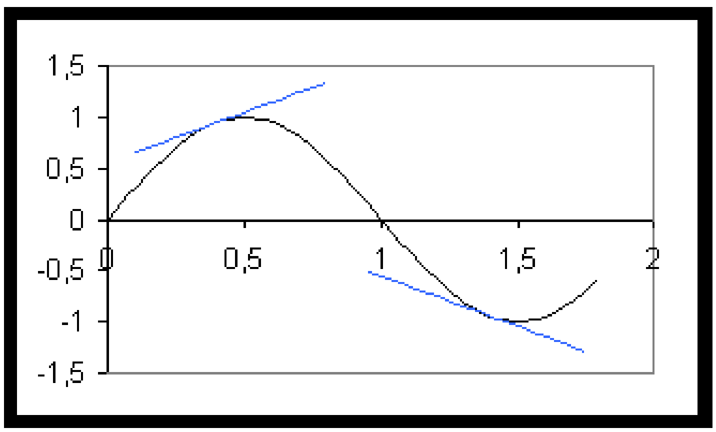
Definition

The derivative of a function f at a point x, written $f\ '(x)$, is given by:

$$f^{'}(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

if this limit exists.

Graphically, the derivative of a function corresponds to **the slope of its tangent line at one specific point.** The following illustration allows us to visualise the tangent line (in blue) of a given function at two distinct points. Note that the slope of the tangent line varies from one point to the next. The value of the derivative of a function therefore depends on the point in which we decide to evaluate it. By abuse of language, we often speak of the slope of the function instead of the slope of its tangent line.



Notation

Here, we represent the derivative of a function by a prime symbol. For example, writing $f'(x)$ represents the derivative of the function f evaluated at point x. Similarly, writing $(3x\ +\ 2)'$ indicates we are carrying out the derivative of the function $3x\ +\ 2$. The prime symbol disappears as soon as the derivative has been calculated.

1. Derivatives of usual functions

Below you will find a list of the most important derivatives. Although these formulas can be formally proven, we will only state them here. We recommend you learn them by heart.

1.1. The constant function

Let $f(x) = k$, where k is some real constant. Then

$$f'(x) = (k)' = 0$$

Examples

$(8)' = 0$

$(-5)' = 0$

$(0{,}2321)' = 0$

1.2. The identity function $f(x) = x$

Let $f(x) = x$, the identity function of x. Then

$$f'(x) = (x)' = 1$$

1.3. A function of the form x^n

Let $f(x) = x^n$, a function of x, and n a real constant. We have

$$f'(x) = (x^n)' = n\,x^{n-1}$$

Examples

$(x^4)' = 4\,x^{4-1} = 4\,x^3$

$(x^{1/2})' = 1/2\,x^{\frac{1}{2}-1} = 1/2\,x^{-1/2}$

$(\boldsymbol{x}^{-2})' = -2\boldsymbol{x}^{-2-1} = -2\,\boldsymbol{x}^{-3}$

$\left(x^{-\frac{1}{3}}\right)' = \left(-\frac{1}{3}\right)\boldsymbol{x}^{-\frac{1}{3}-1} = \left(-\frac{1}{3}\right)\boldsymbol{x}^{-\frac{4}{3}}$

Notes on the $(\boldsymbol{x^n})' = \boldsymbol{n\,x^{n-1}}$ rule:

2. Basic derivation rules

We will generally have to confront not only the functions presented above, but also combinations of these : multiples, sums, products, quotients and composite functions. We therefore need to present the rules that allow us to derive these more complex cases.

2.1. Constant multiples

Let k be a real constant and $f(x)$ any given function. Then

$$(k\, f(x))' = k\, f'(x)$$

In other words, we can forget the constant which will remain unchanged and only derive the function of x.

Examples

$(4x^2)' = 4(x^2)' = 4(2x) = 8x$

$(-5e^x)' = -5(e^x) = -5e^x$

$(12lnx)' = 12(lnx)' = 12\left(\frac{1}{x}\right) = \frac{12}{x}$

2.2. Addition and subtraction of functions

Let $f(x)$ and $g(x)$ be two functions. Then

$$\big(f(x) \pm g(x)\big)' = f'(x) \pm g'(x)$$

When we derive a sum or a subtraction of two functions, the previous rule states that the functions can be individually derived without changing the operation linking them.

Example 1

$(e^x + x^5)' = (e^x)' + (x^5)' = e^x + 5x^4$

Example 2

$$\left(lnx - \frac{1}{x^2} + 8\right)' = (lnx)' - (x^{-2})' + (8)'$$

$$= \frac{1}{x} - (-2x^{-3}) + 0$$

$$= \frac{1}{x} + \frac{2}{x^3}$$

Example 3

$$\left(3\sqrt{x} + 2x - \frac{8}{x}\right)' = \left(3x^{1/2}\right)' + (2x)' - (8x^{-1})'$$

$$= 3\left(x^{\frac{1}{2}}\right)' + 2(x)' - 8(x^{-1})'$$

$$= 3\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + 2(1) - 8(-x^{-2})$$

$$= \frac{3}{2}x^{-\frac{1}{2}} + 2 + 8x^{-2}$$

2.3. Product rule

Let $f(x)$ and $g(x)$ be two functions. Then the derivate of the product

$$(f(x)\ g(x)\ )' = f'(x)\ g(x) + f(x)\ g'(x)$$

We must follow this rule religiously and not succumb to the temptation of writing $\left(f(x)g(x)\right)' = f'(x)g'(x)$; a faulty statement.

Example 1

$$(x^3e^x)' = (x^3)'e^x + x^3(e^x)'$$

$$= 3x^2e^x + x^3e^x$$

Example 2

$$(3\sqrt{x}lnx)' = (3\sqrt{x})'lnx + 3\sqrt{x}(lnx)'$$

$$= 3\left(x^{\frac{1}{2}}\right)' lnx + 3\sqrt{x}(lnx)'$$

$$= 3\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) lnx + 3\sqrt{x}\frac{1}{x}$$

$$= \frac{3}{2}x^{-\frac{1}{2}}lnx + 3x^{-\frac{1}{2}}$$

2.4. Quotient rule

Let $f(x)$ and $g(x)$ be two functions. Then the derivative of the quotient

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Just as with the product rule, the quotient rule must religiously be respected.

Example 1

$$\left(\frac{x^3}{e^x}\right)' = \frac{(x^3)'e^x - x^3(e^x)'}{(e^x)^2}$$

$$= \frac{3x^2e^x - x^3e^x}{(e^x)^2}$$

$$= \frac{x^2e^x(3-x)}{(e^x)^2}$$

$$= \frac{x^2(3-x)}{e^x}$$

Example 2

$$\left(\frac{3\sqrt{x}}{lnx}\right)' = \frac{\left(3\sqrt{x}\right)' lnx - 3\sqrt{x}(lnx)'}{(lnx)^2} = \frac{3\left(x^{\frac{1}{2}}\right)' lnx - 3\sqrt{x}(lnx)'}{(lnx)^2}$$

$$= \frac{3\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)lnx - 3\sqrt{x}\frac{1}{x}}{(lnx)^2} = \frac{3x^{-\frac{1}{2}}lnx - 6x^{-\frac{1}{2}}}{2(lnx)^2}$$

$$= \frac{3x^{-\frac{1}{2}}(lnx-2)}{2(lnx)^2}$$

3 COMPLEX NUMBERS

Objectives

After studying this chapter you should

- understand how quadratic equations lead to complex numbers and how to plot complex numbers on an Argand diagram;
- be able to relate graphs of polynomials to complex numbers;
- be able to do basic arithmetic operations on complex numbers of the form $a+ib$;
- understand the polar form $[r,\theta]$ of a complex number and its algebra;
- understand Euler's relation and the exponential form of a complex number $re^{i\theta}$;
- be able to use de Moivre's theorem;
- be able to interpret relationships of complex numbers as loci in the complex plane.

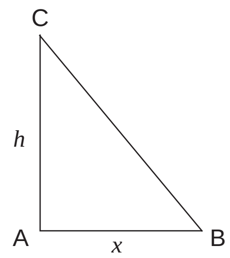
3.0 Introduction

The history of complex numbers goes back to the ancient Greeks who decided (but were perplexed) that no number existed that satisfies

$$\boxed{x^2=-1}$$

For example, *Diophantus* (about 275 AD) attempted to solve what seems a reasonable problem, namely

> 'Find the sides of a right-angled triangle of perimeter 12 units and area 7 squared units.'



Letting $\text{AB}=x$, $\text{AC}=h$ as shown,

then $\quad \text{area} = \frac{1}{2}xh$

and $\quad \text{perimeter} = x+h+\sqrt{x^2+h^2}$

3.1 Complex number algebra

A number such as $3+4i$ is called a **complex number**. It is the sum of two terms (each of which may be zero).

The real term (not containing i) is called the **real** part and the coefficient of i is the **imaginary** part. Therefore the real part of $3+4i$ is 3 and the imaginary part is 4.

A number is real when the coefficient of i is zero and is imaginary when the real part is zero.

e.g. $3+0i=3$ is real and $0+4i=4i$ is imaginary.

Having introduced a complex number, the ways in which they can be combined, i.e. addition, multiplication, division etc., need to be defined. This is termed the **algebra** of complex numbers. You will see that, in general, you proceed as in real numbers, but using

$$i^2=-1$$

where appropriate.

But first **equality** of complex numbers must be defined.

If two complex numbers, say

$$a+bi,\ c+di$$

are **equal**, then both their real and imaginary parts are equal;

$$\boxed{a+bi=c+di \quad \Rightarrow \quad a=c \text{ and } b=d}$$

Addition and subtraction

Addition of complex numbers is defined by separately adding real and imaginary parts; so if

$$z=a+bi,\ w=c+di$$

then $$z+w=(a+c)+(b+d)i\,.$$

Similarly for **subtraction**.

Example

Express each of the following in the form $x+yi$.

(a) $(3+5i)+(2-3i)$

(b) $(3+5i)+6$

(c) $7i-(4+5i)$

Solution

(a) $(3+5i)+(2-3i)=3+2+(5-3)i=5+2i$

(b) $(3+5i)+6=9+5i$

(c) $7i-(4+5i)=7i-4-5i=-4+2i$

Multiplication

Multiplication is straightforward provided you remember that $i^2=-1$.

Example

Simplify in the form $x+yi$:

(a) $3(2+4i)$

(b) $(5+3i)i$

(c) $(2-7i)(3+4i)$

Solution

(a) $3(2+4i) = 3(2)+3(4i) = 6+12i$

(b) $(5+3i)\,i = (5)\,i+(3i)\,i = 5i+3\left(i^2\right) = 5i+(-1)3 = -3+5i$

(c) $$\begin{aligned}(2-7i)(3+4i) &= (2)(3)-(7i)(3)+(2)(4i)-(7i)(4i)\\ &= 6-21i+8i-(-28)\\ &= 6-21i+8i+28\\ &= 34-13i\end{aligned}$$

In general, if

$$z = a+bi, \quad w = c+di,$$

then
$$\begin{aligned}zw &= (a+bi)(c+di)\\ &= ac-bd+(ad+bc)i\end{aligned}$$

Activity 5

Simplify the following expressions:

(a) $(2+6i)+(9-2i)$ (b) $(8-3i)-(1+5i)$

(c) $3(7-3i)+i(2+2i)$ (d) $(3+5i)(1-4i)$

(e) $(5+12i)(6+7i)$ (f) $(2+i)^2$

(g) i^3 (h) i^4

(i) $(1-i)^3$ (j) $(1+i)^2+(1-i)^2$

(k) $(2+i)^4+(2-i)^4$ (l) $(a+ib)(a-ib)$

Division

The **complex conjugate** of a complex number is obtained by changing the sign of the imaginary part. So if $z=a+bi$, its complex conjugate, $\bar{z}$, is defined by

$$\bar{z}=a-bi$$

Note: an alternative notation often used for the complex conjugate is z^*.

Any complex number $a+bi$ has a complex conjugate $a-bi$ and from Activity 5 it can be seen that $(a+bi)(a-bi)$ is a real number. This fact is used in simplifying expressions where the denominator of a quotient is complex.

Example

Simplify the expressions:

(a) $\frac{1}{i}$ (b) $\frac{3}{1+i}$ (c) $\frac{4+7i}{2+5i}$

Solution

To simplify these expressions you multiply the numerator and denominator of the quotient by the complex conjugate of the denominator.

(a) The complex conjugate of i is $-i$, therefore

$$\frac{1}{i}=\frac{1}{i}\times\frac{-i}{-i}=\frac{(1)(-i)}{(i)(-i)}=\frac{-i}{-(-1)}=-i$$

(b) The complex conjugate of $1+i$ is $1-i$, therefore

$$\frac{3}{1+i}=\frac{3}{1+i}\times\frac{1-i}{1-i}=\frac{3(1-i)}{(1+i)(1-i)}=\frac{3-3i}{2}=\frac{3}{2}-\frac{3}{2}i$$

Chapter 13

LIMITS AND DERIVATIVES

13.1 Overview

13.1.1 *Limits of a function*

Let f be a function defined in a domain which we take to be an interval, say, I. We shall study the concept of limit of f at a point 'a' in I.

We say $\lim\limits_{x \to a^-} f(x)$ is the expected value of f at $x = a$ given the values of f near to the left of a. This value is called the *left hand limit* of f at a.

We say $\lim\limits_{x \to a^+} f(x)$ is the expected value of f at $x = a$ given the values of f near to the right of a. This value is called the *right hand limit* of f at a.

If the right and left hand limits coincide, we call the common value as the limit of f at $x = a$ and denote it by $\lim\limits_{x \to a} f(x)$.

Some properties of limits

Let f and g be two functions such that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist. Then

(i) $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(ii) $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(iii) For every real number α

$\lim\limits_{x \to a} (\alpha f)(x) = \alpha \lim\limits_{x \to a} f(x)$

(iv) $\lim\limits_{x \to a} [f(x)\, g(x)] = [\lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x)]$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}, \text{ provided } g(x) \neq 0$$