GDT ON SYNTOMIC COHOMOLOGY AND EULER SYSTEMS: ON p-ADIC PERIODS

NICOLA MAZZARI

Abstract. Informal notes for a seminar on p-adic period rings. Better references are the article by Fontaine on Asterisque 223 (In fact one should read the whole book at some point!) and the survey by Berger or the (unpublished!) book by Brion and Conrad.

A poem

Ich lebe mein Leben in wachsenden Ringen,
die sich über die Dinge ziehn.
Ich werde den letzten vielleicht nicht vollbringen,
aber versuchen will ich ihn.

Ich kreise um Gott, um den uralten Turm,
und ich kreise jahrtausendelang;
und ich weiß noch nicht: bin ich ein Falke, ein Sturm
oder ein großer Gesang.

(Rainer Maria Rilke 1899)[1]

Timeline

- 1967 Tate work on p-divisible groups
- 1967 Serre question on periods of p-divisible groups
- 1972 Illusie Complexe Cotangent et Deformations II
- 1979 Fontaine introduces the formalism of Barsotti-Tate rings
- 1982 Fontaine construct the period rings (Annals paper)
- 2011 Beilinson proof of C_{dR}
- 2011 Scholze Perfectoid Spaces

1. Motivation

1.1. **Complex picture.** Let $K \subset \mathbb{C}$ be a subfield. $\mathbb{G}_{\mathrm{m}} = \mathrm{Spec}(K[T^{\pm}])$ be the affine line minus two points over K. Then $\mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathbb{C}^{\times}$ is homotopy equivalent to the unit circle and the first homology group is $H_1(\mathbb{G}_{\mathrm{m}}(\mathbb{C}), \mathbb{Z}) \cong \pi(\mathbb{G}_{\mathrm{m}}(\mathbb{C}), 1) = \mathbb{Z}\epsilon$ where ϵ is the loop $\{e^{i\theta} : \theta \in [0, 2\pi]\}$. Also we can easily compute the first algebraic

Date: June 28, 2025.

[1] I live my life in ever-widening rings
that stretch themselves out over all the things.
I won't, perhaps, complete the last one,
but I intend on trying.

I circle around God, around the ancient tower,
and I circle for thousands of years;
and I don't know, yet: am I a falcon, a storm,
or a mighty song.

cohomology group

$$H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}/K) = \frac{\Omega^1_{K[T^{\pm}]/K}}{d(K[T^{\pm}])} = K\frac{dT}{T} \ .$$

1.1.1. *The periods' pairing.* The integration along a path gives a natural (bi-linear) pairing

$$H_1(\mathbb{G}_{\mathrm{m}}(\mathbb{C}),\mathbb{Z}) \times H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}/K) \to K(2\pi i) \qquad (\gamma,\omega) \mapsto \int_\gamma \omega$$

and is completely determined by the value on the generators

$$\int_\epsilon \frac{dT}{T} = \int_0^{2\pi} \frac{de^{i\theta}}{e^{i\theta}} = 2\pi i \ .$$

1.1.2. *The periods' isomorphism.* This can be generalised to a (projective) smooth variety X/K. Moreover by Poincaré duality[2] the above paring induces an isomorphism

$$\rho_{\mathrm{dR}} : H^n(X(\mathbb{C}),\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{P} \to H^n(X/K) \otimes_K \mathbb{P}$$

where $\mathbb{P}$ is a sufficiently large ring: in the case of $\mathbb{G}_{\mathrm{m}}$, ρ_{dR} is the multiplication by $1/2\pi i$ so it is enough to take $\mathbb{P} = K(2\pi i)$; in general we could simply take $\mathbb{P} = \mathbb{C}$ but it would be enough to take the extension of K formed by all possible periods of algebraic varieties[3].

1.1.3. *Everything is algebraic.* The de Rham cohomology is an algebraic invariant and we have $H_1(\mathbb{G}_{\mathrm{m}}(\mathbb{C}),\mathbb{Z}) \otimes \mathbb{Z}_p \cong T_p\mathbb{G}_{\mathrm{m}} = \mathbb{Z}_p(1) = \{(u_n)_n \in \mathbb{C}^{\mathbb{N}} : u_{n+1}^p = u_n,\ u_0 = 1\}$ which is again of algebraic nature. We may note that

$$2\pi i = \int_\epsilon \frac{dT}{T} = p^n \int_1^{\epsilon_n} \frac{dT}{T} \ ,$$

where $\epsilon_n = e^{(2\pi i)/p^n}$ is a primitive p^n-root of 1.

1.2. **A p-adic example.** Now let $[K : \mathbb{Q}_p] < +\infty$. We can still compute $T_p\mathbb{G}_{\mathrm{m}} = \mathbb{Z}_p(1) = \{(u^{(n)})_n \in \bar{K}^{\mathbb{N}} : {u^{(n+1)}}^p = u^{(n)},\ u^{(0)} = 1\}$ whose generator is given by a system $\epsilon = (\epsilon^(n))_n$ of primitive p^n-roots of 1. We can compute

$$\int_\epsilon \frac{dT}{T} := p^n \int_{\epsilon^{(n)}} \frac{dT}{T} = p^n \log_p(\epsilon^{(n)}) = \log_p(1) = 0$$

but this does not give any good pairing $T_p\mathbb{G}_{\mathrm{m}} \times H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}/K) \to \mathbb{B}$. One can look for a ring where $t :=$ "$\log(\epsilon)''$ makes sense and it is not 0. Moreover one should expect some compatibility with respect to the Galois action $T_p\mathbb{G}_{\mathrm{m}}$:

$$g \cdot \int_\epsilon \frac{dT}{T} = \int_{g\cdot\epsilon} \frac{dT}{T} \ \text{, or } gt = g\log(\epsilon) = \log(\epsilon^{\chi(g)}) = \chi(g)t \ ,$$

where $g \in G_K = \mathrm{Gal}(\bar{K}/K)$ and $\mathbb{B}$ is assumed to be a G_K-module.

Theorem (Tate [5]). *We have*

$$H^0(G_K, \mathbb{C}_p(k)) = \{c \in \mathbb{C}_p : \forall g \in G_K,\ gc = \chi(g)^k\} = \begin{cases} K & k = 0 \\ 0 & \textit{otherwise} \end{cases} \ .$$

In particular there cannot be such a t in $\mathbb{C}_p$.

[2]Let $d = \dim X$, the there is a canonical isomorphism $\mathrm{Hom}_{\mathbb{Q}}(H_{2d-n}(X(\mathbb{C}),\mathbb{Q}),\mathbb{Q}) \cong H^n(X(\mathbb{C}),\mathbb{Q})$

[3]There is a nice book on "Periods and Nori motives" by Huber and Müller-Stach. Also it is worthwhile to have a look at [4]

2. Barsotti-Tate period rings

[4] Notation: $K/\mathbb{Q}_p$ finite extension, $G = G_K = \mathrm{Gal}(\bar{K}/K)$. A p-adic representation is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous linear action of G_K.

2.1. Admissible representations.

Fontaine [1] constructs several rings of p-adic periods $\mathbb{B}$: they are topological $\mathbb{Q}_p$-algebras with a continuous linear action of G_K (and possibly other structures[5]) such that

(1) $\mathbb{B}$ is a domain and $\mathbb{B}^G = \mathrm{Frac}(\mathbb{B})^G$ is a field.
(2) if $b\mathbb{Q}_p \subset \mathbb{B}$ is G_K-stable, then $b \in \mathbb{B}^\times$.

For instance one can take $\mathbb{B} = \bar{K}$ or $\mathbb{C}_p$, but we already know we need much bigger ring to get periods of a general geometric representation.

Let $F = \mathbb{B}^{G_K}$. We can define, for any p-adic representation[6] V, the F-vector space

$$D_{\mathbb{B}}(V) := (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

and we say that V is $\mathbb{B}$-admissible if the canonical map

$$\alpha_V : \mathbb{B} \otimes_F D_{\mathbb{B}}(V) \to \mathbb{B} \otimes_{\mathbb{Q}_p} V$$

is an isomorphism of $\mathbb{B}[G_K]$-modules. The coefficients of a matrix of this isomorphism are called $\mathbb{B}$-periods of V. In other words V is $\mathbb{B}$-admissible iff $\mathbb{B} \otimes_{\mathbb{Q}_p} V \cong \mathbb{B}^n$ as G-representation.

Alternatively one con work with the contravariant functor

$$D^*_{\mathbb{B}}(V) = D_{\mathbb{B}}(V^*) := (\mathbb{B} \otimes_{\mathbb{Q}_p} V^*)^{G_K} = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, \mathbb{B})$$

and the $\mathbb{B}$-periods of V are images of the maps $f \in \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, \mathbb{B})$.

2.2. Classification of p-adic representations.

For any ring of periods $\mathbb{B}$ we can define $\mathrm{Rep}^{\mathbb{B}}_{\mathbb{Q}_p} G$ to be the full subcategory of $\mathrm{Rep}_{\mathbb{Q}_p} G$ consisting of the $\mathbb{B}$-admissible representations.

For instance when $\mathbb{B} = \bar{K}$ (resp. $\mathbb{C}_p$) one simply get the category of discrete representations (resp. the action of the inertia is discrete).

We are going to define (following Fontaine) three important rings of periods $\mathbb{B}_{\mathrm{crys}} \subset \mathbb{B}_{\mathrm{st}} \subset \mathbb{B}_{\mathrm{dR}}$ (called crystalline, semi-stable and de Rham, respectively) giving a sequence of full embeddings

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p} G \subset \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p} G \subset \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p} G \subset \mathrm{Rep}_{\mathbb{Q}_p} G$$

and the representations arising from algebraic varieties fall in one or the other category according the geometry of the object.

2.3. Conjectures.

Let X/K be a proper and smooth scheme. Let $V := H^n_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ which is a p-adic representation of G_K in natural way. Let $M = H^n_{\mathrm{dR}}(X/K)$ which is a (filtered) finite dimensional K-vector space. We have the following conjectures[7]

---

[4] This notion is introduced first by Fontaine [1]. He explain that the name is due to several reasons: these theory is developed in the contest of Barsotti-Tate groups in order to answer a question by Serre; the only example of such rings was provided by Witt bi-vectors introduced by Barsotti. In fact Fontaine introduced this notion by giving the formal properties: the concrete construction of the rings $\mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{crys}}, \ldots$ was discovered later. Now it is customary to call these rings Fontaine rings.

[5] In fact these rings are always filtered!

[6] A finite dimensional vector space V endowed with a continuous Galois action

[7] In the complex case the comparison theorem can be deduced from the periods pairing via Poincaré duality. In p-adic setting one cannot define such a pairing in general, only for representations associated to Barsotti-Tate groups.

C_{dR} *There is a G_K-equivariant isomorphism*

$$\rho_{\mathrm{dR}} : V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{dR}} \to M \otimes_K \mathbb{B}_{\mathrm{dR}}$$

compatible with the filtrations and the Galois action.
In particular $D_{\mathrm{dR}}(V) := (\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} \cong M$.

C_{st} *If X has semi-stable reduction then the previous isomorphism holds true if we replace $\mathbb{B}_{\mathrm{dR}}$ with the smaller ring $\mathbb{B}_{\mathrm{st}}$. Also $\mathbb{B}_{\mathrm{st}}$ and M are endowed with a monodromy N and a Frobenius ϕ such that $N\phi = p\phi N$ and the isomorphism is compatible with all the extra structures.*
In particular $D_{\mathbb{B}_{\mathrm{st}}}(V) = M$. Mind that the construction of N, ϕ both on M and $\mathbb{B}_{\mathrm{st}}$ depend on the choice of a uniformiser π of $\mathcal{O}_K$.

C_{crys} *If $X/\mathcal{O}_K$ is proper and smooth, then we can replace $\mathbb{B}_{\mathrm{st}}$ by $\mathbb{B}_{\mathrm{crys}} = \mathbb{B}_{\mathrm{st}}^{N=0}$.*

All conjectures have been proved (and generalised) by the work of several authors such as (in order of appearance): Fontaine, Messing, Kato, Faltings, Niziol, Tsuji, Beilinson.

The basic example is that of $\mathbb{G}_{\mathrm{m}}/K$ (which is not proper to be honest, but who cares!) and $n = 1$: $V = \mathbb{Q}_p(-1)$, $M = KdT/T$ and ρ_{dR} is the multiplication by $1/t$, where $t \in \mathbb{B}_{\mathrm{crys}}$.

3. Constructions

Let A be an $\mathcal{O}_K$-algebra such that $F : R/p \to R/p$, $x \mapsto x^p$ is surjective. In fact we can just take $A = \mathcal{O}_{\bar{K}}$ or its p-adic completion $\widehat{A} = \mathcal{O}_{\mathbb{C}_p}$, but in order to prove the comparison theorems it is necessary to consider much bigger rings such as $\mathcal{O}_{\mathbb{C}_p}[T^{1/p^\infty}]$.

3.1. **The first step (Tilting).** We define the tilting[8] $A^\flat := \{(\bar{x}_n)_n \subset A/p : \bar{x}_{n+1}^p = \bar{x}_n\}$. This is a perfect ring of characteristic p and there is an important identification

$$A^\flat = \lim_{F} A/p \longrightarrow \lim_{x \to x^p} \widehat{A} := \{(x^{(n)})_n \subset \widehat{A} : x^{(n+1)^p} = x^{(n)}\}$$

associating to $x = (\bar{x}_n)_n$ the sequence $(x^{(n)})_n$ given by

$$x^{(m)} := \lim_{n\to\infty} x_{n+m}^{p^n} \text{ , for } x_{n+m} \text{ a lift of } \bar{x}_{n+m} \ .$$

For instance if $\epsilon = (\epsilon^{(n)})_n$ is a system of primitive p^n-roots of 1 in $\bar{K}$ then we can define an element $1^\flat \in \mathcal{O}_{\bar{K}}^\flat$ such that $1^{\flat,(n)} = \epsilon^{(n)}$. Also if $\pi \in \mathcal{O}_K$ is a uniformiser we can define $\pi^\flat \in \mathcal{O}_{\bar{K}}^\flat$ by choosing a system $\pi^{\flat,(n)} = \pi^{1/p^n}$ of roots of π.

Lemma. *Let $W(A^\flat)$ be the ring of Witt vectors of $A^\flat$. There is a surjective ring homomorphism*

$$\theta : W(A^\flat) \to \widehat{A} \ , \quad (x_0, x_1, ...) \mapsto \sum_{n\geq 0} p^n x_n^{(n)} \ {}^{9}$$

In particular $\theta([x]) = x_0^{(0)}$ where $[\cdot] : A^\flat \to W(A^\flat)$ is the Teichmüller lift.

[8] This is the name chosen by P. Scholze in 2011 even though the object already appears in construction of Fontaine 30 years earlier. Nevertheless it is thaks to the work of Scholze that this step is the building block of a general theory, that of perfectoid spaces, that clarifies many aspects of p-adic Hodge theory. A nice lecture on this point is The Perfectoid Concept: Test Case for an Absent Theory by Michael Harris

[9] Note that we have $x_k = (\bar{x}_{k,n})_n = (x_k^{(n)})_n$ according to the previous identification.

According to the previous notation we can compute

$$\theta([1^\flat]) = 1^{\flat,(0)} = \epsilon_0 = 1 \qquad \theta([\pi^\flat]) = \pi^{\flat,(0)} = \pi \ .$$

3.2. **The general case.** Let $\widetilde{W}(A^\flat) := \mathcal{O}_K \otimes_W W(A^\flat)$ and $\tilde{\theta}$ the base change of θ. Consider the exact sequence

$$0 \to \ker(\tilde{\theta}) \to \widetilde{W}(A^\flat) \xrightarrow{\tilde{\theta}} \widehat{A} \to 0$$

and define $\mathbb{A}_{\inf}(A/\mathcal{O}_K)$ to be the $(p, \ker(\tilde{\theta}))$-adic completion of $\mathcal{O}_K \otimes_W W(A^\flat)$. This is the universal pro-infinitesimal formal p-adic thickening of A/p, so that

$$H^0_{\inf}(\operatorname{Spec}(A/p)/\operatorname{Spec}(\mathcal{O}_K)) = \mathbb{A}_{\inf}(A/\mathcal{O}_K) = \lim_n \frac{\widetilde{W}(A^\flat)}{(p, \ker(\tilde{\theta}))^{n+1}} \ .$$

By replacing the infinitesimal cohomology by the crystalline (and assuming that $\mathcal{O}_K$ has divided powers) one we get the ring $\mathbb{A}_{\mathrm{crys}}(A/\mathcal{O}_K)$ which the p-adic completion of the PD-envelope of $\widetilde{W}(A^\flat)$ with respect to the ideal $\ker(\tilde{\theta})$. We have the cohomological interpretation

$$H^0_{\mathrm{crys}}(\operatorname{Spec}(A/p)/\operatorname{Spec}(\mathcal{O}_K)) = \mathbb{A}_{\mathrm{crys}}(A/\mathcal{O}_K)^{10} \ .$$

Note that $\mathbb{A}_?(A/\mathcal{O}_K) = \mathbb{A}_?(\widehat{A}/\mathcal{O}_K)$ for both constructions.

3.3. **Classical Fontaine rings.** From now on $A = \mathcal{O}_{\bar{K}}$. Let

$$\xi = \pi + [(-\pi)^\flat] = 1 \otimes [(-\pi)^\flat] + \pi \otimes 1 \in \widetilde{W}(\mathcal{O}^\flat_{\bar{K}}) \ ,$$

then $\ker\tilde{\theta} = \xi\widetilde{W}(\mathcal{O}^\flat_{\bar{K}})$ is principal and $\mathbb{A}_{\inf} := \mathbb{A}_{\inf}(\mathcal{O}_{\bar{K}}/\mathcal{O}_K) = \mathbb{A}_{\inf}(\mathcal{O}_{\mathbb{C}_p}/\mathcal{O}_K)$ is the ξ-adic completion of $\widetilde{W}(\mathcal{O}^\flat_{\bar{K}})$.

Note that the valuation of $\mathcal{O}_{\mathbb{C}_p}$ induces a valuation on $W(\mathcal{O}^\flat_{\mathbb{C}_p})$ by setting $v(x_0, x_1, ...) = ...$

The map θ induces a surjective morphism $\theta_K : \mathbb{A}_{\inf}[1/p] \to \mathbb{C}_p$ and we define[11]

$$\mathbb{B}^+_{\mathrm{dR}} := \lim_n \mathbb{A}_{\inf}[1/p]/(\ker(\theta_K)^n)$$

We can now define the element

$$t := \log([1^\flat]) = \sum_{n>0} (-1)^{n-1} \frac{([1^\flat] - 1)^n}{n}$$

which converges in $\mathbb{B}^+_{\mathrm{dR}}$ since $([1^\flat] - 1)^n \in \ker(\theta_K)^n$. Note that we have to consider $\mathbb{A}_{\inf}[1/p] = \mathbb{A}_{\inf} \otimes K$ in order to have the denominators n of the series and we have to complete to make it convergent.

Also we define[12]

$$\mathbb{A}_{\mathrm{crys}} := \mathbb{A}_{\mathrm{crys}}(\mathcal{O}_{\mathbb{C}_p}/W) = p\text{-adic completion of } W(\mathcal{O}^\flat_{\bar{K}})[\xi^n/n! : n > 0] \ ,$$

where $\xi = p + [(-p)^\flat]$. More concretely we can show that

$$\mathbb{A}_{\mathrm{crys}} = \{b \in \mathbb{B}^+_{\mathrm{dR}} : b = \sum_{n \geq 0} a_n \xi^{[n]}, \ a_n \to 0 \ p\text{-adically in } W(\mathcal{O}_{\mathbb{C}^\flat_p})\}^{13}$$

---

[10] to check Olivier notes!

[11] The first definition of $\mathbb{B}_{\mathrm{dR}}$ appear in [2]. In 2011 Beilinson shows that $\mathbb{B}^+_{\mathrm{dR}} = \widehat{L\Omega^\bullet_{\mathcal{O}_{\bar{K}}/\mathcal{O}_K}} \hat{\otimes} \mathbb{Q}_p$, which means that can be defined directly via the derived de Rham complex (Hodge completed and p-adically completed) of Illusie, already available in 1972.

[12] Note that in the definition of $\mathbb{A}_{\mathrm{crys}}$ the base is W not $\mathcal{O}_K$ is order to be sure to have divided powers

[13] Mind that the developpement $b = \sum_{n \geq 0} a_n \xi^{[n]}$ is not unique. Also note that there elements, such as $\sum p^{-n^2} t^n$ which are in $\mathbb{B}^+_{\mathrm{dR}}$ but not in $\mathbb{B}^+_{\mathrm{crys}}$.

The crystalline periods rings are $\mathbb{B}_{\mathrm{crys}}^+ = \mathbb{A}_{\mathrm{crys}}[1/p]$ and $\mathbb{B}_{\mathrm{crys}} = \mathbb{B}_{\mathrm{crys}}^+[1/t]$. Now let

$$u = u_\pi = \log\left(\frac{1 \otimes [\pi^\flat]}{\pi \otimes 1}\right) \in \mathbb{B}_{\mathrm{dR}}, \text{ since } \theta\left(\frac{1 \otimes [\pi^\flat]}{\pi \otimes 1}\right) = 1$$

and define the semi-stable period rings $\mathbb{B}_{\mathrm{st}}^+ = \mathbb{B}_{\mathrm{crys}}^+[u]$, $\mathbb{B}_{\mathrm{st}} = \mathbb{B}_{\mathrm{st}}^+[u]$. This makes sense since u is transcendental over $\mathbb{B}_{\mathrm{crys}}$. Note that $t \in \mathbb{B}_{\mathrm{crys}}^+$ [14].

3.4. **Properties of period rings.**

(1) the rings $\mathbb{B}_{\mathrm{crys}} \subset \mathbb{B}_{\mathrm{st}} \subset \mathbb{B}_{\mathrm{dR}}$ are Barsotti-Tate rings such that $\mathbb{B}_{\mathrm{dR}}^{G_K} = \bar{K}^{G_K} = K$ and $\mathbb{B}_{\mathrm{crys}}^{G_K} = \mathbb{B}_{\mathrm{st}}^{G_K} = K_0$.
(2) $\mathbb{B}_{\mathrm{dR}}^+$ is a discrete valuation ring[15] with residue field $\mathbb{C}_p$. Its fraction field $\mathbb{B}_{\mathrm{dR}}$ is filtered by the valuation by $\mathrm{Fil}^i \mathbb{B}_{\mathrm{dR}} = \{b : v(b) \geq i\}$.
(3) the canonical embedding $\bar{K} \subset \mathbb{C}_p$ factors trough $\bar{K} \to \mathbb{B}_{\mathrm{dR}}^+ \to \mathbb{B}_{\mathrm{dR}}^+/\mathrm{Fil}^1 = \mathbb{C}_p$ and all maps are G_K-equivariant.[16]
(4) If $\xi = p + [-p^\flat]$ we have[17]
$$\phi(\xi) = \xi^p + p\eta = p((p-1)!\xi[p] + \eta) \in pW(\mathcal{O}_{\mathbb{C}_p}^\flat)[\xi^{[p]}]$$
therefore $\phi(\xi^{[m]}) \in W(\mathcal{O}_{\mathbb{C}_p}^\flat)[\xi^{[p]}]$ and the Frobenius extends to $\mathbb{A}_{\mathrm{crys}}$ and $\mathbb{B}_{\mathrm{crys}}$. We can also compute $\phi(u) = p \cdot u$ so the Frobenius extends to $\mathbb{B}_{\mathrm{st}}$.
(5) We have $gu_\pi = u_{g\pi}$. We can set $Nu = -1$ and we have $N\phi = p\phi N$.
(6) We have the fundamental sequence
$$0 \to \mathbb{Q}_p(r) \to Fil^r \mathbb{B}_{\mathrm{crys}} \xrightarrow{\phi/p^r - 1} \mathbb{B}_{\mathrm{crys}} \to 0$$
(which holds with $\mathbb{B}_{\mathrm{crys}}^+$ if $r \geq 0$).

References

[1] Jean-Marc Fontaine. Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate. In *Journées de Géométrie Algébrique de Rennes. (Rennes, 1978), Vol. III*, volume 65 of *Astérisque*, pages 3–80. Soc. Math. France, Paris, 1979.

[2] Jean-Marc Fontaine. Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate. *Ann. of Math. (2)*, 115(3):529–577, 1982.

[3] Jean-Marc Fontaine. Le corps des périodes p-adiques. *Astérisque*, (223):59–111, 1994. With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

[4] M. Kontsevich. Periods. In *Mathématique et physique*, volume 1999 of *SMF Journ. Annu.*, pages 28–39. Soc. Math. France, Paris, 1999.

[5] J. T. Tate. p-divisible groups. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 158–183. Springer, Berlin, 1967.

Université de Bordeaux
Email address: nicola.mazzari@math.u-bordeaux.fr

[14] since $[1^\flat] - 1 \in \ker(\theta)$ thus $[1^\flat] - 1 = a\xi$ and
$$\frac{([1^\flat]-1)^n}{n} = (n-1)!a^n\xi^{[n]} \ldots.$$

[15] details

[16] details

[17] The Frobenius in characteristic p induces a Frobenius on $\mathcal{O}_{\bar{K}}^\flat$, so on $W(\mathcal{O}_{\bar{K}}^\flat)$.