Decoherence free subspaces for quantum communication in amplitude damping channels

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Any future realization of communication through quantum channels requires attention to transmission rates, which leads to the idea of packing qubits as close to each other as possible. This close packing of qubits may also be desirable for increased efficiency in quantum computation tasks. As the distance between the qubits decreases, they tend to decohere collectively to the same environment. This paper considers collective decoherence in amplitude damping channels modelling energy dissipation. Such channels evolve using a Hamiltonian containing creation and annihilation spin-angular momentum operators. We use *su(2)* algebra and *spinor spaces* formulation to derive irreducible representations of spin-$\frac{1}{2}$ particles. We then find subspaces resistant to collective decoherence; such subspaces show drastic improvements in fidelity of transmitted quantum states in simulation experiments. We further claim the requirement of single-shot quantum communication and propose basic constructions to amplify amplitudes of quantum states at the receiver's end. This achieves higher fidelity than the scheme relying solely on decoherence free subspaces.

I. INTRODUCTION

The biggest challenge in performing computations through quantum methods is the decay of coherence between qubits, causing irrevocable flow of information from the system to the environment[1,2]. Decoherence comes in many forms[3–5], almost all leading to loss in information carried by the qubits, under time evolution. Several quantum error correcting codes have been devised to combat decoherence[6,7]. But for small systems, quantum error correcting codes (QECCs) have high quantum resource overhead: the encoding space is much larger than the space of actual information being transmitted[8]. Such a large encoding space is, however, essential to QECCs because of the syndrome calculation that helps in correction[7,9,10]. Likewise, several quantum error preventing codes (QEPCs) have been developed upon Quantum Zeno effect[13,14] and are useful with quadratic noise[11,12]. QEPCs are also shown to make use of collectivity in decoherence[15–17], and are thus shown to be more effective than QECCs at correction of errors.

Several noise models have been constructed to model decoherence, and an important one among them is the amplitude damping channel[18–20]. Amplitude damping best describes the energy dissipation to the environment. It is a schematic model of the decay of an excited state of a (two-level) atom due to spontaneous emission of a photon. By detecting the emitted photon, i.e. *observing the environment*, measurements can be performed that gives us information about the initial preparation of the atom[7].

For an efficient quantum communication protocol to be designed in future, it is extremely important to consider transmission rates. Increasing transmission rates implies packing more qubits close to one another, leading to the qubits correlating to the same environment[21]. It is observed from the master equation of the amplitude damping channel (section II(B)) that the Hamiltonian controlling the system and causing loss of information contains both *creation* and *annihilation* operators, otherwise termed as ladder operators[22–24]. Such operators have a special treatment using $su(2)$ algebra that generates subspaces resistant to collective decoherence from the amplitude damping channel.

Decoherence free subspaces are of special interest because they do not require excess number of qubits to prevent loss of information. The resistance is, rather, built into the construction of the encoding itself due to symmetry. Decoherence free subspaces have gained traction as research has shown that they can be used as a means for universal quantum computation[25,26,28]. Decoherence free subspaces have been shown to protect fragile information from decoherence in a variety of scenarios[17,26,27] and to perform universal quantum gate computations without moving out of the subspaces initially encoded upon[25]. In such scenarios, decoherence free subspaces are superior to other QECC techniques which may take code-words outside the initial encoded space and procure a large overhead on fault tolerance checks[29,30].

In this paper, we explore the effects of an amplitude damping channel on the decoherence free subspaces constructed by virtue of the Hamiltonian of the channel itself. In section III, we provide a detailed introduction to the Spinor space of a spin $1/2$ particle which can be further used to construct aforementioned subspaces; we construct one such subspace. In section IV, we simulate an amplitude damping channel and observe the behaviour of such spaces. Finally, in Section IV, we move on to consideration of possible schemes for single shot communication. The simulation results are obtained as an average of several runs of the same experiment (minimum number of shots being 1024). This is not true, however, in designing schemes for practical communication. Any scheme or protocol looking to leverage quantum communication must provide high fidelity in a single shot. We attempt at such schemes, looking to expand on them as future research attempts.

II. COLLECTIVE DECOHERENCE IN AMPLITUDE DAMPING CHANNEL

A. Amplitude Damping Channel

The usual mathematics of quantum noise models is as a convex-linear map $\epsilon(\rho)$ that evolves a density operator representation of some input state to that of an output subspace. It is important to note that non-trivial convex transformations are unable to generate the convex points themselves. Considering a pure state p as input to such a channel, $\epsilon(p)$ transforms it to a mixed state. It is easy to note how such an operation over a period of time transforms a pure state into mixed state, thereby losing information.

Such a convex map can model energy dissipation to the environment in an amplitude damping channel. The Harmonic oscillators to model this can be given by the unitary operator[31]:

$$U = e^{-ix(ab^{\dagger}+a^{\dagger}b)}$$

where x is proportional to the coupling constant between the system and the environment. Here a and $a^{\dagger}$ are the creation and annihilation operators of the system, and b and $b^{\dagger}$ are the creation and annihilation operators of the environment.

It is difficult to mathematically manage the effects of the environment on the system. To avoid the details of the environment, the number of open system effects are given by the Kraus representation. From the point of view of quantum computation and information processing, such details of the environment are irrelevant and the Kraus representation keeps them away[32,33]. The Kraus representation $C_m = \langle m_b|U|0_b\rangle$ for a d-dimensional amplitude damping channel is given by:

$$C_m = \sum_{n=m}^{d-1} \sqrt{C(n,m)(1-p)^{n-m}p^m}|n-m\rangle\langle n|$$

for m = 0, 1, ..., d - 1. Where

$$tr(C_m^{\dagger}C_{m'}). = c_m\delta_{m,m'}$$

Considering the specialized case of a two-dimensional amplitude damping channel, evolution of a single qubit $((a|0\rangle_{atom} + b|1\rangle_{atom})|0\rangle_{env})$ occurs as:

$$a|0\rangle_{atom}|0\rangle_{env} + b\sqrt{1-p}|1\rangle_{atom}|0\rangle_{env} + b\sqrt{p}|0\rangle_{atom}|1\rangle_{env}$$

where p is the probability of losing the qubit to the environment (mathematically, probability of conversion from $|1\rangle$ to $|0\rangle$).

B. Collective Decoherence

It has been shown that the Born-Markov approximation holds in the case of amplitude damping channels[21,32]. We refer the reader to Duan *et. al.*[21] for a derivation of the master equation for the amplitude damping channel considering qubits as spin-$\frac{1}{2}$ particles. Under the Born-Markov approximation, the following condition is provided to hold:

$$d << \frac{v_0}{w_0}$$

where v_0 is the velocity of the noise field, w_0 is the usual angular frequency, and the entire fraction denotes the effective wave length of the noise field (under the Born-Markov approximation). From this follows the argument that qubits close to each other suffer collective decoherence since they couple to the same environment.

Under collective decoherence, the probability of a *jump*- transition from a $|1\rangle$ to a $|0\rangle$- is given by[21]:

$$P(t) = \langle\Psi(t)|cS_+S_-|\Psi(t)\rangle$$

where S_+ and S_- are the sum of creation and annihilation operators of all the qubits collectively decohering. The question we ask is: do there exist states which remain resistant to such decoherence, or whose probability of a *jump* is negligible? From $S_-|\Psi(t)\rangle$, it follows that such states exist in the Spinor space created as the carrier space of the irreducible representation of the *su(2)* algebra.

III. PROCEDURE TO CONSTRUCT DECOHERENCE FREE SUBSPACE REPRESENTATIONS

Consider a system of N qubits in the amplitude damping channel, all undergoing collective decoherence. In order to construct decoherence free subspaces, we need to consider the combination of N Spinor spaces.

In the following sections, the total angular momentum of a particle is represented as $J = L + S$, where L is the orbital angular momentum, and S is the spin angular momentum. Any subscript i to these operators represents angular momentum about ith axis.

A. Spinor space of a single spin-$\frac{1}{2}$ particle

We define the *Casimir Operator* J^2 such that $J^2 = J_x^2 + J_y^2 + J_z^2$ having only one eigenvalue and commuting with each of J_x, J_y, and J_z. Next are defined eigenvalue ladder operations such that $J_+ = J_x + iJ_y$ and $J_- = J_x - iJ_y$. The bases for the combined system are indexed as $|\lambda m\rangle$. The following relations hold:

$$J^2|\lambda\, m\rangle = \hbar\lambda|\lambda m\rangle$$

$$J_z|\lambda\, m\rangle = \hbar m|\lambda m\rangle$$

$$J_z(J_+|\lambda m\rangle) = \hbar(m+1)|\lambda\, m+1\rangle$$

$$J_z(J_-|\lambda m\rangle) = \hbar(m-1)|\lambda\, m-1\rangle$$

It is straightforward to see how the defined ladder operators transition between eigenvalues for J_z. Such integral *levels* of eigenvalues are *finite*, implying there exist m_{max} and m_{min} such that the following hold:

$$J_+|\lambda\, m_{max}\rangle = 0$$

$$J_-|\lambda\, m_{min}\rangle = 0$$

Or moving up from the highest possible level and moving down from the lowest possible level is undefined. If the orbital angular momentum is not defined for a certain particle (as is the case with qubits), then $J = S$. We then get

$$S_-|\lambda\, m_{min}\rangle = 0$$

It is immediately contrasted with $S^-|\Psi(t)\rangle$ in the probability relation for a *jump* discussed above. The lowest state in the Spinor space is thus the decoherence free state we are looking for.

B. Spinor space for tensor products

Consider the Spinor space of a single spin-$\frac{1}{2}$ particle having two levels- $+\frac{1}{2}$ and $-\frac{1}{2}$ as the **spin numbers**. Let the representation of such Spinor spaces be D^p where p is the maximum possible level. Note that $D^{\frac{1}{2}}$ is the irreducible representation of the concerned *su(2)* algebra (generators of the $SU(2)$ algebra which is concerned with unitary rotations) in case of a spin-$\frac{1}{2}$ particle.

In the case of two qubits for instance, we combine the Spinor spaces using the Clebsch-Gordan series:

$$D_1^{\frac{1}{2}} \otimes D_2^{\frac{1}{2}} = D^1 \bigoplus D^0$$

The tensor product has 4 states represented by $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$, where $|+\rangle$ represents the spin number $+\frac{1}{2}$ and $|-\rangle$ represents the spin number $-\frac{1}{2}$.

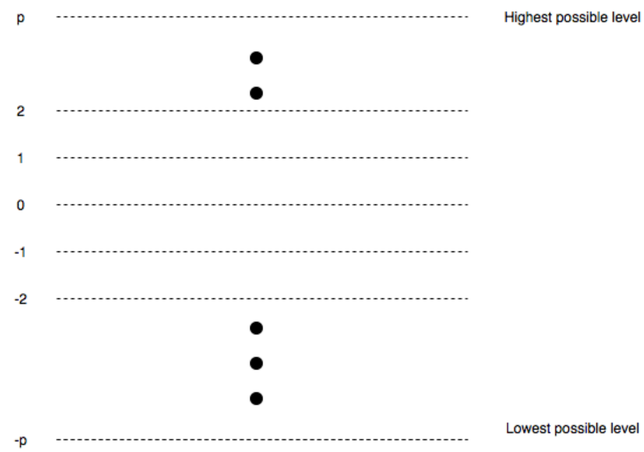

FIG. 1. Generalized Spinor space for $2p+1$ unique possible states. It is to be noted that the construction is always such that if the highest state is p, the lowest state will be $-p$. p is called the **spin number** of the particle.

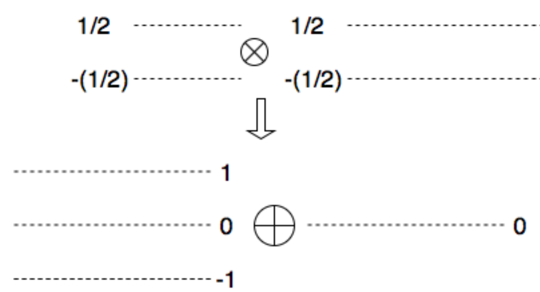

FIG. 2. Conversion of tensor product of Spinor space into direct sum of Spinor space. The case of a system of two spin-$\frac{1}{2}$ particles.

The direct sum also has four states represented by $|1\,1\rangle$, $|1\,0\rangle$, $|1\,-1\rangle$, and $|0\,0\rangle$, where $|1i\rangle$ denotes states in D^1 with $i \in \{-1, 0, 1\}$ and $|0\,0\rangle$ denotes the sole state of D^0. Using the operator S_- repeatedly on $|++\rangle = |11\rangle$ (the highest possible states in both tensor product space and direct sum space are equivalent) and noting that $|0\,0\rangle$ is orthogonal to all states of D^1, we get the following relations:

$$|1\,1\rangle = |++\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt(2)}\{|-+\rangle + |+-\rangle\}$$

$$|1\,-1\rangle = |--\rangle$$

$$|0\,0\rangle = \frac{1}{\sqrt(2)}\{|-+\rangle - |+-\rangle\}$$

From the discussion on Spinor space for a single spin-$\frac{1}{2}$ particle, it is evident that the lowermost states of subspaces D^1 and D^0 are the decoherence free subspaces desired.

IV. SIMULATION RESULTS

From the previous section, it is noted that in order to send one qubit, the encoding is needed in the subspace:

$$Span\{|--\rangle, \frac{1}{\sqrt(2)}\{|-+\rangle - |+-\rangle\}\}$$

Since $|+\rangle$ and $|-\rangle$ are simply representations of the spin of the particle, the above basis can be represented in terms of the standard notation:

$$Span\{|00\rangle, \frac{1}{\sqrt(2)}\{|01\rangle - |10\rangle\}\}$$

The simulation experiments are done for a given number of environment qubits parameterized by θ acting on the encoded quantum state, such that *each* environment qubit tends to individually damp the transmitted state.

A. Amplitude damping channel simulation

The amplitude damping channel simulation can be obtained as follows in Fig. 3[33]:

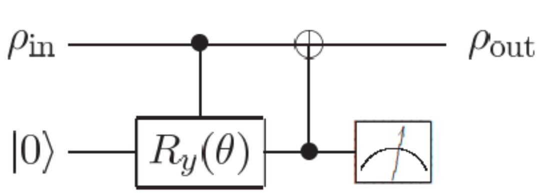

FIG. 3. Circuit model for amplitude damping. θ is the channel parameter signifying the extent of damping done by each environment qubit.

In the given figure, ρ_{in} is the quantum state being transmitted through the amplitude damping channel; the environment qubit is initially $|0\rangle$. Based on the contribution of $|1\rangle$ in the superposition $\rho_{in} = c_1|0\rangle + c_2|1\rangle$, the environment qubit gets rotated about the Y-axis. Considering $c_2 > c_1$, higher is the value of c_2, more is the probability of the environment qubit to flip to $|1\rangle$, and more is the probability of the CNOT being applied which essentially flips the initial quantum state to $\rho_{out} = c_2|0\rangle + c_1|1\rangle$. Since $c_2 > c_1$, it is noted how information about the initial state is lost. Exposure to the channel for an extended period may result in $c_2 >> c_1$ and a near-complete loss of information.

B. Controlled $R_y(\theta)$ simulation

IBM's quantum simulator has been used for the experiments reported here. It is noted that this simulator does not have an explicit defined $R_y(\theta)$ gate; an alternate construction is therefore proposed in Fig. 4.

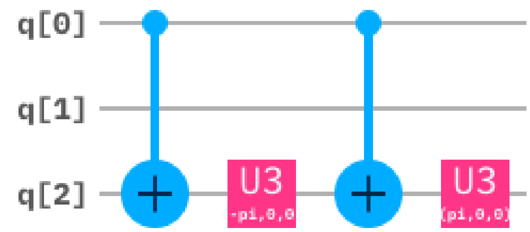

FIG. 4. Controlled R_y operation simulation

In the above construction, to perform $R_y(\theta)$, the circuit moves by considering $q[0]$ as the input quantum state and $q[2]$ as the environment qubit (initially $|0\rangle$). The two U3 gates are $U(-\frac{\theta}{2}, 0, 0)$ and $U(+\frac{\theta}{2}, 0, 0)$.

If the control is $|0\rangle$, the two U3 gates defined cancel each other. The environment qubit ends up having no R_y rotation, implying absence of damping for the control qubit through the model defined in previous subsection (Fig. 3).

If the control is $|1\rangle$, $q[2]$ flips to $|1\rangle$, followed by rotation by $-\frac{\theta}{2}$ about the Y-axis, again a flip, and again a rotation $+\frac{\theta}{2}$ about the Y-axis in theoretically the opposite direction from the first rotation. The difference here is the action of the CNOTs which cause the rotations to add up instead of cancelling.

For instance, consider $\theta = \pi$. The environment qubit flips to $|1\rangle$ because of the first CNOT. A $-\frac{\theta}{2}$ rotation would take the state to $\frac{|0\rangle+|1\rangle}{\sqrt(2)}$. The second CNOT flips this state to $\frac{|0\rangle-|1\rangle}{\sqrt(2)}$. Lastly, the $+\frac{\theta}{2}$ rotation rotates this state to $|1\rangle$, thus simulating the desired $\theta = \pi$ rotation.

C. Encoding circuit

The following circuitry is used to encode an arbitrary quantum state $|\psi\rangle = c_1|0\rangle + c_2|1\rangle$ into the decoherence free subspace $Span\{|00\rangle, \frac{1}{\sqrt(2)}\{|01\rangle - |10\rangle\}\}$

First CNOT operation:

$$|\psi\rangle = c_1|00\rangle + c_2|11\rangle$$

Controlled Hadamard operation:

$$|\psi\rangle = c_1|00\rangle + c_2\{\frac{|01\rangle - |11\rangle}{\sqrt(2)}\}$$

Second CNOT operation:

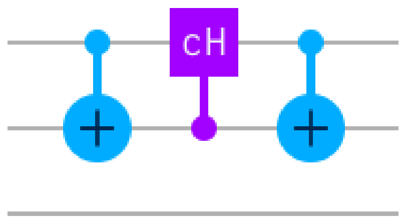

FIG. 5. Initial encoding circuit. The first qubit is the qubit to be communicated, $c_1|0\rangle + c_2|1\rangle$, while the second qubit is the *ancilla* to help encoding, initially $|0\rangle$. Initial composite system state before encoding: $|\psi\rangle = c_1|00\rangle + c_2|10\rangle$

$$|\psi\rangle = c_1|00\rangle + c_2\{\frac{|01\rangle - |10\rangle}{\sqrt(2)}\}$$

where $\psi \in Span\{|00\rangle, \frac{1}{\sqrt(2)}\{|01\rangle - |10\rangle\}\}$ is in the desired subspace.

D. Results

The following results are obtained from averaging results from several runs of the experiment, with each run for 1024 shots on IBM's quantum simulator. Baseline denotes the results from the same experiment but without any encoding. All experiments are done based on the assumption that a $|1\rangle$ is sent to the channel; since $|0\rangle$ is trivial and faces no energy dissipation due to the nature of the construction of the channel, as discussed earlier.

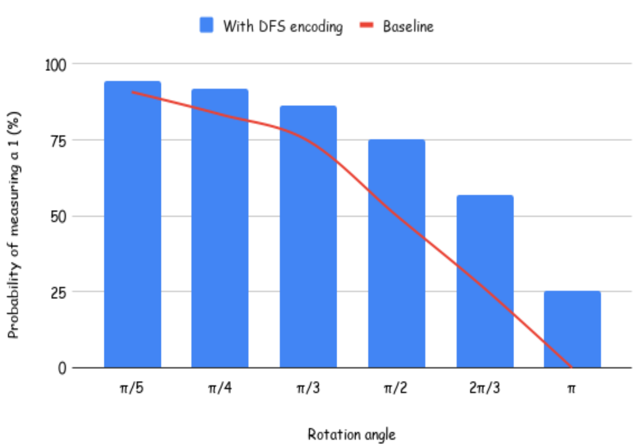

FIG. 6. Decoherence results in presence of a single environment qubit inside the amplitude damping channel.

The resistance of the decoherence free subspaces to damping by the amplitude damping channel is clearly observed here. Significant gains in fidelity over the baseline are obtained.

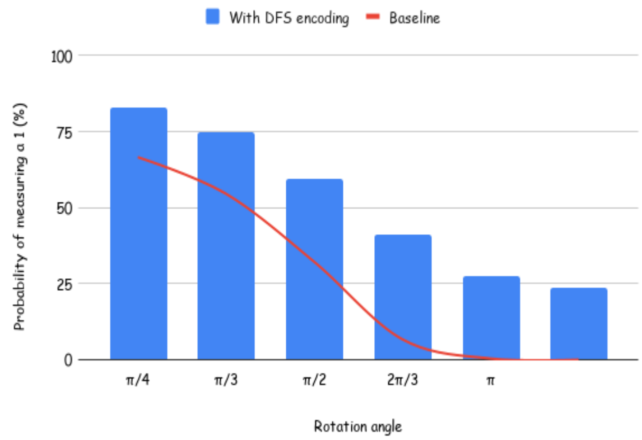

FIG. 7. Decoherence results in presence of five environment qubits inside the amplitude damping channel.

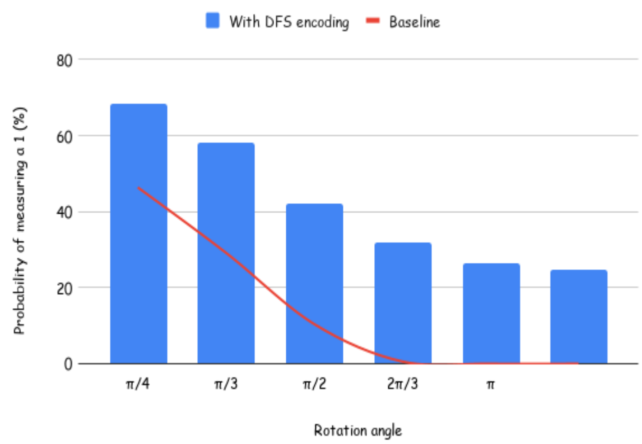

FIG. 8. Decoherence results in presence of ten environment qubits inside the amplitude damping channel.

V. SINGLE SHOT COMMUNICATION

The above results are based on averages over a large number of runs of the same experiment. This is untrue for any communication protocol to be realized. We present here a very basic scheme which can be a baseline for further investigation into schemes improving fidelity for such subspaces. There is always the choice to design a scheme consisting of QECCs, but we refrain from it due to the overhead of implementation.

For any arbitrary input quantum state, the received state can be expressed as $|\psi\rangle = c_1|0\rangle + c_2|1\rangle$ where the probability amplitudes satisfy one of the three cases: $c_1 >> c_2$, $c_1 > c_2$, or $c_1 < c_2$. The first case is when $|0\rangle$ is transmitted through the channel, such that the probability of measuring a $|1\rangle$ is negligible. The second and the third cases occur when $|1\rangle$ is transmitted and energy dissipation occurs: low to moderate dissipation leads to case 3 while high dissipation leads to case 2.

In an experiment with 1024 shots, it is straight-forward to differentiate between cases 1 and 2. In single shot communication, however, differentiation between receiving a $|0\rangle$ and receiving a $|1\rangle$ under high amplitude damping is impossible, since measurement in both the cases is likely to yield a $|0\rangle$. This leads us to the following modification.

A. Amplitude amplification

We claim that there exist rotations based on channel parameters which can amplify the measurement probability of $|1\rangle$ whenever $|1\rangle$ is transmitted, without affecting measurement probability of $|0\rangle$ in case of a $|0\rangle$ transmission. Our observation is based on the behaviour of the ancilla qubit used in encoding, after the received state has been decoded at the receiver end. If $|0\rangle$ is sent, the channel leaves the initially $|0\rangle$ ancilla unchanged. On the other hand, when a $|1\rangle$ is sent, the channel acts such that the ancilla has a higher contribution of $|1\rangle$ in the superposition it exists in at the receiver end.

We propose the following construction to amplify the contribution of $|1\rangle$ whenever $|1\rangle$ is transmitted. We note this construction leaves $|0\rangle$ unchanged up to reasonable levels of tolerance (i.e. $c_1 >> c_2$ instead of the earlier $c_1 \approx 1$). The circuit construction is as follows in Fig. 9.

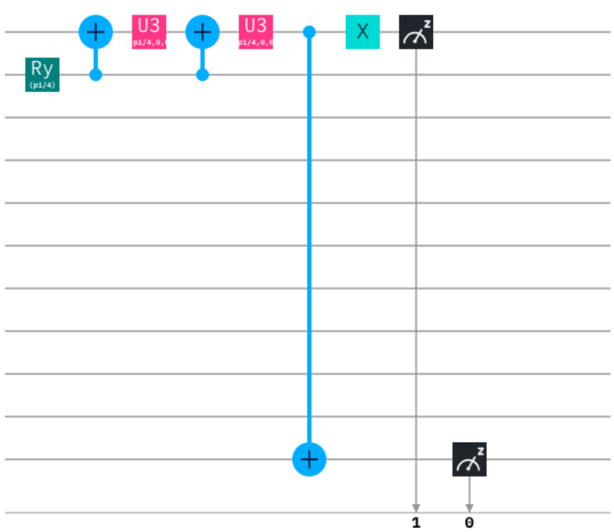

FIG. 9. The circuit construction for amplitude amplification

The first two qubits are the ones used in encoding/decoding. The middle empty qubit lines represent the environment qubits (not of our concern at the receiver end). The bottom-most qubit receiving the CNOT is a special *receiver ancilla* that allows for further tests.

The first R_y gate serves to amplify amplitude of the ancilla qubit used earlier for encoding/decoding. Based on this amplified qubit, controlled rotation about the Y-axis is performed on the first qubit.

The last CNOT on the *receiver ancilla* and the X gate serve to perform further tests to ensure integrity of the state received. On measuring the information qubit and the *receiver ancilla*, if $|10\rangle$ is observed, the observer can be sure the intended information qubit was $|0\rangle$. On the other hand, measuring a $|01\rangle$ implies the information qubit was $|1\rangle$. This construction works because the information qubit and the *receiver ancilla* are not entangled, so they provide independent information about the received quantum system. Such a scheme is also useful in discarding erroneous receptions- $|00\rangle$ or $|11\rangle$, since we are sure the overhead of X and CNOT gates is less enough not to drastically affect the received quantum state.

It is next noted that the initially encoded state at the encoding step $\psi \in Span\{|00\rangle, \frac{1}{\sqrt{(2)}}\{|01\rangle - |10\rangle\}\}$ is an entanglement consisting of two qubits, both of which are *de*-entangled at the receiver end. This *de-entanglement* allows for performing the mentioned operations on the ancilla (second qubit in the construction proposed in this section) without affecting the state of the first qubit (our initial information qubit).

B. Simulation Results

Amplitude amplification scheme as proposed here is contrasted with the results obtained for the earlier decoherence free subspaces experiment in presence of ten environment qubits.

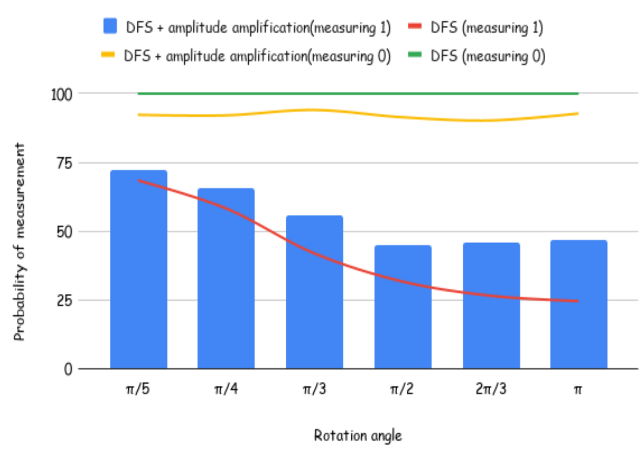

FIG. 10. Simulation results contrasting amplitude amplification enabled DFS and normal DFS, for ten environment qubits. It is noted that by suffering tolerable loss in measurement of $|0\rangle$ in case of $|0\rangle$ transmission, we gain significant improvement in measurement of $|1\rangle$ in case of $|1\rangle$ transmission.

It is to be noted that the rotation angle θ applied on the *ancilla* qubit before controlled R_y on the information qubit needs to be carefully evaluated based on channel properties and intensity of damping present. A large rotation angle tends to overshoot the already high amplitudes (in case of less damping, for instance, 5 qubits and low damping angles), thereby reducing overall measurement probability.

VI. CONCLUSION

In this paper, we studied a situation where qubits will have to be transmitted while being close enough to decohere collectively with the environment. We noted that collective decoherence allows the quantum system to evolve using a Hamiltonian that is controlled by creation and annihilation operators related to spin angular momentum of a spin-$\frac{1}{2}$ particle, and that there exist irreducible representations of such operators. We derived decoherence free states in such irreducible representations that are resistant to actions of the creation and annihilation operators, and claimed that a superposition of any arbitrary quantum state in the span of such decoherence free states will be resistant to collective decoherence. Simulation results seconded this claim. We also attempted schemes for greater fidelity, which is prime requirement for single shot communication. Simulation results seconded the construction of such schemes.

There is one failure point, in the knowledge of the authors, where this construction fails- when the ancilla qubit used in encoding also couples to the same environment as the information qubit. This is however easy to overcome by noting the encoding procedure transforms any arbitrary quantum state to $Span\{|00\rangle, \frac{1}{\sqrt{(2)}}\{|01\rangle - |10\rangle\}\}$, which is in itself an entanglement. Collective decoherence can thus be avoided by increasing physical distance between these two qubits such that the condition- $d << \frac{v_0}{w_0}$- discussed above fails.

Future research prospects include considering other decoherence channels/sources and trying to find common states which can resist decoherence from each of such channels. The authors believe this will take us closer to error-reduced quantum communication; and once we are able to harness quantum error correction in combination with such decoherence free subspaces, higher fidelity of transmissions will be achieved.

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