 4 now has $u = b^{q-j}$ and $q + l < k$, consider

$$uv^i w = b^{q-j} (b^j a^l)^i a^{q-l}$$

This looks very unlikely to be in L_1, but if we subdivide into cases of j being zero or not we get

$$\begin{aligned} j = 0 \Rightarrow & uv^i w = b^q a^{il} a^{q-l} = b^q a^{q+(i-1)l} \in L_1 \iff \\ & q = q + (i-1)l \iff i = 1 \\ j > 0 \Rightarrow & uv^i w \in L_1 \Rightarrow i = 0 \lor i = 1 \end{aligned}$$

The $j > 0$ case clearly breaks down if $i \ge 2$ because we cant have any $(abab)$ in our language, so we simply choose $i > 1$ and this case has been taken care of.
In conclusion we can choose $z = b^q a^q$ where $2q > k$ and the pumping lemma will show that L_1 is non regular. □

2.

Consider $L_2 := \{a^i b^j b^j \mid i \geq 0, j \geq 0\} = \{a^i (bb)^j \mid i \geq 0, j \geq 0\} = \mathcal{L}(a^*(bb)^*)$ so L_2 is regular. After spending (too) many hours trying to make the pumping lemma work on this set, I face palmed so hard that I am unable to formulate a coherent thought about how the above is non trivial.

3.

Let L_3 be an arbitrary finite language, and L_4 be an arbitrary non-regular language. We wish to prove that $L_4 \setminus L_3 \in$NREG. Recall that for any sets we have

$$A = A \setminus B \cup (A \cap B)$$

We take A, remove the bits of B in A and add those bits back in. Now if we assume for contradiction that $L_4 \setminus L_3$ is regular, recall that REG is closed under union and intersection, and that all finite languages are regular. Note that $L_4 \cap L_3$ is finite since L_3 is, consider then

$$L_4 = L_4 \setminus L_3 \cup (L_4 \cap L_3)$$

This is a union of regular languages and a contradiction, since L_4 is non regular. So $L_4 \setminus L_3$ is non regular.