# AN INTRODUCTION TO PERFECTOID SPACES

*by*

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***Abstract.*** — The aim of these notes is to give a short introduction to P. Scholze's theory of perfectoid spaces and their applications. These notes arise from a summer school "Perfectoid Spaces" held in Bressanone (IT) from 31st august to 4th september 2015 in the framework of the Erasmus Mundus master programme ALGANT. The aim was to introduce perfectoid spaces to last year master students.

### Contents



## 1. Introduction

Let $p$ be a prime integer, $K$ a complete discrete valuation field of characteristic 0, with perfect residue field of characteristic $p$, and $A$ an abelian variety with good reduction over $K$. Fix $\overline{K}$ an algebraic closure of $K$, put $G_K = \mathsf{Gal}(\overline{K}/K)$, let $\chi\colon G_K \to \mathbf{Z}_p^\times$ be the cyclotomic character[(1)] and $C$ be the completion of $\overline{K}$. In the seminal paper [46] (which constitutes the birth of $p$-adic Hodge theory), Tate proved that there is a $G_K$-equivariant isomorphism[(2)]

$$\mathsf{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C \simeq \mathsf{H}^1(A, \mathcal{O}_A) \otimes_K C \oplus \mathsf{H}^0(A, \Omega_{A/K}) \otimes_K C(-1)$$

(in fact, his result, more general, encompasses $p$-divisible groups over the ring of integers $\mathcal{O}_K$, *cf* [46, Corollary 2]). This Hodge-Tate decomposition is quite similar to the Hodge decomposition for compact Kähler manifolds, and Tate asked whether such a decomposition exists in general for the $p$-adic étale cohomology of proper smooth schemes or suitable rigid analytic spaces over $K$. This question was made precise and refined by Fontaine, who constructed appropriate period

F. Andreatta, B. Chiarellotto and M. Seveso were sponsored by MIUR-PRIN 2010-11 grant "Arithmetic Algebraic Geometry and Number Theory".
(1) Characterized by the property that $g(\zeta) = \zeta^{\chi(g)}$ for all $g \in G_K$ and $\zeta \in \mu_{p^\infty}$.
(2) As usual, if $M$ is a $G_K$-module and $i \in \mathbf{Z}$, the Tate twist $M(i)$ denotes the module $M$ with the $G_K$-action twisted by $\chi^i$.

rings and stated conjectures on the existence of *comparison isomorphisms* relating the $p$-adic étale cohomology and (refinements of) the de Rham cohomology of proper and smooth varieties over $K$ (*cf* [8]).

A key point of [46] is the computation of Galois cohomology[(3)]:

***Theorem 1.1*** **(*cf*** [46, Theorem 1 & 2]). — *For $q \in \{0,1\}$ and $i \in \mathbf{Z}$, we have*

$$\dim_K \mathsf{H}^q(G_K, C(i)) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

*(in fact, $\chi$ may be replaced by a slightly more general type of character, cf* [46, Theorem 2]*: we restrict to the cyclotomic case for simplicity).*

Even in the case $q = i = 0$, this result is not obvious, in particular it does not follow from Galois theory. Using ramification theory[(4)], one first shows that $\mathsf{H}^0(H_K, C) = \widehat{K}_\infty$ (the completion of the cyclotomic extension $K_\infty = K(\mu_{p^\infty})$) and $\mathsf{H}^1(H_K, C) = 0$ (where $H_K = \mathsf{Ker}(\chi)$). This follows from Tate's observation that the highly ramified extension $K_\infty/K$ kills almost all the ramification, *i.e.* that the extension $\overline{K}/K_\infty$ almost behaves as if it was unramified. The second step consists in an "uncompletion" along the cyclotomic tower, using "normalized Tate's traces". This two steps process, which has been used in many situations in $p$-adic Hodge theory, has been systematized by Colmez (the Tate-Sen formalism *cf* [9, §3]).

The control on the ramification of $\overline{K}/K_\infty$ was also used by Fontaine and Wintenberger to develop the "field of norms" theory (*cf* [51]) who showed that $K_\infty$ has the same Galois theory as some fields of characteristic $p$. Later on, the fields of norms theory gave rise to Fontaine's $(\varphi, \Gamma)$-theory, providing a complete classification of $p$-adic representations in terms of semi-linear algebra (*cf* section 2). This theory has been of central importance in many recent developments in $p$-adic Hodge theory and $p$-adic Langlands correspondence.

Generalizing Tate's ideas, Faltings, in a series of papers, developed the theory of "almost mathematics", of which we introduce the bases in section 3, culminating with the proof of his "almost purity theorem" in [15]. He used then this result to prove comparison theorems mentioned above (*cf* [8]). Later, almost mathematics has been completely axiomatized by Gabber and Ramero in [21] and the almost purity theorem has been proved in a great generality by Scholze in [36], using perfectoid spaces (*cf* theorem 6.15).

Perfectoid spaces theory, introduced by Scholze in [36] is a vast generalization of Tate's observation and the "perfect field of norms theory". Roughly speaking, perfectoid spaces is a class of adic spaces (*cf* section 4) that are ramified enough so that the Frobenius on their "reduction modulo $p$" is surjective (*cf* section 6). They usually are non noetherian, but from a certain point of view, these are the natural framework for certain constructions (that is the case for instance of the isomorphism between the Lubin-Tate and Drinfeld towers, constructed by Faltings in [16], detailed by Fargues in [17], and revisited in a very natural way by Scholze, *cf* [41] & [39]).

The main feature is that there is a functor called tilting on the category of perfectoid spaces into that of perfectoid spaces of characteristic $p$ (*cf* section 6). This functor retains much information on the original space. In particular, a perfectoid space and its tilt have equivalent étale topoi (*cf* theorem 6.22). As usual, thanks to the Frobenius map, spaces in characteristic $p$ are easier to deal with. Perfect fields of norms basically correspond to tilts of perfectoid points. One of the main interests of this construction is also that starting from a smooth variety, one can always construct locally a tower of explicit finite extensions whose "completion" is perfectoid. Of course, this local constructions were used before perfectoid spaces theory together with Faltings almost purity theorem (*cf* theorem 6.15) for local cohomological computations. In [38], Scholze introduced the *pro-étale* site of a perfectoid space, wich allows to sheafify those local computations. A natural application is the proof of the de Rham comparison theorem for proper and smooth adic spaces, whose main steps are explained in section 7.

(3) Here cohomology refers to *continuous cohomology*.
(4) Ax proved that $\mathsf{H}^0(G_K, C) = K$ in a more general context without using ramification theory (*cf* [3]).

Interestingly, the first application of perfectoid spaces and their tilt was the proof of new cases of the monodromy-weight conjecture, using the known case in characteristic $p$ (*cf* section 8). Of course, there are many more applications of perfectoid spaces theory: structure of Rapoport-Zink spaces of infinite level, classification of $p$-divisible groups over $\mathcal{O}_C$, full faithfulness of the Dieudonné functor over semi-perfect rings (*cf* [41]), $p$-adic geometry of Shimura varieties (*cf* [40]), $p$-adic cohomology of the Lubin-Tate tower (*cf* [35]), integral $p$-adic Hodge theory and integral comparison theorems (*cf* [5]), *etc.*

Existing surveys on perfectoid spaces and their applications include Fontaine's Bourbaki talk (*cf* [19]) and Scholze's (*cf* [37]).

## 2. Motivating problems and constructions

One of the first opportunities to introduce the perfectoid technique has been offered by the theory of Galois representations and more specifically about $p$-adic local Galois representations. We would like in this part to give some insight on this subject and to see how the "tilting" method (think at it, in the first instance, as to pass from characteristic 0 to characteristic $p$), which is at the basis of Scholze's construction, has been used in the framework of the $p$-adic representations.

Some notations: we indicate as usual $\mathbf{Q}$ the field of rational numbers, by $\overline{\mathbf{Q}}$ the algebraic closure and by $G_{\mathbf{Q}} = \mathsf{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ the absolute Galois group. For every prime in $\mathbf{Z}$ we can define an absolute value $|.|_p$. We may complete $\mathbf{Q}$ with respect to such an absolute value and we get the field of $p$-adic numbers $\mathbf{Q}_p$: a totally disconnected and locally compact topological field. Naturally we can take its algebraic closure $\overline{\mathbf{Q}}_p$ and its Galois group $G_{\mathbf{Q}_p}$. We can have $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ (not unique and here we need the axiom of choice), and for any such an embedding we have a closed embedding of groups $G_{\mathbf{Q}_p} \to G_{\mathbf{Q}}$. Even if we don't have a unique inclusion they form a class under conjugation [48, p.144].

Representations of $G_{\mathbf{Q}}$ (hence of $G_{\mathbf{Q}_p}$) arise naturally from arithmetic algebraic geometry. Suppose that we have $X/\mathbf{Q}$ a projective and smooth variety defined over $\mathbf{Q}$. Take an embedding of $\overline{\mathbf{Q}}$ in $\mathbf{C}$. Then we may consider the topological space $X(\mathbf{C})$ associated to the complex points of $X$. It is a topological space (actually a complex manifold). Hence we can define its singular cohomology $\mathsf{H}^i(X(\mathbf{C}), \mathbf{Z})$. If we restrict the coefficients to $\mathbf{Z}/\ell^n\mathbf{Z}$ (where $\ell$ a prime of $\mathbf{Z}$), we can consider:

$$\mathsf{H}^i(X(\mathbf{C}), \mathbf{Z}/\ell^n\mathbf{Z})$$

We have another interpretation of such a cohomology groups via étale cohomology. In fact, by a theorem of Artin [2, Exp. XI, Th. 4.4]:

***Theorem 2.1.*** — *There is an isomorphism for each $n \in \mathbf{N}_{>0}$,*

$$\mathsf{H}^i(X(\mathbf{C}), \mathbf{Z}/\ell^n\mathbf{Z}) \simeq \mathsf{H}^i_{\text{ét}}(X \times_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Z}/\ell^n\mathbf{Z}).$$

*It follows that*

$$\mathsf{H}^i(X(\mathbf{C}), \mathbf{Z}_\ell) := \varprojlim_n \mathsf{H}^i(X(\mathbf{C}), \mathbf{Z}/\ell^n\mathbf{Z}) \simeq \varprojlim_n \mathsf{H}^i_{\text{ét}}(X \times_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Z}/\ell^n\mathbf{Z}) =: \mathsf{H}^i_{\text{ét}}(X \times_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Z}_\ell).$$

Because we have such an interpretation via étale cohomology we have as a reward an action of $G_{\mathbf{Q}}$, hence a natural continuous representation on $\mathbf{Z}_\ell$-modules $\mathsf{H}^i_{\text{ét}}(X \times_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Z}_\ell)$. We are not going to discuss here étale cohomology. We can only indicate some references as [32].

***Remark 2.2.*** — If $X = E$ is an elliptic curve, then we may describe

$$\mathsf{H}^1_{\text{ét}}(E \times_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Z}_\ell) \simeq \mathsf{Hom}_{\mathbf{Z}_\ell}(\varprojlim_r E[\ell^r](\overline{\mathbf{Q}}), \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell^2\,.$$

This is $\ell$-adic Galois representation of $G_{\mathbf{Q}}$. Note that in general étale $\mathbf{Z}_\ell$-cohomology can have torsion.

A representation of $G_{\mathbf{Q}}$ as before will be indicated as a global representation. If we specialize to $G_{\mathbf{Q}_p}$ we then have a *local $\ell$-adic representation* ($p$ can be equal to $\ell$) *i.e.* with values in a $\mathbf{Q}_\ell$-vector space (of course we could have replaced $\mathbf{Q}$ with a number field). Because it is a continuous representation, it is possible to find a $\mathbf{Z}_\ell$-lattice of our vector space in such a way we have a representation on a $\mathbf{Z}_\ell$-free module. The latter will be called a $\ell$-adic representation of $G_{\mathbf{Q}_p}$ as well (*cf* [11, Part I, Lemma 1.2.6]).

We are led to discuss different kinds of representations of $G_{\mathbf{Q}_p}$: the $\ell$-adic with $\ell \neq p$ and the $p$-adic ones ($\ell = p$). Remember that $G_{\mathbf{F}_p}$ is the profinite completion of $\mathbf{Z}$. We have a natural surjection $G_{\mathbf{Q}_p} \twoheadrightarrow G_{\mathbf{F}_p}$. The approaches are different (we remind that we work with a local field with finite residue field).

(**i**) $\ell \neq p$, $\ell$-adic Galois representations. In this case one uses the decomposition of $G_{\mathbf{Q}_p}$ using the inertia. The Grothendieck Monodromy theorem tells us that the inertia acts in a unipotent way on an open subset (profinite topology!). Hence the Galois representation can be studied as smooth representation of the Weil-Deligne group endowed with a nilpotent operator [20, § 1.3] (up to the restriction to the Galois group of a finite extension of $\mathbf{Q}_p$).

(**ii**) $\ell = p$, $p$-adic Galois representations. Here the problem is more involved. We cannot use anymore the decomposition of the Galois group via the inertia: this is exactly where the theory of *perfectoid fields (spaces)* is coming in action! One would like to decompose $G_{\mathbf{Q}_p}$ on a part which is the Galois group of a field of characteristic $p$ (and not $\mathbf{F}_p$). This way of acting will be the first instance of *tilting*.

As we said we are interested in study some decomposition of the absolute Galois group $G_{\mathbf{Q}_p}$. Here we want to deal with the first case *i.e.* that one studied by Fontaine-Winterberger [51] (which in turns seems to be connected with Krasner's work, *cf* [42]). We consider, as usual, $\mathbf{Q}_p$ and its algebraic closure $\overline{\mathbf{Q}}_p$ and its absolute Galois group $G_{\mathbf{Q}_p}$.

**The idea**: To consider a sub-Galois extensions $\mathbf{Q}_p \subset \mathbf{Q}_p^\infty \subset \overline{\mathbf{Q}}_p$ whose Galois group $\mathsf{Gal}(\overline{\mathbf{Q}}_p/\,\mathbf{Q}_p^\infty) = G_{\mathbf{Q}_p^\infty}$ is the Galois group of a field of characteristic $p$.

**Why?** Because in this case a Galois representation of $G_{\mathbf{Q}_p}$ can be seen as a representation of $G_{\mathbf{Q}_p^\infty}$ (*i.e.* of a field of characteristic $p$) plus an action of $\mathsf{Gal}(\mathbf{Q}_p^\infty\,/\,\mathbf{Q}_p)$. We then hope that the representations of $G_{\mathbf{Q}_p^\infty}$ admits an interpretation easier to handle. But where to be found such a $\mathbf{Q}_p^\infty$ (The notation with $\infty$ is not chosen *au hasard*...)?

That the representations of a field of characteristic $p$ are easier to handle may be indicated by the following result of Fontaine [18, 1.2]. We introduce some notation to be more compatible with Fontaine's article. Let $E$ be a field of characteristic $p$ and consider $E^{\mathrm{sep}}$ its separable closure. The Galois group of $E$ is denoted by $G_E$ and we take, $\rho$, a $\mathbf{Z}_p$-representation of $G_E$: *i.e.* $G_E$ is acting on a $\mathbf{Z}_p$-module via $\rho$ (or, we may think also as a continuous $\mathbf{Q}_p$-representation of $G_E$).

Let $\mathcal{E}$ be a complete field for a discrete valuation of characteristic 0, whose valuation ring is denoted by $\mathcal{O}_\mathcal{E}$, absolutely unramified (*i.e.* $p$ is a generator of the maximal ideal) such that its residue field is $E = \mathcal{O}_\mathcal{E}/p\mathcal{O}_\mathcal{E}$. This is called a *Cohen field/ring* for $E$ [22, § 18]. In case $E$ is perfect of characteristic $p$ then *the* Cohen ring is called "ring of the Witt vectors". Note that the construction of the Cohen rings is not functorial, while in the perfect case is.

***Remark 2.3.*** — When $E = \mathbf{F}_p(\!(u)\!)$, it is easy to describe a Cohen ring for it:

$$\mathcal{O}_\mathcal{E} = \Big\{ f = \sum_{i=-\infty}^{+\infty} a_i t^i \,\Big|\, a_i \in \mathbf{Z}_p,\ \lim_{i\to -\infty} |a_i| = 0 \Big\}$$

and $\mathcal{E} = \mathcal{O}_\mathcal{E} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ ($t$ is an indeterminate). The valuation on $\mathcal{O}_\mathcal{E}$ is given by $|f| = \max|a_i|_p$: hence the reduction of $t$ is $u$. The Cohen ring for $\mathbf{F}_p$ is $\mathbf{Z}_p$, which is also the ring of the Witt vectors.

We then denote by $\mathcal{E}_{\mathsf{nr}}$ the maximal unramified algebraic extension of $\mathcal{E}$. The residue field of $\mathcal{E}_{\mathsf{nr}}$ is a separable closure of the residue field $E$. We indicate by $\widehat{\mathcal{E}}_{\mathsf{nr}}$ its completion. Then we have $G_E = \mathsf{Gal}(\mathcal{E}_{\mathsf{nr}}/\mathcal{E}) = \mathsf{Gal}(\widehat{\mathcal{E}}_{\mathsf{nr}}/\mathcal{E})$ (see [44]).

Let us suppose that on the ring $\mathcal{O}_\mathcal{E}$ there is an endomorphism $\sigma$ which we will call Frobenius: it has the property that its reduction modulo $p$ is the usual Frobenius $\sigma(x) = x^p$. We denote its unique extension to $\widehat{\mathcal{E}}_{\mathsf{nr}}$ by $\varphi$: it commutes with $G_E$. Denote by $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ the category of modules of finite type over $\mathbf{Z}_p$ endowed with a linear and continuous action of $G_E$. If $V \in \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$, we put:

$$\mathsf{D}_{\mathcal{E}}(V) = (\mathcal{O}_{\hat{\mathcal{E}}_{\mathsf{nr}}} \otimes_{\mathbf{Z}_p} V)^{G_E}.$$

Then $\mathsf{D}_{\mathcal{E}}(V)$ is a $\mathcal{O}_{\mathcal{E}}$-module of finite type endowed with a $\varphi$-action (semilinear with respect the Frobenius lifting in $\mathcal{E}$). This action is étale (*i.e.* its linearization $\Phi\colon \mathsf{D}_{\mathcal{E}}(V) \otimes_\sigma \mathcal{E} \to \mathsf{D}_{\mathcal{E}}(V)$ is bijective, we can call them *unit-root* isocrystals). We denote by $\mathbf{\Phi M}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ the category of étale (finite) $\mathcal{O}_{\mathcal{E}}$-modules of finite type. Then

***Theorem 2.4.*** — *([18, Théorème 3.4.3]) The two functors*

$$\mathsf{D}_{\mathcal{E}}\colon \mathrm{Rep}_{\mathbf{Z}_p}(G_E) \to \mathbf{\Phi M}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$$

*and $\mathsf{V}_{\mathcal{E}}$, which for every étale $\varphi$-module $M$ is given by*

$$\mathsf{V}_{\mathcal{E}}(M) = (\mathcal{O}_{\widehat{\mathcal{E}}_{\mathsf{nr}}} \otimes_{\mathcal{O}_{\widehat{\mathcal{E}}}} M)^{\varphi=1}$$

*give quasi-inverse equivalences of categories.*

So we have linearized our category of representations $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ in a category of modules over a ring in characteristic 0 endowed with a Frobenius action $\mathbf{\Phi M}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$. But what about our problem? If we start with a $p$-adic representation of $G_{\mathbf{Q}_p}$, then one may hope to have an intermediate field $\mathbf{Q}_p^\infty$ as before, such that the Galois group $\mathsf{Gal}(\overline{\mathbf{Q}}_p/\,\mathbf{Q}_p^\infty) = G_{\mathbf{Q}_p^\infty}$ is the Galois group of a field $E$ of characteristic $p$ (for brevity we put $\Gamma = \mathsf{Gal}(\mathbf{Q}_p^\infty\,/\,\mathbf{Q}_p)$). If we take a Cohen ring for $E$ *i.e.* $\mathcal{O}_{\mathcal{E}}$ (its fraction field $\mathcal{E}$) endowed with a lifting of the Frobenius $\varphi$ which commutes with the action of $\Gamma$ then to a $p$-adic representation of $G_{\mathbf{Q}_p}$ we can associate a $\varphi$-$\mathcal{O}_{\mathcal{E}}$ module endowed with a $\Gamma$ action: a $(\varphi,\Gamma)$-module *i.e.* an object of $\mathbf{\Phi M}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ plus an action of $\Gamma$ which is linked to the fact that we started from a representation of $G_{\mathbf{Q}_p}$.

But where to find such an intermediate field, how to characterize it? When does it exist? And then, how to attach to it a field of characteristic $p$ having the same Galois group (*tilting* for perfectoid fields)? Moreover one can go even further in a more geometric direction. The spectrum of a field is a point and the Galois group can be seen as its fundamental group... what about generalization to *a kind* of affine schemes? *i.e.* to perfectoids algebras? These questions are at the basis of the theory of perfectoid spaces.

To give the flavour of the tilting method, we give some details about its ancestor, the so-called *field of norms* (*cf* [51]).

Let $\rho$ be a $\mathbf{Z}_p$-representation of $G_{\mathbf{Q}_p} = \mathsf{Gal}(\overline{\mathbf{Q}}_p/\,\mathbf{Q}_p)$. Put

$$\mathbf{Q}_p^\infty = \cup_n \mathbf{Q}_p(\zeta_n)$$

(where $(\zeta_n)_{n\in\mathbf{N}}$ is a compatible sequence of primitive $p^n$-roots of unity in $\overline{\mathbf{Q}}_p$). Such a field appears in [51] as an example of Arithmetic Profinite Extension (APF) (*i.e.* the upper ramification groups $\mathsf{Gal}(\mathbf{Q}_p^\infty\,/\,\mathbf{Q}_p)^x$ are open in $\mathsf{Gal}(\mathbf{Q}_p^\infty\,/\,\mathbf{Q}_p)$ for all values of $x$). This definition is closely connected with the property of being a deeply ramified extension[(5)]. A reference for all of this is [34].

To an APF extension Fontaine and Winterberger were able to associate a *field of norms* in characteristic $p$. In fact consider, in full generality, $L$ an APF extensions of a local field $K$. We denote by $\mathcal{E}_{L/K}$ the set of finite extensions of $K_0$ (where $K_0$ is the maximal unramified extension of $K$ inside $L$) and then we define $X_K(L)$ as the inverse limit:

$$X_K(L) = \varprojlim_{\mathcal{E}_{L/K}} E$$

where the transition maps are given by the norms $N_{E'/E}\colon E' \to E$ between $E \subset E'$.

(5) A Galois extension $L/K$ is *deeply ramified* if its set of upper ramification breaks is unbounded

This is the reason why it is called field of norms, but where is the field structure? That it has a multiplicative structure is obvious by definition, what is surprising is the fact that we put also an additive one. The law is given by $(\alpha_E) + (\beta_E) = (\gamma_E)$ where

$$\gamma_E = \varinjlim_{E \subset M} N_{M/E}(\alpha_M + \beta_M)$$

where the limit is taken all over the field extension $E \subset M$. Then we have:

***Theorem 2.5.*** — *If $L$ an APF extension of $K$, then $X_K(L)$ is a complete discrete valuation field of characteristic $p$ (an absolute value being given by $v(\alpha) = v_E(\alpha_E)$ for $\alpha = (\alpha_E)_{E \in \mathcal{E}_{L/K}} \in X_K(L)$). Its residue field $k_L$ is isomorphic to the residue field of $L$ (they are* perfect*). Moreover we have an identification between the separable finite extensions of $L$ and those of $X_K(L)$ (not only for the Galois ones).*

***Remark 2.6.*** — Note that the valuation $v_E$ is the normalized valuation in $E$ and $v$ does not depend upon $E$. We may think of the norms as a kind of iterated $p^{\text{th}}$-powers of Frobenius if the extensions are of degrees powers of $p$.

***Remark 2.7.*** — Every complete discrete valuation field $E$ of characteristic $p$ whose residue field is $k_E$ is isomorphic to $k_E(\!(t)\!)$ where $t$ is an indeterminate and seen as a uniformizer for the $t$-adic valuation. And $t$ corresponds to a norm compatible system of uniformizers. When $L = \mathbf{Q}_p^{\infty}$, the field of norms is $\mathbf{F}_p(\!(t)\!)$, where $t = (\zeta_n - 1)_{n \in \mathbf{N}}$.

So, we started with $\rho$, a $p$-adic Galois representation of $\mathsf{Gal}(\overline{\mathbf{Q}}_p/\,\mathbf{Q}_p)$. We may then define $H = \mathsf{Gal}(\overline{\mathbf{Q}}_p/\,\mathbf{Q}_p^{\infty})$ and we can remark that $\mathsf{Gal}(\mathbf{Q}_p^{\infty}/\,\mathbf{Q}_p) = \Gamma \simeq \mathbf{Z}_p^{\times}$ (We follow in this part [4]). $\mathbf{Q}_p^{\infty}$ is an APF extension. By the theory of the field of norms, its Galois group $H$ is the Galois group of a field of characteristic $p$: $\mathbf{F}_p(\!(t)\!)$. We may apply this to the restriction of $\rho_{|H}$. It is a representation of a field of characteristic $p$: by theorem 2.4, giving $\rho_{|H}$ is equivalent to giving a module over

$$\mathcal{O}_{\mathcal{E}} = \Big\{ \sum_{i=-\infty}^{+\infty} a_i t^i \,\Big|\, a_i \in \mathbf{Z}_p,\ \lim_{i \to -\infty} |a_i| = 0 \Big\}$$

(see remark 2.3) endowed with a Frobenius $\varphi$. But we had also an action of $\Gamma$ that can be proved to be compatible with the field of norms construction: at the end our $\rho$ is equivalent to a module defined over $\mathcal{O}_{\mathcal{E}}$ endowed with an action of Frobenius $\varphi$ and $\Gamma$: a $(\varphi, \Gamma)$-module. This is the starting point of the so called $p$-adic Hodge theory.

## 3. Almost mathematics

In the seminal paper [46], Tate realized that there are extensions of local fields that, even being possibly ramified, behave like they are unramified, in the sense that one can perform certain cohomological computations as in the étale case. Generalizing these ideas, Faltings, in a series of papers, developed the theory of “almost mathematics”, culminating with the proof of his “almost purity theorem” in [15]. Faltings used this result to prove a fundamental comparison theorem in $p$-adic Hodge theory (*cf* section 7). Later, almost mathematics has been completely axiomatized by Gabber and Ramero in [21] and the almost purity theorem has been proved more generally for perfectoid spaces by Scholze in [36].

Let us now sketch the main points of the theory. Detailed proofs can be found in [21, §2 and §3]. Let $K^{\circ}$ be a rank 1 *non*-discrete valuation ring, with fraction field $K$ and maximal ideal $\mathfrak{m}$. We assume that $|\cdot|\colon K \to \mathbf{R}_{\geq 0}$ has dense image. The main example is the ring of integers of a perfectoid field (*cf* definition 5.9). Note that as consequence of the non discreteness of the valuation, we have that $\mathfrak{m}$ is not finitely generated, so there is no uniformizer. Moreover, we have $\mathfrak{m}^2 = \mathfrak{m}$.

***Remark 3.1.*** — Eamples of such $K$ include (the completion of) $\mathbf{Q}_p(p^{1/p^{\infty}})$ or $\mathbf{Q}_p(\mu_{p^{\infty}})$. Note that $K$ can be of characteristic $p$, for example $K = \mathbf{F}_p(\!(t^{1/p^{\infty}})\!)$.

***Definition 3.2.*** — Let $M$ be a $K^\circ$-module. We say that $M$ is *almost zero* if for all $x \in \mathfrak{m}$ and for all $m \in M$ we have $xm = 0$. The full subcategory of $K^\circ$-**mod** given by almost zero modules is a Serre subcategory. The quotient, denoted $K^{\circ\mathrm{a}}$-**mod**, is by definition the category of *almost modules*. We shall write $M \mapsto M^{\mathrm{a}}$ for the localization functor.

***Remark 3.3.*** — In practice, what we are doing is "to force" morphisms whose kernel and cokernel are almost zero to become isomorphisms in the quotient category, so we are not changing the objects, but there are more isomorphisms in the category of almost modules (the terminology is rather unfortunate, as an almost module must come from an actual module). More precisely, objects of $K^{\circ\mathrm{a}}$-**mod** are the $M^{\mathrm{a}}$, for a $K^\circ$-module $M$, and one can easily prove that

$$\mathsf{Hom}_{K^{\circ\mathrm{a}}}(M^{\mathrm{a}}, N^{\mathrm{a}}) = \mathsf{Hom}_{K^\circ}(\mathfrak{m} \otimes_{K^\circ} M, N).$$

It follows that $\mathsf{Hom}_{K^{\circ\mathrm{a}}}(M^{\mathrm{a}}, N^{\mathrm{a}})$ is naturally a $K^\circ$-module and one can show that it has no almost zero elements.

***Notation 3.4.*** — (*cf* [38, §2]) Put $\Gamma = |K^\times|$ and let $\pi \in \mathfrak{m} \setminus \{0\}$. The logarithm with base $|\pi|$ identifies $\mathbf{R}_{>0}$ with $\mathbf{R}$, inducing a valuation $v\colon K \to \mathbf{R} \cup \{\infty\}$. For all $\varepsilon \in \log \Gamma := v(K^\times)$, we fix an element $\pi^\varepsilon \in K$ such that $v(\pi^\varepsilon) = \varepsilon$, so that $\mathfrak{m}$ is generated by $\{\pi^\varepsilon \mid \varepsilon \in \mathbf{R}_{>0} \cap \log \Gamma\}$. If $M$ and $N$ are two $K^\circ$-modules and $\varepsilon \in \mathbf{R}_{>0} \cap \log \Gamma$, we write $M \approx_\varepsilon N$ if there exist maps $f_\varepsilon\colon M \to N$ and $g_\varepsilon\colon N \to M$ such that $g_\varepsilon \circ f_\varepsilon = \pi^\varepsilon \,\mathsf{Id}_M$ and $f_\varepsilon \circ g_\varepsilon = \pi^\varepsilon \,\mathsf{Id}_N$. If $M \approx_\varepsilon N$ for all $\varepsilon \in \mathbf{R}_{>0} \cap \log \Gamma$, we write $M \approx N$.

***Remark 3.5.*** — If $M$ and $N$ are $K^\circ$-modules such that $M^{\mathrm{a}} \simeq N^{\mathrm{a}}$, then $M \approx N$, but the converse does not hold. For instance, if $r \in \mathbf{R}_{\geq 0}$, the ideal $I_r := \bigcup_{\substack{\varepsilon \in \log \Gamma \\ \varepsilon > r}} \pi^\varepsilon K^\circ$ satisfies $I_r \approx K^\circ$, but $I_r^{\mathrm{a}}$ is not isomorphic to $K^{\circ\mathrm{a}}$ if $r \notin \log \Gamma$. Nevertheless, we have $M^{\mathrm{a}} = 0 \Leftrightarrow M \approx \{0\}$.

The category $K^{\circ\mathrm{a}}$-**mod** is an abelian tensor category and the localization functor is compatible with kernels, cokernels, and tensor products. This just means that we have the notions of kernel and cokernel of a morphism and we also have a natural tensor product, denoted $- \otimes_{K^{\circ\mathrm{a}}} -$. All these notions behave as usual and moreover, for all $K^\circ$-modules $M$ and $N$ and all morphism $f\colon M \to N$, we have

$$\mathsf{Ker}(f^{\mathrm{a}}\colon M^{\mathrm{a}} \to N^{\mathrm{a}}) = \mathsf{Ker}(f\colon M \to N)^{\mathrm{a}}$$
$$\mathsf{Coker}(f^{\mathrm{a}}\colon M^{\mathrm{a}} \to N^{\mathrm{a}}) = \mathsf{Coker}(f\colon M \to N)^{\mathrm{a}}$$
$$M^{\mathrm{a}} \otimes_{K^{\circ\mathrm{a}}} N^{\mathrm{a}} = (M \otimes_{K^\circ} N)^{\mathrm{a}}$$

We also have an internal Hom functor, denoted $\mathsf{alHom}_{K^{\circ\mathrm{a}}}(-,-)$ and given by

$$\mathsf{alHom}_{K^{\circ\mathrm{a}}}(M^{\mathrm{a}}, N^{\mathrm{a}}) = \mathsf{Hom}_{K^\circ}(M, N)^{\mathrm{a}}.$$

***Remark 3.6.*** — Recall that by "internal Hom functor" we mean that $\mathsf{alHom}_{K^{\circ\mathrm{a}}}(M^{\mathrm{a}}, N^{\mathrm{a}})$ is an almost module and that we have, canonically for all $K^{\circ\mathrm{a}}$-modules $M^{\mathrm{a}}$, $N^{\mathrm{a}}$, and $P^{\mathrm{a}}$,

$$\mathsf{Hom}_{K^{\circ\mathrm{a}}}(M^{\mathrm{a}} \otimes_{K^{\circ\mathrm{a}}} N^{\mathrm{a}}, P^{\mathrm{a}}) = \mathsf{Hom}_{K^{\circ\mathrm{a}}}(M^{\mathrm{a}}, \mathsf{alHom}(N^{\mathrm{a}}, P^{\mathrm{a}})).$$

Given an almost module $M \in K^{\circ\mathrm{a}}$-**mod**, we write $M_*$ for $\mathsf{Hom}_{K^{\circ\mathrm{a}}}(K^{\circ\mathrm{a}}, M)$. The functor $M \mapsto M_*$ is then right adjoint to the localization functor. Moreover the adjunction morphism $(M_*)^{\mathrm{a}} \to M$ is an isomorphism. Note that if $M$ is a $K^\circ$-module then $(M^{\mathrm{a}})_* = \mathsf{Hom}_{K^{\circ\mathrm{a}}}(K^{\circ\mathrm{a}}, M^{\mathrm{a}}) = \mathsf{Hom}_{K^\circ}(\mathfrak{m}, M)$. The localization functor has also a left adjoint, given by $M \mapsto M_! = M_* \otimes_{K^\circ} \mathfrak{m}$. Again, for an almost module $M$, the adjunction morphism $M \to (M_!)^{\mathrm{a}}$ is an isomorphism.

***Remark 3.7.*** — One convenient way of thinking about the category of almost modules is the following: we have the open immersion $\mathsf{Spec}(K) \hookrightarrow \mathsf{Spec}(K^\circ)$ and the corresponding functor $K^\circ$-**mod** $\to K$-**mod**. We also have a functor $K^{\circ\mathrm{a}}$-**mod** $\to K$-**mod** and the composition with the localization functor corresponds to taking the generic fiber. The situation is the same as if we had morphisms

$$K^\circ \to K^{\circ\mathrm{a}} \to K$$

such that the corresponding morphisms

$$\mathsf{Spec}(K) \hookrightarrow \mathsf{Spec}(K^{\circ a}) \hookrightarrow \mathsf{Spec}(K^{\circ})$$

are all open immersions. (Indeed, open immersions can be characterized categorically, and one can check the this abstract properties are verified by the localization functor and his right adjoint, even if we do not have the actual ring $K^{\circ a}$.)

In particular, we see that the generic fiber functor factors through the localization functor, and one should think the latter as a “slightly generic fiber”. In general, proprieties over $\mathsf{Spec}(K^{\circ})$ stay true, by restriction, over $\mathsf{Spec}(K)$ (and over $\mathsf{Spec}(K^{\circ a})$, that is, after localization). The converse is of course usually not true: we can not extend properties from the generic fiber to the whole $K^{\circ}$ and in general not even to the smaller $K^{\circ a}$. In the “perfectoid world” by the way, proprieties over $\mathsf{Spec}(K)$ “almost extend” to $\mathsf{Spec}(K^{\circ})$, in the sense that they are true over $\mathsf{Spec}(K^{\circ a})$ (*cf* section 6).

We want now to move on from modules to algebras: the natural approach would be to introduce the notion of an almost algebra, and then the notion of an almost module over an almost algebra. This can be done without too much trouble, but it is rather abstract. Since we are not interested in the abstract theory by itself, but rather in properties of actual algebras that are “almost true”, we follow a different approach. Let $R$ be a $K^{\circ}$-algebra and let $M$ be an $R$-module.

***Definition 3.8.*** — We say that $M$ is *almost flat* if $\mathsf{Tor}_i^R(M, X)$ is almost zero for all $i > 0$ and for any $R$-module $X$. This is the same as asking that, given an injective map of $R$-modules $f\colon M_1 \hookrightarrow M_2$, then $\mathsf{Ker}(f \otimes_R M\colon M_1 \otimes_R M \to M_2 \otimes_R M)$ is almost zero. Dually, we say that $M$ is *almost projective* if $\mathsf{Ext}_i^R(M, X)$ is almost zero for all $i > 0$ and for any $R$-module $X$. This is the same as asking that, given a surjective map of $R$-modules $f\colon M_1 \twoheadrightarrow M_2$, then the cokernel of the induced map $\mathsf{Hom}(M, M_1) \to \mathsf{Hom}(M, M_2)$ is almost zero.

We have that the localization functor preserve flatness, in the sense that flat $R$-module are almost flat. Moreover, the tensor product of almost flat (resp. almost projective) modules is again almost flat (resp. almost projective). Finally, almost projective modules are automatically almost flat.

We now define some almost finiteness condition. The fact that $\mathfrak{m}$ is not finitely generated as $K^{\circ}$-module causes some trouble and complicates a little bit the definitions.

***Definition 3.9.*** — Let $R$ and $M$ be as above.

(i) We say that $M$ is *almost finitely generated* if, for all $\varepsilon \in \mathbf{R}_{>0} \cap \log \Gamma$, there exist a finitely generated $R$-module $M_\varepsilon$ such that $M \approx_\varepsilon M_\varepsilon$. This is the same as asking that, for all finitely generated submodule $\mathfrak{m}_0 \subset \mathfrak{m}$, there exist a finitely generated $R$-module $M_0 \subset M$ such that $\mathfrak{m}_0 M \subset M_0$.

(ii) We say that $M$ is *almost finitely presented* if, for all $\varepsilon \in \mathbf{R}_{>0} \cap \log \Gamma$, there exist a finitely presented $R$-module $M_\varepsilon$ such that $M \approx_\varepsilon M_\varepsilon$. This is the same as asking that, for all finitely generated submodule $\mathfrak{m}_0 \subset \mathfrak{m}$, there exist a complex

$$R^m \xrightarrow{f} R^n \xrightarrow{g} N \to 0$$

such that $\mathfrak{m}_0 \,\mathsf{Coker}(g) = 0$ and $\mathfrak{m}_0 \,\mathsf{Ker}(g) \subset \mathsf{Im}(f)$.

***Remark 3.10.*** — Note that $M_\varepsilon$ depends on $\varepsilon$ in general. More importantly, the number of generators of $M_\varepsilon$ is not necessarily bounded when $\varepsilon$ varies: if $M_\varepsilon$ can be chosen to be generated by $n$ elements, for all $\varepsilon$, then $M$ is said to be *uniformly almost finitely generated*.

***Example 3.11.*** — (*cf* [38, Theorem 2.5]). If $M$ is an almost finitely generated $K^{\circ}$-module, there exists a unique $n \in \mathbf{N}$ and a unique sequence $r_1 \geq r_2 \geq \cdots \geq 0$ of real numbers such that $\lim\limits_{i \to \infty} r_i = 0$ and

$$M \approx (K^{\circ})^n \oplus (K^{\circ}/I_{r_1}) \oplus (K^{\circ}/I_{r_2}) \oplus \cdots$$

(*cf* remark 3.5).

We will need the following technical result in section 7.

***Lemma 3.12.*** — *(cf* [38, Lemma 2.12]*). Assume that $K$ is algebraically closed of characteristic $p$. For all $k \in \mathbf{N}_{>0}$, let $M_k$ be a $K^\circ/\langle \pi^k \rangle$-module. Assume that :*

1. *$M_1$ is almost finitely generated ;*
2. *for all $k \in \mathbf{N}_{>0}$, there are maps $p_k \colon M_{k+1} \to M_k$ and $q_k \colon M_k \to M_{k+1}$ such that $p_k \circ q_k = \pi \operatorname{Id}_{M_k}$ and the sequence $M_1 \xrightarrow{q_k \circ \cdots \circ q_1} M_{k+1} \xrightarrow{p_k} M_k$ is exact ;*
3. *for all $k \in \mathbf{N}_{>0}$, there are isomorphisms $\varphi_k \colon M_k \otimes_{K^\circ/\langle \pi^k \rangle, \varphi} K^\circ/\langle \pi^{pk} \rangle \xrightarrow{\sim} M_{pk}$ that are compatible with $p_k$ and $q_k$ (where $\varphi$ is the Frobenius map).*

*Then there exist $r \in \mathbf{N}$ and isomorphisms $M_k^{\mathrm{a}} \simeq (K^{\circ \mathrm{a}}/\langle \pi^k \rangle)^r$ of $K^{\circ \mathrm{a}}$-modules, such that $p_k$ (resp. $q_k$, resp. $\varphi_k$) is carried to the projection (resp. the multiplication by $\pi$, resp. the coordinatewise Frobenius map).*

We have that an almost finitely generated module that is also almost projective is automatically almost finitely presented and, for almost flat modules, almost finitely generated implies almost finitely presented. We say that a module that is almost flat and almost finitely presented is almost finite projective. It is the same as almost projective and almost finitely generated.

***Definition 3.13.*** — Let $f \colon R \to S$ be a morphism of $K^\circ$-algebras. We say that $S$ is an *almost finite étale* $R$-algebra if

1. $S$ is an almost finite projective $R$-module;
2. $S$ is an almost projective $S \otimes_R S$-module via the diagonal morphism.

We write $A_{\text{fét}}$ for the category of almost finite étale $A$-algebras.

***Remark 3.14.*** — There exists a more general notion of an almost étale $A$-algebra, and a morphism $A \to B$ is almost finite étale if and only if it is almost étale and $B$ is almost finitely presented as $A$-module, see [36, Section 4].

Let $A$ be a $\varpi$-adically complete and flat $K^\circ$-algebra, where $\varpi \in K^\circ$ be an element of positive valuation. As in the classical case, we have that almost finite étale algebras lift uniquely over nilpotent thickenings, in the sense that the base change functor is an equivalence of categories $A_{\text{fét}} \cong (A/\varpi A)_{\text{fét}}$.

## 4. Adic spaces

The notion of adic space has been introduced by Roland Huber during the nineties in order to construct a category containing both the category of rigid analytic spaces and locally noetherian formal schemes. In fact, the construction of Berthelot's generic fibre produces a rigid analytic space attached to a formal scheme, but the formal scheme and the generic fibre live in different categories. The formalism of adic spaces gives a category in which both a locally noetherian formal scheme and its generic fibre define "the same kind of space". After their introduction, adic spaces have been a bit forgotten for about twenty years. Interest in this new kind of geometrical objects rekindled during the publication of Scholze's first works on perfectoid spaces. In fact, with the introduction of perfectoid spaces, it becomes immediately clear that the formalism of adic spaces is the right language to define them. The construction of adic spaces is a typical geometric construction. As in the case of schemes, we will develop the construction of adic spaces in two steps. First we define affine adic spaces and their points, and we equip them with a suitable topology. Finally, we introduce the structure presheaf, which we will see that a priori is not a sheaf. Maybe this last fact is the main reason why the geometry of adic spaces has not been really developed before the introduction of perfectoid spaces. In fact, it can be proved that, in the case of perfectoid spaces, the structure presheaf of the associated adic space satisfies sheaf axioms, and so it gives rise to a reasonable geometry.

**4.1. Affinoid adic spaces.** — As in the construction of schemes and formal schemes, the first object we need to introduce is the algebraic counterpart of an adic spaces or, equivalently, the notion of affinoid adic spaces. The main point is that, in Huber geometry, valuations play the role of prime ideals in scheme theory. The first well known example of valuations are the usual discrete valuations attached to discrete valuation rings. We begin with a some general definitions.

***Definition 4.2.*** — (1) A *totally ordered group* is an abelian group $(\Gamma, \cdot)$, usually written multiplicatively, with a total order "$\leq$" compatible with the multiplication, *i.e.* such that

$$\gamma \leq \delta \Rightarrow \gamma\gamma' \leq \delta\gamma' \quad \forall \gamma, \gamma', \delta \in \Gamma$$

*Morphisms of totally ordered groups* are group homomorphisms compatible with the total orders. Given a totally ordered group $\Gamma$ it is possible to construct a totally ordered monoid, usually denoted $\Gamma \cup \{0\}$ with the conditions that $0 \leq \gamma$ and $0\gamma = 0$ for all $\gamma \in \Gamma$.

(2) An element $\gamma \in \Gamma$ totally ordered group is called *cofinal* if for all $\delta \in \Gamma$ there exists $n \in \mathbf{N}$ such that $\gamma^n \leq \delta$.

(3) We put a topology over the totally ordered monoid $\Gamma \cup \{0\}$ simply saying that a subset $U$ is open if and only if $0 \notin \Gamma$ or there exists $\gamma \in \Gamma$ such that $\Gamma_{<\gamma} := \{\delta \in \Gamma \mid \delta < \gamma\} \subseteq U$.

(4) Let $\Gamma$ be a totally ordered group and let $\Delta \leq \Gamma$ be a subgroup. Then $\Delta$ is called *convex* if for every $\gamma \in \Gamma$ such that $\delta \leq \gamma \leq \delta'$, for some $\delta, \delta' \in \Delta$, $\gamma \in \Delta$.

(5) Let $A$ be a ring. A *valuation* over $A$ with values in a totally ordered monoid $\Gamma \cup \{0\}$ is a function $|\cdot| \colon A \to \Gamma \cup \{0\}$ satisfying the following properties:
   - (i) $|a + b| \leq \max(|a|, |b|)$ for all $a, b \in A$;
   - (ii) $|ab| = |a||b|$ for all $a, b \in A$;
   - (iii) $|0| = 0$ and $|1| = 1$.

(6) Two surjective[6] valuations $|\cdot|_1$ and $|\cdot|_2$ are called *equivalent* if one of the following equivalent conditions are satisfied:
   - (i) There exists an isomorphism of totally ordered monoids $f \colon \Gamma_1 \cup \{0\} \to \Gamma_2 \cup \{0\}$ such that $f \circ |\cdot|_1 = |\cdot|_2$;
   - (ii) $\mathsf{supp}(|\cdot|_1) = \mathsf{supp}(|\cdot|_2)$ and $A(|\cdot|_1) = A(|\cdot|_2)$, where $\mathsf{supp}(|\cdot|) := \{a \in A \mid |a| = 0\}$ and $A(|\cdot|) := \{a \in A \mid |a| \leq 1\}$;
   - (iii) for all $a, b \in A$, we have $|a|_1 \leq |b|_1$ if and only if $|a|_2 \leq |b|_2$.

(7) Given $|\cdot| \colon A \to \Gamma \cup \{0\}$ a valuation, the *characteristic subgroup* $c\Gamma_{|\cdot|}$ of $\Gamma$ is the group of all the elements in the image of $|\cdot|$ with valuation greater or equal than 1.

We want to point out that the previous definition, also given in Bourbaki, allows to consider a very general class of valuations, surely bigger than the usual discrete valuations. Moreover, the notion of totally ordered group does not require an embedding of the value group of a valuation in a "concrete" group.

In Huber geometry, the role played by rings in scheme theory is played by a particular class of topological rings. Recall that by a topological ring we mean a ring equipped with a topology compatible with ring operations. A well known example is a ring with $I$-adic topology, for $I$ an ideal. We now introduce the crucial notion of Huber rings, or $f$-adic rings, as they are called in Huber's papers.

***Definition 4.3.*** — (1) Let $A$ be a topological ring. A subset $B \subseteq A$ is called *bounded* if for all $U \subseteq A$ open neighborhood of 0, there exists a $V$ open neighborhood of 0 such that $BV := \left\{\sum_{i=1}^{N<\infty} b_i v_i \text{ with } b \in B, v \in V\right\} \subseteq U$.

(2) An element $a \in A$ is called *power bounded* if $B = \{a^n \mid n \in \mathbf{N}\}$ is bounded. We denote $A^\circ$ the set of power bounded elements.

(3) An element $a \in A$ is called *topologically nilpotent* if for all $U$ open neighborhood of 0, there exists $N \in \mathbf{N}$ such that $a^n \in U$ for all $n \geq N$. We denote $A^{\circ\circ}$ the set of topologically nilpotent units.

[6] From now on, we will always assume valuations are surjective.

(4) A subring $A_0 \subseteq A$ is called a *ring of definition* if $A_0$ is open in $A$ and the induced topology on $A_0$ is the $I$-adic topology for $I$ an ideal of $A_0$. We call such an $I$ an *ideal of definition.*
(5) A topological ring $A$ is called *Huber* if it admits a ring of definition whose adic topology is given by a finitely generated ideal of definition.
(6) A topological ring $A$ is called *Tate* if it contains a topologically nilpotent unit.

It is not hard to prove that an open subring of an Huber ring is a ring of definition if and only if it is bounded.

***Example 4.4.*** — Let $k$ be a non-archimedean field, *i.e.* a topological field with a basis of open neighborhoods of 0 given by subgroups of the additive group $(k, +)$. Consider an affinoid Tate algebra $A := k\langle X_1, \ldots, X_n\rangle / J$, for $J$ an ideal, and take $\mathscr{O}_k \{X_1, \ldots, X_n\}$ the ring of power series $\sum_I a_I X^I$ with coefficients in the valuation ring $\mathscr{O}_k$ of $k$ such that $a_I \to 0$ as $\|I\| \to \infty$. Now take the image $A_0$ of $\mathscr{O}_k \{X_1, \ldots, X_n\}$ in $A$. It can be proved that it is a ring of definition, where the topology on $A_0$ is the one induced by a pseudo-uniformizer $\varpi$ of $k$, *i.e.* an element such that $0 < |\varpi| < 1$. Notice that this ring is also a Tate ring, as, by definition, $\varpi$ is a topologically nilpotent unit.

With these definitions in mind, we are quite close to define an affinoid adic space, at least as a set. The first object we consider is:

$$\mathsf{Spv}(A) = \{\text{Equivalence classes of valuations of } A\}$$

This object is really too huge to give rise to a reasonable geometry, so the idea of Huber is to focus on particular classes of valuations over a ring. In particular, as we have introduced a topology over totally ordered monoids, we can speak of continuous valuations. Given a topological ring $A$, we call $\mathsf{Cont}(A)$ the set of equivalence classes of continuous valuations over $A$. There is another construction which is in some sense intermediate between $\mathsf{Spv}$ and $\mathsf{Cont}$, and which is useful to state a semi-algebraic characterization of continuity. What we define here is $\mathsf{Spv}(A, J)$.

***Definition 4.5.*** — Given an Huber ring $A$ with a valuation $|\cdot|\colon A \to \Gamma_{|\cdot|} \cup \{0\}$ and an ideal $J$ whose radical is the radical of a finitely generated ideal, we define the subgroup of the value group $c\Gamma(J)$ as the minimal convex subgroup of $\Gamma_{|\cdot|}$ containing $c\Gamma_{|\cdot|}$ and having nonempty intersection with the image of $|\cdot|$. Moreover, we put

$$\mathsf{Spv}(A, J) = \{v \in \mathsf{Spv}(A) \mid c\Gamma_v(J) = \Gamma_v\}$$

It can be shown, and it is really a useful operative criterion, that, in the case of an Huber ring, the continuity of a valuation is controlled essentially by the behaviour of topologically nilpotent elements under valuation. In particular the following holds:

***Theorem 4.6.*** — *Let $A$ be an Huber ring. Then we have:*

$$\mathsf{Cont}(A) = \{v \in \mathsf{Spv}(A, A^{\circ\circ} \cdot A) \mid v(a) \leq 1 \textit{ for all } a \in A^{\circ}\}.$$

Now we are quite done, since an affinoid space is something smaller than the continuous spectrum of an Huber ring.

***Definition 4.7.*** — (1) Let $A$ be an Huber ring. We call a subring $A^+$ of $A$ a *ring of integral elements* if it is open and integrally closed in $A$, and $A^+ \subseteq A^{\circ}$. We call a couple $(A, A^+)$ a *Huber pair.*
(2) We can construct the category of Huber pairs defining morphisms of Huber pairs. Given $(A, A^+)$ and $(B, B^+)$ two Huber pairs, we call morphism of Huber pair $f\colon (A, A^+) \to (B, B^+)$ a continuous ring homomorphism $f\colon A \to B$ such that $f(A^+) \subseteq B^+$.
(3) Given a Huber pair $(A, A^+)$, we call *affinoid adic space* the set

$$\mathsf{Spa}(A, A^+) := \{v \in \mathsf{Cont}(A) \mid v(a) \leq 1 \text{ for all } a \in A^+\}$$

equipped with a suitable topology (see below).

It is possible to show that, for a non-archimedean valuation field $k$ with valuation ring $k^\circ$, $\mathsf{Spa}(k, k^\circ)$ consists of only one point, while there are examples of such fields such that $\mathsf{Cont}(k)$ is infinite. This can at least make reasonable the definition of $\mathsf{Spa}$. It can also be shown that this construction is functorial. Finally, given a Huber pair $(A, A^+)$ with ring of definition $(A_0, I)$, it is possible to construct its Hausdorff completion

$$\widehat{A} := \varprojlim_n A/I^n$$

where the limit must be intended in the sense of additive topological group. It can be shown that this completion has topological ring structure, and moreover $\widehat{A}$ is also complete and separated for this topology. Its ring of definition is the usual $I$-adic completion of $A_0$. On the "+" part, we define the completion in the same way, and then we integrally close again. The adic spectrum does not depend on the fact of the pair being complete or not and of taking $A^+$ integrally closed or not (*cf* [23, Lemma 3.3 & Proposition 3.9]). However, completeness and integral closure will be necessary in the definition of the structure presheaf.

**4.8. Localizations and topology.** — As in scheme theory, where the topology is generated by principal affine opens, also in the context of adic spaces, the topology that can be put on an adic space is strictly related to the idea of localizations of Huber pairs. The point is that in the theory of localization of Huber rings topology must be taken into account. The construction of a right notion of localization sounds very familiar to those who know the theory of rigid analytic spaces. The following result defines the topology we are looking for. Note that it holds, not only for Huber pairs, but for any ring equipped with a non-archimedean ring topology, *i.e.* a topological ring with a fundamental system of neighborhoods of 0 given by subgroups of $(A, +)$.

***Theorem 4.9.*** — *Let $A$ be a non-archimedean ring, $S = (s_i)_{i\in I}$ a family of elements of $A$ and $R \subseteq A$ the multiplicative subset generated by $S$. Let then $T = (T_i)_{i\in I}$ be a family of finite subsets of $A$ such that for all open additive subgroup $U$ of $A$, $T_i \cdot U$ is open in $A$ for all $i \in I$. Then there exists a non-archimedean topology on $R^{-1}A$ making it into a topological ring, denoted*

$$A\left(\tfrac{T}{S}\right) = A\left(\tfrac{T_i}{s_i} \mid i \in I\right)$$

*such that $\{\frac{t_i}{s_i} \mid t_i \in T_i, i \in I\}$ is power bounded in $A\left(\frac{T}{S}\right)$ and such that the canonical continuous homomorphism $\phi\colon A \to A\left(\frac{T}{S}\right)$ satisfies the following universal property. If $B$ is a non-archimedean topological ring and $f\colon A \to B$ a continuous homomorphism such that $f(s_i)$ is invertible in $B$ for all $i \in I$ and such that the set $\{f(t)f(s_i)^{-1} \mid i \in I, t \in T\}$ is power bounded in $B$, then there exists a unique continuous ring homomorphism $g\colon A\left(\frac{T}{S}\right) \to B$ with $f = g \circ \phi$.*

Notice that in the theory of localization becomes evident the fact that topology makes the structure richer. In fact, as a ring $A\left(\frac{T}{S}\right) = R^{-1}A$ for every choice of $T$, but topology adds a dependence on $T$, and so the same localizations with different $T$'s generating the topology may be different in the category of Huber rings. The condition on the $T_i$'s via the Huber hypothesis, can be weakened to ask that just $T_i \cdot A$ is open in $A$ for all $i$. We can also define as before the completion of a localization, which we denote as

$$A\langle \tfrac{T}{S} \rangle := \widehat{A\left(\tfrac{T}{S}\right)}.$$

Notice that, by universal property, $A\langle T/S\rangle$ is universal among complete non-archimedean $A$-algebras in which each element in $S$ is a unit and each fraction $t/s$ is power bounded. This yelds the consistency with the analogue construction in rigid geometry, where the universal property is the same but among Banach algebras over a non-archimedean ground field. Now we want to see how localizations are related to a suitable topology on the adic spectra. First of all, we have just discussed localizations of Huber rings, but we usually deal with Huber pairs of the form $(A, A^+)$. In this situation, we interpret the localization of a Huber pair as the couple $\left(A\left(\frac{T}{S}\right), A\left(\frac{T}{S}\right)^+\right)$, where the second component is the integral closure in $A(T/S)$ of the open $A^+$-subalgebra generated by fractions $t/s$ with $t \in T_i$ and $s \in S_i$ for some $i$.

Now we introduce the base for the topology we want put on affinoid adic spaces.

***Theorem 4.10.*** — *Let $(A, A^+)$ be a Huber pair and define $X := \mathsf{Spa}(A, A^+)$.*

(1) *The space $X$ has a base of quasi compact open subsets given by*

$$X\left(\tfrac{T}{s}\right) := \{v \in X \mid v(t_i) \leq v(s) \neq 0 \text{ for all } t_i \in T\}$$

*where $s \in A$ and $T \subset A$ is a finite nonempty subset such that $T \cdot A$ is open in $A$. This base is stable under finite intersections:*

$$X\left(\tfrac{T}{s}\right) \cap X\left(\tfrac{T'}{s'}\right) = X\left(\tfrac{TT'}{ss'}\right)$$

(2) *The natural map*

$$\mathsf{Spa}\left(A\left(\tfrac{T}{s}\right), A\left(\tfrac{T}{s}\right)^+\right) \to X$$

*is a homeomorphism onto $X\left(\frac{T}{s}\right)$, respecting rational domains in both directions.*

We call the open sets as in the theorem *rational domains*.

**4.11. Structure (pre)-sheaves.** — In this section, we construct the structure presheaf attached to an adic space, and we discuss its sheaf property which turns out to not be always satisfied. Let $(A, A^+)$ be a Huber pair and $X := \mathsf{Spa}(A, A^+)$, which has a base of opens given by rational domains $X(T/s)$ as in the previous section. In order to define in a suitable way a structure presheaf, it should be well understood how a rational domain $X(T/s)$ completely determines the completed Huber pair $\left(A\langle T/s\rangle, A\langle T/s\rangle^+\right)$. The relation is completely contained in the following universal property.

***Proposition 4.12.*** — *For any morphism of Huber pairs $\phi\colon (A, A^+) \to (B, B^+)$ with $B$ complete, $\mathsf{Spa}(\phi)\colon \mathsf{Spa}(B, B^+) \to \mathsf{Spa}(A, A^+)$ has image in $X(T/s)$ if and only if there is a factorization as morphisms of Huber pairs:*

$$\begin{tikzcd}
(A, A^+) \arrow[r] \arrow[d] & (B, B^+) \\
(A\langle T/s\rangle, A\langle T/s\rangle^+) \arrow[ur] &
\end{tikzcd}$$

This proposition is the first step to the definition of the structure presheaf. In fact, it says that it is possible to define in a unique way a Huber pair attached to a rational domain $U$ of $X$. For any presentation of $U$ as $X(T/s)$, we can put $\mathscr{O}_X(U) = (A\langle T/s\rangle, A\langle T/s\rangle)^+)$ : this definition is meaningful by proposition 4.12. Take for example $\mathscr{O}_X(X)$. Even if $A$ is not assumed to be complete, we immediately get $\mathscr{O}_X(X) = (\widehat{A}, \widehat{A}^+)$. We can also compute the image of the empty set under this association: we have $\varnothing = X(1/0)$, so by the previous universal property, we immediately get $\mathscr{O}_X(\varnothing) = 0$. In order to correctly define a presheaf, we need functoriality in the pair $(A, U)$. Consider a map of Huber pairs $\phi\colon (A', A'^+) \to (A, A^+)$, which induces the continuous map $f\colon X \to X'$, and let $U$ and $U'$ be rational subsets of $X$, respectively $X'$, such that $f(U) \subseteq U'$. Then we can consider the composition $(A', A'^+) \to (A, A^+) \to (\mathscr{O}_X(U), \mathscr{O}_X(U)^+)$. It satisfies the topological condition required to apply the universal property: it factors uniquely through a map

$$\mathscr{O}_X(\phi)\colon (\mathscr{O}_X(U'), \mathscr{O}(U')^+) \to (\mathscr{O}_X(U), \mathscr{O}(U)^+)$$

By uniqueness, it can be shown that this construction is transitive in $\phi$ and so it is really a functor. On the other side, if we take $(A', A'^+) = (A, A^+)$, and we consider an inclusion $i\colon V' \subseteq V$ of rational domains inside $X$, we get in a unique way a restriction map

$$\mathscr{O}_X(i)\colon (\mathscr{O}_X(V), \mathscr{O}(V)^+) \to (\mathscr{O}_X(V'), \mathscr{O}(V')^+).$$

It can also be proved that the homeomorphism $\mathsf{Spa}(A, A^+) \simeq \mathsf{Spa}(\widehat{A}, \widehat{A}^+)$ preserves rational domains, and this last fact allows to define the structure presheaf $\mathscr{O}_X$ following a standard procedure in sheaf theory: if $U \subseteq X$ is open, we put

$$\mathscr{O}_X(U) = \varprojlim_{\substack{W \text{ rational} \\ W \subset U}} \mathscr{O}_X(W)$$

In the same spirit as for $\mathscr{O}_X$, we also put

$$\mathscr{O}_X^+(U) = \varprojlim_{\substack{W \text{ rational} \\ W \subset U}} \mathscr{O}_X^+(W).$$

After having introduced the structure presheaf, we can compute the stalks, and it is not so difficult to prove the following

***Proposition 4.13.*** — *For $x \in X$, $\mathscr{O}_{X,x}$ is a local ring.*

Notice that the stalks can also be equipped with valuations. In fact, let $x \in X$ and $\mathscr{O}_{X,x}$ be the stalk. Consider two rational domains of $X$, $W' \subseteq W$, both containing the point $x$. Then the following diagram is commutative:

$$\begin{tikzcd} \mathscr{O}_X(W) \arrow[r, "v_x"] \arrow[d] & \Gamma_x \cup \{0\} \arrow[d] \\ \mathscr{O}_X(W') \arrow[r, "v'_x"] & \Gamma'_x \cup \{0\} \end{tikzcd}$$

and so, since $\mathscr{O}_{X,x}$ is a direct limit, which can be computed over rational domains, we get a valuation $v_x$ on $\mathscr{O}_{X,x}$.

Now, in order to define global adic spaces, the strategy is the usual one, also used in the definition of schemes, or formal schemes or rigid analytic space. We construct a category which plays, for adic spaces, the role of locally ringed spaces for schemes. The solution to this problem is given by the following definition, which also states a “sheafy” notion of affinoid adic space.

***Definition 4.14.*** — (1) Let $\mathscr{V}$ be the category of triples $(X, \mathscr{O}_X, (v_x)_{x\in X})$, where $X$ is a topological space, $\mathscr{O}_X$ is a sheaf of topological rings such that each stalk $\mathscr{O}_{X,x}$ is a local ring, and $v_x$ is a valuation on the residue field $k(x)$ of $\mathscr{O}_{X,x}$. Morphisms $f : X \to Y$ in $\mathscr{V}$ are maps $(f, f^\flat)$ of locally topologically ringed spaces.

(2) Given $(X, \mathscr{O}_X, (v_x)_{x\in X}) \in \mathscr{V}$, we define the sheaf of integral structures $\mathscr{O}_X^+$ as follows. For any open $U \subseteq X$, we set

$$\mathscr{O}_X^+(U) = \{f \in \mathscr{O}_X(U) \mid v_x(f) \leq 1 \text{ for all } x \in U\}.$$

(3) An affinoid adic space is an object of $\mathscr{V}$ isomorphic to $\mathsf{Spa}(A, A^+)$ with its associated sheaf and local valuation. An adic space is an object $X$ of $\mathscr{V}$ such that there exists an open covering $\{U_i\}$ of $X$ such that each $U_i$ viewed as an object in $\mathscr{V}$ with the obvious restrictions of sheaf and valuations, is an affinoid adic space.

***Remark 4.15.*** — Our previous construction only provide structure presheaves, while in the definition of the category $\mathscr{V}$, objects are triples whose second component is a sheaf. *Not all Huber pairs give rise to a structure sheaf.* Those giving a sheaf are called *sheafy.* Now, there are essentially three criteria to determine wether a Huber pair is sheafy. Fortunately, these criteria asserts that, in the most useful contexts, the presheaf is a sheaf.

***Theorem 4.16.*** — *Let $(A, A^+)$ be a Huber pair. Then $\mathcal{O}_A$ is a sheaf if either one of the following holds:*

(i) *$\widehat{A}$ has a noetherian ring of definition (Huber, for locally noetherian formal schemes);*
(ii) *$A$ is a strongly noetherian Tate ring (i.e. all the rings $A\langle X_1, \ldots, X_n\rangle$ are noetherian, Huber, for rigid analytic spaces);*
(iii) *$(A, A^+)$ is a stably uniform (i.e. for all rational $U \subseteq \mathsf{Spa}(A, A^+)$ the ring $\mathcal{O}_A(U)$ is such that $\mathcal{O}_A(U)^\circ$ is bounded) with $A$ Tate (cf [7] for perfectoid spaces).*

**4.17. Complements.** — We finish this brief introduction to the theory of adic spaces by mentioning some constructions which are typical in scheme theory. One of the first constructions available in scheme theory is the fiber product of two schemes over a common one. Firstly, we must say that in the context of adic spaces fiber products don't exist in general. One can realize this fact by trying to compute a suitable fiber product for

$$\mathsf{Spa}(\mathbf{Z}[T_1, T_2, \ldots], \mathbf{Z}) \times_{\mathsf{Spa}(\mathbf{Z},\mathbf{Z})} \mathsf{Spa}(K, \mathcal{O}_K)$$

where $K$ is a non-archimedean field, and $\mathcal{O}_K$ is its ring of integers, which means a suitable adic space representing the corresponding functor. It's not difficult to see that this functor is not represented by an adic space.

Moreover, even when the fiber product exists, its behaviour can be very different from the case of schemes. For example, fiber product of affinoid adic spaces may not be affinoid. In fact, it can be proved (*cf* [50, §5.4]) that:

$$\mathsf{Spa}(\mathbf{Z}[T], \mathbf{Z}) \times_{\mathsf{Spa}(\mathbf{Z},\mathbf{Z})} \mathsf{Spa}(K, \mathcal{O}_K)$$

is covered by an infinite union of affinoid adic spaces, connected with adic morphisms (see below), but it is not affinoid by itself. Notice that this example is a kind of "generic fiber construction".

In general, one would like to define $\mathsf{Spa}(A, A^+) \times_{\mathsf{Spa}(C,C^+)} \mathsf{Spa}(B, B^+) = \mathsf{Spa}(\widehat{A \otimes_C B}, -)$, with a suitable choice of ring of integral elements. The problem is that there is no a canonical way to topologize that fiber product. A condition related to compatibility of topologies is necessary. The natural condition to require is the following.

***Definition 4.18.*** — Let $\phi\colon (C, C^+) \to (A, A^+)$ be a morphism of adic spaces. Then $\phi$ is called *adic* if there exist a ring of definition $C_0 \subseteq C$, an ideal of definition $I \subseteq C_0$ and a ring of definition $A_0 \subseteq A$ such that $\phi(C_0) \subseteq A_0$ and $\phi(I)$ is an ideal of definition for $C_0$.

When both morphisms $\phi\colon (C, C^+) \to (A, A^+)$ and $\psi\colon (C, C^+) \to (B, B^+)$ are adic, with rings of definition (resp. ideals of definition) $C_0$, $A_0$, $B_0$ (resp. $I_C$, $I_A$, $I_B$) then it is possible to topologize $\widehat{A \otimes_C B}$ using $A_0 \otimes_{C_0} B_0$ as a ring of definition, and $\phi(I_C)A_0 \otimes_{C_0} B_0$ as ideal of definition. Using this, and the tensor product of the integral part, one can properly define a notion of fiber product of adic spaces (*cf* [49, §8.6]). Notice that in this case, fiber product of affinoid adic spaces is affinoid, hence a "generic fiber construction", like the one we mentioned above cannot be described so easily in general. Fortunately, in the case that interests us, *i.e.* when $f$ is a morphism between Huber pairs given by Tate rings, then $f$ is automatically adic (*cf* [50, Proposition 5.1.3]).

We end this section simply giving a dictionary for properties which are useful in the context of schemes, and that can be translated in the situation of adic spaces. In particular, we want to give a series of definitions which contains the notions of finite and finite type, smooth, proper, unramified and étale morphisms of adic spaces. We collect everything in a unique long definition in order to set up a kind of glossary of these notions. In the following definitions, angle brackets mean affinoid Tate algebras, *i.e.* algebras of convergent power series.

***Definition 4.19.*** —

(i) Let $f\colon (A, A^+) \to (B, B^+)$ be a morphism of Huber pairs. Then $f$ is *of topologically finite type* if it factors as

$$(A, A^+) \to (A\langle T_1, \ldots, T_n\rangle, A^+\langle T_1, \ldots, T_n\rangle) \xrightarrow{\pi} (B, B^+)$$

where $\pi$ is surjective on the first component, continous, open, and such that $B^+$ is the integral closure of $A^+\langle T_1, \ldots, T_n\rangle$ in $B$ (*cf* [24, §3]).

(ii) Let $\phi\colon X \to Y$ be a morphism of adic spaces. Then $\phi$ is called *locally of finite type* if for all $x \in X$ there exist $U$, $V$ open affinoids of $X$ and $Y$ respectively, such that $x \in U$, $\phi(U) \subseteq V$, and such that the Huber pair homomorphism $(\mathscr{O}_Y(V), \mathscr{O}_Y)^+(V)) \to (\mathscr{O}_X(U), \mathscr{O}_X)^+(U))$ induced by $\phi$ is of topologically finite type. If moreover, $\phi$ is quasi compact, then it is called finite type (*cf* [25, Definition 1.2.1 (iii)]).

(iii) If $\phi\colon X \to Y$ is a morphism of adic spaces locally of finite type, then we say that $\phi$ is *separated* if the diagonal is closed in $X \times_Y X$ (the fiber product is well defined because $f$ is locally of finite type, *cf* [25, Lemma 10.1.6]). $\phi$ is *universally closed* if for all adic morphism

$Y' \to Y$, the map $X \times_Y Y' \to Y'$ is closed. Finally, $\phi$ is *proper* if it is separated, of finite type and universally closed (*cf* [25, Definitions 1.10.13 & 1.10.14]).

(iv) A morphism $\phi\colon X \to Y$ which is locally of finite type is called *unramified* (resp. *smooth*, resp. *étale*) if for all Huber pair $(A, A^+)$, and all ideal $I \subseteq A$ such that $I^2 = 0$, and $\psi\colon \mathsf{Spa}(A, A^+) \to Y$, the map $\mathsf{Hom}_Y(\mathsf{Spa}(A, A^+), X) \to \mathsf{Hom}_Y(\mathsf{Spa}(A/I, \overline{(A^+/(A^+ \cap I))})$ (where the overline denotes the integral closure), is injective (resp. surjective, resp. bijective, *cf* [25, Definition 1.6.5]).

## 5. Perfectoid fields and their tilt

**5.1. The fundamental construction.** — The tilting construction provides a functor defined on a "good" class of spaces (namely "perfectoid spaces", *cf* definitions 6.2 & 6.19) with values in "good" spaces in characteristic $p$. It relies on the following fundamental construction, which generalizes the "perfect field of norms" theory (*cf* Remark 5.24).

Let $A$ be a ring, $\varpi \in A$ such that $\varpi^N A \subseteq pA \subseteq \varpi A$ for some $N \in \mathbf{N}_{>0}$, and $\widehat{A} = \varprojlim_{m} A/p^m A$ the completion of $A$ for the $p$-adic topology. We put

$$\mathscr{R}(A) = \varprojlim_{x \mapsto x^p} A/\varpi A = \big\{(x_n)_{n\in\mathbf{N}} \in (A/\varpi A)^{\mathbf{N}} \,\big|\, (\forall n \in \mathbf{N})\ x_{n+1}^p = x_n\big\} \subset (A/\varpi A)^{\mathbf{N}}$$

This is a ring of characteristic $p$. If $\underline{x} = (x_n)_{n\in\mathbf{N}} \in \mathscr{R}(A)$, we have $\varphi(\underline{x}) = (x_0^p, x_1^p, x_2^p, \ldots) = (x_0^p, \underbrace{x_0, x_1, \ldots}_{\underline{x}})$, so the Frobenius map $\varphi$ on $\mathscr{R}(A)$ is bijective, with inverse given by the truncation of the first term: the ring $\mathscr{R}(A)$ is perfect.

***Proposition 5.2.*** — *The reduction modulo $\varpi$ map*

$$\varprojlim_{x \mapsto x^p} \widehat{A} = \big\{\big(x^{(n)}\big)_{n\in\mathbf{N}} \in \widehat{A}^{\mathbf{N}} \,\big|\, (\forall n \in \mathbf{N})\ \big(x^{(n+1)}\big)^p = x^{(n)}\big\} \to \mathscr{R}(A)$$

*is an isomorphism of multiplicative monoids.*

*Proof.* — Let $x = (x_n)_{n\in\mathbf{N}} \in \mathscr{R}(A)$ and $(\widehat{x}_n)_{n\in\mathbf{N}} \in \widehat{A}^{\mathbf{N}}$ lifting $x$. For $n, m \in \mathbf{N}$, we have $\widehat{x}_{n+m+1}^p \equiv \widehat{x}_{n+m} \mod \varpi\widehat{A}$, so $\widehat{x}_{n+m+1}^{p^{m+1}} \equiv \widehat{x}_{n+m}^{p^m} \mod \varpi^{m+1}\widehat{A}$ (since $p \in \varpi A$). For fixed $n$, the sequence $\big(\widehat{x}_{n+m}^{p^m}\big)_{m\in\mathbf{N}} \in \widehat{A}^{\mathbf{N}}$ is Cauchy for the $p$-adic topology: it converges. Let $x^{(n)} \in \widehat{A}$ be its limit. As $\widehat{x}_{n+m}^{p^m}$ lifts $x_n$ for all $m \in \mathbf{N}$, so does $x^{(n)}$. If $(\widetilde{x}_n)_{n\in\mathbf{N}} \in \widehat{A}^{\mathbf{N}}$ is an other lifting of $x$, we have $\widetilde{x}_{n+m} \equiv \widehat{x}_{n+m} \mod \varpi\widehat{A}$, so $\widetilde{x}_{n+m}^{p^m} \equiv \widehat{x}_{n+m}^{p^m} \mod \varpi^{m+1}\widehat{A}$: the sequences $\big(\widehat{x}_{n+m}^{p^m}\big)_{m\in\mathbf{N}}$ and $\big(\widetilde{x}_{n+m}^{p^m}\big)_{m\in\mathbf{N}}$ have the same limit, *i.e.* $x^{(n)}$ only depends on $x$ and $n$. Furthermore, we have:

$$x^{(n)} = \lim_{m\to\infty} \widehat{x}_{n+m+1}^{p^{m+1}} = \lim_{m\to\infty} \big(\widehat{x}_{(n+1)+m}^{p^m}\big)^p = \big(x^{(n+1)}\big)^p$$

for all $n \in \mathbf{N}$. Finally, if $y \in \mathscr{R}(A)$, the sequence $\big(x^{(n)}y^{(n)}\big)_{n\in\mathbf{N}} \in \widehat{A}^{\mathbf{N}}$ lifts $xy = (x_n y_n)_{n\in\mathbf{N}} \in \mathscr{R}(A)$, so $(xy)^{(n)} = \lim\limits_{m\to\infty} \big(x^{(n+m)}y^{(n+m)}\big)^{p^m} = x^{(n)}y^{(m)}$ for all $n \in \mathbf{N}$. □

***Remark 5.3.*** — (1) In what follows, we will identify elements $x \in \mathscr{R}(A)$ with a sequence $(x^{(n)})_{n\in\mathbf{N}}$ as in the proposition.

(2) For $x, y \in \mathscr{R}(A)$ and $n \in \mathbf{N}$, we have

$$(x+y)^{(n)} = \lim_{m\to\infty} \big(x^{(n+m)} + y^{(n+m)}\big)^{p^m}.$$

(3) It follows that the ring $\mathscr{R}(A)$ does not depend on the choice of $\varpi$.

(4) This construction is functorial.

***Notation 5.4.*** — In what follows, we put $x^\sharp = x^{(0)}$ for all $x \in \mathscr{R}(A)$: we have $x^{(n)} = \big(x^{1/p^n}\big)^\sharp$ for all $n \in \mathbf{N}$. The map $\mathscr{R}(A) \to \widehat{A}$; $x \mapsto x^\sharp$ is multiplicative. It is an isomorphism when $A$ is perfect of characteristic $p$.

Assume from now on that $A/pA$ is *semi-perfect*, *i.e.* that the Frobenius map $A/pA \to A/pA$; $x \mapsto x^p$ is surjective. Under this assumption, $\widehat{A}$ and $\mathscr{R}(A)$ are also related as follows. As $\mathscr{R}(A)$ is perfect of characteristic $p$, the Witt vectors ring $\mathsf{W}(\mathscr{R}(A))$ is $p$-adically separated and complete, and $\mathsf{W}(\mathscr{R}(A))/p\,\mathsf{W}(\mathscr{R}(A)) \xrightarrow{\sim} \mathscr{R}(A)$. For any $a = (a_0, a_1, a_2, \ldots) \in \mathsf{W}(\mathscr{R}(A))$, we can write

$$a = \sum_{n\in\mathbf{N}} V^n([a_n]) = \sum_{n\in\mathbf{N}} p^n([a_n^{1/p^n}])$$

Put $\theta(a) = \sum\limits_{n=0}^{\infty} p^n a_n^{(n)} = \sum\limits_{n=0}^{\infty} p^n (a_n^{1/p^n})^\sharp \in \widehat{A}$.

***Proposition 5.5.*** — *The map $\theta\colon \mathsf{W}(\mathscr{R}(A)) \to \widehat{A}$ is a surjective ring homomorphism.*

*Proof.* — For $n \in \mathbf{N}_{>0}$, let $\theta_n\colon \mathsf{W}_n(\mathscr{R}(A)) \to A/p^nA$; $(x_0, x_1, \ldots, x_{n-1}) \mapsto \sum\limits_{i=0}^{n-1} p^i x_i^{(i)}$ be the composite of $\theta$ and the reduction modulo $p^n\,\mathsf{W}(\mathscr{R}(A))$. Recall that the $n-1$-th phantom component

$$\begin{aligned}\Phi_{n-1}\colon \mathsf{W}_n(A/p^nA) &\to A/p^nA\\ (x_0, x_1, \ldots, x_{n-1}) &\mapsto \sum_{i=0}^{n-1} p^i x_i^{p^{n-i-1}}\end{aligned}$$

is a ring homomorphism (by definition of Witt vectors). If $x_i \equiv y_i \mod pA$ then $x_i^{p^{n-i-1}} \equiv y_i^{p^{n-i-1}} \mod p^{n-i}A$ so $\sum\limits_{i=0}^{n-1} p^i x_i^{p^{n-i-1}} \equiv \sum\limits_{i=0}^{n-1} p^i y_i^{p^{n-i-1}} \mod p^nA$. Thus we have a ring homomorphism factorization

$$\begin{tikzcd}
\mathsf{W}_n(A/p^nA) \arrow[rr, "\Phi_{n-1}"] \arrow[dr, two heads] & & A/p^nA \\
& \mathsf{W}_n(A/pA) \arrow[ur, "\widetilde{\Phi}_{n-1}"'] &
\end{tikzcd}$$

where the left arrow is induced by the map $A/p^nA \to A/pA$. We deduce the commutative diagram

$$\begin{tikzcd}
\mathsf{W}_n(\mathscr{R}(A)) \arrow[rr, "\theta_n"] \arrow[d, "\varphi^{1-n}"'] & & A/p^nA \\
& \mathsf{W}_n(A/p^nA) \arrow[ur, "\Phi_{n-1}"'] \arrow[dr, two heads] & \\
\mathsf{W}_n(\mathscr{R}(A)) \arrow[rr, two heads, "\beta"] \arrow[ur, "\alpha"] & & \mathsf{W}_n(A/pA) \arrow[uu, "\widetilde{\Phi}_{n-1}"']
\end{tikzcd}$$

where $\alpha(x_0, x_1, \ldots, x_{n-1}) = (x_0^{(0)} \mod p^nA, \ldots)$ and $\beta(x_0, x_1, \ldots, x_{n-1}) = (x_0^{(0)} \mod pA, \ldots)$. The map $\alpha$ is not a ring homomorphism, but $\beta = \mathsf{W}_n(\mathrm{pr}_0)$ is. This implies that $\theta_n = \widetilde{\Phi}_{n-1} \circ \beta \circ \varphi^{1-n}$ is a ring homomorphism. As this holds for all $n \in \mathbf{N}_{>0}$, the map $\theta$ is a ring homomorphism as well. As both sides are $p$-adically separated and complete, surjectivity is checked modulo $p$: it reduces to that of the map $\mathrm{pr}_0 = \theta_1\colon \mathscr{R}(A) \to A/pA$, which follows from the semi-perfectness of $A$. $\square$

***Remark 5.6.*** — In general, the map $\mathscr{R}(A) \to \widehat{A}$; $a \mapsto a^\sharp$ is neither surjective nor a ring homomorphism.

In particular we have $\mathsf{W}(\mathscr{R}(A))/\operatorname{\mathsf{Ker}}(\theta) \xrightarrow{\sim} \widehat{A}$. In good situations, the ideal $\operatorname{\mathsf{Ker}}(\theta)$ is principal.

We apply the preceding construction to (the ring of integers) of appropriate extensions of $\mathbf{Q}_p$.

***Definition 5.7.*** — In what follows, a *local field* will be a complete discrete valuation field with perfect residue field of positive characteristic.

## 5.8. Perfectoid fields. —

***Definition 5.9*** **(*cf*** [36, Definition 1.2]**).** — A *perfectoid field* is a topological field $K$ satisfying:

- it is complete for a non-archimedean valuation $|.|\colon K \to \mathbf{R}_{\geq 0}$ whose image is dense;
- its residue field has characteristic $p$, and the Frobenius map

$$\varphi\colon \mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K$$

  is surjective.

***Remark 5.10.*** — (1) In characteristic $p$, perfectoid implies perfect.
(2) The density condition on the image of $|.|$ is equivalent to the fact that $|.|$ is not discrete.
(3) As the residue field of $K$ has characteristic $p$, we have $|p| < 1$.

***Example 5.11.*** — Let $F$ be a local field. The completions of

- the cyclotomic extension $F_\infty^{\mathrm{cycl}} := \cup_{n=0}^{\infty} F(\mu_{p^n})$ (when $F$ has characteristic 0);
- the Breuil-Kisin extension $F_\infty^{\mathrm{BK}} := \cup_{n=0}^{\infty} F(\pi^{1/p^n})$ (where $\pi \in F$ is a uniformizer)

are perfectoid.

***Lemma 5.12*** **(*cf*** [36, 3.2]**).** — *The group $|K^\times|$ is $p$-divisible.*

From now on, we fix a perfectoid field $K$. We denote by $|.|\colon K \to \mathbf{R}_{\geq 0}$ its valuation, and $\mathfrak{m}$ the maximal ideal of its ring of integers $\mathcal{O}_K$.

**5.13. The tilt.** — Choose $\varpi \in \mathcal{O}_K$ such that $|p| \leq |\varpi| < 1$. The ring $\mathscr{R}(\mathcal{O}_K) \simeq \varprojlim_{x\mapsto x^p} \mathcal{O}_K$ is integral, perfect of characteristic $p$.

***Lemma 5.14*** **(*cf*** [36, Lemma 3.4 (ii)]**).** — *There exists $\varpi^\flat \in \mathscr{R}(\mathcal{O}_K)$ such that $|(\varpi^\flat)^\sharp| = |\varpi|$.*

As the constructions do not depend on the choice of $\varpi$, it is convenient to replace $\varpi$ by $(\varpi^\flat)^\sharp$ (whose valuation is the same): from now on, we assume that $(\varpi^\flat)^\sharp = \varpi$, *i.e.* $\varpi^\flat = (\varpi^{(n)})_{n\in\mathbf{N}} \in \varprojlim_{x\mapsto x^p} \mathcal{O}_K$ with $\varpi^{(0)} = \varpi$.

For $i \in \mathbf{N}$, the map

$$\begin{aligned} \pi_i\colon \mathscr{R}(\mathcal{O}_K) &\to \mathcal{O}_K/\varpi\mathcal{O}_K \\ (x_n)_{n\in\mathbf{N}} &\mapsto x_i \end{aligned}$$

is a surjective ring homomorphism (this follows from the surjectivity of the Frobenius map on $\mathcal{O}_K/\varpi\mathcal{O}_K$). We have $\pi_0 = \pi_i \circ \varphi^i$. If $x \in \mathsf{Ker}(\pi_0)$, then $|x^{(0)}| \leq |\varpi|$: for all $n \in \mathbf{N}$, we have $|x^{(n)}| \leq |\varpi^{(n)}|$, so there exists $y^{(n)} \in \mathcal{O}_K$ such that $x^{(n)} = \varpi^{(n)} y^{(n)}$. As $\mathcal{O}_K$ is integral, this implies that $y = (y^{(n)})_{n\in\mathbf{N}} \in \mathscr{R}(\mathcal{O}_K)$ and $x = \varpi^\flat y$, so $\mathsf{Ker}(\pi_0) = \varpi^\flat \mathscr{R}(\mathcal{O}_K)$ whence $\mathsf{Ker}(\pi_i) = (\varpi^\flat)^{p^i} \mathscr{R}(\mathcal{O}_K)$ for all $i \in \mathbf{N}$. We deduce isomorphisms

$$\begin{tikzcd} \mathscr{R}(\mathcal{O}_K)/\varpi^\flat\mathscr{R}(\mathcal{O}_K) \arrow[d, "\varphi^i"'] \arrow[dr, "\overline{\pi}_0"] & \\ \mathscr{R}(\mathcal{O}_K)/(\varpi^\flat)^{p^i}\mathscr{R}(\mathcal{O}_K) \arrow[r, "\overline{\pi}_i"'] & \mathcal{O}_K/\varpi\mathcal{O}_K \end{tikzcd}$$

which imply that the $\varpi^\flat$-topology coincides with the inverse limit topology on $\mathscr{R}(\mathcal{O}_K)$.
Put

$$K^\flat := \mathscr{R}(\mathcal{O}_K)\big[\tfrac{1}{\varpi^\flat}\big]$$

The multiplicative monoid bijection $\mathscr{R}(\mathcal{O}_K) \simeq \varprojlim_{x\mapsto x^p} \mathcal{O}_K$ extends to

$$K^\flat \simeq \varprojlim_{x\mapsto x^p} K$$

In particular, $K^\flat$ is a field: this is the fraction field of $\mathscr{R}(\mathcal{O}_K)$. It is perfect. We call it the *tilt* of $K$. Of course, the map $x \mapsto x^\sharp$ extends to a multiplicative map

$$\begin{aligned} K^\flat &\to K \\ x &\mapsto x^\sharp \end{aligned}$$

For $x \in K^\flat$, we put

$$|x|^\flat := |x^\sharp|$$

***Lemma 5.15* (*cf* [36, Lemma 3.4]).** — (1) *The map* $|.|^\flat \colon K^\flat \to \mathbf{R}_{\geq 0}$ *is a valuation.*

(2) $\mathcal{O}_{K^\flat} = \mathscr{R}(\mathcal{O}_K)$ *and* $\mathcal{O}_{K^\flat}/\varpi^\flat \mathcal{O}_{K^\flat} \simeq \mathcal{O}_K/\varpi\mathcal{O}_K$*. If* $\mathfrak{m}^\flat$ *is the maximal ideal in* $\mathcal{O}_{K^\flat}$*, then* $\mathcal{O}_{K^\flat}/\mathfrak{m}^\flat \xrightarrow{\sim} \mathcal{O}_K/\mathfrak{m}$.

***Proposition 5.16* (*cf* [36, Lemma 3.4 (iii)]).** — *The field* $K^\flat$ *is perfectoid.*

*Proof.* — As $|K^\flat|^\flat = |K|$ is dense in $\mathbf{R}_{\geq 0}$, it remains to see that $K^\flat$ is complete for $|.|^\flat$. The ring $(\mathcal{O}_K/\varpi\mathcal{O}_K)^{\mathbf{N}}$, endowed with the product topology (where each copy of $\mathcal{O}_K/\varpi\mathcal{O}_K$ is endowed with the discrete topology), is complete. As $\mathscr{R}(\mathcal{O}_K) = \varprojlim\limits_{x\mapsto x^p} \mathcal{O}_K/\varpi\mathcal{O}_K$ is closed in this product, it is complete for the induced topology. It is thus enough to show that the topology defined by $|.|^\flat$ on $\mathscr{R}(\mathcal{O}_K)$ (*i.e.* the $\varpi^\flat$-adic topology) coincides with that induced by the product topology. As a basis of open sets for the product topology is given by $(U_i)_{i\in\mathbf{N}}$ where $U_i = \{x \in (\mathcal{O}_K/\varpi\mathcal{O}_K)^{\mathbf{N}} \mid (\forall n \leq i)\, x_n = 0\}$, it follows from the equality $U_i \cap \mathscr{R}(\mathcal{O}_K) = \mathsf{Ker}(\pi_i) = (\varpi^\flat)^{p^i}\mathscr{R}(\mathcal{O}_K)$ for all $i \in \mathbf{N}$. □

***Remark 5.17.*** — (1) The product topology and the $\varpi^\flat$-adic topology differ on $(\mathcal{O}_K/\varpi\mathcal{O}_K)^{\mathbf{N}}$ (the latter is strictly finer than the former): only their restrictions to the sub-space $\mathscr{R}(\mathcal{O}_K)$ coincide.

(2) When $K$ has characteristic $p$, we have $K^\flat = K$.

***Example 5.18.*** — Let $F$ be an absolutely unramified local field of characteristic 0 and $K$ be the completion of $F_\infty^{\mathrm{BK}} = F(p^{1/p^\infty})$, then $\mathcal{O}_K$ is the completion of $\mathcal{O}_F[p^{1/p^\infty}]$, so that

$$\mathscr{R}(\mathcal{O}_K) = \varprojlim_{x\mapsto x^p} k[t^{1/p^\infty}]/(t) = \varprojlim_n k[t^{1/p^\infty}]/(t^{p^n}) = k[\![t]\!]\widehat{[t^{1/p^\infty}]}$$

where $k$ is the residue field of $F$ and $t = p^\flat = (p, p^{1/p}, p^{1/p^2}, \ldots)$.

**5.19. The purity theorem for perfectoid fields.** —

***Proposition 5.20* (*cf* [36, Proposition 3.8]).** — *$K$ is algebraically closed if and only if $K^\flat$ is algebraically closed.*

*Proof.* — Assume $K^\flat$ algebraically closed and let $P(X) = X^d + a_1X^{d-1} + \cdots + a_d \in \mathcal{O}_K[X]$ be irreducible. There exists $c \in K$ such that $|c|^d = |a_0|$ (because $|K^\times| = |K^{\flat\times}|^\flat$ is a $\mathbf{Q}$-vector space): to show that $P$ has a root in $K$, we may replace $P(X)$ by $c^{-d}P(cX)$ (which still has coefficients in $\mathcal{O}_K$ because the Newton polygon of $P$ is a line since it is irreducible), and assume that $|a_d| = 1$. Let $Q(X) = X^d + b_1X^{d-1} + \cdots + b_d \in \mathcal{O}_{K^\flat}[X]$ with same image as $P(X)$ in $(\mathcal{O}_{K^\flat}/\varpi^\flat\mathcal{O}_{K^\flat})[X] \simeq (\mathcal{O}_K/\varpi\mathcal{O}_K)[X]$. As $K^\flat$ is algebraically closed, $Q$ has a root $y_0 \in \mathcal{O}_{K^\flat}$, so $P(y_0^\sharp) \in \varpi\mathcal{O}_K$. Let $c_0 \in K$ such that $|c_0|^d = |P(y_0^\sharp)| \leq |\varpi|$ and $P_1(X) = c_0^{-d}P(c_0X + y_0^\sharp) \in \mathcal{O}_K[X]$ (because $P$ is irreducible). The constant term of $P_1$ is invertible: we can repeat the preceding procedure. By induction, this provides sequences $(c_n)_{n\in\mathbf{N}} \in \mathcal{O}_K^{\mathbf{N}}$, $(y_n)_{n\in\mathbf{N}} \in \mathcal{O}_{K^\flat}^{\mathbf{N}}$ and $(P_n(X))_{n\in\mathbf{N}} \in (\mathcal{O}_K[X])^{\mathbf{N}}$ such that $P_0 = P$ and

$$\begin{cases} P_{n+1}(X) = c_n^{-d}P_n(c_nX + y_n^\sharp) \\ |c_n|^d = |P_n(y_{n+1}^\sharp)| \leq |\varpi| \end{cases}$$

for all $n \in \mathbf{N}$. We have $P_{n-1}(y_n^\sharp) = (c_0c_1\cdots c_{n-1})^{-d}P(x_n)$ with

$$x_n = y_0^\sharp + c_0y_1^\sharp + c_0c_1y_2^\sharp + \cdots + c_0c_1\cdots c_{n-1}y_n^\sharp \in \mathcal{O}_K$$

so $|P(x_n)| = |(c_0c_1\cdots c_{n-1})^d P_{n-1}(y_n^\sharp)| \leq |\varpi|^n$, hence $P(x) = 0$ for $x = \sum\limits_{n=0}^{+\infty} c_0c_1\cdots c_{n-1}y_n^\sharp$ (the series converges for the $\varpi$-adic topology). The proof of the converse is similar. $\square$

***Theorem 5.21 (cf*** [36, Theorem 3.7]). — *Assume that $K$ is perfectoid.*

(1) *If $L/K$ be a finite extension, then:*
  (i) *$L$ is perfectoid (endowed with the topology of finite dimensional $K$-vector space);*
  (ii) (Tate, Faltings, Ramero-Gabber) *the map*

$$\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \xrightarrow{\text{mult}} \mathcal{O}_L \xrightarrow{\mathsf{Tr}_{L/K}} \mathcal{O}_K$$

*is an almost perfect pairing, i.e. the map*

$$\begin{aligned} t_L\colon \mathcal{O}_L &\to \mathcal{O}_L^\vee := \mathsf{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_K) \\ x &\mapsto \big(y \mapsto \mathsf{Tr}_{L/K}(xy)\big) \end{aligned}$$

*is an almost isomorphism (= its kernel and cokernel are killed by $\mathfrak{m}$, cf definition 3.2).*

(2) *The functor $L \mapsto L^\flat$ is an equivalence between the category of finite extensions of $K$ and that of finite extensions of $K^\flat$. This equivalence preserves degrees.*

*Proof.* — If $L/K$ is a finite extension, the valuation on $K$ extends uniquely to $L$, and the latter is complete (topologically isomorphic to $K^{[L:K]}$). The image of $|.|\colon L \to \mathbf{R}_{\geq 0}$ contains that of its restriction to $K$, it is dense in $\mathbf{R}_{\geq 0}$.

• Case $\mathrm{car}(K) = p$ The finite extension $L/K$ is separable, and $L$ perfect as well. The field $L$ is thus perfectoid, which proves (1-i). We have the commutative square:

$$\begin{tikzcd} \mathcal{O}_L \arrow[r, "t_L"] \arrow[d, hook] & \mathcal{O}_L^\vee \arrow[d, hook] \\ L \arrow[r, "\sim"] & L^\vee \end{tikzcd}$$

Where $L^\vee = \mathsf{Hom}_K(L, K)$ (the vertical lines are localizations). The bottom arrow is an isomorphism because $L/K$ is separable: this shows that $t_L$ is injective, and that $t_L(\mathcal{O}_L)$ spans the $K$-vector space $L^\vee$. As $L/K$ is separable, there exists $\alpha \in \mathcal{O}_L$ such that $L = K(\alpha)$: we have $\mathcal{O}_K[\alpha] \subseteq \mathcal{O}_L$. By functoriality, there is a map $\mathcal{O}_L^\vee \to (\mathcal{O}_K[\alpha])^\vee$. As $\mathcal{O}_L^\vee \subset L^\vee$, the map $\mathcal{O}_L^\vee \to (\mathcal{O}_K[\alpha])^\vee$ is injective: as the $\mathcal{O}_K$-module $\mathcal{O}_K[\alpha]$ is free of rank $d := [L:K]$, so is $(\mathcal{O}_K[\alpha])^\vee$. Let $(f_i)_{1\leq i\leq d}$ be a basis. There exists $x \in \mathcal{O}_K \setminus \{0\}$ such that $xf_i \in t_L(\mathcal{O}_L)$ for all $i \in \{1,\ldots,d\}$. We have $x(\mathcal{O}_K[\alpha])^\vee \subseteq t_L(\mathcal{O}_L)$, thus $x\mathcal{O}_L^\vee \subseteq t_L(\mathcal{O}_L)$. For all $m \in \mathbf{N}$, we get

$$x^{1/p^m}\mathcal{O}_L^\vee = \varphi^{-m}(x\mathcal{O}_L^\vee) \subseteq \varphi^{-m}(t_L(\mathcal{O}_L)) = t_L(\varphi^{-m}(\mathcal{O}_L)) = t_L(\mathcal{O}_L)$$

*i.e.* $x^{1/p^m}\,\mathsf{Coker}(t_L) = 0$, hence $\mathfrak{m}\,\mathsf{Coker}(t_L) = 0$: this proves (1-ii). As $L^\flat = L$, (2) is trivial.

• Case $\mathrm{car}(K) = 0$ *First step.* One constructs a fully faithful functor

$$\big\{\text{finite étale } K^\flat\text{-algebras}\big\} \to \big\{\text{finite étale } K\text{-algebras}\big\}$$

(quasi-inverse of tilt). One proceeds as follows:

$$\begin{tikzcd} \big\{\text{finite étale } K^\flat\text{-algebras}\big\} \arrow[r, hook, "(*)"] \arrow[d, "①"'] & \big\{\text{finite étale } K\text{-algebras}\big\} \\ \big\{\text{afe } \mathcal{O}_{K^\flat}\text{-algebras}\big\} \arrow[d, "②"'] & \big\{\text{afe } \mathcal{O}_K\text{-algebras}\big\} \arrow[u, hook, "(**)"'] \arrow[d, "④"'] \\ \big\{\text{afe } \mathcal{O}_{K^\flat}/\varpi^\flat\mathcal{O}_{K^\flat}\text{-algebras}\big\} \arrow[r, "③"] & \big\{\text{afe } \mathcal{O}_K/\varpi\mathcal{O}_K\text{-algebras}\big\} \end{tikzcd}$$

(where afe means "almost finite étale"). Functor ① is an equivalence (characteristic $p$ case). Functors ② and ④ are equivalences: this is the generalization to almost mathematics of the usual equivalence (topologically nilpotent deformation of étale extensions). Functor ③ is an equivalence because $\mathcal{O}_{K^\flat}/\varpi^\flat\mathcal{O}_{K^\flat} \simeq \mathcal{O}_K/\varpi\mathcal{O}_K$ (*cf* Lemma 5.15 (2)). Finally, the functor $(**)$ (generic fiber) is fully faithful (on the essential image, a quasi-inverse is given by taking the ring of integers).

*Second step.* The functor $(*)$ is essentially surjective, which proves (2) and implies that $(**)$ is an equivalence (which proves (1-ii)). One reduces to the case where $K^\flat$ is algebraically closed, where this is obvious thanks to the preceding proposition.

*Third step.* Let $L/K$ be a finite extension: the extension $\mathcal{O}_L/\mathcal{O}_K$ is afe: the idempotent $e \in L \otimes_K L$ corresponding to the diagonal immersion $\mathsf{Spec}(L) \to \mathsf{Spec}(L \otimes_K L)$ satisfies $(\forall \varepsilon \in \mathfrak{m})\ \varepsilon e \in \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L$. On the other hand, we have

$$\operatorname{mult}(f) = (\mathsf{Tr}_{L/K} \otimes \mathsf{Id}_L)(fe)$$

for all $f \in L \otimes_K L$ (where $\operatorname{mult}\colon L \otimes_K L \to L$ is the multiplication[(7)]). If $x \in \mathcal{O}_L$ and $\varepsilon \in \mathfrak{m}$, we deduce $\varepsilon^p x = \operatorname{mult}(x \otimes \varepsilon^p) = (\mathsf{Tr}_{L/K} \otimes \mathsf{Id}_L)((x \otimes 1)(\varepsilon e)^p)$. Write $\varepsilon e = \sum\limits_{i=1}^r x_i \otimes y_i \in \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L$: we have $(\varepsilon e)^p \equiv \sum\limits_{i=1}^r x_i^p \otimes y_i^p \mod p\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L$, thus $\varepsilon^p x \equiv \sum\limits_{i=1}^r \mathsf{Tr}_{L/K}(x x_i^p) y_i^p \mod p\mathcal{O}_L$. As the Frobenius is surjective on $\mathcal{O}_K/p\mathcal{O}_K$, there exists $z_i \in \mathcal{O}_K$ such that $\mathsf{Tr}_{L/K}(x x_i^p) \equiv z_i^p \mod p\mathcal{O}_K$ for all $i \in \{1, \ldots, r\}$. We get $\varepsilon^p x \equiv \Big(\sum\limits_{i=1}^r z_i y_i\Big)^p \mod p\mathcal{O}_L$. This shows that the Frobenius is almost surjective on $\mathcal{O}_L/p\mathcal{O}_L$. In fact, it is surjective (which shows (1-i)): let $y \in \mathcal{O}_L$ and $\varepsilon \in \mathcal{O}_K$ such that $|\varepsilon|^{p^2} = |p|$ (possible since $|K^\times|$ is $p$-divisible). We know that there exist $x_0, y_0 \in \mathcal{O}_L$ such that $\varepsilon^p y = x_0^p + \varepsilon^{p^2} y_0$. This shows that $x_0 = \varepsilon x_1$ with $x_1 \in \mathcal{O}_L$: we have $y = x_1^p + \varepsilon^{p^2-p} y_0$. Similarly, there exists $x_2, y_1 \in \mathcal{O}_L$ such that $y_0 = x_2^p + \varepsilon^{p^2-p} y_1$, and we have $y = x_1^p + (\varepsilon^{p-1} x_2)^p + \varepsilon^{2p(p-1)} y_1$. As $p \geq 2$, we have $2p(p-1) \geq p^2$, hence $y \equiv (x_1 + \varepsilon^{p-1} x_2)^p \mod p\mathcal{O}_L$. □

***Remark 5.22.*** — In the preceding proof, the characteristic $p$ case illustrates an easy but fundamental fact: if $A$ is a perfect $\mathcal{O}_K$-algebra and "something" is true up to $\varpi^N$, then it is true up to $\varpi^{N/p^n}$ for all $n \in \mathbf{N}$, so it is almost true, hence true after inverting $\varpi$.

If $L/K$ is a finite and Galois extension, elements of $\mathsf{Gal}(L/K)$ act on $L$ hence on $L^\flat \simeq \varprojlim\limits_{x \mapsto x^p} L$. As the field of invariants of $L^\flat$ under $\mathsf{Gal}(L/K)$ is $K^\flat$, the extension $L^\flat/K^\flat$ is Galois and the map $\mathsf{Gal}(L/K) \to \mathsf{Gal}(L^\flat/K^\flat)$ is surjective. It is an isomorphism since $[L : K] = [L^\flat : K^\flat]$ (*cf* Theorem 5.21 (2)). Passing to the limit on $L$ provides a natural group homomorphism.

***Corollary 5.23.*** — *The natural map $G_K \overset{\sim}{\to} G_{K^\flat}$ is an isomorphism of topological groups.*

***Remark 5.24.*** — Let $F$ be a local field of characteristic 0 and $K$ the completion of the cyclotomic extension $F_\infty^{\mathrm{cycl}}$ (*cf* example 5.11). If $L/K$ is a finite extension, then $L^\flat$ is naturally isomorphic to the completion of the perfect closure the field of norms $X_F(L)$. This follows, at the level of rings of integers, from the commutativity of the diagram:

$$\begin{tikzcd}
\mathcal{O}_{L_n}/\mathfrak{a}_n \arrow[r, "\mathsf{N}_{L_n/L_{n-1}}"] \arrow[d, hook] & \mathcal{O}_{L_{n-1}}/\mathfrak{a}_{n-1} \arrow[d, hook] \\
\mathcal{O}_{\overline{K}}/\mathfrak{a} \arrow[r, "x \mapsto x^p"] & \mathcal{O}_{\overline{K}}/\mathfrak{a}
\end{tikzcd}$$

where $\mathfrak{a} = (\zeta_1 - 1)\mathcal{O}_{\overline{K}}$ and $\mathfrak{a}_n = (\zeta_1 - 1)\mathcal{O}_L$ for all $n \in \mathbf{N}$ (*cf* [18, 3.1.7] and [9, Remarque 4.4]). This shows that what precedes is a generalization of the "perfect field of norms" theory. Note that it does not cover the (imperfect) fields of norms theory of Fontaine-Wintenberger, which is the starting point of $(\varphi, \Gamma)$-modules theory (*cf* [51] and section 2).

(7) Assume $L/K$ is Galois with group $G$. Then $L \otimes_K L \to \oplus_{\sigma \in G} L$; $x \otimes y \mapsto (\sigma(x)y)_\sigma$ is an isomorphism. Note that $\mathsf{Tr}_{L/K} \otimes \mathsf{Id}_L$ corresponds to summing the components on the right. The idempotent $e$ corresponds to $(\delta_{\sigma, \mathsf{Id}_L})_\sigma$ (where $\delta$ denotes the Kronecker symbol), so $(x \otimes y)e$ corresponds to $(xy, 0, \ldots, 0)$ whence $(\mathsf{Tr}_{L/K} \otimes \mathsf{Id}_L)((x \otimes y)e) = xy = \operatorname{mult}(x \otimes y)$.

**5.25. Application to Galois cohomology.** —

***Proposition 5.26.*** — *Let $G$ be a finite group, $A$ a ring, $A \to B$ an $A$-algebra with $G$-action (i.e. the action is $A$-linear) and $M$ a $B$-module endowed with a semilinear action of $G$. Let $b \in B$ be such that $\mathsf{Tr}(b) := \sum\limits_{g \in G} g(b)$ is the image of some $a \in A$. Then $\mathsf{H}^n(G, M)$ is killed by $a$ for all $n \in \mathbf{N}_{>0}$.*

*Proof.* — We use *homogeneous* cocycles. Let $f \in \mathsf{C}^{n+1}_{\mathrm{cont}}(G, M)^G$ be a cocycle. As the homogeneous cochain complex is acyclic, there exists an homogeneous cochain $\widehat{f} \in \mathsf{C}^n_{\mathrm{cont}}(G, M)$ such that $f = \partial_n(\widehat{f})$ (of course $\widehat{f}$ is not invariant under $G$ in general). Put $h = \mathsf{Tr}(b\widehat{f}) = \sum\limits_{g \in G} g(b) g\widehat{f} \in \mathsf{C}^n_{\mathrm{cont}}(G, M)$. By construction, $h \in \mathsf{C}^n_{\mathrm{cont}}(G, M)^G$ is a cocycle, and

$$\partial_n(h) = \sum_{g \in G} g(b) \partial_n(g\widehat{f}) = af$$

because $\partial_n(g.\widehat{f}) = g.\partial_n(\widehat{f}) = g.f = f$ since $f$ is invariant under $G$. This implies that $af$ is a coboundary. □

***Example 5.27.*** — For instance, if $L/F$ is an unramified[(8)] Galois extension of local fields, the trace $\mathsf{Tr}_{F_1/F_2} \colon \mathcal{O}_{F_1} \to \mathcal{O}_{F_2}$ is surjective, so $\mathsf{H}^i(\mathsf{Gal}(L/F), \mathcal{O}_L) = \{0\}$ when $i > 0$.

If $K$ is a perfectoid field with maximal ideal $\mathfrak{m}$ and $L/K$ a finite Galois extension, then $\mathfrak{m} \subset \mathsf{Tr}_{L/K}(\mathcal{O}_L)$, so the modules $\mathsf{H}^i\big(\mathsf{Gal}(L/K), \mathcal{O}_L/p^n\mathcal{O}_L\big)$ are almost zero if $i, n > 0$. This implies that $\mathsf{H}^i\big(\mathsf{Gal}(\overline{K}/K), \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}\big)$ is almost zero if $i, n > 0$. Using results of Tate-Jannsen (*cf* [47] and [30, 2.1 & Theorem 2.2]) on passing to the limit on cohomology, we deduce that $\mathsf{H}^i_{\mathrm{cont}}\big(\mathsf{Gal}(\overline{K}/K), \mathcal{O}_{\widehat{\overline{K}}}\big)$ is almost zero hence[(9)] $\mathsf{H}^i_{\mathrm{cont}}\big(\mathsf{Gal}(\overline{K}/K), \widehat{\overline{K}}\big) = 0$ for $i > 0$.

***Remark 5.28.*** — When $K$ is the completion of the cyclotomic extension of a local field $F$ of characteristic 0, this gives the the first step of the proof of theorem 1.1. Of course, the cohomology groups $\mathsf{H}^i_{\mathrm{cont}}(G_F, \mathcal{O}_F)$ (for $i > 0$) are *not* almost zero: they are killed by some power of $p$.

## 6. The relative case: perfectoid spaces and their tilt

**6.1. Affinoid perfectoid spaces.** — The appropriate framework to globalize what precedes is that of adic spaces (*cf* section 4). As these are unions of affinoid spaces, we have first to define what perfectoid affinoid spaces are.

In what follows $K$ is a fixed perfectoid field.

***Definition 6.2 (cf** [36, Definition 5.1]**).*** — A perfectoid $K$-algebra is a Banach $K$-algebra[(10)] $R$ such that:

- $R^\circ := \{x \in R,\ x \text{ is power bounded}\}$ is bounded (*i.e.* $R$ is *uniform*);
- the Frobenius map $R^\circ/pR^\circ \to R^\circ/pR^\circ$ is surjective.

A perfectoid $\mathcal{O}_K$-algebra (resp. $\mathcal{O}_K/\varphi\mathcal{O}_K$-algebra) is a $\varpi$-adically complete and flat $\mathcal{O}_K$-algebra (resp. $\mathcal{O}_K/\varphi\mathcal{O}_K$-algebra) $A$ such that the Frobenius induces an isomorphism $A/\varpi^{1/p}A \simeq A/\varpi A$.

***Example 6.3.*** — If $R = K\big\langle T^{1/p^\infty}\big\rangle = \Big\{ \sum\limits_{i \in \mathbf{N}[p^{-1}]} a_i T^i \,\Big|\, a_i \in K,\ \lim a_i = 0 \Big\}$ (where the limits iare taken with respect to the cofinite filter on $\mathbf{N}[p^{-1}]$), then $R^\circ = \mathcal{O}_K\big\langle T^{1/p^\infty}\big\rangle$ and $R^\circ/pR^\circ \simeq (\mathcal{O}_K/p\mathcal{O}_K)\big[T^{1/p^\infty}\big]$, so $R$ is perfectoid.

(8) *I.e.* the pairing $\mathcal{O}_L \otimes_{\mathcal{O}_F} \mathcal{O}_L \to \mathcal{O}_F$; $x \otimes y \mapsto \mathsf{Tr}_{L/F}(xy)$ is perfect, or equivalently the different ideal $\mathfrak{D}_{L/F}$ is $\mathcal{O}_L$ where $\mathfrak{D}_{L/F}^{-1} := \{x \in L \,|\, (\forall y \in \mathcal{O}_L)\, \mathsf{Tr}_{L/F}(xy) \in \mathcal{O}_F\} = \mathcal{O}_L$.

(9) This does *not* follow from $\mathsf{H}^i_{\mathrm{cont}}\big(\mathsf{Gal}(\overline{K}/K), \overline{K}\big) = 0$ for $i > 0$ (Hilbert 90).

(10) *I.e.* a $K$-algebra $R$ equipped with a map $|.| \colon R \to \mathbf{R}_{\geq 0}$ satisfying $|1| \leq 1$, $|x| = 0 \Leftrightarrow x = 0$, $|x+y| \leq \max\{|x|, |y|\}$, $|xy| \leq |x||y|$ and $|\lambda x| = |\lambda||x|$ for all $x, y \in R$ and $\lambda \in K$, and such that $R$ is complete with respect to the metric deduced from $|.|$.

*Remark 6.4*. — It is possible to generalize the notion to a more general class of rings, that do not necessarily live on a field (*cf* [50, §6.1]).

*Definition 6.5*. — Let $R$ be a perfectoid $K$-algebra. The *tilt* of $R$ is

$$R^\flat := \varprojlim_{x\mapsto x^p} R$$

endowed with the inverse limit topology.

Choose an element $\varpi \in \mathcal{O}_K$ such that $|p| \leq |\varpi| < 1$ and $\varpi = (\varpi^\flat)^\sharp$ with $\varpi^\flat \in \mathscr{R}(\mathcal{O}_K)$. If $R_0$ is the unit ball (for some norm defining the Banach $K$-algebra structure), then $R = R_0\big[\frac{1}{\varpi}\big]$, hence $R = R^\circ\big[\frac{1}{\varpi}\big]$.

***Proposition 6.6* (*cf* [36, Proposition 5.17]).** — *$R^\flat$ is a perfectoid $K^\flat$-algebra, whose ring of power-bounded elements is $\mathscr{R}(R^\circ)$. Moreover, $R^{\flat\circ}/\varpi^\flat R^{\flat\circ} \xrightarrow{\sim} R^\circ/\varpi R^\circ$.*

*Proof.* — As $R$ is uniform, $R^\circ$ is a ring of definition (*cf* definition 4.3 (4)): it is separated and complete for the $\varpi$-adic topology. By proposition 5.2, $\varprojlim_{x\mapsto x^p} R^\circ \xrightarrow{\sim} \mathscr{R}(R^\circ)$ is a perfect $\mathscr{R}(\mathcal{O}_K)$-algebra. Moreover, the morphism $\overline{\pi}_0\colon \mathscr{R}(R^\circ) \to R^\circ/\varpi R^\circ$; $(x_n)_{n\in\mathbf{N}} \mapsto x_0$ is surjective. As above, we have $\mathsf{Ker}(\overline{\pi}_0) = \varpi^\flat\mathscr{R}(R^\circ)$: it induces the isomorphism $\mathscr{R}(R^\circ)/\varpi^\flat\mathscr{R}(R^\circ) \xrightarrow{\sim} R^\circ/\varpi R^\circ$.
If $x = (x^{(n)})_{n\in\mathbf{N}} \in R^\flat$, there exists $m \in \mathbf{N}$ such that $\varpi^m x^{(0)} \in R^\circ$: if $y = (\varpi^\flat)^m x = (y^{(n)})_{n\in\mathbf{N}}$, we have $y^{(n)} \in R^\circ$ for all $n \in \mathbf{N}$, *i.e.* $y \in \mathscr{R}(R^\circ)$. This implies that

$$R^\flat = \mathscr{R}(R^\circ)\big[\tfrac{1}{\varpi^\flat}\big]$$

is a perfect $K^\flat$-algebra.
By proposition 5.2 again, we have

$$\mathscr{R}(R^\circ) \xrightarrow{\sim} \mathscr{R}(R^\circ/\varpi R^\circ) = \varprojlim_{x\mapsto x^p}(R^\circ/\varpi R^\circ) \simeq \varprojlim_{x\mapsto x^p}(\mathscr{R}(R^\circ)/\varpi^\flat\mathscr{R}(R^\circ)) \xrightarrow{\sim} \varprojlim_{n\in\mathbf{N}_{>0}} \mathscr{R}(R^\circ)/(\varpi^\flat)^n\mathscr{R}(R^\circ)$$

which shows that $\mathscr{R}(R^\circ)$ is separated and complete for the $\varpi^\flat$-topology (exercise: show that the $\varpi^\flat$-topology and the inverse limit topology coincide). This implies that $R^\flat$ is a Banach $K^\flat$-algebra. Moreover, $x = (x^{(n)})_{n\in\mathbf{N}} \in R^\flat$ is power bounded if and only if all its components $x^{(n)}$ are, so that $R^{\flat\circ} = \mathscr{R}(R^\circ)$, which is bounded, so that $R^\flat$ is uniform, and $R^{\flat\circ}/\varpi^\flat R^{\flat\circ} \xrightarrow{\sim} R^\circ/\varpi R^\circ$ by what precedes. □

*Example 6.7*. — (*cf* [36, Proposition 5.20]) If $R = K\langle T^{1/p^\infty}\rangle$ (*cf* example 6.3), then $R^\flat = K^\flat\langle T^{\flat 1/p^\infty}\rangle$ where $T^\flat = (T, T^{1/p}, T^{1/p^2}, \ldots)$.

Recall that the additive law on $R^\flat$ is given by $(x+y)^{(n)} = \lim\limits_{m\to\infty}(x^{(n+m)} + y^{(n+m)})^{p^m}$ for all $x, y \in R^\flat$ and $n \in \mathbf{N}$.

***Theorem 6.8* (*cf* [36, Theorem 5.2]).** — *The tilt provides an equivalence between the category of perfectoid $K$-algebras and that of perfectoid $K^\flat$-algebras.*

*Proof.* — A quasi-inverse is given by

$$S \mapsto \mathsf{W}(S^\circ) \otimes_{\mathsf{W}(\mathcal{O}_{K^\flat})} K$$

where the map $\mathsf{W}(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K$ is $\theta$ (*cf* proposition 5.5). □

*Remark 6.9*. — One can prove Theorem 6.8 using the same procedure as that of the proof of Theorem 5.21 (the deformation argument is more subtle).

Let $R$ be a Banach $K$-algebra. Recall that a *ring of integral elements* in $R$ is a sub-ring $R^+ \subset R^\circ$ which is open and integrally closed (*cf* definition 4.7). The pair $(R, R^+)$ is then called an affinoid $K$-algebra. When $R$ is perfectoid, the pair $(R, R^+)$ is called a *perfectoid affinoid $K$-algebra*.

*Remark 6.10*. — As $R^+ \subset R^\circ$ is open, we have $\varpi^N R^\circ \in R^+$ for $N \gg 0$: this implies that $\mathfrak{m}R^\circ \subset R^+$ since $R^+$ is integrally closed. In particular, $R^+$ is almost equal to $R^\circ$.

***Lemma 6.11.*** — (*cf* [36, Lemma 6.2]) *Assume $R$ is perfectoid. The map*

$$R^+ \mapsto R^{\flat +} := \varprojlim_{x \mapsto x^p} R^+$$

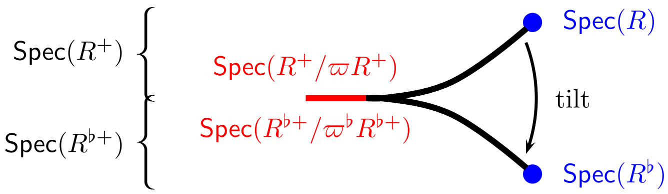
*is a bijection between the set of rings of integral elements in $R$ and that of rings of integral elements in $R^\flat$. Moreover, the map $x \mapsto x^\sharp$ induces an isomorphism $R^{\flat +}/\varpi^\flat R^{\flat +} \xrightarrow{\sim} R^+/\varpi R^+$.*

*Proof.* — If $R^+$ is a ring of integral elements and $y \in R^+$, there exist $x, z \in R^\circ$ such that $y = x^p + pz$. As $x^p = y - pz \in R^+$ (since $\mathfrak{m}R^\circ \subset R^+$ by the above remark) and $R^+$ is integrally closed, we have $x \in R^+$, so that the Frobenius map on $R^+/pR^+$ is surjective.
By the remark above, there is a bijection between the rings of integral elements in $R$ and the integrally closed subrings of $R^\circ/\mathfrak{m}R^\circ \simeq (R^\flat)^\circ/\mathfrak{m}^\flat(R^\flat)^\circ$, hence with the rings of integral elements in $R^\flat$. The last statement is proved as in the case $R^+ = R^\circ$. □

What precedes allows to extend the theory to the affinoid case: under the tilt, the categories of perfectoid affinoid $K$-algebras and perfectoid affinoid $K^\flat$-algebras are equivalent.

***Theorem 6.12.*** — (*cf* [37, Theorem 6.3]) *Let $(R, R^+)$ be a perfectoid affinoid $K$-algebra, $X = \mathsf{Spa}(R, R^+)$ and $X^\flat = \mathsf{Spa}(R^\flat, R^{\flat +})$ its tilt.*

(i) *There is a canonical homeomorphism $X \cong X^\flat$; $x \mapsto x^\flat$, defined by $|f(x^\flat)| = |f^\sharp(x)|$ for all $f \in R^\flat$. This homeomorphism identifies rational subsets.*
(ii) *If $U \subset X$ is a rational subset, the pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is perfectoid affinoid with tilt $(\mathcal{O}_{X^\flat}(U), \mathcal{O}_{X^\flat}^+(U))$ (where $U$ is seen as a rational subset in $X^\flat$, cf (i)).*
(iii) *The presheaves $\mathcal{O}_X$, $\mathcal{O}_{X^+}$, $\mathcal{O}_{X^\flat}$ and $\mathcal{O}_{X^\flat}^+$ are sheaves.*
(iv) *$\mathsf{H}^i(X, \mathcal{O}_X^+)$ is almost zero (cf definition 3.2) so $\mathsf{H}^i(X, \mathcal{O}_X) = \{0\}$ for all $i > 0$. Idem with $X^\flat$.*




*Proof.* — If $f, g \in R^\flat$, we have

$$\begin{aligned}
|(f+g)(x^\flat)| = |(f+g)^\sharp(x)| &= \lim_{m\to\infty} |((f^{1/p^m})^\sharp + (g^{1/p^m})^\sharp)(x)|^{p^m} \\
&\leq \max\big(\lim_{n\to\infty} |(f^{1/p^m})^\sharp(x)|^{p^m}, \lim_{n\to\infty} |(g^{1/p^m})^\sharp(x)|^{p^m}\big) \\
&= \max\big(|f(x^\flat)|, |g(x^\flat)|\big)
\end{aligned}$$

As $f \mapsto f^\sharp$ is multiplicative, $f \mapsto |f(x^\flat)|$ is thus a valuation: this shows that the map in (i) is well defined.
To show its continuity, let $|.|$ be a valuation, $f_1, \ldots, f_n, g \in R^\flat$ such that $f_1, \ldots, f_n$ generate the unit ideal in $R^\flat$, and $U\big(\frac{f_1, \ldots, f_n}{g}\big) = \big\{x \in X \,|\, (\forall i \in \{1, \ldots, n\})\, |f_1(x)| \leq |g(x)|\big\}$ be the corresponding rational subset[(11)]. Its preimage by $X \to X^\flat$; $x \mapsto x^\flat$ is nothing but[(12)] $U\big(\frac{f_1^\sharp, \ldots, f_n^\sharp}{g^\sharp}\big)$.

The proof is then a rather delicate chain of arguments with several going back and forth between characteristic 0 and characteristic $p$. The first step is to describe $\mathcal{O}_X$ on some rational subsets:

(11) Without loss of generality, we may assume that $f_n$ is a power of $\varpi^\flat$ (if not, we can put $f_{n+1} = \varpi^{\flat N}$ with $N \in \mathbf{N}$ large enough so that $|\varpi^N(x)| \leq |g(x)|$ for all $x \in U\big(\frac{f_1, \ldots, f_n}{g}\big)$, *cf* [36, Remark 2.8].
(12) Assuming that $f_n = \varpi^{\flat N}$, the family $f_1^\sharp, \ldots, f_n^\sharp$ generates the unit ideal in $R$.

***Lemma 6.13.*** — *([36, Lemma 6.4]). Let $U = U\left(\frac{f_1,\dots,f_n}{g}\right) \subset X^\flat$ be rational with preimage $U^\sharp \subset \mathsf{Spa}(R,R^+)$. Assume $f_1,\dots,f_n,g \in R^{\flat\circ}$ and $f_n = \varpi^{\flat N}$ for some $N$ (we can always assume this). The $K$-algebra $\mathcal{O}_X(U^\sharp)$ is perfectoid, with associated perfectoid $\mathcal{O}_K^a$-algebra*[13]

$$\mathcal{O}_X(U^\sharp)^{\circ a} = R^\circ\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\dots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right\rangle^a.$$

*Moreover, its tilt is $\mathcal{O}_{X^\flat}(U)$.*

The proof of the lemma itself has several steps: one first proves that $R^\circ\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\dots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right\rangle^a$ is perfectoid when $K$ has characteristic $p$. Then one show that in any characteristic, the statement on $\mathcal{O}_X(U^\sharp)$ follows from the fact that $R^\circ\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\dots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right\rangle^a$ is perfectoid. From this, one proves that $R^\circ\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\dots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right\rangle^a$ is perfectoid in the general case, and that $\mathcal{O}_X(U^\sharp)^\flat = \mathcal{O}_{X^\flat}(U)$.

A tricky point is that the map $g \mapsto g^\sharp$ is not surjective: it is circumvented thanks to an approximation lemma (*cf* [36, Lemma 6.5]) that imply that any $f \in R$ can be approximated by an element of the form $g^\sharp$ so that the maps $x \mapsto |f(x)|$ and $x \mapsto |g^\sharp(x)|$ are close. From this one can prove (i) and (ii).

Statements (iii) & (iv) are first proved in the characteristic $p$ case, when $(R,R^+)$ is $p$-finite (*i.e.* the $\varpi$-adic completion of the perfection of a $K$-algebra which is topologically of finite type): if $X = \cup_{i=1}^n U_i$ is a rational covering and

$$\mathsf{C}^\bullet\colon\ 0 \to R^+ \to \prod_{i=1}^n \mathcal{O}_X^+(U_i) \to \prod_{1\le i,j\le n} \mathcal{O}_X^+(U_i\cap U_j) \to \cdots$$

the associated Cech complex, then $\mathsf{C}^\bullet[\varpi^{-1}]$ is exact (this follows from Tate's acyclicity theorem, *cf* [6, Theorem 8.2.1]), and the Banach open mapping theorem shows that each cohomology group of $\mathsf{C}^\bullet$ is killed by some power of $\varpi$. As $R$ is perfect, the Frobenius map induces isomorphisms on these groups: they are in fact killed by $\varpi^{1/p^m}$ for all $m \in \mathbf{N}$.
This implies statements (iii) & (iv) for perfectoid affinoid $K$-algebra $(R,R^+)$ of characteristic $p$, for which $R^+$ is a $K^\circ$-algebra, because these are completions of filtered direct limits of $p$-finite perfectoid affinoid $K$-algebras (*cf* [36, Lemma 6.13]). The general case in characteristic $p$ is reduced to the latter by replacing $K$ by the $\varpi$-adic completion of $\mathbf{F}_p(\!(\varpi)\!)[\varpi^{-1}]$.
Finally the characteristic 0 case follows from the characteristic $p$ case: let $X = \cup_{i=1}^n U_i$ be a rational covering and consider the sequence

$$(*)\qquad 0 \to \mathcal{O}_X(X)^{\circ a} \to \prod_{i=1}^n \mathcal{O}_X(U_i)^{\circ a} \to \prod_{1\le i,j\le n} \mathcal{O}_X(U_i\cap U_j)^{\circ a} \to \cdots$$

The characteristic $p$ case shows that the tilted sequence is exact. By flateness, the reduction mod $\varpi^\flat$ of the latter is still exact. But this reduction is just the sequence $(*)$ mod $\varpi$. This implies that $(*)$ is exact by flatness and $\varpi$-adic completeness (*cf* [36, Proposition 6.14]). ☐

***Remark 6.14.*** — As mentioned in theorem 4.16, the proof of (iii) was simplified by Buzzard-Verberkmoes, who showed that if $(R,R^+)$ is Tate and stably uniform, then it is sheafy.

Theorem 5.21 generalizes to the relative case as follows:

***Theorem 6.15 (cf* [36, Proposition 5.23 & Theorem 5.25]). — *Let $R$ be a perfectoid $K$-algebra.*

(13) Where $R^\circ\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\dots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right\rangle$ is the the $\varpi$-adic completion of $R^\circ\left[\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\dots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right] \subset R^\circ\left[\frac{1}{g^\sharp}\right]$

(i) *The tilt induces an equivalence*

$$\begin{aligned}\{\textit{finite étale } R\textit{-algebras}\} &\to \{\textit{finite étale } R^\flat\textit{-algebras}\}\\ S &\mapsto S^\flat\end{aligned}$$

(ii) *If $S/R$ is finite étale, then $S$ is perfectoid.*

(iii) (Faltings' almost purity theorem) *If $S/R$ is finite étale, then $S^\circ$ is almost finite étale over $R^\circ$ (cf definition 3.13).*

*Proof.* — If $x \in X := \mathsf{Spa}(R, R^\circ)$, we have $K(x)^\flat = K(x^\flat)$ (residue fields). Theorem 5.21 implies the equivalence (i) at every point. To glue these local equivalences, one uses the following approximation theorem à la Elkik.

***Theorem 6.16.*** — (*cf* [14, Théorème 5], [21, Proposition 5.4.53]) *Let $A$ be a $K$-algebra which is topologically henselian, i.e. the pair $(A_0, \varpi A_0)$ is henselian*[(14)] *for some ring of definition $A_0$. Then the functor*

$$\begin{aligned}\{\textit{finite étale } A\textit{-algebras}\} &\to \{\textit{finite étale } \widehat{A}\textit{-algebras}\}\\ B &\mapsto B \otimes_A \widehat{A}\end{aligned}$$

*is an equivalence.*

If $A = \varinjlim_{i\in I} A_i$ is a filtered direct limit of Banach $K$-algebras, this implies that[(15)]

$$2\text{-}\varinjlim \{\text{finite étale } A_i\text{-algebras}\} = \{\text{finite étale } A\text{-algebras}\} = \{\text{finite étale } \widehat{A}\text{-algebras}\}$$

As $K(x) = \widehat{\varinjlim_{x\in U} \mathcal{O}_X(U)}$, the horizontal arrows in the following diagram are equivalences:

$$\begin{tikzcd}
2\text{-}\varinjlim_{x\in U} \{\text{finite étale } \mathcal{O}_X(U)\text{-algebras}\} \arrow[r] \arrow[d, "\text{tilt}"'] & \{\text{finite étale } K(x)\text{-algebras}\} \arrow[d, "\text{tilt}"] \\
2\text{-}\varinjlim_{x\in U} \{\text{finite étale } \mathcal{O}_{X^\flat}(U)\text{-algebras}\} \arrow[r] & \{\text{finite étale } K(x^\flat)\text{-algebras}\}
\end{tikzcd}$$

By using the sheaf property of $\mathcal{O}_X$, one can glue compatible families of finite étale covers of coverings of $X$ by finitely many rational subsets, and get finite étale covers of $X$: this allows to glue the equivalences.
(ii) and (iii) are easily proved when $pR = 0$: they follow in general thanks to the tilting equivalence as in (i). $\square$

***Example 6.17.*** — Assume $K$ has characteristic 0 and contains all $p$-power roots of unity, and put $A = K\langle T^{\pm 1}\rangle$ (corresponding to the one dimensional analytic torus) and $A_\infty = \cup_n A[T^{1/p^n}]$, so $R := \widehat{A_\infty}$ is perfectoid. Denote by $\overline{A}$ the integral closure of $A$ in an algebraic closure of $\mathsf{Frac}(A)$ (corresponding to a geometric generic point $\bar{x}$ of $\mathsf{Spec}(A)$) and $G = \pi_1(\mathsf{Spec}(R), \bar{x})$, so that

$$\begin{tikzcd}
\overline{A} \arrow[dd, dash, "G"'] \arrow[dr, dash, "H"] & \\
& A_\infty \arrow[dl, dash, "\Gamma\simeq \mathbf{Z}_p(1)"] \\
A &
\end{tikzcd}$$

(14) Recall that a pair $(A, I)$ is henselian if $I$ is contained in the Jacobson ideal of $A$, and for all $P \in A[X]$ monic whose reduction modulo $I$ factors as a product of two monic polynomials $Q_0$ and $R_0$ which generate the unit ideal, then $P = QR$ with $Q, R \in A[X]$ lift $Q_0$ and $R_0$ respectively. This is the case, for instance, when $A$ is $I$-adically complete.

(15) Given a filtered category $I$ and a direct system of categories $(\mathcal{C}_i, F_{i,j})$, the 2-limit $\mathcal{C} = 2\text{-}\varinjlim_I \mathcal{C}_i$ is the category whose objects are objects in any of the $\mathcal{C}_i$ and if $X_i \in \mathcal{C}_i$, $X_j \in \mathcal{C}_j$, we have $\mathsf{Hom}_{\mathcal{C}}(X_i, X_j) = \varinjlim_{i,j\to k} \mathsf{Hom}_{\mathcal{C}_k}(F_{i,k}(X_i), F_{j,k}(X_j))$.

Then $\mathsf{H}^i_{\mathrm{cont}}(G,\widehat{\overline{A}}) = \begin{cases} A & \text{if } i = 0 \\ A(-1) & \text{if } i = 1 \\ \{0\} & \text{if } i > 1 \end{cases}.$

Indeed, using Faltings' almost purity, we have $\mathsf{H}^i_{\mathrm{cont}}(H,\widehat{\overline{A}}) \approx \begin{cases} R & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$, so $\mathsf{H}^i_{\mathrm{cont}}(G,\widehat{\overline{A}}) \simeq \mathsf{H}^i_{\mathrm{cont}}(\Gamma, R)$, which is very easy to compute because the extension $A_\infty/A$ is quite explicit. Let $\gamma$ be a topological generator. The Banach space $R$ is topologically free over $A$, with basis $\{T^\alpha\}_{\alpha\in[0,1[\cap\mathbf{Z}[p^{-1}]}$. The contribution of the eigenspace $AT^\alpha$ to the cohomology is given by the complex

$$AT^\alpha \xrightarrow{\gamma-1} AT^\alpha.$$

Write $\alpha = \frac{a}{p^n}$, so that $\gamma(T^\alpha) = \zeta_{p^n}^a T^\alpha$: the complex identifies with

$$A \xrightarrow{\zeta_{p^n}^a - 1} A$$

so this contribution is zero if $\alpha \neq 0$ or in degrees $> 1$, and is a free $A$-module of rank 1 in degrees 0 and 1 if $\alpha = 0$.

**6.18. Perfectoid spaces.** — An adic space $X$ as in Theorem 6.12 is called a *perfectoid affinoid space* (over $K$). By theorem 6.12, one can glue them to get perfectoid spaces:

***Definition 6.19* (*cf* [36, Definition 6.15]).** — A *perfectoid space* over $K$ is an adic space over $K$ covered by perfectoid affinoid spaces.

The tilt on perfectoid affinoid spaces glues into a functor $X \mapsto X^\flat$ between perfectoid spaces over $K$ and perfectoid spaces over $K^\flat$.

***Definition 6.20* (*cf* [36, Definition 7.1]).** — Let $f\colon X \to Y$ be a morphism between perfectoid spaces.

(1) $f$ is *finite étale* if for every open affinoid $\mathsf{Spa}(A,A^+) \subset Y$, we have $X \times_Y \mathsf{Spa}(A,A^+) \simeq \mathsf{Spa}(B,B^+)$ where $B$ is a finite étale $A$-algebra, and $B^+$ is the integral closure of $A^+$ in $B$.

(2) $f$ is *étale* if it is locally the composite of an open immersion and a finite étale morphism, *i.e.* if for any $x \in X$, there exist open $x \in U \subset X$ and $f(U) \subset V \subset Y$ such that there is a factorization

$$\begin{tikzcd} U \arrow[d, "f_{|U}"'] \arrow[r, hook] & W \arrow[dl, "\text{finite étale}"] \\ V & \end{tikzcd}$$

An *étale cover* of $Y$ is a family $\{f_i\colon X_i \to Y\}_{i\in I}$ of étale maps such that $Y = \cup_{i\in I} f(X_i)$.

These notions enjoy the usual properties:

***Proposition 6.21.*** — (i) *The property of being (finite) étale is stable under composition and pullback.*

(ii) *If $g$ and $g\circ f$ are étale, so is $f$.*

(iii) *$f$ is étale if and only if $f^\flat$ is.*

If $X$ is perfectoid, one defines the *étale site* $X_{\text{ét}}$ as usual[(16)]. The presheaf $U \mapsto \mathcal{O}_U(U)$ is a sheaf $\mathcal{O}_X$ on $X_{\text{ét}}$.

***Theorem 6.22.*** — (*cf* [36, Theorem 7.12 & Proposition 7.13]) *Let $X$ be a perfectoid space over $K$. The tilt induces an equivalence between the étale topoi $X_{\text{ét}}$ and $X^\flat_{\text{ét}}$. Moreover $\mathsf{H}^i(X_{\text{ét}}, \mathcal{O}_X^+)$ is almost zero if $X$ is affinoid perfectoid and $i > 0$.*

(16) The underlying category is that of étale maps $U \to X$ (with morphisms given by $X$-morphisms), and covering families of $U \to X$ are families $\{f_i\colon U_i \to U\}$ such that $U = \cup_i f_i(U_i)$.

## 7. Comparison theorem for rigid analytic varieties

In this section, we give an overview of [38]. Let $K$ be a complete valuation extension of $\mathbf{Q}_p$, and $K^+ \subset K$ an open and bounded sub-valuation ring. Scholze proves (among other things) the $\mathsf{C}_{\mathrm{dR}}$-conjecture for adic spaces, answering a question of Tate in this context (*cf* [46, p.180]).

***Theorem 7.1* (*cf* [38, Theorem 8.4]).** — *Assume that the valuation on $K$ is discrete and that its residue field is perfect. Let $X$ be a proper and smooth adic space over $\mathsf{Spa}(K, \mathcal{O}_K)$ and $V = \mathsf{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$. Then $V$ is de Rham, and $\mathsf{D}_{\mathrm{dR}}(V) \cong \mathsf{H}^n_{\mathrm{dR}}(X/K)$. In particular, $V$ is Hodge-Tate and the Hodge to de Rham spectral sequence*

$$\mathsf{H}^i(X, \Omega^j_{X/K}) \Rightarrow \mathsf{H}^{i+j}_{\mathrm{dR}}(X/K)$$

*degenerates.*

***Remark 7.2.*** — In fact, Scholze shows a relative version with coefficients of this result ([38, Theorem 8.8], *cf* theorems 7.44 & 7.45).

To prove this theorem, he shows the following intermediate results (that are fundamental by themselves).

***Theorem 7.3* (*cf* [38, Theorem 5.1]).** — *Assume $K$ algebraically closed. If $X$ is proper and smooth*[17]*, and $\mathbb{L}$ a local system in $\mathbf{F}_p$-vector spaces on $X_{\text{ét}}$, then $\mathsf{H}^i(X_{\text{ét}}, \mathbb{L})$ is a finite dimensional $\mathbf{F}_p$-vector spaces, and vanishes for $i > 2\dim(X)$.*

***Remark 7.4.*** — The proof of this theorem, inspired by that, due to Kiehl, of the finiteness of coherent cohomology of proper rigid analytic varieties, uses the Artin-Schreier sequence.

***Theorem 7.5* (*cf* [38, Theorem 4.9]).** — *Assume $K$ algebraically closed. If $X$ is connected affinoid, $x \in X(K)$ and $\mathbb{L}$ a $p$-torsion local system, the natural map*

$$\mathsf{H}^i_{\mathrm{cont}}(\pi_1(X, x), \mathbb{L}_x) \to \mathsf{H}^i(X_{\text{ét}}, \mathbb{L})$$

*is an isomorphism ($X$ is locally a $K(\pi, 1)$ for $p$-torsion coefficients).*

***Remark 7.6.*** — (1) The proof of this theorem, which also uses the Artin-Schreier sequence, is based on [36].
(2) In Faltings' works, in the algebraic case, this result is proved under smallness hypotheses, which are not needed here.

The strategy, which is common (more or less explicitly) to most of the proofs of comparison theorems, relies on the use of relative (*i.e.* sheafified) versions of period rings. A major difficulty is the problem of passing to the limit. In the present case, this is crucial since, in contrast with the crystalline or semi-stable cases, one cannot use $p$-adic dévissage with the ring $\mathsf{B}_{\mathrm{dR}}$ (which is a direct limit of an inverse limit of a direct limit). One of the main contributions of [38] is the definition of the *pro-étale* site, which provides the right framework for local computations, globalization and passages to the limit.

### 7.7. The pro-étale site. —

***Definition 7.8.*** — Let $\mathscr{C}$ be a category and $\mathscr{C} \to \widehat{\mathscr{C}} := \mathsf{Fonct}(\mathscr{C}, \mathbf{Sets})^{\mathrm{op}}$ its Yoneda extension. The category pro-$\mathscr{C}$ of *pro-objects* of $\mathscr{C}$ is the full subcategory of $\widehat{\mathscr{C}}$ whose objects are inverse limits of representable objects, indexed by small cofiltrant categories, *i.e.* functors $F \colon I \to \mathscr{C}$ with $I$ small cofiltrant category. If $F \colon I \to \mathscr{C}$ and $G \colon J \to \mathscr{C}$ are objects of pro-$\mathscr{C}$, we have:

$$\mathsf{Hom}_{\text{pro-}\mathscr{C}}(F, G) = \varprojlim_{j \in J} \varinjlim_{i \in I} \mathsf{Hom}_{\mathscr{C}}(F(i), G(j))$$

[17] In fact, thanks to the resolution of singularities for proper adic spaces, the smoothness assumption is superfluous (*cf* [37, Theorem 3.17]).

Let $X$ be a locally noetherian scheme or adic space. There are categories $X_{\text{ét}}$ and $X_{\text{fét}}$, whose objects are spaces that are (finite) étale over $X$ (*cf* definition 6.20). An object $U$ in pro-$X_{\text{ét}}$ is a functor $i \to U_i$ on a small cofiltrant category $I$: we denote this by $U = \varprojlim_{i \in I} U_i$. We have $|U| = \varprojlim_{i \in I} |U_i|$ as topological spaces.

***Definition 7.9*** **(*cf*** [38, Definitions 3.3 & 3.4 & 3.9] **&** [43]**).** — (0) A map $V \to U$ in pro-$X_{\text{fét}}$ has the property $(*)$ if it can be written as an inverse limit $\varprojlim_{\mu<\lambda} V_\mu \to U$ of $V_\mu \in$ pro-$X_{\text{fét}}$ ($V_0 = U$) indexed by ordinals $\mu$ less that some ordinal $\lambda$, with the property that for all $\mu < \lambda$, $V_\mu \to V_{<\mu} := \varprojlim_{\mu'<\mu} V_{\mu'}$ is the pullback of a finite étale and surjective map in $X_{\text{fét}}$.

(1) The *pro-finite étale* site $X_{\text{profét}}$ of $X$ is the site whose underlying category is pro-$X_{\text{fét}}$, and whose coverings are families of morphisms $\{U_i \xrightarrow{f_i} U\}$ having the property $(*)$, such that $|U| = \cup_i f_i(|U_i|)$.
If $X$ is connected and $\overline{x}$ a geometric point of $X$, there is an equivalence of sites:

$$X_{\text{profét}} \xrightarrow{\approx} \pi_1(X, \overline{x})\text{-}\,\mathbf{pfSets}$$

where $\pi_1(X,\overline{x})$- **pfSets** is the site of profinite $\pi_1(X,\overline{x})$-sets (*i.e.* of profinite sets endowed with a continuous action of $\pi_1(X,\overline{x})$), and whose coverings are given by families of $\pi_1(X,\overline{x})$-equivariant maps $\{S_i \xrightarrow{f_i} S\}$ (having the property analogue to $(*)$) such that $S = \cup_i f_i(S_i)$.

(2) A morphism $U \to V$ in pro-$X_{\text{ét}}$ is called *étale* (resp. *finite étale*) if it comes from an étale (resp. finite étale) morphism $U_0 \to V_0$ in $X_{\text{ét}}$ by a base-change $V \to V_0$.

(3) A morphism $U \to V$ in pro-$X_{\text{ét}}$ is called *pro-étale* if it admits a *pro-étale presentation* $U = \varprojlim_{i \in I} U_i \to V$, cofiltrant inverse limit of étale morphisms $U_i \to V$, whose transition morphisms $U_i \to U_j$ are finite étale for large $i > j$[18].

(4) The *pro-étale* site $X_{\text{proét}}$ of $X$ is the site whose underlying category is the full subcategory of pro-$X_{\text{ét}}$ made of pro-étale objects over $X$, and whose coverings are families of pro-étale morphisms $\{U_i \xrightarrow{f_i} U\}$ (again, having the property similar to $(*)$, *cf* [43]) such that $|U| = \cup_i f_i(|U_i|)$.

***Remark 7.10.*** — The fact that $X_{\text{proét}}$ is indeed a site is not trivial (*cf* [38, Lemma 3.10]).

***Proposition 7.11.*** — *(1) In $X_{\text{proét}}$, pro-étale maps are open (cf* [38, Lemma 3.10]*).*

*(2) A surjective (finite) étale map $U \to V$ in $X_{\text{proét}}$ comes from a surjective (finite) étale map $U_0 \to V_0$ in $X_{\text{ét}}$ by a base change $V \to V_0$ (cf* [38, Lemma 3.10]*).*

*(3) There is a fully faithful functor $X_{\text{profét}} \to X_{\text{proét}}$ which induces a morphism of sites $X_{\text{proét}} \to X_{\text{profét}}$ (cf* [38, Lemma 3.11]*).*

*(4) The site $X_{\text{proét}}$ has enough points: a conservative family is given by profinite coverings of geometric points*[19] *(cf* [38, Proposition 3.13]*).*

Let $\nu\colon X_{\text{proét}} \to X_{\text{ét}}$ be the projection morphism.

***Proposition 7.12*** **(*cf*** [38, Lemma 3.16, Corollary 3.17]**).** — *Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{ét}}$ and $U = \varprojlim_{i \in I} U_i \in X_{\text{proét}}$. Then $\mathsf{H}^j(U, \nu^*\mathcal{F}) = \varinjlim_{i\in I} \mathsf{H}^j(U_i, \mathcal{F})$. This implies that the adjuction morphism $\mathcal{F} \to \mathsf{R}\,\nu_*\nu^*\mathcal{F}$ is an isomorphism, as is the base change morphism $\nu_Y^* \mathsf{R}\, f_{\text{ét}\,*}\mathcal{F} \to \mathsf{R}\, f_{\text{proét}\,*}\nu_X^*\mathcal{F}$ if $f\colon X \to Y$ is a quasi-compact and quasi-separated morphism.*

***Definition 7.13.*** — The structure sheaf of $X_{\text{proét}}$ and its subring of integral elements are:

$$\mathcal{O}_X = \nu^*\mathcal{O}_{X_{\text{ét}}} \quad \text{and} \quad \mathcal{O}_X^+ = \nu^*\mathcal{O}_{X_{\text{ét}}}^+$$

[18] $U = \varprojlim_{i \in I} U_i \in$ pro-$X_{\text{ét}}$ since $U_i \in$ pro-$X_{\text{ét}}$ for all $i$ (because pro-$X_{\text{ét}}$ is stable by inverse limits indexed by small cofiltrant categories).
[19] Geometric points alone are not enough.

There are also completed versions:

$$\widehat{\mathscr{O}}_X^+ = \varprojlim_n \mathscr{O}_X^+/p^n\mathscr{O}_X^+ \quad \text{and} \quad \widehat{\mathscr{O}}_X = \widehat{\mathscr{O}}_X^+[1/p]$$

***Remark 7.14.*** — If $U \in X_{\text{proét}}$ and $x \in |U|$, we have the continuous valuation $f \mapsto |f(x)|$ on $\mathscr{O}_X(U)$. It extends into a continuous valuation on $\widehat{\mathscr{O}}_X(U)$, and

$$\widehat{\mathscr{O}}_X^+(U) = \big\{ f \in \widehat{\mathscr{O}}_X(U),\ (\forall x \in |U|)\ |f(x)| \leq 1 \big\}$$

(*idem* with the non complete version). Furthermore, the morphism of sheaves

$$\mathscr{O}_X^+/p^n\mathscr{O}_X^+ \to \widehat{\mathscr{O}}_X^+/p^n\widehat{\mathscr{O}}_X^+$$

is an isomorphism for all $n \in \mathbf{N}_{>0}$ (*cf* [38, Lemma 4.2]).

In this paragraph and the following two, we assume that $K$ is perfectoid, and that $X$ is a locally noetherian adic space over $\mathsf{Spa}(K, K^+)$. In what follows, perfectoid spaces play a key role: their definition has to be extended to objects of $X_{\text{proét}}$.

***Definition 7.15 (cf** [38, Definition 4.3]**).*** — Let $U \in X_{\text{proét}}$.

1. $U$ is called *affinoid perfectoid* if there exists a pro-étale presentation $U = \varprojlim_{i\in I} U_i \to X$ with $U_i = \mathsf{Spa}(R_i, R_i^+)$ affinoid for all $i \in I$, such that $(R, R^+)$ is an affinoid perfectoid $(K, K^+)$-algebra, where $R^+$ is the $p$-adic completion of $\varinjlim_{i\in I} R_i$ and $R = R^+[1/p]$. The affinoid perfectoid $\widehat{U} = \mathsf{Spa}(R, R^+)$ space is independent from the presentation, $U \mapsto \widehat{U}$ is functorial, and $|\widehat{U}| = |U|$.
2. $U$ is called *perfectoid* if it admits an open covering by affinoid perfectoid[20].

**Basic example.** If $X = \mathbb{T}^n = \mathsf{Spa}\left(K\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle, K^+\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle\right)$, then

$$\widetilde{\mathbb{T}}^n := \varprojlim_{i\in\mathbf{N}} \mathsf{Spa}\left(K\langle T_1^{\pm 1/p^i}, \ldots, T_n^{\pm 1/p^i}\rangle, K^+\langle T_1^{\pm 1/p^i}, \ldots, T_n^{\pm 1/p^i}\rangle\right)$$

is affinoid perfectoid.

***Proposition 7.16 (cf** [38, Lemma 4.6]**).*** — *If $V \in X_{\text{proét}}$ is perfectoid and $U \to V$ is pro-étale, then $U$ is perfectoid.*

***Proposition 7.17 (cf** [38, Proposition 4.8]**).*** — *The $U \in X_{\text{proét}}$ that are affinoid perfectoid form a basis of the topology, i.e. $X$ is locally perfectoid for the pro-étale topology.*

***Remark 7.18.*** — When $X$ is smooth over $\mathsf{Spa}(K, K^+)$, this follows from proposition 7.16. Indeed, $X$ locally admits a map to $\mathbb{T}^n$: we can reduce to the case $X = \mathbb{T}^n$, and every $U \in X_{\text{proét}}$ is covered by[21] $U \times_{\mathbb{T}^n} \widetilde{\mathbb{T}}^n$, which is pro-étale over $\widetilde{\mathbb{T}}^n$ (*cf* [38, Corollary 4.7] and its proof).

**7.19. Proof of the finiteness and vanishing theorem.** — The following general lemma will be useful for passing to the limit.

***Lemma 7.20 (cf** [38, Lemma 3.18]**).*** — *Let $\{\mathcal{F}_i\}_{i\in\mathbf{N}}$ be an inverse system of sheaves on a site $\mathscr{T}$. Assume there exists a basis $\mathscr{B}$ of $\mathscr{T}$ such that for all $U \in \mathscr{B}$, one has $\mathsf{R}^1 \varprojlim_i \mathcal{F}_i(U) = 0$ and $\mathsf{H}^n(U, \mathcal{F}_i) = 0$ for all $i \in \mathbf{N}$ and $n \in \mathbf{N}_{>0}$. Then:*

1. *$\mathsf{R}^n \varprojlim_i \mathcal{F}_i = 0$ for all $n \in \mathbf{N}_{>0}$ and $(\varprojlim_i \mathcal{F}_i)(U) = \varprojlim_i \mathcal{F}_i(U)$ for all $U \in \mathscr{T}$ ;*
2. *$\mathsf{H}^n(U, \varprojlim_i \mathcal{F}_i) = 0$ for all $U \in \mathscr{B}$ and $n \in \mathbf{N}_{>0}$.*

[20] A quasi-compact open of $U \in X_{\text{proét}}$ gives rise an object of $X_{\text{proét}}$: if $W \subset |U|$ is a quasi-compact open subset, there exists $V \in X_{\text{proét}}$ and an étale map $V \to U$ inducing an homeomorphism $|V| \to W$ such that for any $V' \in X_{\text{proét}}$ and any étale map $V' \to U$ that factors over $W$ on topological spaces, the map $V' \to U$ factors over $V$ (*cf* [38, Lemma 3.10 (iii)]).

[21] This construction of a perfectoid tower is used constantly for local cohomological computations.

In what follows, this lemma will be applied to $\mathscr{T} = X_{\text{proét}}$ and $\mathscr{B}$ the affinoid perfectoid objects, whose cohomology has good properties of almost vanishing, as we shall now see.

***Lemma 7.21** (**cf** [38, Lemma 4.10, Lemma 4.12]).* — *Let $U \in X_{\text{proét}}$ be affinoid perfectoid. Then $\widehat{\mathcal{O}}_X^+(U) = R^+$ is the $p$-adic completion of $\mathcal{O}_X^+(U)$. Furthermore, the cohomology groups $\mathsf{H}^n(U, \widehat{\mathcal{O}}_X^+)$ are almost zero for $n \in \mathbf{N}_{>0}$. In particular,*

$$\big(|U|, \mathcal{O}_X|_{|U|}, \{|\bullet(x)|\}_{x\in|U|}\big)$$

*is a perfectoid adic space, which is nothing but $\widehat{U}$. It follows*[22] *that for all local system of $\mathbf{F}_p$-vector spaces $\mathbb{L}$ on $X_{\text{ét}}$, the cohomology group*

$$\mathsf{H}^j\big(U, \mathbb{L}\otimes(\mathcal{O}_X^+/p\mathcal{O}_X^+)\big)$$

*is almost zero for $j \in \mathbf{N}_{>0}$, and an almost projective $R^+/pR^+$-module of finite type if $j = 0$.*

The local ingredient for the proof of theorem 7.3 is the following:

***Lemma 7.22** (**cf** [38, Lemma 5.6]).* — *Assume that $K$ contains the $p^m$-th roots of unity for all $m \in \mathbf{N}$. Let $V$ be a smooth affinoid adic space over $\mathsf{Spa}(K, \mathcal{O}_K)$ and $\mathbb{L}$ local system of $\mathbf{F}_p$-vector spaces on $V_{\text{ét}}$. Assume there is an étale map $V \to \mathbb{T}^n$ which is a composite of rational embeddings and finite étale maps. Then:*

*(1) $\mathsf{H}^j(V_{\text{ét}}, \mathbb{L}\otimes\mathcal{O}_V^+/p\mathcal{O}_V^+)$ is almost zero for $j > n (= \dim(V))$ ;*
*(2) if $V' \subsetneq V$ is rational open, the image of*

$$\mathsf{H}^j\big(V_{\text{ét}}, \mathbb{L}\otimes(\mathcal{O}_V^+/p\mathcal{O}_V^+)\big) \to \mathsf{H}^j\big(V'_{\text{ét}}, \mathbb{L}\otimes(\mathcal{O}_{V'}^+/p\mathcal{O}_{V'}^+)\big)$$

*is almost of finite type over $\mathcal{O}_K$.*

*Sketch of the proof.* — Thanks to proposition 7.12, we can compute in the pro-étale site. Consider $\widetilde{V} = V\otimes_{\mathbb{T}^n}\widetilde{\mathbb{T}}^n$, the perfectoid tower built on $V$: we know that $\mathsf{H}^j\big(\widetilde{V}, \mathbb{L}\otimes(\mathcal{O}_V^+/p\mathcal{O}_V^+)\big)$ is almost zero for $j \in \mathbf{N}_{>0}$ and almost projective of finite type over $S^+/pS^+$ (where $\widehat{\widetilde{V}} = \mathsf{Spa}(S, S^+)$). As $\widetilde{V} \to V$ is Galois with group $\mathbf{Z}_p^n$, (1) follows from explicit computations (Koszul complex). The proof of (2), more subtle, requires the construction of a chain $V' = V^{(n+2)} \subsetneq V^{(n+1)} \subsetneq \cdots \subsetneq V^{(1)} = V$. One computes the cohomology using the Hochschild-Serre spectral sequence[23], and one has to control finiteness of images of maps between spectral sequences (*cf* [38, Lemma 5.4]). ☐

***Lemma 7.23** (**cf** [38, Lemma 5.9]).* — *Under the assumptions of lemma 7.22, if $X$ is a proper and smooth adic space over $\mathsf{Spa}(K, \mathcal{O}_K)$, and if $\mathbb{L}$ is a local system of $\mathbf{F}_p$-vector spaces on $X_{\text{ét}}$, then the $\mathcal{O}_K$-module $\mathsf{H}^j\big(X_{\text{ét}}, \mathbb{L}\otimes(\mathcal{O}_X^+/p\mathcal{O}_X^+)\big)$ is almost projective of finite type, and almost zero for $j > 2\dim(X)$.*

*Sketch of the proof.* — The preceding lemma shows that if $\lambda\colon X_{\text{ét}} \to X_{\text{an}}$ is the projection[24], then $\mathsf{R}^j\lambda_*(\mathbb{L}\otimes\mathcal{O}_X^+/p\mathcal{O}_X^+)$ is almost zero for $j > \dim(X)$. As $X_{\text{an}}$ has cohomological dimension $\dim(X)$, the vanishing property follows. For finiteness, one proceeds similarly as above, by constructing $j+2$ coverings of $X$ (having appropriate properties of inclusion and smallness, *cf* [38, Lemma 5.3]), and controlling the finiteness through the corresponding $j+2$ spectral sequences. ☐

To prove theorem 7.3, one shows[25] first that modulo an almost isomorphism, we may reduce to the case $K^+ = \mathcal{O}_K$. Then, one tilts[26]. One has $\mathcal{O}_{X^\flat}/p^\flat\mathcal{O}_{X^\flat} \xrightarrow{\sim} \mathcal{O}_X^+/p\mathcal{O}_X^+$, so

$$\mathsf{H}^j\big(X_{\text{ét}}, \mathbb{L}\otimes(\mathcal{O}_{X^\flat}^+/p\mathcal{O}_{X^\flat}^+)\big)$$

(22) Thanks to the almost purity theorem almost faithfully flat descent.
(23) For the exact sequence $0 \to p^m\,\mathbf{Z}_p^n \to \mathbf{Z}_p^n \to (\mathbf{Z}_p/p^m\,\mathbf{Z}_p)^n \to 0$.
(24) Where $X_{\text{an}}$ is the site of opens in $X$.
(25) By considering $X \times_{\mathsf{Spa}(K,K^+)} \mathsf{Spa}(K, \mathcal{O}_K)$, a simplicial covering of $X$ and using lemma 7.21.
(26) So some results above have to be extended to the characteristic $p$ case...

is almost projective of finite type, and almost zero for $j > 2\dim(X)$ (by the preceding lemma). Let $p^\flat \in \mathcal{O}_{K^\flat}$ be such that $(p^\flat)^\sharp = p$. If $k \in \mathbf{Z}_{>0}$, the $\mathcal{O}_{K^\flat}/p^{\flat k}\mathcal{O}_{K^\flat}$-module

$$M_k = \mathsf{H}^i\big(X_{\text{proét}}, \mathbb{L} \otimes \widehat{\mathcal{O}}^+_{X^\flat}/p^{\flat k}\widehat{\mathcal{O}}^+_{X^\flat}\big)$$

satisfies the hypothesis of lemma 3.12 (thanks to lemma 7.23, since $\widehat{\mathcal{O}}^+_{X^\flat}$ is a sheaf of perfect flat $\mathcal{O}_{K^\flat}$-algebras and $\widehat{\mathcal{O}}^+_{X^\flat}/p^\flat\widehat{\mathcal{O}}^+_{X^\flat} \simeq \mathcal{O}^+_X/p\mathcal{O}^+_X$, note also that $K^\flat$ is algebraically closed because $K$ is, *cf* proposition 5.20): there exists an integer $r$ such that $M_k^a \simeq (\mathcal{O}^a_{K^\flat}/p^{\flat k}\mathcal{O}^a_{K^\flat})^r$ as almost $\mathcal{O}_{K^\flat}$-modules, compatibly with the Frobenius action. Applying lemma 7.20 (using proposition 7.17), we get $\mathsf{R}\varprojlim\big(\mathbb{L}\otimes\widehat{\mathcal{O}}^+_{X^\flat}/p^{\flat k}\widehat{\mathcal{O}}^+_{X^\flat}\big)^a = \big(\mathbb{L}\otimes\widehat{\mathcal{O}}^+_{X^\flat}\big)^a$ so $\mathsf{H}^i(X_{\text{proét}}, \mathbb{L}\otimes\widehat{\mathcal{O}}^+_{X^\flat})^a \simeq (\mathcal{O}^a_{K^\flat})^r$. Inverting $p^\flat$ implies that

$$\mathsf{H}^i(X_{\text{proét}}, \mathbb{L}\otimes\widehat{\mathcal{O}}_{X^\flat}) \simeq (K^\flat)^r$$

is a finite dimensional $K^\flat$-vector space for all $i$, zero if $i > 2\dim(X)$. Artin-Schreier morphism

$$\begin{aligned} \mathbb{L}\otimes\widehat{\mathcal{O}}_{X^\flat} &\to \mathbb{L}\otimes\widehat{\mathcal{O}}_{X^\flat} \\ v\otimes x &\mapsto v\otimes(x^p - x) \end{aligned}$$

provides the long exact sequence:

$$\cdots \to \mathsf{H}^i(X_{\text{proét}}, \mathbb{L}) \to \mathsf{H}^i(X_{\text{proét}}, \mathbb{L}\otimes\widehat{\mathcal{O}}_{X^\flat}) \to \mathsf{H}^i(X_{\text{proét}}, \mathbb{L}\otimes\widehat{\mathcal{O}}_{X^\flat}) \to \cdots$$

and

$$\mathsf{H}^i(X_{\text{ét}}, \mathbb{L}) = \mathsf{H}^i(X_{\text{proét}}, \mathbb{L}) = \big(\mathsf{H}^i(X_{\text{proét}}, \mathbb{L}\otimes\widehat{\mathcal{O}}_{X^\flat})\big)^{\varphi=1} \cong \mathbf{F}_p^r$$

Note that the proof provides an almost isomorphism

$$\mathsf{H}^i(X_{\text{ét}}, \mathbb{L}) \otimes (K^+/pK^+) \to \mathsf{H}^i\big(X_{\text{ét}}, \mathbb{L}\otimes(\mathcal{O}^+_X/p\mathcal{O}^+_X)\big)$$

for all $i \in \mathbf{N}$. Finally, theorem 7.3 implies[(27)] its own relative version:

***Theorem 7.24* (*cf* [38, Corollary 5.11]).** — *Let $f\colon X \to Y$ be a proper and smooth morphism of locally noetherian adic spaces over $\mathsf{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, and $\mathbb{L}$ a local system of $\mathbf{F}_p$-vector spaces on $X_{\text{ét}}$. Then for all $i \in \mathbf{N}$, there is an almost isomorphism of sheaves of $\mathcal{O}^+_Y$-modules*

$$(\mathsf{R}^i f_{\text{ét}\,*}\mathbb{L}) \otimes (\mathcal{O}^+_Y/p\mathcal{O}^+_Y) \to \mathsf{R}^i f_{\text{ét}\,*}(\mathbb{L}\otimes(\mathcal{O}^+_X/p\mathcal{O}^+_X))$$

***Remark 7.25.*** — Scholze has extended this result to the case where $f$ comes by adification from a proper morphism of schemes of finite type over a complete algebraically closed extension of $\mathbf{Q}_p$, and $\mathbb{L}$ from a *constructible* $\mathbf{F}_p$-sheaf (*cf* [37, Theorem 3.13]).

**7.26. The period sheaves.** — Assume $K$ contains the $p^m$-th roots of unity for all $m \in \mathbf{N}$. On $X_{\text{proét}}$, there are the following sheaves:

$$\mathbf{A}_{\text{inf}} = \mathsf{W}(\widehat{\mathcal{O}}^+_{X^\flat}) \quad\text{ and }\quad \mathbb{B}_{\text{inf}} = \mathbf{A}_{\text{inf}}[1/p]$$

endowed with ring morphism $\theta\colon \mathbb{A}_{\text{inf}} \to \widehat{\mathcal{O}}^+_X$ and $\theta\colon \mathbb{B}_{\text{inf}} \to \widehat{\mathcal{O}}_X$; put:

$$\mathbb{B}^+_{\text{dR}} = \varprojlim_{n\in\mathbf{N}_{>0}} \mathbb{B}_{\text{inf}}/(\mathsf{Ker}(\theta))^n \quad\text{ and }\quad \mathbb{B}_{\text{dR}} = \mathbb{B}^+_{\text{dR}}[1/t]$$

filtered by $\mathsf{Fil}^i\,\mathbb{B}^+_{\text{dR}} = (\mathsf{Ker}(\theta))^i$ and $\mathsf{Fil}^i\,\mathbb{B}_{\text{dR}} = \sum\limits_{j\in\mathbf{Z}} t^{-j}\,\mathsf{Fil}^{i+j}\,\mathbb{B}^+_{\text{dR}}$. As usual, $t = \log[\varepsilon] = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}([\varepsilon]-1)^n$, where $[\varepsilon]$ is the Teichmüller lift of a section $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$ of $\widehat{\mathcal{O}}^+_{X^\flat}$ such that $\zeta_{p^m}$ is a primitive $p^m$-th root of unity for all $m \in \mathbf{N}$ (so that $\mathbf{Z}_p(1) = \mathbf{Z}_p\,t$). The element $t$ exists locally and is unique up to a unit. It is not a zero divisor.
Fix $\pi \in K^\flat$ such that $\pi^\sharp/p \in (K^+)^\times$. The kernel of $\theta\colon \mathbf{A}(K, K^+) \to K^+$ is principal, generated by an element $\xi \equiv [\pi] \mod p\mathbf{A}(K, K^+)$. One can compute global sections of these sheaves on affinoid perfectoid spaces:

(27) By looking at the fibers at geometric points $Y$.

***Theorem 7.27*** (*cf* [38, Theorem 6.5]). — *Let $U \in X_{\text{proét}}$ affinoid perfectoid over $\mathsf{Spa}(K, K^+)$: write $\widehat{U} = \mathsf{Spa}(R, R^+)$.*

(1) *There is a canonical isomorphism:*

$$\mathbf{A}_{\text{inf}}(U) \cong \mathbf{A}_{\text{inf}}(R, R^+) := \mathsf{W}(R^{\flat +})$$

*and similar isomorphisms*[28] *with $\mathbb{B}_{\text{inf}}$, $\mathbb{B}_{\text{dR}}^+$ and $\mathbb{B}_{\text{dR}}$.*

(2) *$\mathsf{H}^i(U, \mathbf{A}_{\text{inf}})$ and $\mathsf{H}^i(U, \mathbb{B}_{\text{inf}})$ are almost zero*[29] *for $i \in \mathbf{N}_{>0}$.*

(3) *$\mathsf{H}^i(U, \mathbb{B}_{\text{dR}}^+)$ and $\mathsf{H}^i(U, \mathbb{B}_{\text{dR}})$ are zero for $i \in \mathbf{N}_{>0}$.*

*Sketch of the proof.* — By induction on $n$, one has the description of global sections of $(\mathbf{A}_{\text{inf}}/p^n\mathbf{A}_{\text{inf}})(R, R^+)$ and the almost vanishing of cohomology. Using lemma 7.20 to pass to the limit, one deduces the statements on $\mathbf{A}_{\text{inf}}$, those on $\mathbb{B}_{\text{inf}}$ follow. For $\mathbb{B}_{\text{dR}}^+$, one uses the exact sequence $0 \to \mathbb{B}_{\text{inf}} \xrightarrow{\xi^n} \mathbb{B}_{\text{inf}} \to \mathbb{B}_{\text{inf}}/(\mathsf{Ker}(\theta))^n \to 0$ and lemma 7.20. The vanishing of the cohomology results from the invertibility of $\theta([\pi])$. □

***Proposition 7.28*** (*cf* [38, Proposition 6.7]). — $\mathsf{gr}^i\,\mathbb{B}_{\text{dR}} \cong \widehat{\mathcal{O}}_X(i)$ *for all $i \in \mathbf{Z}$.*

Assume $K$ has discrete valuation and a perfect residue field. Let $X$ be a smooth adic space over $\mathsf{Spa}(K, \mathcal{O}_K)$. Put $\mathcal{O}\mathbb{B}_{\text{inf}} = \mathcal{O}_X \otimes_{\nu^*\mathsf{W}(k)} \mathbb{B}_{\text{inf}}$ and $\Omega_X^1 = \nu^*\Omega_{X_{\text{ét}}}^1$: these are sheaves on $X_{\text{proét}}$, and there is a ring morphism $\theta \colon \mathcal{O}\mathbb{B}_{\text{inf}} \to \widehat{\mathcal{O}}_X^+$, and a $\mathbb{B}_{\text{inf}}$-linear connection $\nabla \colon \mathcal{O}\mathbb{B}_{\text{inf}} \to \mathcal{O}\mathbb{B}_{\text{inf}} \otimes_{\mathcal{O}_X} \Omega_X^1$. Let $\mathcal{O}\mathbb{B}_{\text{dR}}^+$ be the sheafification of the presheaf sending $U = \varprojlim_i U_i$ (with $U_i = \mathsf{Spa}(R_i, R_i^+)$) to

$$\varinjlim_i \Big( \varprojlim_n \big(R_i^+ \widehat{\otimes}_W \mathbf{A}_{\text{inf}}(R, R^+)\big)[p^{-1}]/(\mathsf{Ker}(\theta))^n \Big)$$

(where $(R, R^+)$ is the affinoid perfectoid completed direct limit of the $(R_i, R_i^+)$, and $\widehat{\otimes}$ is the completed tensor product for the $p$-adic topology, *cf* [43]). Put:

$$\mathcal{O}\mathbb{B}_{\text{dR}} = \mathcal{O}\mathbb{B}_{\text{dR}}^+[1/t]$$

These are sheaves on $X_{\text{proét}}$. They are filtered by $\mathsf{Fil}^i\,\mathcal{O}\mathbb{B}_{\text{dR}}^+ = (\mathsf{Ker}(\theta))^i$ and $\mathsf{Fil}^i\,\mathcal{O}\mathbb{B}_{\text{dR}} = \sum_{j \in \mathbf{Z}} t^{-j}\,\mathsf{Fil}^{i+j}\,\mathcal{O}\mathbb{B}_{\text{dR}}^+$ pour $i \in \mathbf{Z}$. The connection $\nabla$ extends into a $\mathbb{B}_{\text{dR}}^+$-linear connection

$$\nabla \colon \mathcal{O}\mathbb{B}_{\text{dR}}^+ \to \mathcal{O}\mathbb{B}_{\text{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^1$$

One can describe these sheaves locally: assume there is an étale morphism $X \to \mathbb{T}^n$: put $\widetilde{X} = X \otimes_{\mathbb{T}^n} \widetilde{\mathbb{T}}^n$. The field $C = \widehat{\widetilde{K}}$ is perfectoid: so is the base change $\widetilde{X}_C \in X_{\text{proét}}/X_C$. In what follows, we restrict to the localized site $X_{\text{proét}}/\widetilde{X}$. For $1 \leq i \leq n$, we have:

$$u_i = T_i \otimes 1 - 1 \otimes [T_i^\flat] \in \mathsf{Ker}(\theta) \subset \mathcal{O}\mathbb{B}_{\text{inf}}|_{\widetilde{X}}$$

so there is a morphism of $\mathbb{B}_{\text{dR}}^+|_{\widetilde{X}}$-algebras $\mathbb{B}_{\text{dR}}^+|_{\widetilde{X}}[\![X_1, \ldots, X_n]\!] \to \mathcal{O}\mathbb{B}_{\text{dR}}^+|_{\widetilde{X}}$ mapping $X_i$ to $u_i$.

***Proposition 7.29*** (*cf* [38, Proposition 6.10, Corollary 6.15]). — *The map*

$$\mathbb{B}_{\text{dR}}^+|_{\widetilde{X}}[\![X_1, \ldots, X_n]\!] \to \mathcal{O}\mathbb{B}_{\text{dR}}^+|_{\widetilde{X}}$$

*is an isomorphism of sheaves on $X_{\text{proét}}/\widetilde{X}$. In particular, one has*

$$\mathsf{gr}^i\,\mathcal{O}\mathbb{B}_{\text{dR}}|_{\widetilde{X}_C} \cong \xi^i \widehat{\mathcal{O}}_X\left[\frac{X_1}{\xi}, \ldots, \frac{X_n}{\xi}\right]$$

*so that*

$$\mathsf{gr}^\bullet\,\mathcal{O}\mathbb{B}_{\text{dR}}|_{\widetilde{X}_C} \cong \widehat{\mathcal{O}}_X[\xi^{\pm 1}, X_1, \ldots, X_n]$$

*(with $\xi, X_1, \ldots, X_n$ of degree $1$).*

[28] Note that $\mathsf{Ker}(\theta \colon \mathbf{A}_{\text{inf}}(R, R^+) \to R^+) = \xi\mathbf{A}_{\text{inf}}(R, R^+)$.

[29] With respect to the ideal generated by $\{[\pi^{1/p^r}]\}_{r \in \mathbf{N}}$ in $\mathbf{A}_{\text{inf}}(K, K^+)$ (note that all the rings considered are $\mathbf{A}_{\text{inf}}(K, K^+)$-algebras).

***Corollary 7.30* (Poincaré lemma, *cf* [38, Corollary 6.13]).** — *If $X$ is smooth of dimension $n$ over $\mathsf{Spa}(K, \mathcal{O}_K)$, the sequence*

$$0 \to \mathbb{B}_{\mathrm{dR}}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^n \to 0$$

*of sheaves on $X_{\mathrm{pro\acute{e}t}}$ is exact.*

***Remark 7.31.*** — The first graded piece of this exact sequence provides the exact sequence:

$$0 \to \widehat{\mathcal{O}}_X(1) \to \mathsf{gr}^1\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \to \widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X} \Omega_X^1 \to 0$$

called *Faltings' extension.*

We now describe the cohomology of these period sheaves.

***Proposition 7.32* (*cf* [38, Proposition 6.16]).** — *Assume $X = \mathsf{Spa}(R, R^+)$ affinoid of finite type over $\mathsf{Spa}(K, \mathcal{O}_K)$, endowed with a map $X \to \mathbb{T}^n$, which is a composite of rational embeddings and finite étale maps. Then:*

(7.32.1)
$$\mathsf{H}^q(X_C, \mathsf{gr}^0\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}) = \begin{cases} R\widehat{\otimes}_K C & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

(7.32.2)
$$\mathsf{H}^q(X, \mathsf{gr}^i\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}) = \begin{cases} R & \text{if } i = 0 \text{ and } q = 0 \\ R\log(\chi) & \text{if } i = 0 \text{ and } q = 1 \\ 0 & \text{if } i \neq 0 \text{ or } q > 1 \end{cases}$$

*(recall that $\chi\colon \mathsf{Gal}(\overline{K}/K) \to \mathbf{Z}_p^\times$ is the cyclotomic character, and $\log(\chi)$ is its logarithm).*

*Sketch of the proof.* — (1) Consider the covering $\widetilde{X}_C \to X_C$: it is Galois of group $\mathbf{Z}_p^n$. As $\widetilde{X}_C$ is perfectoid, its cohomology is zero in positive degrees (*cf* lemma 7.21), which reduces to the computation of the cohomology of $\mathbf{Z}_p^n$ with values in $\widetilde{R}[V_1, \ldots, V_d]$, where $\widehat{\widetilde{X}} = \mathsf{Spa}(\widetilde{R}, \widetilde{R}^+)$ and $V_i = t^{-1}\log\big(\frac{[T_i^\flat]}{T_i}\big)$. The action of the $i$-th basis vector $\gamma_i$ of $\mathbf{Z}_p^n$ is given by $\gamma_i(V_j) = V_j + \delta_{ij}$: the cohomology is computed by a very simple Koszul complex. (2) follows, from Tate results. □

***Corollary 7.33* (*cf* [38, Corollary 6.19]).** — *If $X$ is smooth $\mathsf{Spa}(K, \mathcal{O}_K)$, then*[(30)] *:*

$$\nu_*\mathcal{O}\mathbb{B}_{\mathrm{dR}} = \mathcal{O}_{X_{\mathrm{\acute{e}t}}}$$

$$\mathsf{R}^i\nu_*\widehat{\mathcal{O}}_X(j) = \begin{cases} \Omega^i_{X_{\mathrm{\acute{e}t}}} & \text{if } i = j \\ \Omega^i_{X_{\mathrm{\acute{e}t}}}\log(\chi) & \text{if } i = j+1 \\ 0 & \text{if } i \notin \{j, j+1\} \end{cases}$$

***Remark 7.34.*** — The isomorphism $\mathsf{R}^1\nu_*\widehat{\mathcal{O}}_X(1) \cong \Omega^1_{X_{\mathrm{\acute{e}t}}}$ is deduced from Faltings' extension.

**7.35. Filtered modules with integrable connection.** — Henceforth, $X$ is assumed to be smooth over $\mathsf{Spa}(K, \mathcal{O}_K)$ (recall that $K$ has discrete valuation). The local structure theorem of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ implies[(31)] the following result:

***Proposition 7.36* (*cf* [38, Theorem 7.2]).** — *The functor $\mathbb{M} \mapsto (\mathcal{M} := \mathbb{M} \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+, \nabla_{\mathcal{M}} = \mathsf{Id} \otimes \nabla)$ induces an equivalence of categories between the category of $\mathbb{B}_{\mathrm{dR}}^+$-local systems*[(32)] *and that of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$-modules with an integrable connection*[(33)] *(one has $\mathbb{M} = \mathcal{M}^{\nabla_{\mathcal{M}}=0}$).*

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(30) This generalizes computations of Tate (*cf* theorem 1.1) and of Hyodo [26, Theorem 1].

(31) Thanks to the fact that for a $\mathbf{Q}$-algebra $A$, a module with integrable connection over $A[\![X_1, \ldots, X_n]\!]$ has enough horizontal sections (*cf* [31, Proposition 8.9] when $A$ is a field, the proof is valid, *mutatis mutandis*, in the general case).

(32) Made of $\mathbb{B}_{\mathrm{dR}}^+$-modules $\mathbb{M}$ that are locally free of finite rank over $X_{\mathrm{pro\acute{e}t}}$.

(33) Made of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$-modules $\mathcal{M}$ that are locally free of finite rank over $X_{\mathrm{pro\acute{e}t}}$, endowed with an integrable connection $\nabla_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X^1$.

***Definition 7.37*** (*cf* [38, Definition 7.5]). — (1) A *filtered $\mathcal{O}_X$-module with integrable connection* is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ over $X$, endowed with a decreasing, separated and exhaustive filtration by sub-modules $\mathsf{Fil}^\bullet \mathcal{E}$ that are locally direct factors, and a connection $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ satisfying Griffith transversality.
(2) A filtered $\mathcal{O}_X$-module with integrable connection $\mathcal{E}$ and a $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$-module with integrable connection $\mathcal{M}$ are *associated* if there exists an isomorphism

$$\mathcal{M} \otimes_{\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \cong \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}}$$

compatible with filtrations and connections.

***Proposition 7.38*** (*cf* [38, Theorem 7.6]). — (1) *If $\mathcal{M}$ and $\mathcal{E}$ are associated, then*[34]

$$\mathbb{M} := \mathcal{M}^{\nabla_{\mathcal{M}}=0} = \mathsf{Fil}^0(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}})^{\nabla=0}$$
$$\mathcal{E}_{\text{ét}} = \lambda^*\mathcal{E} = \nu_*(\mathbb{M} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}_{\mathrm{dR}})$$

(2) *If $\mathcal{E}$ is a filtered $\mathcal{O}_X$-module with integrable connection, then $\mathbb{M} = \mathsf{Fil}^0(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}})^{\nabla=0}$ is a $\mathbb{B}^+_{\mathrm{dR}}$-local system; furthermore, $\mathcal{E}$ and $\mathcal{M} = \mathbb{M} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ are associated. In particular, there is a fully faithful functor from the category of filtered $\mathcal{O}_X$-modules with integrable connection to that of $\mathbb{B}^+_{\mathrm{dR}}$-local systems.*

***Theorem 7.39*** (*cf* [38, Theorem 7.11]). — *Let $\mathcal{E}$ be a filtered $\mathcal{O}_X$-module with integrable connection and $\mathbb{M}$ the associated $\mathbb{B}^+_{\mathrm{dR}}$-local system. There is a canonical isomorphism*[35] *:*

$$\mathsf{H}^i(X_{\overline{K}}, \mathbb{M}) \otimes_{\mathsf{B}^+_{\mathrm{dR}}} \mathsf{B}_{\mathrm{dR}} \cong \mathsf{H}^i_{\mathrm{dR}}(X, \mathcal{E}) \otimes_K \mathsf{B}_{\mathrm{dR}}$$

*compatible with filtrations and the action of $G_K = \mathsf{Gal}(\overline{K}/K)$. Furthermore, there is a $G_K$-equivariant isomorphism:*

$$\mathsf{H}^i(X_{\overline{K}}, \mathsf{gr}^0\,\mathbb{M}) \cong \bigoplus_{j\in\mathbf{Z}} \mathsf{H}^{i-j,j}_{\mathsf{h}}(X, \mathcal{E}) \otimes_K C(-j)$$

*where $\mathsf{H}^{i-j,j}_{\mathsf{h}}(X, \mathcal{E}) = \mathsf{H}^i(X, \mathsf{gr}^j\,\mathrm{DR}(\mathcal{E}))$ is the Hodge cohomology*[36] *in bidegree $(i-j, j)$.*

*Sketch of the proof.* — There is a morphism of filtered complexes

$$\mathrm{DR}(\mathcal{E}) \to \mathrm{DR}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}}$$

which induces a morphism

$$\mathsf{R}\Gamma(X_{\overline{K}}, \mathrm{DR}(\mathcal{E})) \otimes_{\overline{K}} \mathsf{B}_{\mathrm{dR}} \to \mathsf{R}\Gamma(X_{\overline{K}}, \mathrm{DR}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}})$$

in the filtered derived category. One checks it is a quasi-isomorphism by looking at the graded. One can replace $\mathrm{DR}(\mathcal{E})$ by a locally free $\mathcal{O}_X$-module, and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ by $\mathsf{gr}^0\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}$, which reduces the computation to coherent cohomology, and one can use the fact that coherent cohomology on a proper adic space has finite dimension and commutes with base field change (*cf* [38, Lemma 7.13]). □

**7.40. The comparison theorem.** —

***Definition 7.41.*** — Denote by $\widehat{\mathbf{Z}}_p$ the sheaf $\varprojlim_n(\mathbf{Z}/p^n\,\mathbf{Z})$ on $X_{\text{proét}}$. A *lisse $\widehat{\mathbf{Z}}_p$-sheaf* over $X_{\text{proét}}$ is a sheaf $\mathbb{L}$ locally (over $X_{\text{proét}}$) of the form $\widehat{\mathbf{Z}}_p \otimes_{\mathbf{Z}_p} M$ where $M$ is $\mathbf{Z}_p$-module of finite type.

***Proposition 7.42*** (*cf* [38, Proposition 8.2]). — *There is an equivalence of categories $\mathbb{L}_\bullet \mapsto \widehat{\mathbb{L}_\bullet} := \varprojlim_n \nu^*\,\mathbb{L}_n$ between the category of lisse $\mathbf{Z}_p$-sheaves over $X_{\text{ét}}$ and that of lisse $\widehat{\mathbf{Z}}_p$-sheaves over $X_{\text{proét}}$. Furthermore, $\mathsf{R}^j \varprojlim_n \nu^*\,\mathbb{L}_n = 0$ for all $j \in \mathbf{N}_{>0}$.*

[34] Recall that $\lambda\colon X_{\text{ét}} \to X_{\mathrm{an}}$ is the projection, where $X_{\mathrm{an}}$ is the site of opens in $X$ (*cf* proof of lemma 7.23).
[35] Where, of course $\mathsf{B}_{\mathrm{dR}} = \mathbb{B}_{\mathrm{dR}}(C, \mathcal{O}_C)$.
[36] $\mathrm{DR}(\mathcal{E}) = (0 \to \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} \cdots)$ denoting the de Rham complex of $\mathcal{E}$, endowed with its filtration $\mathsf{Fil}^m\,\mathrm{DR}(\mathcal{E}) = (0 \to \mathsf{Fil}^m\,\mathcal{E} \xrightarrow{\nabla} \mathsf{Fil}^{m-1}\,\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} \cdots)$.

***Definition 7.43.*** — Assume $X$ proper[37] over $\mathsf{Spa}(K, \mathcal{O}_K)$. A lisse $\widehat{\mathbf{Z}}_p$-sheaf $\mathbb{L}$ over $X_{\text{proét}}$ is said to be *de Rham* if the $\mathbb{B}_{\text{dR}}^+$-local system $\mathbb{M} = \mathbb{L} \otimes_{\widehat{\mathbf{Z}}_p} \mathbb{B}_{\text{dR}}^+$ is associated to a filtered $\mathcal{O}_X$-module with integrable connection $(\mathcal{E}, \nabla, \mathsf{Fil}^\bullet \mathcal{E})$.

***Theorem 7.44*** **(*cf* [38, Theorem 8.4]).** — *Let $\mathbb{L}$ be a lisse $\widehat{\mathbf{Z}}_p$-sheaf over $X_{\text{proét}}$ and $\mathbb{M} = \mathbb{L} \otimes_{\widehat{\mathbf{Z}}_p} \mathbb{B}_{\text{dR}}^+$. Then there is a $G_K$-equivariant isomorphism*

$$\mathsf{H}^i(X_{\overline{K}}, \mathbb{L}) \otimes_{\mathbf{Z}_p} \mathsf{B}_{\text{dR}} \cong \mathsf{H}^i(X_{\overline{K}}, \mathbb{M})$$

*If $\mathbb{L}$ is de Rham, associated to the filtered $\mathcal{O}_X$-module with integrable connection $\mathcal{E}$, then the Hodge to de Rham spectral sequence*

$$\mathsf{H}_{\mathsf{h}}^{i-j,j}(X, \mathcal{E}) \Rightarrow \mathsf{H}_{\text{dR}}^i(X, \mathcal{E})$$

*degenerates, and there is a $G_K$-equivariant isomorphism*

$$\mathsf{H}^i(X_{\overline{K}}, \mathbb{L}) \otimes_{\mathbf{Z}_p} \mathsf{B}_{\text{dR}} \cong \mathsf{H}_{\text{dR}}^i(X, \mathcal{E}) \otimes_K \mathsf{B}_{\text{dR}}$$

*which is compatible with filtrations and whose graded provides a $G_K$-equivariant isomorphism*

$$\mathsf{H}^i(X_{\overline{K}}, \mathbb{L}) \otimes_{\mathbf{Z}_p} C \cong \bigoplus_{j \in \mathbf{Z}} \mathsf{H}_{\mathsf{h}}^{i-j,j}(X, \mathcal{E}) \otimes_K C(-j)$$

*Sketch of the proof.* — Using lemma 7.20, one shows by induction on $n$ that $\mathsf{H}^i(X_{\overline{K}}, \mathbb{L}_n)$ is of finite type and that there is an almost isomorphism

$$\mathsf{H}^i(X_{\overline{K}}, \mathbb{L}_n) \otimes_{\widehat{\mathbf{Z}}_p} \mathrm{A}_{\text{inf}} \to \mathsf{H}^i(X_{\overline{K}}, \mathbb{L}_n \otimes_{\mathbf{Z}_p} \mathbf{A}_{\text{inf}})$$

(the case $n = 1$ being theorem 7.3). Inverting $p$ and reducing modulo $\xi^n$, one gets an isomorphism[38]

$$\mathsf{H}^i(X_{\overline{K}}, \mathbb{L}_n) \otimes_{\mathbf{Z}_p} (\mathsf{B}_{\text{inf}} / \mathsf{Ker}(\theta)^m) \to \mathsf{H}^i(X_{\overline{K}}, \mathbb{L}_n \otimes_{\widehat{\mathbf{Z}}_p} (\mathbb{B}_{\text{inf}} / \mathsf{Ker}(\theta))^m)$$

for all $m \in \mathbf{N}_{>0}$. Using lemma 7.20 again, one deduces the first isomorphism. The degeneracy of the Hodge to de Rham spectral sequence then follows by counting dimensions, and the second isomorphism from theorem 7.39. □

Here again, theorem 7.44 implies its relative version. Let $f\colon X \to Y$ be a smooth morphism of smooth adic spaces over $\mathsf{Spa}(K, \mathcal{O}_K)$, of relative dimension $d$. The projection $\Omega_X^1 \to \Omega_{X/Y}^1$ induces a relative connection

$$\nabla_{X/Y}\colon \mathcal{O}\mathbb{B}_{\text{dR},X} \to \mathcal{O}\mathbb{B}_{\text{dR},X} \otimes_{\mathcal{O}_X} \Omega_{X/Y}^1$$

Moreover, the sequence

$$0 \to \mathbb{B}_{\text{dR},X} \otimes_{f_{\text{proét}}^* \mathbb{B}_{\text{dR},Y}} f_{\text{proét}}^* \mathcal{O}\mathbb{B}_{\text{dR},Y} \to \mathcal{O}\mathbb{B}_{\text{dR},X} \xrightarrow{\nabla_{X/Y}} \mathcal{O}\mathbb{B}_{\text{dR},X} \otimes_{\mathcal{O}_X} \Omega_{X/Y}^1 \xrightarrow{\nabla_{X/Y}} \cdots$$
$$\cdots \xrightarrow{\nabla_{X/Y}} \mathcal{O}\mathbb{B}_{\text{dR},X} \otimes_{\mathcal{O}_X} \Omega_{X/Y}^d \to 0$$

is exact.

***Theorem 7.45*** **(*cf* [38, Theorem 8.8]).** — *Assume $f\colon X \to Y$ proper and smooth. Let $\mathbb{L}$ be a lisse $\widehat{\mathbf{Z}}_p$-sheaf over $X_{\text{proét}}$ and $\mathbb{M} = \mathbb{L} \otimes_{\widehat{\mathbf{Z}}_p} \mathbb{B}_{\text{dR},X}^+$. Assume*[39] *that $\mathsf{R}^i f_{\text{proét}\,*} \mathbb{L}$ is a lisse $\widehat{\mathbf{Z}}_p$-sheaf over $Y_{\text{proét}}$.*

*(1) there is a canonical isomorphism:*

$$\mathsf{R}^i f_{\text{proét}\,*} \mathbb{M} \cong \mathsf{R}^i f_{\text{proét}\,*} \mathbb{L} \otimes_{\widehat{\mathbf{Z}}_p} \mathbb{B}_{\text{dR},Y}^+$$

(37) Recall it is assumed to be smooth.
(38) Not "almost", because the ideal generated by $\{[\pi^{1/p^r}]\}_{r \in \mathbf{N}}$ is the unit ideal of $\mathsf{B}_{\text{inf}} / \mathsf{Ker}(\theta)^m$.
(39) This holds true when $f$ and $\mathbb{L}$ are analytifications of algebraic objects.

*(2) If $\mathbb{L}$ is de Rham, associated to the filtered $\mathcal{O}_X$-module with integrable connection $(\mathcal{E}, \nabla, \mathsf{Fil}^\bullet \mathcal{E})$, then the relative Hodge cohomology $\mathsf{R}_{\mathsf{h}}^{i-j,j} f_{\mathsf{h}*}\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, and the Hodge to de Rham spectral sequence*

$$\mathsf{R}^{i-j,j} f_{\mathsf{h}*}\mathcal{E} \Rightarrow \mathsf{R}^i f_{\mathrm{dR}}\mathcal{E}$$

*degenerates. Furthermore, $\mathsf{R}^i f_{\mathrm{pro\acute{e}t}*}\mathbb{L}$ is de Rham, associated to the filtered $\mathcal{O}_Y$-module with integrable connection $\mathsf{R}^i f_{\mathrm{dR}}\mathcal{E}$.*

**7.46. The Hodge-Tate spectral sequence.** — Let $X$ be a proper and smooth rigid analytic variety over $C$. In [37], Scholze constructs a (*Hodge-Tate*) spectral sequence

$$\mathsf{H}^i(X, \Omega_X^j)(-j) \Rightarrow \mathsf{H}_{\mathrm{\acute{e}t}}^{i+j}(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C$$

(*cf* [37, Theorem 3.20]). It is constructed as follows. The projection $\nu\colon X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ provides the spectral sequence

$$\mathsf{H}^i(X_{\mathrm{\acute{e}t}}, \mathsf{R}^j \nu_* \widehat{\mathcal{O}}_X) \Rightarrow \mathsf{H}^{i+j}(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}_X)$$

The abutment is isomorphic to $\mathsf{H}_{\mathrm{\acute{e}t}}^{i+j}(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C$. Indeed, by the proof of theorem 7.3, there is an almost isomorphism

$$\mathsf{H}^i(X_{\mathrm{\acute{e}t}}, \mathbf{F}_p) \otimes_{\mathbf{F}_p} (\mathcal{O}_C/p\mathcal{O}_C) \to \mathsf{H}^i(X_{\mathrm{\acute{e}t}}, \mathcal{O}_X^+/p\mathcal{O}_X^+)$$

thus an almost isomorphism

$$\mathsf{H}^i(X_{\mathrm{\acute{e}t}}, \mathbf{Z}/p^n\mathbf{Z}) \otimes_{\mathbf{Z}/p^n\mathbf{Z}} (\mathcal{O}_C/p^n\mathcal{O}_C) \to \mathsf{H}^i(X_{\mathrm{\acute{e}t}}, \mathcal{O}_X^+/p^n\mathcal{O}_X^+)$$

by induction on $n$. Passing to the limit (thanks to lemma 7.20), we get an almost isomorphism

$$\mathsf{H}^i(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathbf{Z}}_p) \otimes_{\mathbf{Z}_p} \mathcal{O}_C \to \mathsf{H}^i(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}_X^+)$$

hence the isomorphism searched for by inverting $p$. Then, local computations (similar to those of proposition 7.32, relying on the smoothness of $X$) show that:

$$\mathcal{O}_{X_{\mathrm{\acute{e}t}}} \xrightarrow{\sim} \nu_* \widehat{\mathcal{O}}_X$$
$$\Omega^1_{X_{\mathrm{\acute{e}t}}}(-1) \xrightarrow{\sim} \mathsf{R}^1 \nu_* \widehat{\mathcal{O}}_X$$

which induces an isomorphism

$$\Omega^j_{X_{\mathrm{\acute{e}t}}}(-j) \xrightarrow{\sim} \mathsf{R}^j \nu_* \widehat{\mathcal{O}}_X$$

for all $j \in \mathbf{N}$ (*cf* [37, Proposition 3.23]).

this spectral sequence degenerates if $X$ is defined over $K$. In this case, it provides a decreasing filtration on $\mathsf{H}_{\mathrm{\acute{e}t}}^\bullet(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C$, such that

$$\mathsf{Fil}^q\, \mathsf{H}_{\mathrm{\acute{e}t}}^i(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C / \mathsf{Fil}^{q+1}\, \mathsf{H}_{\mathrm{\acute{e}t}}^i(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C \cong \mathsf{H}^q(X, \Omega_X^{i-q})(q-i)$$

## 8. The monodromy-weight conjecture

In this section we state the MWC, recall the result of Deligne and state the theorem of Scholze.

**8.1. Setting.** — We fix a local field $k$ with finite residue field $\mathbf{F}_q$, where $q = p^n$ for some prime $p > 0$ (*e.g.* $k = \mathbf{Q}_p, \mathbf{F}_q((t))$). We have the exact sequence of Galois groups

$$1 \to I \to \mathsf{Gal}(\bar{k}/k) \to \mathsf{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q) \to 1$$

and, for any prime $\ell \neq p$, the inertia $I = \mathsf{Gal}(\bar{k}/k^{\mathrm{ur}})$ fits in the following exact sequence

$$1 \to P_\ell \to I \to \mathbf{Z}_\ell(1) \to 1$$

where $\mathbf{Z}_\ell(1)$ denotes the inverse limit $\varprojlim_n \mu_{\ell^n}(\overline{\mathbf{F}}_q)$[40]; if we denote by $k_\ell = k^{\mathrm{ur}}(\pi^{1/\ell^n} \,|\, n \in \mathbf{N})$, then $P_\ell = \mathsf{Gal}(\bar{k}/k_\ell)$.

We denote by and $\tau_\ell\colon I \to \mathbf{Z}_\ell(1)$ the projection defined by $\pi^{1/\ell^n}\tau_{\ell,n}(g) = g(\pi^{1/\ell^n})$.

[40] $\mu_{\ell^n}(\overline{\mathbf{F}}_q) = \{x \in \overline{\mathbf{F}}_q \,|\, x^{\ell^n} = 1\}$ and by Hensel's lemma $\mu_{\ell^n}(\overline{\mathbf{F}}_q) = \mu_{\ell^n}(\bar{k})$.

We can identify $\mathbf{Z}_\ell(1)$ with $\mathsf{Gal}(k_\ell/k^{\mathrm{ur}})$. This is compatible with the action of $\mathsf{Gal}(\bar{k}/k)$: it acts in a natural way on the roots of unity so on $\mathbf{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}(k^{\mathrm{sep}})$: it acts by inner automorphism on $\mathsf{Gal}(k_\ell/k^{\mathrm{ur}})$. So if $t \in \mathbf{Z}_\ell(1)$ is a topological generator, we have the $\ell$-adic cyclotomic character $\chi_\ell \colon G_k \to \mathbf{Z}_\ell^*$ defined by

$$t^{\chi(g)} = g(t) = gtg^{-1}.$$

(see [44] for more details).

**8.2. The Local Monodromy Theorem.** — Let us recall an important result due to Grothendieck. It appears for the first time in [10, letter 24 Sept. 1964 p.183][(41)], see also [45, Appendix] or [27, 1, 13].

***Theorem 8.3*** **(Grothendieck).** — *Let $\rho\colon \mathsf{Gal}(\bar{k}/k) \to \mathsf{GL}(V)$ be a finite dimensional $\overline{\mathbf{Q}}_\ell$-representation. Then there exists an open subgroup $I_1 \subset I$ such that $\rho$ restricted to $I_1$ is unipotent.*[(42)]

*Proof.* — First notice that, after replacing $k$ by a finite extension, $\rho_{|I}$ factors through $\mathbf{Z}_\ell(1)$ and we can reduce to prove that $\rho(t)$ is quasi-unipotent, for $t \in \mathbf{Z}_\ell(1)$ a generator.

One easily shows that if $a$ is an eigenvalue of $\rho(t)$, *i.e.* $\rho(t)\cdot v = av$, then

$$\rho(t)(w) = a^{\chi(g)}w \qquad \text{for } w := \rho(g^{-1})v$$

for all $g$. It follows that $a^{\chi(g)}$ is an eigenvalue of $\rho(t)$ for all $g$ in $G_k$. Since we assume that the residue field is finite we get that the image of the cyclotomic character is infinite. Since $V$ is finite dimensional we deduce that $a$ is a root of unity and $\rho(t)$ is quasi-unipotent as expected. □

**8.4. Monodromy filtration.** — Let $V(1) := V \otimes_{\mathbf{Z}} \mathbf{Z}_\ell(1)$.

Let $T \in \mathsf{End}_{\mathbf{Q}_\ell}(V)$ be unipotent, we can define the nilpotent endomorphism

$$\log(T) := -\sum_{n=1} \frac{(\mathsf{Id} - T)^n}{n}.$$

Now let $t \in \mathbf{Z}_\ell(1)$ be a generator and define

$$N\colon V \otimes_{\mathbf{Q}_\ell(G_k)} \mathbf{Q}_\ell(1) \to V \ , \ \ v \otimes t \mapsto \log(\rho(t))(v)$$

called the *logarithm of the unipotent part of the monodromy.*

It is easy to verify that $N$ is a morphism of $\ell$-adic representations: it is $G_k$-equivariant and $\mathbf{Q}_\ell$-linear.

Moreover

$$(\forall\ g \in I_1) \quad \rho(g) = \exp(N(- \otimes \tau_\ell(g)))$$

A formal consequence of the above theorem is the existence of a unique nilpotent operator $N\colon V(1) \to V$ such that $\rho(g) = \exp(N \cdot \tau_\ell(g))$ for all $g \in I_1$. The map $N$ is called the *logarithm of the nilpotent part of the local monodromy.*

***Lemma 8.5*** **(Jacobson-Morozov).** — *Given a nilpotent endomorphism $N$ on a vector space $V$, then there is a unique (increasing, separated exhaustive) filtration $V_i \subset V_{i+1} \subset \cdots \subset V$ such that $N(V_i) \subset V_{i-2}$ and $N^i$ induces an isomorphism $\mathsf{gr}_i^N V \to \mathsf{gr}_{-i}^N V$, where $\mathsf{gr}_i^N V := V_i/V_{i-1}$.*

*Proof.* — We construct $V_i$ by induction on $d$ such that $N^{d+1} = 0$: if $d = 0$, then $N = 0$ and we set $V_{-1} = 0 \subset V_0 = V$; if $d > 0$ we set

$$V_{-d-1} = 0 \subset V_{-d} = \mathsf{Im}(N^d) \subset V_{d-1} = \mathsf{Ker}(N^d) \subset V_d = V,$$

so that the following sequence is exact

$$0 \to V_{-d} \to V_{d-1} \to \mathsf{Ker}(N^d)/\,\mathsf{Im}(N^d) \to 0$$

(41) Serre answers Grothendieck saying: *Ton théorème sur l'action du groupe d'inertie est rupinant – si tu l'as vraiment montré.*

(42) *i.e.* for any $g \in I_1$, $(\rho(g) - \mathsf{Id})^r = 0$ for some positive integer $r$ (depending on $g$).

and $N$ induces nilpotent endomorphism $\overline{N}$ on the latter term such that $\overline{N}^d = 0$. Now by induction there exists a filtration on $\mathsf{Ker}(N^d)/\,\mathsf{Im}(N^d)$ (associated with $\overline{N}$): taking the pull-back of this filtration we get a filtration with the desired properties on $V$. $\square$

In our setting we can identify $V$ and $V(1)$ after choosing a topological generator of $\mathbf{Q}_\ell(1)$ and we get the so called the *local monodromy filtration*. Mind that this implies that $I_1$ acts trivially on the graded quotients $\mathsf{gr}_i^N V$.

*8.5.1. Tate Elliptic curve.* — Let $E/\,\mathbf{Q}_p$ be the Tate elliptic curve whose group of $\overline{\mathbf{Q}}_p$-rational points is $E(\overline{\mathbf{Q}}_p) = \overline{\mathbf{Q}}_p^\times/p^{\mathbf{Z}}$. The $\ell$-adic Tate module of $E$ fits in the following exact sequence

$$1 \to \mathbf{Z}_\ell(1) \to T_\ell E \to \mathbf{Z}_\ell \to 0.$$

Hence the rational Tate module $V_\ell(E) = T_\ell E \otimes \mathbf{Q}_\ell = \mathsf{H}^1(E)^\vee$ is isomorphic to $\mathbf{Q}_\ell(1)\cdot t \oplus \mathbf{Q}_\ell \cdot 1$ and the matrix of $\rho(t)$ in that basis is

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{so that} \quad N = \log(M) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the monodromy filtration is $V_{-1} = \mathsf{Im}(N) \subset V_0 = \mathsf{Ker}(N) \subset V_1 = V$. In particular one gets

$$\mathsf{gr}_1 V = \mathbf{Q}_\ell \ , \ \mathsf{gr}_{-1} V = \mathbf{Q}_\ell(1)$$

where $\mathbf{Q}_\ell$ is the trivial 1-dimensional representation of $G_{\mathbf{F}_p}$, so the Frobenius $\mathsf{F}$ acts as $1 = p^0$; the action of $\mathsf{F}$ on $\mathbf{Q}_\ell(1)$ is the multiplication by $p^{-1}$. If we pass to the dual representation $\mathsf{H}^1(E)$ we find out that the Frobenius acts (on the dual of the graded pieces above) by 1 and $p$, in accordance with the following conjecture.

**8.6. A conjecture and two results.** —

***Conjecture*** **(Monodromy-weight, [12]).** — [43] *Assume $X$ to be proper and smooth over $k$ and $V := \mathsf{H}^n_{\text{ét}}(X_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$. For all $i$ and for any geometric Frobenius element $\phi \in \mathsf{Gal}(\bar{k}/k)$, the eigenvalues of $\phi$ on $\mathsf{gr}_i^N V$ are Weil numbers of weight $n+i$, i.e. algebraic numbers $\alpha$ such that $|\alpha|_\sigma = q^{(i+n)/2}$ for any embedding $\sigma\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$.*

***Theorem 8.7*** **(Deligne, [13]).** — *The MWC holds true for $X := Y \times_{C\setminus\{x\}} \mathsf{Spec}\, k$, where*

(1) *$C$ is a smooth curve over $\mathbf{F}_q$;*
(2) *$k$ is the (equal characteristic) local field of $C$ at $x \in C(\mathbf{F}_q)$, i.e. $k$ is the quotient field of the henselian local ring $\mathcal{O}^h_{C,x}$;*
(3) *$Y$ is a proper and smooth scheme over $C\setminus\{x\}$.*

If char$(k) = 0$ the MWC was proved only in a few cases before the work of Scholze. Namely for $X/k$ satisfying at least one of the following condtions:

- $X$ has a proper and smooth model over $\mathcal{O}_k$ (*cf* [**2**, **13**]);
- $X$ is either an abelian variety or a curve (*cf* [1, Exp. IX]);
- $X$ is a surface by [33, Satz 2.13] and de Jong alteration theorem;
- $X$ is a certain kind of threefold with strictly semistable reduction (*cf* [28]);
- $X$ is a $p$-adically uniformized variety (*cf* [29]).

***Theorem 8.8*** **(Scholze** [36, Theorem 9.6]**).** — *Let $X$ be a geometrically connected smooth and proper variety over a local field $k$ of characteristic 0. Assume that $X$ is a set theoretic complete intersection in a projective toric variety $Y$ over $k$. Then the MWC is true for $X$.*

*Sketch of the proof.* — We will just consider the case where $Y = \mathbf{P}^d_K$ is the $d$-dimensional projective space and $X$ is an hypersurface defined by a homogeneous polynomial $f$ of degree $m$. All the main ingredients already appear in this case.

[43] There are several formulations. We give the original one. See Ito [29] for a detailed account. Sometimes people use *weight-monodromy*. The original conjecture is in French under the name *monodromie-poids*.

**FW.** — We can identify the algebraic closures $\bar{k} = \overline{\mathbf{Q}}_p$ since $k/\mathbf{Q}_p$ is finite. Let $\mathbf{C}_p$ be the completion of $\bar{k}$, then the action of $G_k = \mathsf{Gal}(\bar{k}/k)$ on $\bar{k}$ can be extended, by continuity, to an action on $\mathbf{C}_p$. Since the tilt is functorial we get an action of $G_k$ on $\mathbf{C}_p^\flat$. Now let $\varpi \in k$ be a uniformizer, then $K = \widehat{k(\varpi^{1/p^\infty})}$ is a perfectoid field and $G_K \subset G_k$ acts on $\mathbf{C}_p^\flat$. The tilt $K^\flat$ of $K$ is the completion of the perfection[(44)] of the field $E = \mathbf{F}_q(\!(\varpi^\flat)\!)$ (*cf* example 5.18). Since $\overline{K}^\flat \subset \mathbf{C}_p^\flat$ is dense we have an action $G_{K^\flat}$ on $\mathbf{C}_p^\flat$ giving the identification of Galois groups $G_K = G_{K^\flat}$. Also $\overline{E} \subset \mathbf{C}_p^\flat$ is dense and stable by the action of $G_{K^\flat}$, thus

$$G_K = G_{K^\flat} = G_E$$

(*cf* Fontaine-Wintenberger, *cf* section 2).

**Go perfectoid.** — We base change $X$ and $Y$ to the perfectoid field $K$. We want to prove that the MWC holds for $V = \mathsf{H}^i_{\text{ét}}(X_{\mathbf{C}_p})$ as representation of $G_K$. Note that $\mathsf{H}^i_{\text{ét}}(X_{\mathbf{C}_p}) = \mathsf{H}^i_{\text{ét}}(X_{\bar{k}})$ as $\mathbf{Q}_\ell$-vector spaces but $G_K$ (acting on both terms) is just a subgroup of $G_k$ (acting on the second term). Nevertheless the monodromy filtration is the same since $K/k$ is a totally ramified pro-$p$-extension.

**From $p$ to 0.** — The adic space $\lim_\phi(\mathbf{P}^N_K)^{\text{ad}}$ is perfectoid with tilt

$$\lim_\phi(\mathbf{P}^N_{K^\flat})^{\text{ad}}\ , \text{ where } \phi(x_0 : \ldots : x_N) = (x_0^p : \ldots : x_N^p).$$

By almost purity (*cf* theorem 6.22) there is an equivalence of topoi $(\mathbf{P}^N_{K^\flat})^{\text{ad},\sim}_{\text{ét}} \cong (\lim_\phi(\mathbf{P}^N_K)^{\text{ad}})^\sim_{\text{ét}}$. From this we get a projection map of topological spaces (and also étale topoi) $\pi\colon (\mathbf{P}^N_{K^\flat})^{\text{ad}} \to (\mathbf{P}^N_K)^{\text{ad}}$, which is given on coordinates $\pi(x_0 : \ldots : x_N) = (x_0^\sharp : \ldots : x_N^\sharp)$. This gives an isomorphism of $\ell$-adic étale cohomology.

The same holds true for any toric variety $Y$, any how since the conjecture needs a projective toric variety, the map $\pi$ is defined as above on projective coordinates.

**Approximation.** — Note that the preimage $\pi^{-1}(X_K) \subset (\mathbf{P}^d_{K^\flat})^{\text{ad}}$ is not algebraic: for instance let $X = V(x_0 + x_1 + x_2) \subset \mathbf{P}^2_k$, then $\pi^{-1}(X_K)$ is given by the inverse limit (over $n \geq 0$) of

$$V(x_0^{p^n} + x_1^{p^n} + x_2^{p^n}) \subset (\mathbf{P}^2_K)^{\text{ad}}.$$

Anyhow we have the following result.

***Proposition 8.9* (*cf* [36, Proposition 8.7]).** — *Let $K$ be a perfectoid field and $X \subset \mathbf{P}^d_K$ be an hypersurface defined by an homogeneous polynomial $f$ of degree $m$, then for a small neighbourhood $\widetilde{X} \supset X_K^{\text{ad}}$ there exists an algebraic hypersurface $Z \subset \mathbf{P}^d_{K^\flat}$ such that $Z$ is defined over a dense subfield $F \subset K^\flat$ and $Z^{\text{ad}} \subset \pi^{-1}(\widetilde{X})$.*
*Moreover we have $H^n(\widetilde{X}) = H^n(X_K^{\text{ad}})$.*

*Proof.* — Recall that $\mathbf{P}^d_K = \mathsf{Proj}(K[T_0, \ldots, T_d])$ is constructed by gluing the affine schemes

$$U_i = \mathsf{Spec}\left(K\left[\tfrac{T_j}{T_i}\right]_{j \neq i}\right)$$

and there is a line bundle $\mathcal{O}(m)$ whose zero sections are

$$\mathsf{H}^0(\mathbf{P}^d_K, \mathcal{O}(m)) = \text{homogeneous polynomial of degree } m \text{ in } K[T_0, \ldots, T_d]$$

which is the $K$-vector space generated by monomials of degree $m$. Also

$$K[T_0, \ldots, T_d] = \bigoplus_{m \geq 0} \mathsf{H}^0(\mathbf{P}^d_K, \mathcal{O}(m)) \text{ [(45)]}.$$

(44) In French: *clôture radicielle*.
(45) We emphasise these well known facts since any projective toric variety share the same features: an explicit construction by gluing nice affine schemes and an explicit basis for the coherent cohomology (independent of the characteristic of $K$). These properties make possible to extend the current proof to the general case.

We can now consider the perfectoidization of $\mathbf{P}_K^d$, namely $\mathbf{P}_K^{d,\mathrm{perf}}$ which is obtained by gluing the perfectoid affinoids

$$U_i^{\mathrm{perf}} := \mathsf{Spa}\left(K\big\langle\big(\tfrac{T_j}{T_i}\big)^{1/p^\infty}, j \neq i\big\rangle, K^\circ\big\langle\big(\tfrac{T_j}{T_i}\big)^{1/p^\infty}, j \neq i\big\rangle\right).$$

We denote by $\mathcal{O}^{\mathrm{perf}}(m)$ the pull-back of (the adification of) $\mathcal{O}(m)$ via canonical map of adic spaces $\mathbf{P}_K^{d,\mathrm{perf}} \to \mathbf{P}_K^{d,\mathrm{ad}}$ and we get that

$$\mathsf{H}^0(\mathbf{P}_K^{d,\mathrm{perf}}, \mathcal{O}^{\mathrm{perf}}(m))$$

is the completion of the $K$-vector space generated by monomials of degree $m/p^s$, for some $s$, in the $T_i$. This is a subspace of the affinoid algebra $R = K\langle T_0^{1/p^\infty}, \dots, T_d^{1/p^\infty}\rangle$ whose tilt $R^\flat$ is $K^\flat\langle T_0^{1/p^\infty}, \dots, T_d^{1/p^\infty}\rangle$ and contains $\mathsf{H}^0(\mathbf{P}_{K^\flat}^{d,\mathrm{perf}}, \mathcal{O}^{\mathrm{perf}}(m))$.

We have that the neighbourhood $\widetilde{X}$ is of the form

$$\widetilde{X} = \left\{x \in \mathbf{P}_K^{d,\mathrm{ad}} \mid |f(x)| \le |\varpi^{\flat,(0)}(x)|\right\} \quad (46)$$

with $\varpi^\flat$ running over the pseudo-uniformizer of $K^\flat$.

Now we need the following technical result.

***Lemma 8.10***. — *For any $\epsilon > 0$ and $N \in \mathbf{N}$ there exists $g \in R^\flat$, homogeneous of degree $m$ and such that for all $x \in \mathsf{Spa}(R, R^\circ)$, we have*

$$|f(x) - g^{(0)}(x)| \ge |\varpi(x)|^{1-\epsilon} \max\{|f(x)|, |\varpi(x)|^N\}.$$

From the lemma we get

$$\pi^{-1}(\widetilde{X}) = \left\{x \in \mathbf{P}_K^{d,\mathrm{ad}} \mid |g(x)| \le |\varpi^\flat(x)|\right\}.$$

Now we can modify the coefficients of $g$ so that they are defined over $F$ a dense subfield of $K^\flat$. Mind that $g$ is a linear combination of homogeneous elements of degree $m$ in $K^{\flat,\circ}\langle T_0^{1/p^\infty}, \dots, T_d^{1/p^\infty}\rangle$ (for instance $T_0^{m^2/p^5} \cdot T_1^{m(p^5-m)/p^5}$), but for $N$ big enough we get

$$g^{p^N} \in \mathsf{H}^0(\mathbf{P}_F^d, \mathcal{O}(p^N m))$$

and the zero locus $Z$ of $g^{p^N}$ has the desired properties. □

**End of the proof.** — First we recall a result of Huber: for any algebraic variety $X$ there is a canonical isomorphism of étale cohomology groups $\mathsf{H}^n(X^{\mathrm{ad}}) = \mathsf{H}^n(X)$.

Then we obtain the following commutative diagram

$$\begin{tikzcd}
\mathsf{H}^n_{\text{ét}}(\mathbf{P}_K^N) \arrow[r] \arrow[d, "\sim"] & \mathsf{H}^n_{\text{ét}}(\widetilde{X}_K) \arrow[r, "\sim"] \arrow[d] & \mathsf{H}^n_{\text{ét}}(X_K) \\
\mathsf{H}^n_{\text{ét}}(\mathbf{P}_{K^\flat}^N) \arrow[r] & \mathsf{H}^n_{\text{ét}}(\pi^{-1}(\widetilde{X})_{K^\flat}) \arrow[r] & \mathsf{H}^n_{\text{ét}}(Z_{K^\flat})
\end{tikzcd}$$

and we get a map $\alpha^{(n)} \colon \mathsf{H}^n(X_K) \to \mathsf{H}^n(Z)$. Up to alteration, we can assume $Z$ to be smooth over (a finite extension of) $E$ which is dense in $K^\flat$ so that the result of Deligne applies to $H^n(Z)$. Then we can easily conclude in the following way. First one proves that $\alpha^{(2d)}$ is an isomorphism for $d = \dim Z = \dim X$: it is sufficient to prove that it is non zero and this follows from the non vanishing of the first Chern class of the canonical line bundle on $\mathbf{P}^n$. To complete the proof we can use Poincaré duality to deduce that $\alpha^{(n)}$ is injective for any $n$. □

(46) To make sense of $|f(x)|$ we need $f$ to be a function, not a section of a line bundle. Nevertheless we can trivialize over the affinoids $U_i^{\mathrm{perf}}$ where $f$ become a function, well defined up to an invertible element, so that $|f(x)|$ make sense.

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