ASSIGNMENT - 2

REPORT

- Nimish Manware (22B0944)
- Prasanna Nage (22B0953)
- Jayesh Vinod Jadhav (22B1056)

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1. Solution:

We are given $Y_1 = \max(X_1, X_2, ---, X_n)$

Now, for cdf of Y_1

$$F_{Y_1}(x) = P(Y_1 \le x)$$
$$= P(\max(X_1, X_2, ---, X_n) \le x)$$

For $\max(X_1, X_2, ---, X_n) \le x$ we need to have each one of X_i to be less than x,

so, $F_{Y_1}(x) = P(X_1 \le x, X_2 \le x, ---, X_n \le x)$

$$= P(X_1 \le x) \cdot P(X_2 \le x) --- P(X_n \le x)$$

[$\because$ $X_1, X_2, --, X_n$ are independent]

so, $F_{Y_1}(x) = F_{X_1}(x) \cdot F_{X_2}(x) -- F_{X_n}(x)$ [$\because$ $P(X_i \le x) = F_{X_i}(x)$]

$$= F_X(x) \cdot F_X(x) --- F_X(x)$$ [$F_{X_i}(x) = F_X(x)$ because X_i are identically distributed]

so, $\boxed{F_{Y_1}(x) = [F_X(x)]^n}$

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Now for pdf of ~~F_X~~ Y_1.

$$f_{Y_1}(x) = \frac{d}{dx}[F_{Y_1}(x)] = \frac{d}{dx}[F_X(x)]^n = n[F_X(x)]^{n-1}\frac{d}{dx}F_X(x)$$

$$\boxed{f_{Y_1}(x) = n[F_X(x)]^{n-1} \cdot F_X'(x)}$$

Now, for Y_2, we are given $Y_2 = \min(X_1, X_2, ---, X_n)$

Now, for cdf of Y_2

$$F_{Y_2}(x) = P(Y_2 \le x)$$
$$= P(\min(X_1, X_2, ---, X_n) \le x)$$

For $\min(X_1, X_2, ---, X_n) \le x$, we need to have at least one of X_i to be less than x, which is the negation of every $X_i > x$. So, $F_{Y_2}(x)$ can be written as

$$F_{Y_2}(x) = 1 - P(X_1 > x, X_2 > x, ---, X_n > x)$$
$$= 1 - P(X_1 > x) \cdot P(X_2 > x) ---- P(X_n > x)$$

[$\because$ $X_1, X_2, --, X_n$ are independent]

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so, $F_{Y_2}(x) = 1 - (1 - F_{X_1}(x)).(1 - F_{X_2}(x)) --- (1 - F_{X_n}(x))$

$[\because P(X > x) = 1 - F_X(x)]$

$F_{Y_2}(x) = 1 - (1 - F_X(x)).(1 - F_X(x)) --- (1 - F_X(x))$

[as X_i are identically distributed].

$$\boxed{F_{Y_2}(x) = 1 - (1 - F_X(x))^n}$$

Now, for pdf of Y_2,

$$f_{Y_2}(x) = \frac{d}{dx} F_{Y_2}(x) = \frac{d}{dx}[1 - (1 - F_X(x))^n]$$

$$= -n(1 - F_X(x))^{n-1} \cdot \left[-\frac{d}{dx} F_X(x)\right]$$

$$\boxed{f_{Y_2}(x) = n(1 - F_X(x))^{n-1} F_X'(x)}$$

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2. Solution: According to the definition of GMM, we are selecting the i^{th} distribution with probability p_i, then we draw a value for a random variable with pdf as that of the selected distribution.

So, the equation of pdf for X becomes,

$$f_X(x) = \sum_{i=1}^{K} p_i f_{X_i}(x)$$

Now, $f_{X_i}(x) = \mathcal{N}(\mu_i, \sigma_i^2)$

so, $f_X(x) = \sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)$

Mean :-

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \left(\sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)\right) dx$$

$$= \sum_{i=1}^{K}\left(p_i \int_{-\infty}^{\infty} x \mathcal{N}(\mu_i, \sigma_i^2)\, dx\right)$$

But, $\int_{-\infty}^{\infty} x \mathcal{N}(\mu_i, \sigma_i^2)\, dx$ is nothing but μ_i.

so, $\boxed{E[X] = \sum_{i=1}^{K}(p_i \mu_i)}$

Variance :-

we have, $Var(X) = E[X^2] - (E[X])^2$

For $E[X^2]$, we have,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_{-\infty}^{\infty} x^2 \left(\sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)\right) dx$$

$$= \sum_{i=1}^{K}\left(p_i \int_{-\infty}^{\infty} x^2 \mathcal{N}(\mu_i, \sigma_i^2)\, dx\right)$$

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But, $\int_{-\infty}^{\infty} x^2 \mathcal{N}(\mu_i, \sigma_i^2)\, dx = \sigma_i^2 + \mu_i^2$

$$E[x^2] = \sum_{i=1}^{k} p_i(\sigma_i^2 + \mu_i^2)$$

So, we get finally,

$$\boxed{Var(X) = \sum_{i=1}^{k} p_i(\sigma_i^2 + \mu_i^2) - \left(\sum_{i=1}^{k} p_i \mu_i\right)^2}$$

MGF :-

$$\begin{aligned}
\phi_X(t) &= E[e^{tX}] \\
&= \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} e^{tx} \left(\sum_{i=1}^{k} p_i \mathcal{N}(\mu_i, \sigma_i^2)\right) dx \\
&= \sum_{i=1}^{k} \left(p_i \int_{-\infty}^{\infty} e^{tx} \mathcal{N}(\mu_i, \sigma_i^2)\, dx \right) \\
&= \sum_{i=1}^{k} \left(p_i\, \phi_{X_i}(t) \right) \quad [\text{where } \phi_{X_i}(t) \text{ is MGF of normal distribution } \mathcal{N}(\mu_i, \sigma_i^2)] \\
&= \sum_{i=1}^{k} \left(p_i \cdot e^{\mu_i t + \sigma_i^2 t^2/2} \right)
\end{aligned}$$

so, $\boxed{\phi_X(t) = \sum_{i=1}^{k} \left(p_i e^{\mu_i t + \sigma_i^2 t^2/2}\right)}$

Now, $Z = \sum_{i=1}^{k} p_i X_i$

so, $E[Z] = E\left[\sum_{i=1}^{k} p_i X_i\right] = \sum_{i=1}^{k} p_i E[X_i] = \sum_{i=1}^{k} p_i \mu_i$

so, $\boxed{E[Z] = \sum_{i=1}^{k} p_i \mu_i}$

$$Var(Z) = \cancel{E}\ Var\left(\sum_{i=1}^{K} p_i x_i\right)$$
$$= \sum_{i=1}^{K} Var(p_i x_i) \quad [\text{as } x_i \text{ are independent}]$$
$$= \sum_{i=1}^{K} p_i^2\, Var(X_i) = \sum_{i=1}^{K} p_i^2 \sigma_i^2$$

$$\boxed{Var(Z) = \sum_{i=1}^{K} p_i^2 \sigma_i^2}$$

MGF of Z :

$$\phi_Z(t) = \prod_{i=1}^{K} \phi_{p_i x_i}(t) \quad [\text{as } x_i\text{'s are independent random variables}]$$
$$= \prod_{i=1}^{K} \phi_{x_i}(p_i t)$$
$$= \prod_{i=1}^{K} e^{p_i \mu_i t + p_i^2 \sigma_i^2 t^2/2}$$

$$\boxed{\phi_Z(t) = \exp\left\{ t \sum_{i=1}^{K} p_i \mu_i + \frac{t^2}{2} \sum_{i=1}^{K} p_i^2 \sigma_i^2 \right\}}$$

PDF of Z,

Consider a random variable Y, with pdf $\mathcal{N}\left(\sum_{i=1}^{K} p_i \mu_i, \sum_{i=1}^{K} p_i^2 \sigma_i^2\right)$.

$$\phi_Y(t) = \exp\left\{ t \sum_{i=1}^{K} p_i \mu_i + \frac{t^2}{2} \sum_{i=1}^{K} p_i^2 \sigma_i^2 \right\} = \phi_Z(t)$$

Now, we know that for a ~~discrete rand.~~ continuous random variable, pdf and MGF uniquely determine each other.

Thus,

$$\boxed{f_Z(x) = f_Y(x) = \mathcal{N}\left(\sum_{i=1}^{K} p_i \mu_i, \sum_{i=1}^{K} p_i^2 \sigma_i^2\right)}$$

So,

$$\boxed{f_Z(x) = \frac{1}{\sqrt{Var(Z) \cdot 2\pi}} \exp\left\{ -\frac{1}{2} \frac{(x - E[Z])^2}{Var(Z)} \right\}}$$

3. Solution :-

Given an independent random variable X with mean μ and variance σ^2.

We define a random variable $Y = X - \mu$.

It is clear that $E[Y] = E[X] - E[\mu] = 0$.

and, Variance of $Y = E[[Y-0]^2] = E[x^2 - 2\mu X + \mu^2]$
$$= E[X^2] - 2\mu^2 + \mu^2 = E[(X-\mu)^2] = \sigma^2$$

Now, for any $u \geq 0$, we have for $\tau > 0$,

$$P(X - \mu \geq \tau) = P(Y \geq \tau) = P(Y + u \geq \tau + u)$$
$\nleq$
$$\leq P((Y+u \geq \tau + u) \cup (Y+u \leq -(\tau+u)))$$
$$\leq P((Y+u)^2 \geq (\tau+u)^2) \quad [\because \tau + u > 0]$$
$$\leq \frac{E[(Y+u)^2]}{(\tau+u)^2} \quad [\text{By Markov's inequality}]$$
$$= \frac{\sigma^2 + u^2}{(\tau+u)^2} \quad [\because E[(Y+u)^2] = E[Y^2 + 2uY + u^2] = \sigma^2 + u^2]$$

This is true for any $u \geq 0$. So, choose $u = \frac{\sigma^2}{\tau}$.

We get,

$$P(X - \mu \geq \tau) \leq \frac{\sigma^2 + \sigma^4/\tau^2}{(\tau + \sigma^2/\tau)^2} = \frac{\sigma^2}{\sigma^2 + \tau^2}$$

so, $$P((X-\mu) \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}, \quad \text{for } \tau > 0 \quad -(1)$$

Now, for any $u \leq 0$, we have for $\tau < 0$,

$$P(X - \mu \geq \tau) = P(Y \geq \tau) = P(Y+u \geq \tau + u)$$
$$\leq P((Y+u \geq \tau+u) \cup (Y+u \leq -(\tau+u)))$$
$$\leq P((Y+u)^2 \leq (\tau+u)^2) \quad [\because \tau + u < 0]$$
$$= 1 -$$

$$P(X - \mu \geq \tau) = P(Y \geq \tau) = P(Y+u \geq \tau+u)$$
$$\geq P((Y+u)^2 \leq (\tau+u)^2) \quad [\because \tau + u < 0]$$
$$\geq 1 - P((Y+u)^2 \geq (\tau+u)^2) \quad [\because P(E) = 1 - P(E)]$$

so, $P(X-\mu \geq \tau) \geq 1 - P((Y+u)^2 \geq (\tau+u)^2)$

Now, we have by Markov's inequality

$$P((Y+u)^2 \geq (\tau+u)^2) \leq \frac{E[(Y+u)^2]}{(\tau+u)^2}$$

so $P(X-\mu \geq \tau) \geq 1 - \frac{E[(Y+u)^2]}{(\tau+u)^2}$

$$= 1 - \frac{\{E[Y^2 + 2Y\mu + u^2]\}}{(\tau+u)^2}$$

$$= 1 - \frac{\{\sigma^2 + u^2\}}{(\tau+u)^2}$$

since, this is true for any $u \leq 0$, choose $u = \frac{\sigma^2}{\tau}$ $(\tau < 0)$

so, we get, $P(X-\mu \geq \tau) \geq 1 - \frac{\{\sigma^2 + \sigma^4/\tau^2\}}{(\tau + \sigma^2/\tau)^2}$

$$= 1 - \frac{\sigma^2}{\sigma^2 + \tau^2}$$

so, $P(X-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$ for $\tau < 0$.

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4. Solution :

we have,

$$\phi_x(t) = \int_{-\infty}^{\infty} e^{tz} f_x(z)\, dz \geq \int_{x}^{\infty} e^{tz} f_x(z)\, dz$$

$$\geq e^{tx} \int_{x}^{\infty} f_x(z)\, dz \quad [\text{as } e^{tx} \text{ is minimum value of } e^{tz} \text{ when } z \in [x,\infty)\,]$$

so, $\phi_x(t) \geq e^{tx}\, P(X \geq x)$

so, $P(X \geq x) \leq e^{-tx} \phi_x(t)$

Now, we prove it for discrete random variables.

$$\phi_x(t) = \sum_{z} e^{tz} f_x(z) \geq \sum_{z \geq x} e^{tz} f_x(z)$$

$$\geq e^{tx} \cdot \sum_{z \geq x} f_x(z) \quad [\text{as } e^{tx} \text{ is minimum value of } e^{tz} \text{ when } z \geq x\,]$$

so, $\phi_x(t) \geq e^{tx}\, P(X \geq x)$

so, $P(X \geq x) \leq e^{-tx} \phi_x(t)$

Now, given n independent Bernoulli random variables $X_1, X_2, \ldots, X_n$, where $E[X_i] = p_i$ and $\mu = \sum_{i=1}^{n} p_i$

We have, by using the inequality proved above,

$$P(X > (1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \phi_x(t)$$

$$\leq \frac{\phi_x(t)}{e^{(1+\delta)t\mu}}$$

Given, x_i are independent,

$$P(X > (1+\delta)\mu) \leq \frac{\phi_{x_1}(t)\, \phi_{x_2}(t) \cdots \phi_{x_n}(t)}{e^{(1+\delta)t\mu}}$$

We know, that as X_i's are Bernoulli random variables,

so, $\phi_{x_i}(t) = 1 - p_i + p_i e^t = 1 + p_i(e^t - 1)$

So, $\phi_{x_i}(t) = 1+p_i(e^t-1) \le e^{p_i(e^t-1)}$ [given $1+x \le e^x$]

so, we get,

$$P(X>(1+\delta)\mu) \le \frac{e^{p_1(e^t-1)} \cdot e^{p_2(e^t-1)} \cdots e^{p_n(e^t-1)}}{e^{(1+\delta)t\mu}}$$

$$\le \frac{e^{(\sum_{i=1}^{n} p_i)(e^t-1)}}{e^{(1+\delta)t\mu}}$$

$$\le \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}} \quad \left[\text{given } \sum_{i=1}^{n} p_i = \mu\right]$$

Hence, $P(X>(1+\delta)\mu) \le \dfrac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}}$

To tighten this bound, we should make the right hand side of the inequality minimum.

So, let $g(t) = \dfrac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}} = e^{\mu(e^t-1)-(1+\delta)t\mu}$

$$g'(t) = e^{\mu(e^t-1)-(1+\delta)t\mu} \cdot (\mu e^t - (1+\delta)\mu)$$

we can see that for $t < \log_e(1+\delta)$, $g'(t)$ is negative and for $t > \log_e(1+\delta)$, $g'(t)$ is positive. So, $g(t)$ decreases for $t < \log_e(1+\delta)$ and increases for $t > \log_e(1+\delta)$.

So, the minimum value for $g(t)$ is attained at $t = \log_e(1+\delta)$

So, the appropriate value of $t = \log_e(1+\delta)$

6. 1) Non-inverted :

The correlation coefficient is positive, which implies that the images are not inverted relative to each other.

2) Inverted :

The correlation coefficient will be minimum when the two images overlap each other and the shift is 0 ; this can be seen in the figure. Also, we can see from the graphs that the quadratic mutual information and correlation coefficient are inversely related. Also, the curves are not symmetric, so we can infer that the images are also not symmetric about Y axis.

When the joint histogram is more concentrated in a small number of bins, then the QMI is also higher.

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7. Solution: $\text{cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j]$ $(i \neq j)$

We know that, MGF of multinomial is

$$\phi_X(\vec{t}) = (p_1 e^{t_1} + p_2 e^{t_2} + \cdots + p_k e^{t_k})^n$$

Now, $\cancel{E[X_i \cdot X_j] = \frac{\partial^2}{\partial x \partial y}}$

$$E[X_i \cdot X_j] = \frac{\partial^2 (\phi_X(\vec{t}))}{\partial t_i \partial t_j}\Big|_{\cancel{t_i = t_j = 0}\ \vec{t}=0} \quad (i \neq j)$$

$$= \frac{\partial}{\partial t_j} n(p_1 e^{t_1} + p_2 e^{t_2} + \cdots + p_k e^{t_k})^{n-1} p_i e^{t_i}\Big|_{\vec{t}=0}$$

$$= n(n-1)(p_1 e^{t_1} + p_2 e^{t_2} + \cdots + p_k e^{t_k})^{n-2} p_i e^{t_i} \cdot p_j e^{t_j}\Big|_{\vec{t}=0}$$

$$= n(n-1)\left(\sum_{i=1}^{k} p_i\right)^{n-2} \cdot p_i p_j$$

$$= n(n-1) \cdot (1)\, p_i p_j$$

So,

$$E[X_i X_j] = n(n-1) p_i p_j$$

So, $\text{cov}(X_i, X_j) = n(n-1) p_i p_j - n p_i \cdot n p_j$

So, $\text{Cov}(X_i, X_j) = -n p_i p_j$

So, the covariance matrix is

$$\text{cov}(X_i, X_j) = \begin{cases} -n p_i p_j & \text{if } i \neq j \\ \text{Var}(X_i) & \text{if } i = j \end{cases}$$

So,

$$\text{cov}(X_i, X_j) = \begin{cases} -n p_i p_j & (\text{if } i \neq j) \\ n p_i (1 - p_i) & (\text{if } i = j) \end{cases}$$