CSC210: Data Structures and Algorithms

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Sore additional points by presenting in class the proofs of the numbered equations.

Standard notations and common functions (ch. 3.2)

Monotonicity
$f(n)$ is:

- monotonically increasing if $m \leq n$ implies $f(m) \leq f(n)$
- monotonically decreasing if $m \leq n$ implies $f(m) \geq f(n)$
- strictly increasing if $m < n$ implies $f(m) < f(n)$
- strictly decreasing if $m < n$ implies $f(m) > f(n)$

Floor and ceiling are monotonically increasing

floor of x, $\lfloor x \rfloor$, is the greatest integer y such that $y \leq x$

ceiling of x, $\lceil x \rceil$, is the smallest integer y such that $y \geq x$

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1 \tag{1}$$

For any integer n:

$$\lceil n/2 \rceil + \lfloor n/2 \rfloor = n \tag{2}$$

For any real number $x \geq 0$ and integers $a, b > 0$:

$$\lceil \frac{\lceil \frac{x}{a} \rceil}{b} \rceil = \lceil \frac{x}{ab} \rceil \tag{3}$$

$$\lfloor \frac{\lfloor \frac{x}{a} \rfloor}{b} \rfloor = \lfloor \frac{x}{ab} \rfloor \tag{4}$$

$$\lceil \frac{a}{b} \rceil \leq \frac{a + (b-1)}{b} \tag{5}$$

$$\lfloor \frac{a}{b} \rfloor \geq \frac{a - (b-1)}{b} \tag{6}$$

Modular arithmetic

For any integer a and a positive integer n, $a \mod n$ is a remainder of the quotient $\frac{a}{n}$

$$a \mod n = a - n\lfloor \frac{a}{n} \rfloor \tag{7}$$

$$0 \leq a \bmod n < n \tag{8}$$

We say $a \equiv b \pmod{n}$ iff $n \mid (b-a)$ (n is a divisor of $(b-a)$)

$$a \equiv b \pmod{n} \iff a \bmod n = b \bmod n \tag{9}$$

Exponentials

For all real constants a, b such that $a > 1$:

$$\lim_{n \to \infty} \frac{n^b}{a^n} = 0 \tag{10}$$

$$n^b = o(a^n) \tag{11}$$

For real x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \Sigma_{i=0}^{\infty} \frac{x^i}{i!} \tag{12}$$

$$e^x \geq 1 + x \tag{13}$$

For $|x| \leq 1$

$$1 + x \leq e^x \leq 1 + x + x^2 \tag{14}$$

For all x

$$\lim_{n \to \infty} (1 + \frac{x}{n})^n = e^x \tag{15}$$

Logarithms

Notation: $\lg n = \log_2 n$, $\ln n = \log_e n$, $\lg^k n = (lgn)^k$, $\lg\lg n = \lg(\lg n)$

Notice how logarithm binds its argument: $\lg n + k = (\lg n) + k$ and thus $\lg n + k \neq \lg(n + k)$

For $b > 1$, $n > 0$, $\log_b n$ is strictly increasing.

For all real $a > 0, b > 0, c > 0$ and n (and the base of the logarithms is not 1):

$$a = b^{\log_b a} \tag{16}$$

$$\log_c(a, b) = \log_c a + \log_c b \tag{17}$$

$$\log_b a^n = n \log_b a \tag{18}$$

$$\log_b a = \frac{\log_c a}{\log_c b} \tag{19}$$

$$\log_b(1/a) = -\log_b a \tag{20}$$

$$\log_b a = \frac{1}{\log_a b} \tag{21}$$

$$a^{\log_b c} = c^{\log_b a} \tag{22}$$

For $|x| < 1$:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \tag{23}$$

For $x > -1$:

$$\frac{x}{1 + x} \leq \ln(1 + x) \leq x \tag{24}$$

For any constant $a > 0$:

$$\lg^b n = o(n^a) \tag{25}$$

Factorials

Note the recursive definition of the factorial function:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n > 0 \end{cases}$$

The **Stirling's approximation**:

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n(1 + \Theta(\frac{1}{n}))$$

$$n! = o(n^n) \tag{26}$$

$$n! = \omega(2^n) \tag{27}$$

$$\lg(n!) = \Theta(n \lg n) \tag{28}$$

Fibonacci numbers:

Definition:

$$\begin{aligned} &F_0 = 0 \\ &F_1 = 1 \\ &F_i = F_{i-1} + F_{i-2} \quad \text{for } i \geq 2 \end{aligned}$$

Golden ratio

$$\phi = \frac{1 + \sqrt{5}}{2}$$

conjugate of golden ratio:

$$\phi = \frac{1 - \sqrt{5}}{2}$$

Golden ration and its conjugate are two roots of the equation:

$$x^2 = x + 1$$

$$F_i = \frac{\phi^i - \hat{\phi}_i}{\sqrt{5}} \tag{29}$$