School of Engineering and Applied Science
Ahmedabad University

B. Tech. (ICT): Semester II, EMT
Problem Set: 01

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1. A static charge distribution produces a radial electric field
$$\vec{E} = Ae^{-br}\ \hat{r}\ ,$$
where A and b are constants.
   (a) What is the charge density? Sketch it.
   (b) What is the total charge?

2. Prove that $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ by writing out both the sides in component form.

3. Consider a rigid body undergoing plane rotation with a constant angular velocity $\vec{\omega}$. The linear velocity at any point with position vector $\vec{r}$ is $\vec{v} = \vec{\omega} \times \vec{r}$. What is the curl of the linear velocity field? What is the divergence of the linear velocity field?

4. Just like the current density vector field, it is often convenient to define with the thermal current density vector field $\vec{J}(\vec{r})$ whose magnitude measures the time rate of flow of heat per unit area at the location of interest. If T is the temperature field inside a solid, we have
$$\vec{J} = -\kappa \vec{\nabla} T\ ,$$
where κ is a constant (called thermal conductivity). If the temperature inside a spherical metal ball is proportional to the square of the distance from the centre of the ball, find the rate of heat flow across a sphere of radius R with centre at the centre of the ball.

5. Consider three straight, infinitely long, equally spaced parallel wires (with zero radius) each carrying a current I in the same direction. Let d be the separation between the nearby wires.
   (a) Calculate the location of the two zeros in the magnetic field.
   (b) Sketch the magnetic field line pattern.
   (c) If the middle wire is rigidly displaced a very small distance x ($x \ll d$) upward while the other two wires are held fixed, what will be the subsequent motion of the middle wire?

6. The centre of a circular loop of radius R kept in the $x-y$ plane is at the origin. It has a uniform linear charge density λ and is rotating with an angular velocity $\vec{\omega} = \omega\hat{z}$. Find the electrictrostatic field and magnetostatic field at a point with coordinate $(0, 0, z)$. What happens when $z \ll R$? What happens when $z \gg R$?

7. The centre of a circular disk of radius R kept in the $x-y$ plane is at the origin. It has a uniform surface charge density λ and is rotating with an angular velocity $\vec{\omega} = \omega\hat{z}$. Find the expression for the surface current density vector field. Find the electrictrostatic field and magnetostatic field at a point with coordinate $(0, 0, z)$. What happens when $z \ll R$? What happens when $z \gg R$?

8. Using Gauss law, find the electric field vector field inside and outside a uniformly charge solid sphere of radius R. Using this, find the electrostatic potential everywhere. Repeat for the case in which $\rho \propto r$ when $r < R$.

9. Now repeat above problem by first finding out the potential and then calculating the electric field from it (without using Gauss Law).

10. Solve problem 5.9 of Griffiths.

11. In all situations involving electromagnetism, the electric field $\vec{E}(t, \vec{r})$ and magnetic field $\vec{B}(t, \vec{r})$ can always be found from the scalar potential $V(t, \vec{r})$ and the vector potential $\vec{A}(t, \vec{r})$ using the following expressions:

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}\,.$$

E.g. let's consider what's called a **dipole antenna**, kept at the origin of the coordinate system, with its axis along the z-axis and radiating electromagnetic waves. For points far away from the antenna s.t. $r \gg c/\omega$ (ω being the angular frequency of radiation emitted and c, the speed of em radiation), one can solve Maxwell's equations to get

$$V(t, r, \theta) = -\frac{\alpha\omega}{c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)]\ , \quad \vec{A}(t, r, \theta) = -\frac{\beta\omega}{r}\sin[\omega(t - r/c)]\hat{z}\ ,$$

here, α and β, ω and c are constants and r, θ, ϕ are spherical polar coordinates.

(a) Find the components of the vector potential in spherical polar coordinates,

(b) Find the electric field due to the antenna,

(c) Use the fact that $r \gg c/\omega$ to simplify the expression for $\vec{E}$ by retaining only the largest contribution,

(d) Find the magnetic field due to the antenna,

(e) Use the fact that $r \gg c/\omega$ to simplify the expression for $\vec{B}$ by retaining only the largest contribution.

(f) An observer sits at the location with coordinates $(r, \theta = \pi/2, \phi = 0)$, with a detector. Draw a graph of the electric field detected by the observer as a function of time.

(g) Redraw fig(1) in your answer sheet. If the dipole antenna sits at the origin of the coordinate system and points in the z direction and the observer is at the point P, draw arrows to indicate the direction of the electric field and the magnetic field detected by the observer.

(h) A quantity named Poynting vector, denoted by $\vec{S}$, is defined by $\vec{S} = \vec{E} \times \vec{B}$. Using the expression for $\vec{E}$ and $\vec{B}$ obtained above, find the expression for the Poynting vector.

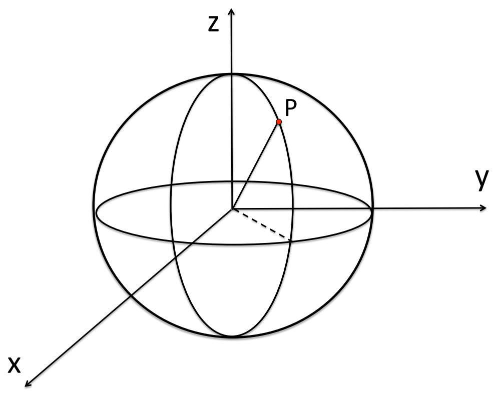

Figure 1

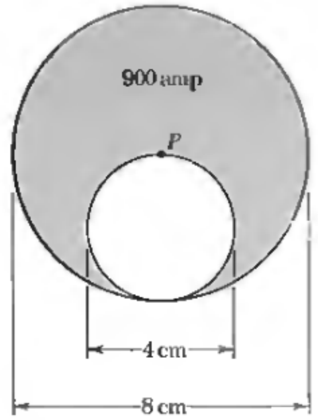

Figure 2

(i) The Poynting vector has the property that its surface integral through any surface S species the rate at which energy flows out of that surface i.e. $\int_S \vec{P} \cdot d\vec{a} = \frac{dE}{dt}$. In the above situation, find the rate at which the antenna radiates energy. HINT: choose S to be a spherical surface centred at the origin and perform the surface integral.

Feel free to make use of

$$\vec{\nabla} \times \vec{v} = \frac{\hat{r}}{r \sin\theta} \left[\frac{\partial}{\partial\theta}(\sin\theta \; v_\phi) - \frac{\partial v_\theta}{\partial\phi} \right] + \frac{\hat{\theta}}{r} \left[\frac{1}{\sin\theta} \frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi) \right] + \frac{\hat{\phi}}{r} \left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta} \right] .$$

12. **Electrostatic energy:** In this problem, we'll learn how to calculate the energy of a system of charges. Consider a system which has two point charges Q_1 and Q_2 at rest at locations $\vec{r}_1$ and $\vec{r}_2$. Let us understand how much energy is needed to assemble this system. Imagine the first charge is already kept at $\vec{r}_1$ and we are bringing the second charge from infinity to $\vec{r}_2$.

(a) What is the force on Q_2 (due to Q_1) when it is at infinity? As we bring Q_2 from infinity to $\vec{r}_2$, when we are at any intermediate location $\vec{r}$, what is the expression for vector force on Q_2?

(b) If $d\vec{l}$ is a small infinitesimal displacement vector, write the infinitesimal work done by the electrostatic force as the charge Q_2 goes from $\vec{r}$ to $\vec{r} + d\vec{l}$. Does this seem to depend on the path $d\vec{l}$? Is it actually dependent on the path? Why?

(c) The work done by the electrostatic force to go from $\vec{r}_i$ to $\vec{r}_f$ along a path $\mathcal{P}$ will be

$$W = \int_{\vec{r}_i \ \ \text{along}\mathcal{P}}^{\vec{r}_f} \vec{F} \cdot d\vec{l} \,,$$

will this be path independent? Justify your answer.

(d) Suppose an external agency (say you) does the following: at every instant, as the charge is brought from $\vec{r}_i$ to $\vec{r}_f$, the external agency applies a force equal and opposite to the electrostatic force so that the charge never experiences any net force and hence never accelerates. Argue that the net work done by the external agency in bringing the charge Q_2 from infinity to $\vec{r}_2$ is going to be given by $Q_2 V(\vec{r}_2)$, where $V(\vec{r}_2)$ is the potential at the location $\vec{r}_2$ because of the source Q_1 at $\vec{r}_1$.

(e) Let us suppose I also bring a third charge Q_3 at location $\vec{r}_3$. How much energy is needed to build this system?

(f) Write the expression for the total amount of work needed to assemble a system consisting of N point charges. Show that this can be written as $W = \frac{1}{2}\sum_{i=1}^{N} Q_i V(\vec{r}_i)$.

(g) The above expression can be generalized to the case of continuous distribution of charge

$$W = \frac{1}{2}\int d\tau \rho(\vec{r})V(\vec{r}) \ .$$

Use this to derive an expression for the electrostatic energy required to assemble a uniformly charged solid sphere.

13. Use the divergence theorem to prove that

$$\int_{\mathcal{V}} (\vec{\nabla} T) \ d\tau = \oint_{\mathcal{S}} T d\vec{a} \ ,$$

where T is a scalar field.

14. A long copper rod 8 cm in diameter has an off-centre cylindrical hole as shown in Fig (2), down its full length. This conductor carries a current of 900 A flowing in the direction into the paper. Find the direction and strength in Gauss, of the magnetic field at the point P which lies on the axis of the outer cylinder.

15. Write down the expression (in vector form, using cylindrical polar coordinates) of magnetic due to an infinitely long an infinitesimally thick wire carrying a steady current I. Find a vector potential $\vec{A}$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$. With reasons explain: could you have obtained this $\vec{A}$ using the one dimensional equivalent of

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{d\tau' \vec{J}(\vec{r'})}{|\vec{r} - \vec{r'}|} \quad ?$$

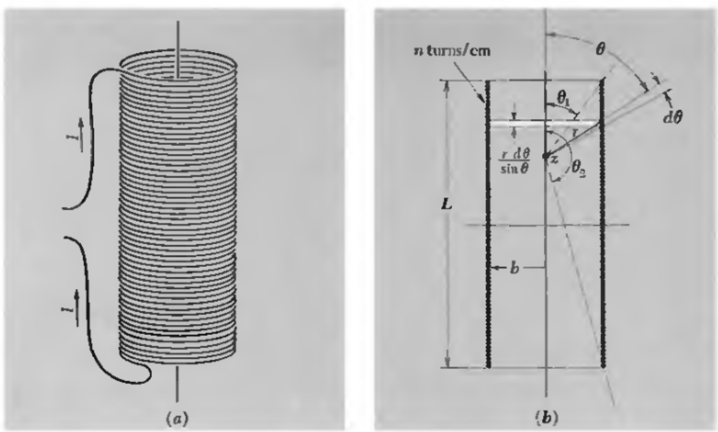

Figure 3

16. Consider the solenoid of finite size as shown in Fig (3). Find the z component of the magnetic field, B_z at the location z on the axis as shown in the Figure. Notice that the lines joining the point z to the ends of solenoid make angles θ_1 and θ_2 from the z-axis. Using this, find the magnetic field for the case in which the coil has infinite length, denote it as B_z^{∞}. Plot a graph of B_z/B_z^{∞} against z/b for a coil for which $L = 4b$.

17. **Electric dipoles in electric fields:** Read section 4.1 of Griffiths to convince yourself that when a dielectric material is placed in an external electric field, it gets "polarized." Moreover, when a dipole (with dipole moment $\vec{p}$) is placed in a uniform electric field of strength $\vec{E}$, it experiences zero force and a torque given by $\vec{p} \times \vec{E}$. Show that the energy of an ideal dipole $\vec{p}$ in an electric field is $U = -\vec{p} \cdot \vec{E}$.

18. (a) For a configuration of charges and currents confined within a volume $\mathcal{V}$, show that

$$\int_{\mathcal{V}} \vec{J} d\tau = \frac{d\vec{p}}{dt} ,$$

where $\vec{p}$ is the total electric dipole moment of the system.

(b) Recall that, for a localized current distribution, the vector potential can be chosen to satisfy

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{d\tau' \vec{J}(\vec{r'})}{|\vec{r} - \vec{r'}|} .$$

Expand $\frac{1}{|\vec{r}-\vec{r'}|}$ in powers of (r'/r) and retain only the first two terms. Show that the monopole term is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d\tau' \vec{J} ,$$

and also, show that this is zero.

(c) Show that the next term in the expansion is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} ,$$

where, $m = \frac{1}{2} \int d\tau' \left[\vec{r'} \times \vec{J}(\vec{r'})\right]$ is the magnetic dipole moment.

19. Consider the fields

$$\vec{E} = \frac{5\hat{y}}{1 + (x + ct)^2} \quad , \quad \vec{B} = \frac{-5\hat{z}}{1 + (x + ct)^2} .$$

Show that these fields satisfy the source free Maxwell's equations. Show that they also satisfy the wave equations for $\vec{E}$ and $\vec{B}$. What is the direction of propagation of the EM wave? Is it a plane wave? Justify. Draw a graph showing the profile of the electric field and magnetic field at $t = 0$. Redraw them at $t = -3$ nanoseconds.

20. Reading assignments:

(a) Read about and learn how one could use the concept of electrostatic energy to find the heat of vaporization plus the the energy required to dissociate the molecules into ions for table salt (NaCl). One reference: section 8-3 of volume II of The Feynman Lectures on Physics.

(b) Read section 2.5 of Griffiths (Conductors).

(c) Recall how the following devices work: an electric motor, a dynamo (or generator), a transformer, electromagnet.

(d) Read about self inductance and mutual inductance from Griffiths.