Assignment GG

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27. marts 2019

1 Countable and Uncountable Sets

Following exercise 8.42 in the book: We know that $2^{\mathbb{N}}$ is uncountable. Of course its complement, $\emptyset$ is countable. Question: Define a subset $S \subseteq 2^{\mathbb{N}}$, such that both S and its complement $S^c := 2^{\mathbb{N}} \setminus S$ are uncountable.
Here follows a sneaky solution.
We know that $\mathbb{R}$ is uncountable, so there exists a bijection $b : \mathbb{R} \to 2^{\mathbb{N}}$, we also know that all non empty open real intervals are uncountable, for instance $(0, 1) \subset \mathbb{R}$, as we can map $\mathbb{R}$ bijectively onto $(0, 1)$ via a bijection $r : \mathbb{R} \to (0, 1)$. Consider the commutative diagram

$$\begin{tikzcd}
2^{\mathbb{N}} \arrow[r, "b^{-1}"] & \mathbb{R} \arrow[d, "r"] \\
S \arrow[u, "\subseteq"] \arrow[r, "f"] & (0,1)
\end{tikzcd}$$

Choose then the set $S = b((0, 1)) \subsetneq 2^{\mathbb{N}}$ and then notice how

$$\begin{aligned} b(r^{-1}(b^{-1}(S))) &= b(r^{-1}(b^{-1}(b((0, 1))))) = b(r^{-1}((0, 1))) \\ &= b(\mathbb{R}) = 2^{\mathbb{N}} \end{aligned}$$

So S stands in bijection to $2^{\mathbb{N}}$ via the bijective composition of bijections

$$f = b \circ r^{-1} \circ b^{-1}|_S : S \to 2^{\mathbb{N}}$$

Where $|_S$ denotes the restriction of domain to S, or equivivalently composition with the inclusion injection. The restriction of a bijection is clearly injective, and above we saw that the function is surjective, so f is a bijection from S to $2^{\mathbb{N}}$. On the other hand $b^{-1}(2^{\mathbb{N}} \setminus S) = \mathbb{R} \setminus (0, 1)$ which clearly is uncountable as well, for instance the interval (2,3) is in there and we saw that those are uncountable. $\square$

Using Cantor's diagonal argument on the other hand, here follows a not so sneaky solution: We choose S as the power set of all the even numbers, i.e. the set 2^X where $X := \{n \mid n \in \mathbb{N} \wedge n \text{ } even\}$ this is clearly a subset of $2^{\mathbb{N}}$. Using the same approach as in Cantor's diagonal argument we assume for contradiction, that our chosen set S is countable. We can now visually represent a list of all the subsets of S as seen in the lectures, but instead of denoting for every natural number, we only use the even numbers, (including 0 as this makes no difference for the argument.), observe the following table, where the top row denotes the number in the subsets.

	0	2	4	6	8	$\cdots$
A_0	1	0	1	0	1	
A_1	0	1	0	0	0	
A_2	1	1	0	0	0	
A_3	0	0	0	0	0	
A_4	1	1	1	1	1	
$\vdots$						

As we have just "written", the list of all subsets of our chosen S, we look at the diagonal and make a new subset changing every number, from a 1 to a 0 and a 0 to a 1 in this new subset. In this way we have found a new subset, that doesn't match any of the previous subsets, thereby showing by contradiction that S is not countable, as we have found a subset, that is not already in our list of all the subsets of S. And we are done proving that $S = 2^{\{n|n\in\mathbb{N}\wedge n\ even\}}$ is uncountable. We now look at the power set of all the odd numbers, observing that $2^{\mathbb{N}} \setminus S = S^c \supset 2^{\{n|n\in\mathbb{N}\wedge n\ odd\}}$ so the power set of all the odd numbers is a subset of S^c and therefore we just have to show that this subset is uncountable, and we will have shown the desired property. But this is almost exactly the same argument as above, just now choosing all the odd numbers, but again we are able to make a new subset not already in the list of all the subsets, and this proves that $2^{\{n|n\in\mathbb{N}\wedge n\ odd\}}$ is uncountable, and therefore as it is a subset of S^c that S^c is uncountable.

2 Enumeration and Recursively Enumerable

Assume that TM M_1 enumerates on a tape all words in L "in the weak sense". That is, it is possibly that duplicate words appear on the tape, but you can at least assume that M1 writes every word from L to the tape at least once (and no other words).
Question: Prove that in this case, L must be recursively enumerable, i.e. there exists a TM M_2 that accepts the same language L.
Proof: Theorem 2 lecture 13 slide 6 says that if there exists a TM that enumerates in this weak sense, there exists another TM M_2 that accepts L, it is possible that there is hanging, so L is recursively enumerable. This is achieved by M_2 simply looks for the input word within the enumeration provided by M_1. □

3 Halting Problem Variant

Question: Assuming $L(M_0)$ is not recursive, prove that the language, $L = \{w|M_0\ halts\ on\ input\ w\}$ is not recursive
Proof: The assumption that $L(M_0)$ is non recursive is equivalent to saying that $L(M_0)$ is recursive enumerable, and not recursive, i.e. has at least one input w that it hangs on. Assume for contradiction that there exists a TM H such that $L = \mathcal{L}(H)$ and H always halts, i.e. that L is recursive.
Now M_0 hangs on w, so H should reject w because H accepts exactly the inputs that M_0 halts on. But since M_0 hangs on w H cannot decide if it should reject w in finite time, so it will also hang on w, which is a contradiction, since we assumed that H always halts. □