Handin G7

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1 Natural deduction proof

Prove the following sequents by natural deduction. You have to show the complete proof, with boxes or a numbering scheme that shows the administration of all assumptions. You may use derived rules, but only if you show their derivation as well.

1.1

Claim: $(r \to s) \vdash ((s \to t) \vee r) \to (s \vee t)$

1.	$r \to s$		$(premise)$	$line$
	$a.$	$((s \to t) \vee r)$	$(assumption)$	
	$b.$	r	$\vee_{e_1}$	a
	$c.$	$(s \to t)$	$\vee_{e_2}$	a
	$d.$	s	$\to e$	$1, b$
	$e.$	t	$\to e$	c, d
	$f.$	$s \vee t$	$\vee i$	c, d
2	$((s \to t) \vee r) \to (s \vee t)$		$\to i$	$a. - f.$

1.2

Claim: $(p \to q) \vee (s \to t) \vdash (p \to t) \vee (s \to q)$

Overall structure of the proof below. We want to use $\vee e$ as to show the desired and therefore assume $p \to q$ in case 1 and $s \to t$ in case 2. In case 1 we use example 1.24 (page 26 in the book) to rewrite $p \to q$ and once again use $\vee e$, into subcases 1.1 and 1.2, in these subcases we reach the desired $(p \to t) \vee (s \to q)$ and therefore as well in case 1. We repeat the above procedure in case 2.
After consulting Jaco we were allowed to just reference example 1.24 used in the below proof. The example is found on page 26 in the book, but as this example uses the law of excluded middle, we reference the derivation of this rule as well, this can be found on page 25.

1.$(p \to q) \vee (s \to t)$ (premise)

2. Case 1

   $a.$ $p \to q$ (assume for the sake of $\vee e$)

   $b.$ $\neg p \vee q$ (example 1.24)

   $c.$ case 1.1

      $\alpha.$ $\neg p$ (assume for the sake of $\vee e$)

      $\beta.$ $\neg p \vee t$ ($\vee i_1$)

      $\gamma.$ $(p \to t)$ (* shown below)

      $\delta.$ $(p \to t) \vee (s \to q)$ ($\vee i_1$)

   $d.$ case 1.2

      $\alpha.$ q (assume for the sake of $\vee e$)

      $\beta.$ $\neg s \vee q$ ($\vee i_2$)

      $\gamma.$ $(s \to q)$ (*shown below)

      $\delta.$ $(p \to t) \vee (s \to q)$ ($\vee i_2$)

   $e.$ $(p \to t) \vee (s \to q)$ ($\vee e, a - d$)

3. Case 2

   $a.$ $s \to t$ (assume for the sake of $\vee e$)

   $b.$ $\neg s \vee t$ (example 1.24)

   $c.$ case 2.1

      $\alpha.$ $\neg s$ (assume for the sake of $\vee e$)

      $\beta.$ $\neg s \vee q$ ($\vee i_1$)

      $\gamma.$ $(s \to q)$ (* shown below)

      $\delta.$ $(p \to t) \vee (s \to q)$ ($\vee i_2$)

   $d.$ case 2.2

      $\alpha.$ t (assume for the sake of $\vee e$)

      $\beta.$ $\neg p \vee t$ ($\vee i_2$)

      $\gamma.$ $(p \to t)$ (*shown below)

      $\delta.$ $(p \to t) \vee (s \to q)$ ($\vee i_1$)

   $e.$ $(p \to t) \vee (s \to q)$ ($\vee e, a - d$)

4.$(p \to t) \vee (s \to q)$ ($\vee e, 1 - 3$)

In the above proof we use a lemma $*$, observe the following proof:
Claim: $\neg p \vee q \vdash p \to q$

$$
\begin{array}{ll}
1.(\neg p \vee q) & \text{(premise)} \\
\quad a.\ p & \quad \text{(assume for the sake of } \to i\text{)} \\
\quad b.\ \text{case 1} & \\
\quad\quad \alpha.\ \neg p & \quad\quad \text{(assume for the sake of } \vee\, e\text{)} \\
\quad\quad \beta.\ \perp & \quad\quad (\neg e) \\
\quad\quad \gamma.\ q & \quad\quad (\perp e) \\
\quad c.\ \text{case 2} & \\
\quad\quad \alpha.\ q & \\
\quad d.\ q & \quad (\vee e, a - c) \\
2.p \to q & (\to i, a, d)
\end{array}
$$

2

2.1

Let $\phi_1 = p \to q$ and $\phi_2 = \phi_1 \to p$ and $\phi = \phi_2 \to p$, we give the full truth table using semantics propositional logic

p	q	ϕ_1	ϕ_2	ϕ
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Notice that ϕ is true for all valuations of its variables, and as such is a tautologi i.e. $\vDash \phi$ as wanted.

2.2

We wish to prove each line in the truth table from 2.1, in each of the sub cases.
In the following we make the four ϕ_1 steps corresponding to this part of the truth table

p	q	ϕ_1
T	T	T
T	F	F
F	T	T
F	F	T

$$
\begin{array}{ll}
\text{Case 1: } p, q \vdash (p \to q) & \\
\quad 1. p & \text{(pre)} \\
\quad 2. q & \text{(pre)} \\
\quad 3. (p \to q) & \to i, 1-2 \\
\text{Case 2: } \neg p, q \vdash (p \to q) & \\
\quad 1. \neg p & \text{(pre)} \\
\quad 2. q & \text{(pre)} \\
\quad 3. \neg p \vee q & \vee i_1 \\
\quad 4. p \to q & \text{Lemma } * \\
\text{Case 3: } p, \neg q \vdash \neg(p \to q) & \\
\quad 1. p & \text{(pre)} \\
\quad 2. \neg q & \text{(pre)} \\
\quad 3. \quad a. p \to q & \text{(Assume for } \neg i) \\
\quad \qquad b. q & (\to e, 1) \\
\quad \qquad c. \bot & (\neg e\ 2, 3.b) \\
\quad 4. \neg(p \to q) & (\neg i, 3) \\
\text{Case 4: } \neg p, \neg q \vdash (p \to q) & \\
\quad 1. \neg p & \text{(pre)} \\
\quad 2. \neg q & \text{(pre)} \\
\quad 3. \quad a. p & \text{(Assume for } \to i) \\
\quad \qquad b. \bot & (\neg e\ 1, 3.a) \\
\quad \qquad c. q & (\bot e) \\
\quad 4. (p \to q) & (\to i, 3a - 3c)
\end{array}
$$

Now we can procced to prove the three unique lines corresponding to ϕ_2, assuming the relevant truth values of p and ϕ_1

p	ϕ_1	ϕ_2
T	T	T
T	F	T
F	T	F
F	T	F

$p, (p \to q) \vdash (p \to q) \to p$

$1. p$	(pre)
$2. (p \to q)$	(pre)
$3. (p \to q) \to p$	$\to (i, 1-2)$

$\neg p, (p \to q) \vdash \neg((p \to q) \to p)$

$1. \neg p$	(pre)
$2. (p \to q)$	(pre)
$3. \quad a. (p \to q) \to p$	(Assume for $\neg i$)
$\quad b. \neg(p \to q) \vee p$	(1.24)
$\quad c. \quad \alpha.$case 1	
$\quad\quad \beta. \neg(p \to q)$	(Assume for $\neg i$)
$\quad\quad \gamma. \perp$	$(\neg e, 2, 3.c.\beta)$
$\quad d. \quad \alpha.$case 2	
$\quad\quad \beta. p$	(Assume for $\neg i$)
$\quad\quad \gamma. \perp$	$(\neg e, 2, 3.d.\beta)$
$4. \neg((p \to q) \to p)$	$(\neg i, 3a - 3d)$

$p, \neg(p \to q) \vdash (p \to q) \to p$

$1. p$	(pre)
$2. \neg(p \to q)$	(pre)
$3. \quad a. p \to q$	(Assume for $\to i$)
$\quad b. \perp$	$(\neg e\ 2, 3.a)$
$\quad c. p$	$(\perp e)$
$4. (p \to q) \to p$	$(\to i, 3a - 3c)$

Finally we combine p and ϕ_2 to get ϕ, notice that there are only two relevant cases left

p	ϕ_2	ϕ
T	T	T
T	T	T
F	F	T
F	F	T

$$
\begin{array}{rll}
p, (p \to q) \to p \vdash ((p \to q) \to p) \to p & & \\
& 1. p & \text{(pre)} \\
& 2. (p \to q) \to p & \text{(pre)} \\
& 3. ((p \to q) \to p) \to p & \to i, 1-2 \\
\neg p, \neg((p \to q) \to p) \vdash ((p \to q) \to p) \to p & & \\
& 1. \neg p & \text{(pre)} \\
& 2. \neg((p \to q) \to p) & \text{(pre)} \\
& 3. \quad a. ((p \to q) \to p) & \text{(Assume for } \to i) \\
& \qquad b. \bot & (\neg e\ 1, 3.a) \\
& \qquad c. p & (\bot e) \\
& 4. ((p \to q) \to p) \to p & (\to i, 3a - 3c)
\end{array}
$$

In total we now have a proof starting with every assumption of p and q in accordance with the uniformity of lemma 1 as follows

$$
\begin{array}{c}
p, q \vdash p, \phi_1 \vdash p, \phi_2 \vdash \phi \\
p, \neg q \vdash p, \neg\phi_1 \vdash p, \phi_2 \vdash \phi \\
\neg p, q \vdash \neg p, \phi_1 \vdash \neg p, \neg\phi_2 \vdash \phi \\
\neg p, \neg q \vdash \neg p, \phi_1 \vdash \neg p, \neg\phi_2 \vdash \phi
\end{array}
$$

Combining this with lemma 2 we then get that

$$\vdash \phi = ((p \to q) \to p) \to p$$

is a theorem, as wanted.

If left wanting for a direct proof of lemma 2, observe:
Proving $\vdash \phi$ without any assumptions should be fairly obvious by using the law of excluded middle.

$1.p \vee \neg p$				LEM
2.	$a.p$			(assume)
	$b.q \vee \neg q$			LEM
	$c.$ $\alpha.q$	$c'.$	$\alpha'.\neg q$	(assume, both individually)

The above $2.c.\alpha.$ is clearly case 1, and we are then able to just use the introduced assumptions as the premises in the above case 1 on page 4 of this assignment. The same goes for $2.c'.\alpha'.$ as this is case 3. By also assuming $\neg p$ in line 2, we are able to make all four cases (not shown above). By following the above proof for each case we end up with the desired $((p \to q) \to p) \to p$ and are then able to use $\vee e$ first for $q \vee \neg q$ and then for $p \vee \neg p$, the later not shown.
All in all, we are able to prove ϕ without any assumptions. As lemma 1, gives us that if each line in the truth table is true, then we are able to prove it, with the variables as found in the truth table as premises. This was shown in 2.2. This coincided with Lemma 2, that stated, if ϕ can be proven from all combinations of its n variables being negated or not, then ϕ can be proven from scratch. This was then shown, perhaps a bit schematically, by using LEM, and reusing the proof showing that ϕ could be proved with any combination of its variables negated or not. This shows that $\vdash \phi$ as wanted.