Calculus short note - Oshanath

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Limit

$$\lim_{x \to a} f(x) = L \in \mathbb{R} \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x,\ |f(x) - L| < \varepsilon$$

2D Limit

$$\lim_{(x,y) \to (a,b)} f(x,y) = L \in \mathbb{R}$$

$$\Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x,y),\ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon$$

Disproving a limit

If a limit exists,

$$\lim_{(x,y) \to (a,b)} f(x) = L \in \mathbb{R} = \lim_{y \to b} \left(\lim_{x \to a} f(x,y) \right)$$

$$= \lim_{x \to a} \left(\lim_{y \to b} f(x,y) \right)$$

Chain rule

- $f = f(x,y) \qquad x = x(t) \qquad y = y(t)$
- $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

- $f = f(x,y) \qquad x = x(u,v) \qquad y = y(u,v)$
- $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
- $$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Taylor series

- $f(x) = T_n(x,a) + R_n(x,a)$

Taylor Polynomial

- $T_n(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$

Error

- $R_n(x,a) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt$ ← Integral version

- $= \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$ ← Derivative version

Second derivative test for 1D

$f(x) - f(a) = \frac{f''(\xi)}{2} (x-a)^2$ ← From Taylor series when $f'(x) = 0$

If $f''(\xi) > 0 \Rightarrow$ local minimum

Hessian Matrix

- $Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$

Trace of a matrix

- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad tr\, A = a + d$

$tr\, Hf(\underset{\sim}{a}) > 0$ and $\det Hf(\underset{\sim}{a}) > 0 \Rightarrow \underset{\sim}{a}$ is a local minimum

$tr\, Hf(\underset{\sim}{a}) < 0$ and $\det Hf(\underset{\sim}{a}) > 0 \Rightarrow \underset{\sim}{a}$ is a local maximum

$\det Hf(\underset{\sim}{a}) < 0 \Rightarrow \underset{\sim}{a}$ is a saddle point.

Taylor series until n=2

2D → $f(\underset{\sim}{a}+\underset{\sim}{h}) = f(\underset{\sim}{a}) + \nabla f(\underset{\sim}{a})\,\underset{\sim}{h} + \frac{1}{2!}\,\underset{\sim}{h}^T\, Hf(\underset{\sim}{a})\,\underset{\sim}{h}$

1D → $f(a+h) = f(a) + f'(a)\,h + \frac{1}{2!}h^2 f''(\underset{\sim}{a})$

- Directional derivative $= \nabla f(\underset{\sim}{a}) \cdot \underset{\sim}{u} = \dfrac{\partial f}{\partial \underset{\sim}{u}}$

- Normal vector to a surface

  $\underset{\sim}{n} = \langle f_x(a,b),\ f_y(a,b),\ -1 \rangle$

- Equation of a tangent line

  $z = f(\underset{\sim}{a}) + \nabla f(\underset{\sim}{a}) \cdot (\underset{\sim}{x} - \underset{\sim}{a})$

2D Taylor series

- $f(a+h,\ b+k) = f(x,y) = \sum_{k=0}^{n} \frac{1}{k!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^k f(a,b)$

$$+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+\xi h,\ b+\xi k)$$

Lagrange multipliers

$f, g \in C'$, $\nabla g \neq 0$, then max/min of $f(x,y)$ subjected to $g(x,y)=c$ are included in the solutions of

1. $\det \dfrac{\partial(f,g)}{\partial(x,y)} = 0$ and $g(x,y) = C$
2. $\nabla f(x,y) = \lambda \nabla g(x,y)$ $\quad g(x,y) = c$

Least squares regression

$$\begin{pmatrix} \sum x_k^2 & \sum x_k \\ \sum x_k & \sum(1) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum x_k y_k \\ \sum y_k \end{pmatrix}$$

L'hopital's rule

$\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$

$$\implies \lim_{x \to c} \frac{f(x)}{g(x)} = L$$

$\lim_{x \to c} f(x) = \pm\infty$ and $\lim_{x \to c} g(x) = \pm\infty$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$

$$\implies \lim_{x \to c} \frac{f(x)}{g(x)} = L$$

Fundamental theorem of calculus

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

Second fundamental theorem of calculus

$$\frac{d}{dx} \int f(x)\, dx = f(x)$$

Fubin's theorem

$$\int_{(a,b)\times(c,d)} f(x,y)\, dA = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

$$= \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

Changing variabe of a 2D integral

- $\iint_{A_{xy}} f(x,y)\,dx\,dy = \iint_{A_{uv}} f(x(u,v), y(u,v)) \underbrace{\left| \det \frac{\partial(x,y)}{\partial(u,v)} \right|}_{\text{Jacobian matrix}} du\,dv$

- $\left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| = \| \underset{\sim}{r}_u \times \underset{\sim}{r}_v \|$

Green's thm

$$\oint \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_a^b \int_c^d \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

F is conservative $\Rightarrow \exists\, \phi \in C^2$ s.t. $\underset{\sim}{F} = \nabla \phi$

Stokes' theorem

- $\oint \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint \text{curl}\, \underset{\sim}{F} \cdot d\underset{\sim}{S}$

If $\underset{\sim}{F}$ is conservative,

- $\text{curl}\, \underset{\sim}{F} = \text{curl}\, \nabla\phi = \nabla \times \nabla \phi = 0$

- $d\underset{\sim}{S} = \underset{\sim}{n}\, dS$

- $\underset{\sim}{F}$ is irrotational $\Leftrightarrow$ $\text{curl}\, \underset{\sim}{F} = \underset{\sim}{0}$

- $\underset{\sim}{F}$ is conservative $\Rightarrow$ $\underset{\sim}{F}$ is irrotational

  $\underset{\sim}{F} = \nabla\phi \Rightarrow \nabla \times \underset{\sim}{F} = \underset{\sim}{0}$

- In a simply connected domain,

  $\underset{\sim}{F}$ is irrotational $\Rightarrow$ $\underset{\sim}{F}$ is conservative

Divergence theorem

$$\iint_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \iiint_V \operatorname{div} \underset{\sim}{F} \, dV$$

$$\operatorname{div} \underset{\sim}{F} = \nabla \cdot \underset{\sim}{F}$$

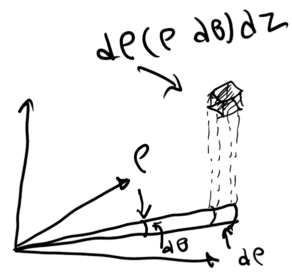
$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Volume integrals - variable change

$$dx\,dy\,dz = \underset{\sim}{r}_u \times \underset{\sim}{r}_v \cdot \underset{\sim}{r}_w \, du\,dv\,dw$$

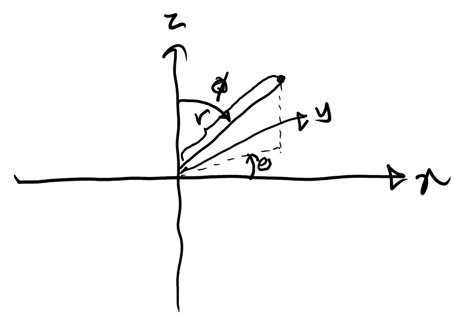
$$= \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \, du\,dv\,dw$$

Cyllindrical polar coordinates



$$\iiint f \, dx\,dy\,dz = \iiint f \rho \, d\rho\, d\theta\, dz$$

spherical polar coordinates



$$dx\,dy\,dz = r^2 \sin\phi \, d\phi\, d\theta\, dr$$

$$\operatorname{curl}(\operatorname{curl} \underset{\sim}{E}) = \nabla \times (\nabla \times \underset{\sim}{E}) = \nabla(\operatorname{div} \underset{\sim}{E}) - \nabla^2 \underset{\sim}{E}$$

$\nabla^2 = \nabla \cdot \nabla$ (Laplacian)

$$\nabla^2 = \sum \frac{\partial^2}{\partial x_i^2}$$

Maxwell's equations

Gauss' law $\nabla \cdot \underset{\sim}{E} = \dfrac{\rho}{\varepsilon_0}$ $\qquad \nabla \cdot \underset{\sim}{B} = 0$

Ampere's law $\nabla \times \underset{\sim}{B} = \mu_0 \left(\underset{\sim}{J} + \varepsilon_0 \dfrac{\partial E}{\partial t} \right)$

Ampere's law $\nabla \times \underset{\sim}{B} = \mu_0 \left(\underset{\sim}{J} + \varepsilon_0 \frac{\partial E}{\partial t} \right)$

$\underset{\sim}{J} = 0$ in vacuum

$$\therefore \quad \nabla \times \underset{\sim}{B} = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \frac{1}{c^2} \frac{\partial E}{\partial t}$$

Farday's law $\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$

wave equation

$$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$$

2D divergence theorem

$$\oint \underset{\sim}{E} \cdot \partial \underset{\sim}{l} = \iint_S \operatorname{div} \underset{\sim}{E} \, dA$$

perpendicular to the line.

Complex analysis

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\theta = \operatorname{Arg} z + 2n\pi$$

$\operatorname{Arg} z$ — principal argument

$\arg z$ — argument

Roots

$$z = e^{i \operatorname{Arg} z}$$

$$z = e^{i \operatorname{Arg} z + i2n\pi} \quad \leftarrow \text{general}$$

$$\sqrt{z} = e^{i \operatorname{Arg} z / 2 + in\pi}$$

$$= \sqrt{z} \, e^{in\pi}$$

$$\sqrt[3]{z} = \sqrt[3]{z} \cdot e^{i \frac{2n\pi}{3}}$$

logarithms

$i \operatorname{Arg} z + i2n\pi$

$$\ln z = \underbrace{\ln|z| + i\,\mathrm{Arg}\,z}_{\text{principal logarithm}} + i2n\pi$$

- $e^{i\theta} = \cos\theta + i\sin\theta$
- $e^{-i\theta} = \cos\theta - i\sin\theta$

- $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$
- $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

- $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$
- $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$
- $\tanh x = \dfrac{\sinh x}{\cosh x}$

$f = u(x,y) + iv(x,y)$

If $f \in D$, 1) u_x, u_y, v_x, v_y exist

2) $u_x = v_y$ and $v_x = -u_y$ (Cauchy Riemann equations)

f is analytic at $a \Rightarrow f$ is differentiable in a nbd of a

Integration

$$\oint_C f(z)\,dz = 2\pi i \,\mathrm{Res}(f,a)$$

$$\mathrm{Res}(f,a) = \frac{1}{(n-1)!} \lim_{z\to a}\left(\frac{d^{n-1}}{dz^{n-1}} f(z)(z-a)^n\right)$$

$$\oint_C f(z)\,dz = 2\pi i \sum_k \mathrm{Res}(f,a)$$ ← for all singularities,

Taylor series

- $f(z) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$

Laurant Series

- $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$

- $a_k = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{k+1}} \, dz$