MA 2073-1953 - Lect 2

Tuesday, August 31, 2021 12:58 PM

Last time - Limits, $f: \mathbb{R}^2 \to \mathbb{R}$

Today - partial derivatives

differentiability

MVT

Taylor series.

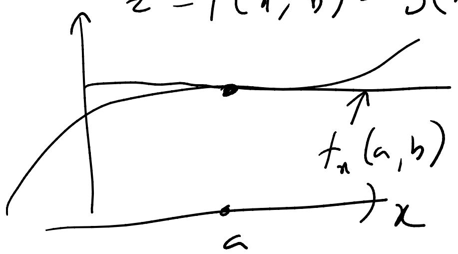
Def: $\lim_{(x,y)\to(a,b)} f(x,y) = L \in \mathbb{R}$ $\qquad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \underline{a} = \begin{pmatrix} a \\ b \end{pmatrix}$

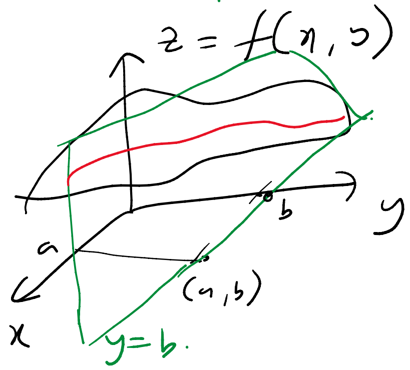
$\Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall (x,y), \; 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon$

$\lim_{\underline{x}\to\underline{a}} f(\underline{a}) = L \in \mathbb{R}$

$\Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall \underline{x}, \; 0 < \|\underline{x} - \underline{a}\| < \delta \Rightarrow |f(\underline{x}) - L| < \varepsilon$

Partial derivatives





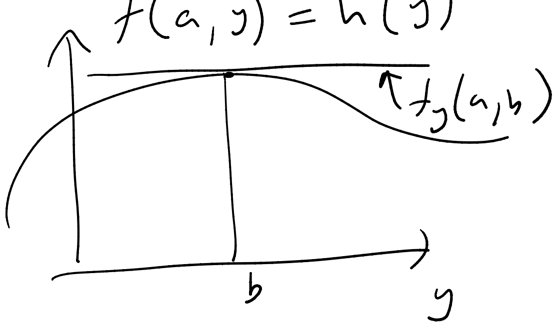
$$\frac{d}{dx} g(x)\Big|_{x=a} = \frac{d}{dx} f(x,b)\Big|_{x=a}$$

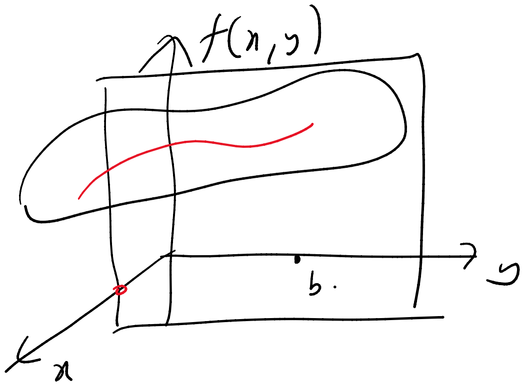
$$= \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{f(x,b)-f(a,b)}{x-a}, \quad x-a=h.$$

$$= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$= \frac{\partial f}{\partial x}(a,b) = f_x(a,b) = f_1(a,b)$$

= partial derivative of f w.r.t. x at (a,b)





$$\frac{d}{dy} h(y)\Big| = \frac{d}{dy} f(a,y)\Big|$$

$$\frac{d}{dy} h(y)\Big|_{y=b} = \frac{d}{dy} f(a,y)\Big|_{y=b}$$

$$= \lim_{y\to b} \frac{h(y)-h(b)}{y-b} = \lim_{y\to b} \frac{f(a,y)-f(a,b)}{y-b}, \quad y-b=k$$

$$= \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$$

$$= f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = f_2(a,b)$$

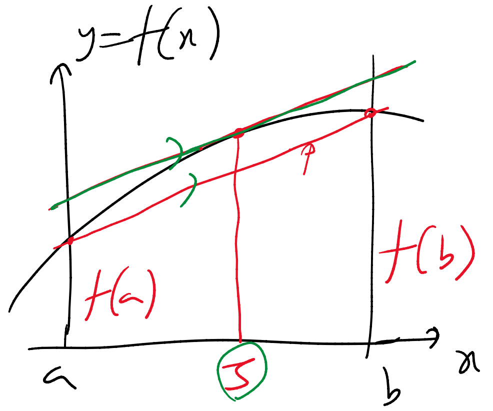
= partial derivative of f w.r.t y at (a,b)

Mean value theorem

1D: $f:\mathbb{R}\to\mathbb{R}$

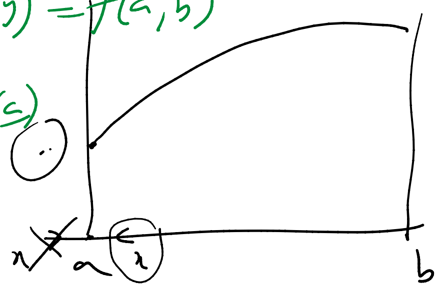
f is cts on $[a,b]$
f is diff on (a,b)

$\exists\, \xi\in(a,b)$ s.t. $\dfrac{f(b)-f(a)}{b-a} = f'(\xi)$



f is cts at a.
$\Leftrightarrow f\in C(a)$
$\Leftrightarrow \lim_{x\to a} f(x) = f(a)$

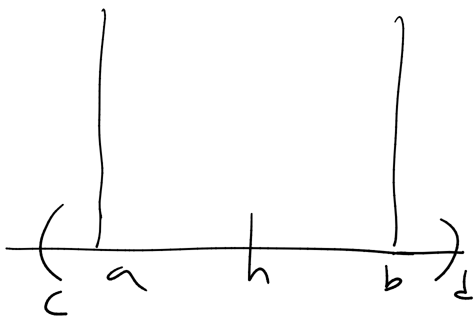
$f:\mathbb{R}^2\to\mathbb{R}$ cts.
$\Leftrightarrow \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
$\Leftrightarrow \lim_{\underline{x}\to\underline{a}} f(\underline{x}) = f(\underline{a})$



f is diff. at a.
$\Leftrightarrow f\in D(a)$
$\Leftrightarrow \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a} = f'(a)\in\mathbb{R}$.

Thm: $f\in D(a) \Rightarrow f\in C(a)$.

If $f\in D(c,d) \Rightarrow f\in C(c,d)$



$\Rightarrow f\in D(a,b)$ and $f\in C(a,b)$

$\Rightarrow \exists\,\xi\in(a,b),\ \dfrac{f(b)-f(a)}{b-a} = f'(\xi)$

$\Rightarrow \xi\in(a,b),\ f(b) = f(a) + (b-a)f'(\xi),\quad b=a+h.$

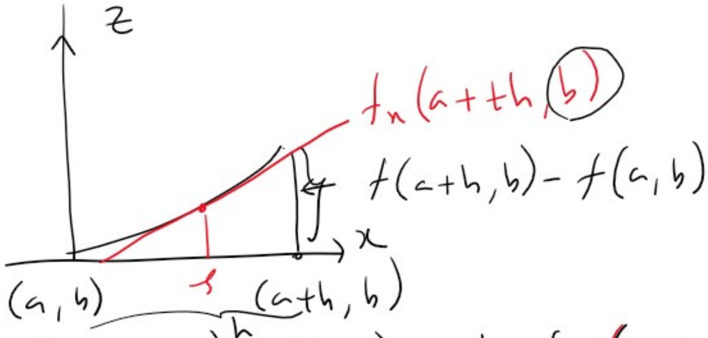
$\Rightarrow \xi \in (a,b)$, $f(b) = f(a) + (b-a) f'(\xi)$, $b = a+h$
$= a + \Delta x$

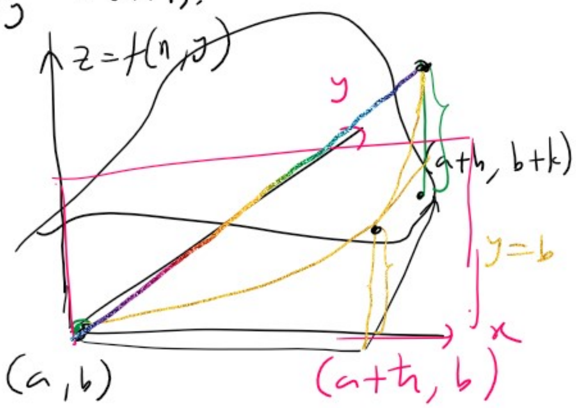
$\boxed{t \in (0,1) \quad f(a+h) = f(a) + h f'(a+th)}$

$\xi \in (a, a+h)$, $\underset{\underset{a+0\cdot h}{\parallel}}{a} < \xi < \underset{\underset{a+1h}{\parallel}}{a+h}$, $\xi = a + th$
$t \in (0,1)$

2D: $f : \mathbb{R}^2 \to \mathbb{R}$ assume f_x, f_y exists.

$f(a+h, b+k) - f(a,b)$.

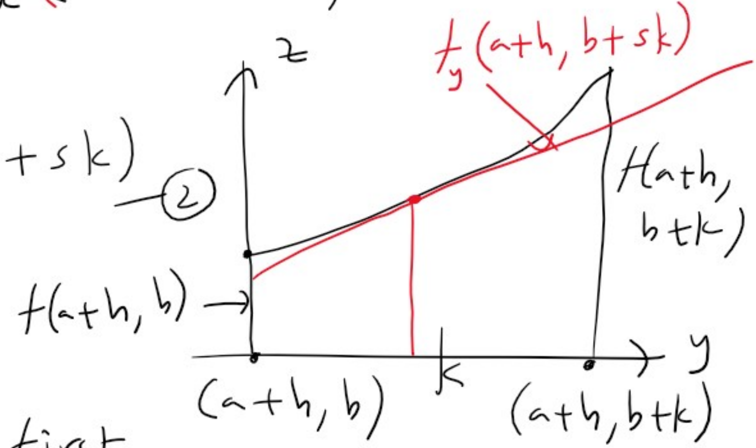




$f(a+h, b) = f(a,b) + h f_x(a+th, b)$ — ① $t \in (0,1)$.

$f(a+h, b+k)$

$= f(a+h, b) + k f_y(a+h, b+sk)$ — ②



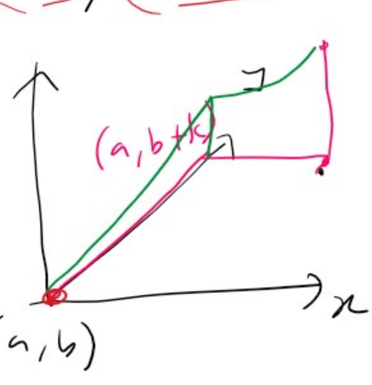
①, ②, going in x axis first

$s \in (0,1)$

$\underline{f(a+h, b+k)} = f(a,b)$
$+ h f_x(a+th, b) + k f_y(a+h, b+sk)$.
$0 < t, s < 1$

going in y axis first

$f(a+h, b+k) = f(a,b)$
$+ k f_y(a, b+\theta k)$
$+ h f_x(a+\alpha h, b+k)$
$0 < \theta, \alpha < 1$



Def: $f : \mathbb{R}^2 \to \mathbb{R}$; f_x, f_y exists

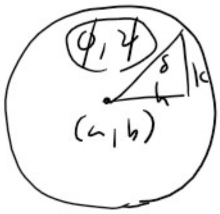
Def: $f: \mathbb{R}^2 \to \mathbb{R}$; f_x, f_y exists

Differentiability of f at (a,b), $f \in D(a,b)$

$\iff \exists \delta > 0 \ \exists \phi, \psi$ (two functions) $\forall (h,k)$

$h^2 + k^2 < \delta^2 \Rightarrow f(a+h, b+k)$

$$= f(a,b) + h f_x(a,b) + k f_y(a,b)$$



h^2

$$+ h\phi(h,k) + k\psi(h,k)$$

and $\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \psi(h,k) = 0$

$\phi, \psi \sim h$

Thm: $f_x, f_y \in \mathcal{C} \Rightarrow f \in D$.

Proof: MVT.

$$f(a+h, b+k) = f(a,b) + h f_x(a+th, b) + k f_y(a+h, b+sk)$$

$0 < s, t < 1$.

$$f(a+h, b+k)$$
$$= f(a,b) + h f_x(a,b) + k f_y(a,b)$$
$$+ h\left[\underbrace{f_x(a+th, b) - f_x(a,b)}_{\phi\,?}\right] + k\left[\underbrace{f_y(a+h, b+sk) - f_y(a,b)}_{\psi\,?}\right].$$

$$\lim_{(h,k)\to(0,0)} \phi(h,k)$$
$$= \lim_{(h,k)\to(0,0)} \left[f_x(a+th, b) - f_x(a,b)\right], \quad f_x \in \mathcal{C}$$
$$= f_x(a+0, b) - f_x(a,b) = 0.$$

$$\lim_{(h,k)\to(0,0)} \psi(h,k)$$
$$= \lim_{(h,k)\to(0,0)} f_y(a+h, b+sk) - f_y(a,b)$$
$$= f_y(a+0, b+0) - f_y(a,b) = 0. \quad \blacksquare$$

$(h,k)\to(0,0)$

$= f_y(a+0, b+0) - f_y(a,b) = 0.$ ☒

$f_x, f_y \in C$ (also $f \in C^{(1)}$) $\Rightarrow f \in D.$

what is f'.

Def: Frechet derivative

$$\lim_{\underline{h}\to\underline{0}} \frac{|f(\underline{a}+\underline{h}) - f(\underline{a}) - f'(\underline{a})\cdot\underline{h}|}{\|\underline{h}\|}$$

find a f' satisfies above

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)-hf'(a)}{h} = 0$$

$f: \mathbb{R}\to\mathbb{R}.$

$\underline{h} = \begin{pmatrix} h \\ k \end{pmatrix}$

Diff. at $\underline{a}$.

$f(a+h, b+k)$ $\quad f(\underline{a}+\underline{h})$

$= f(a,b)$ $\quad + f(\underline{a})$

$+ hf_x(a,b) + kf_y(a,b) \to (f_x(\underline{a}), f_y(\underline{a}))\cdot(h,k)$

$= Df(\underline{a})\cdot\underline{h}.$

$+ h\phi(h,k) + k\psi(h,k).$

$(\phi(\underline{h}), \psi(\underline{h}))\cdot\underline{h} = \underline{\phi}(\underline{h})\cdot\underline{h}.$

$$\lim_{\underline{h}\to\underline{0}} \frac{|f(\underline{a}+\underline{h}) - f(\underline{a}) - f'(\underline{a})\cdot\underline{h}|}{\|\underline{h}\|},$$

$\boxed{\text{claim } f' = Df}$

$$= \lim_{\underline{h}\to\underline{0}} \frac{|\underline{\phi}(\underline{h})\cdot\underline{h}|}{\|\underline{h}\|}$$

$\underline{a}\cdot\underline{b} = \|\underline{a}\|\|\underline{b}\|\cos\theta$

$\le \|a\|\,\|\underline{b}\|.$

$$\le \lim_{\underline{h}\to\underline{0}} \frac{\|\underline{\phi}(\underline{h})\|\,\cancel{\|\underline{h}\|}}{\cancel{\|\underline{h}\|}}$$

$$\le \lim_{(h,k)\to(0,0)} \sqrt{\phi(h,k)^2 + \psi(h,k)^2} = 0 + 0 = 0.$$

Chain rules $f = f(x,y) \in C^1$, $x = x(t) \in C^1$, $y = y(t) \in C^1$

$f_x, f_y \in C \Rightarrow f \in D.$

ie. $f = f(x(t), y(t)) = g(t) \in C^1$

$h = \Delta x$

$k = \Delta y.$

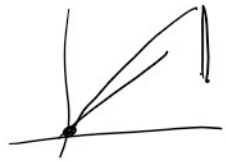
$f = f(x(t), y(t)) = g(t)$ | $k = \Delta y$.

$\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b) \leftarrow$ total f change.

$= \Delta x\, f_x(a,b) + \Delta y\, f_y(a,b) + \Delta x\, \phi(\Delta x, \Delta y) + \Delta y\, \psi(\Delta x, \Delta y)$

$a = x(0)$, $b = y(0)$, $\Delta x = x(a+\Delta t) - x(a)$

$\Delta y = y(b+\Delta t) - y(b)$.



$x, y \in C^1 \Rightarrow \Delta x, \Delta y \to 0$ as $\Delta t \to 0$.

$$\frac{\Delta f}{\Delta t} = \frac{\Delta x}{\Delta t} f_x(a,b) + \frac{\Delta y}{\Delta t} f_y(a,b) + \frac{\Delta x}{\Delta t}\phi(\Delta x, \Delta y) + \frac{\Delta y}{\Delta t}\psi(\Delta x, \Delta y)$$

$\downarrow$ as $\Delta t \to 0$

$$\frac{df}{dt} = \frac{dx}{dt} f_x + \frac{dy}{dt} f_y + \frac{dx}{dt}\cdot 0 + \frac{dy}{dt}\cdot 0.$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Ex. $f = f(x, y) = xy^2 + e^{xy} \in C^1$

$x = t^2$, $y = \sin t \in C^1$.

$g(t) = f(t^2, \sin t) = t^2(\sin t)^2 + e^{t^2 \sin t}$.

$$g'(t) = \frac{df}{dt} = t^2 \cdot 2\sin t \cdot \cos t + \sin^2 t \cdot 2t + e^{t^2 \sin t}\left(t^2 \cos t + \sin t \cdot 2t\right)$$

chain rule.

$\frac{\partial f}{\partial x} = 1 \cdot y^2 + e^{xy} \cdot y$, $\quad \frac{\partial f}{\partial y} = x \cdot 2y + e^{xy} x$

$= \sin^2 t + e^{t^2 \sin t} \cdot \sin t \qquad = 2t^2 \sin t + e^{t^2 \sin t} \cdot t^2$.

$x = t^2$, $\frac{dx}{dt} = 2t$ | $y = \sin t$, $\frac{dy}{dt} = \cos t$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left(\sin^2 t + e^{t^2 \sin t} \sin t\right) 2t + \left(2t^2 \sin t + e^{t^2 \sin t} t^2\right)\cos t$$

$$= \left(\sin^2 t + e^{t^2\sin t}\sin t\right)2t + \left(2t^2\sin t + e^{t^2\sin t}t^2\right)\cos t$$

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chane rule. $f = f(x,y) \in C^1$, $x = x(t) \in C^1$, $y = y(t) \in C^1$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left(\frac{\partial f}{\partial x} \;\; \frac{\partial f}{\partial y}\right)\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix}$$

$$\frac{d}{dt}\left(f(\underline{x}(t))\right) = f'(x(t), y(t)) \cdot \underline{x}'(t)$$

$$(f\circ \underline{x})'(t) = f'(\underline{x}(t))\,\underline{x}'(t)$$

$$(f\circ x)'(t) = (f'\circ \underline{x})(t)\,\underline{x}'(t)$$

$$(f\circ x)' = (f'\circ \underline{x})\,\underline{x}'$$

again $\dfrac{df}{dt} = \left(\dfrac{\partial f}{\partial x} \;\; \dfrac{\partial f}{\partial y}\right)\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix}$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \cancel{(x,y)}} \cdot \frac{\cancel{\partial (x,y)}}{\partial t}$$

$f = f(x,y)$

$f: \mathbb{R}^2 \to \mathbb{R}$

$f' = (f_x, f_y)$

$\underline{x}(t) = (x(t), y(t))$

$\underline{x}: \mathbb{R} \to \mathbb{R}^2$

$\underline{x}' = \begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix}$

$f = f(x)$

$x = x(t)$

$\dfrac{df}{dt} = \dfrac{d}{dt} f(x(t))$

$= f'(x(t))\,x'(t)$

$= (f\circ x)'(t).$

$\dfrac{df}{dt} = \dfrac{df}{\cancel{dx}}\dfrac{\cancel{dx}}{dt}$

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chain rule 2

$f = f(x,y) \in C^1$, $x = x(s,t) \in C^1$, $y = y(s,t) \in C^1$.

$= f(x(s,t), y(s,t)) = g(s,t).$

$\dfrac{\partial f}{\partial s} = $ ___

$\dfrac{\partial f}{\partial t} = $ ___
