Handin G8

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1 Question 1.1

For this exercise we use conventions in order to make the proof fit on one vertical page, the first convention is that instead of writing

$$\wedge i \frac{\phi \qquad \psi}{\phi \wedge \psi}$$

We write the abbreviation you see on the image, writing how we arrive at ϕ and ψ respectively on "top of them"instead of writing the same line twice. Hopefully this increases readability.
The second convention is that writing IND_ϕ out is tiresome and would fill another horizontal page with letters. For the purposes of these exercises induction is an axiom so we just write the subproof of the induction start on top of the induction start part of the implication and write the subproof of the induction step on top of the induction step part of the induction implication, along with the and introduction convention we feel that this method highlights the induction start and steps parts of the proof, as we are tasked to do.

conventions: $\dfrac{\overset{\vdots}{\varphi} \quad \overset{\vdots}{\psi}}{\varphi \wedge \psi}\wedge i$ " $\equiv$ " $\dfrac{\vdots \quad \vdots}{\varphi \wedge \psi}$, $\dfrac{\varphi \quad \dfrac{\mathrm{Ind}\,\varphi'}{\varphi \to \psi}}{\psi} \to e$ " $\equiv$ " $\dfrac{\varphi \to \psi}{\psi} \to e$

x_0 fresh

z_0 fresh

PA2

$\forall y(x_0+(y+1)=(x_0+y)+1)$ $\forall e$

$(x_0+(z_0+1))=(x_0+z_0)+1$ $\forall e$ $\quad$ $a+1=a+1$ $=i$

$(x_0+(z_0+1))+1=((x_0+z_0)+1)+1$ $=e$ $\quad$ $(x_0+z_0)+1=(x_0+1)+z_0$ $\to$ assumption

$(x_0+(z_0+1))+1=((x_0+1)+z_0)+1$ $=e$

PA2

$\forall x\forall y(x+(y+1)=(x+y)+1)$ $\forall e$

$\forall y((x_0+1)+(y+1)=((x+1)+y)+1)$ $\forall e$

$(x_0+1)+(z_0+1)=(x+1)+z_0)+1$ $=e$

$(x_0+(z_0+1))+1=(x_0+1)+(z_0+1)$

$(x_0+z_0)+1=(x_0+1)+z_0) \to ((x_0+(z_0+1))+1=(x_0+1)+(z_0+1))$ $\to i$

$\forall i$

PA1

$\forall x(x+0=x)$ $\forall e$

$x_0=x_0+0$

$x_0+1=x_0+1$ $=i$

PA1

$\forall x(x+0=x)$ $\forall e$

$x_0+1=(x_0+1)+0$ $=e$

$x_0+1=(x_0+1)+0$ $=e$

$((x_0+0)+1=(x_0+1)+0) \wedge \forall z((x_0+z)+1=(x+1)+z) \to (x_0+(z+1))+1=(x_0+1)+(z+1)) \to \forall y((x_0+y)+1=(x_0+1)+y$ $\to e$

$\forall y((x_0+y)+1=(x_0+1)+y$

$\forall i$

$\forall x\,\forall y((x+y)+1=(x+1)+y)$

2 Question 1.2

Denote by $\phi \equiv x_0 + y = y + x_0$
In the following proof we let x_0 be fresh, show an induction start, by simultaneously using premise PA1 and Lemma 1.0. We then show the induction step, by letting z_0 be fresh and using premise PA2 and Q1.1 simultaneously, putting them together to show the desired induction step. (This calculation has been moved below the main proof for greater readability). We then use the fresh z_0 for an forall introduction, ending the induction step. Then putting together the induction start and step, before finally using the fresh x_0 for another forall introduction proving the desired.

$PA \vdash \forall x \forall y(x + y = y + x)$

$\mathfrak{A}. \forall x(x + 0 = x)$ (Premise PA1)

$\mathfrak{B}. \forall x \forall y\ x + (y + 1) = (x + y) + 1$ (Premise PA2)

$\mathfrak{C}. \big(\psi[0/y] \wedge (\forall z \psi[z/y] \to \psi[z + 1/y])\big) \to \forall y\ \psi$ (Premise INDψ)

$\mathfrak{D}. \forall x(0 + x = x)$ (Premise Lemma 1.0)

$\mathfrak{E}. \forall x \forall y(x + y) + 1 = (x + 1) + y$ (Premise Q1.1)

1.let x_0 be fresh

induction start $\phi[0/y]$, start:

$1.a.PA1$

$1.b.\forall x(x + 0 = x)$

$1.c.x_0 + 0 = x_0$ $(\forall e, 1b)$

simultaneously:

$2.a$ Lemma 1.0

$2.b$ $\forall x(0 + x = x)$

$2.c$ $0 + x_0 = x_0$ $(\forall e, 2b)$

$\aleph.x_0 + 0 = 0 + x_0$ (=e, 1.c 2.c)

induction start $\phi[0/y]$, finished

induction step $\forall z \phi[z/y] \to \phi[z + 1/y]$, start

α.let z_0 be fresh (for sake of $\forall i$)

see calculations (*) below

$\beta.\phi[z_0/y] \to \phi[z_0 + 1/y]$ $(\to i)$

$\beth.\forall z \phi[z/y] \to \phi[z + 1/y]$ $(\forall i, \alpha - \beta)$

induction step $\forall z \phi[z/y] \to \phi[z + 1/y]$, finished

2.$\forall y(x_0 + y) = y + x_0)$ (IND$_\phi$ using $\aleph, \beth$)

$\mathfrak{F}.\forall x \forall y(x + y) = y + x)$ $(\forall i, 1, 2)$

(*) calculations(*):

1. $x_0 + z_0 = z_0 + x_0$ (assume for $\to i$)

   $a.Q1.1$

   $b.\forall x \forall y (x + y) + 1 = (x + 1) + y$

   $c.\forall y (z_0 + y) + 1 = (z_0 + 1) + y$ $(\forall e, x = z_0)$

   $d.(z_0 + x_0) + 1 = (z_0 + 1) + x_0$ $(\forall e, y = x_0)$

   simultaneously:

   $e.PA2$

   $f.\forall x \forall y\ x + (y + 1) = (x + y) + 1$

   $g.\forall y\ x_0 + (y + 1) = (x_0 + y) + 1$ $(\forall e)$

   $h.x_0 + (z_0 + 1) = (x_0 + z_0) + 1$ $(\forall e, y = z_0)$

   $i.x_0 + (z_0 + 1) = (z_0 + x_0) + 1$ $(= e, g$ and $1)$

2. $x_0 + (z_0 + 1) = (z_0 + 1) + x_0$ $(= e, d, i)$

3. $\phi[z_0/y] \to \phi[z_0 + 1/y]$ $(\to i, 1 - 2)$