Software Implementation of (Post-Quantum) Public-Key Cryptography

Curve-Based

Patrick Longa

Microsoft Research

SPACE 2020

Virtual Conference, December 2020

Outline

- Preliminaries: public-key cryptography
- (Post-quantum) public-key cryptography
  - Motivation: the quantum menace
- Implementation aspects of PKC
  - Preliminaries: CISC vs RISC, x64 ISA
  - Side-channel attacks: timing and cache attacks
  - Computer representation
  - Computer arithmetic (cases with ECC and SIKE)
  - Modular arithmetic
- The case of ECC
- The case of SIKE

Cryptography in use today

Symmetric-key cryptography

Public-key cryptography

Cryptography in use today

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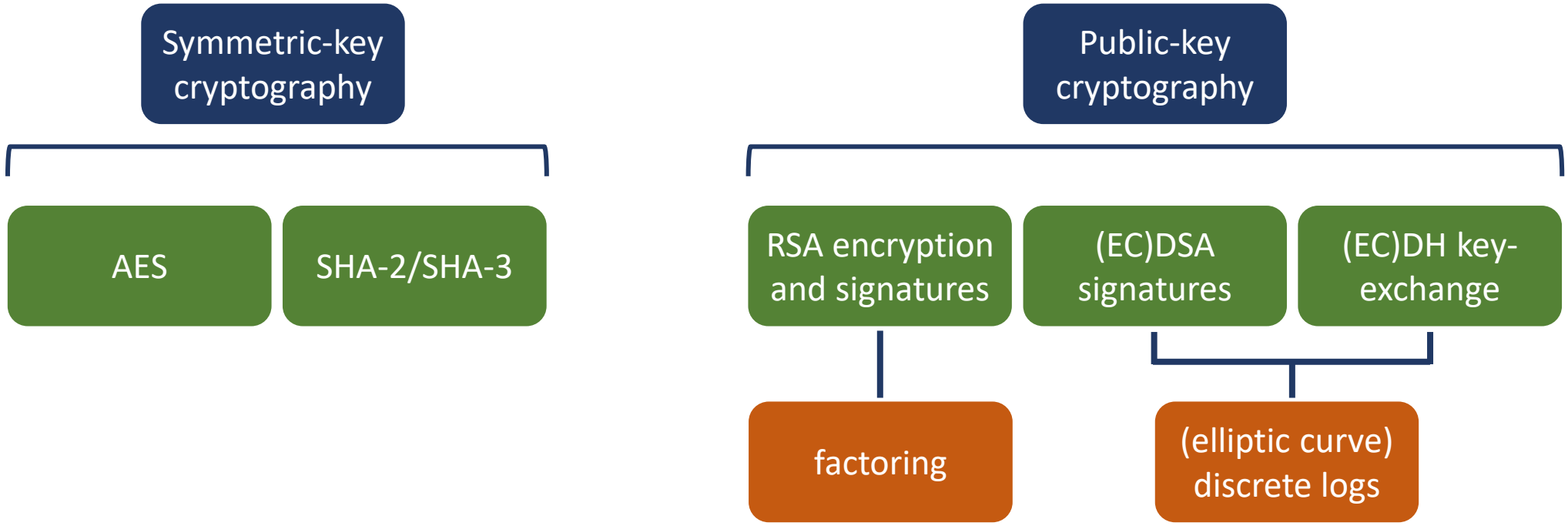

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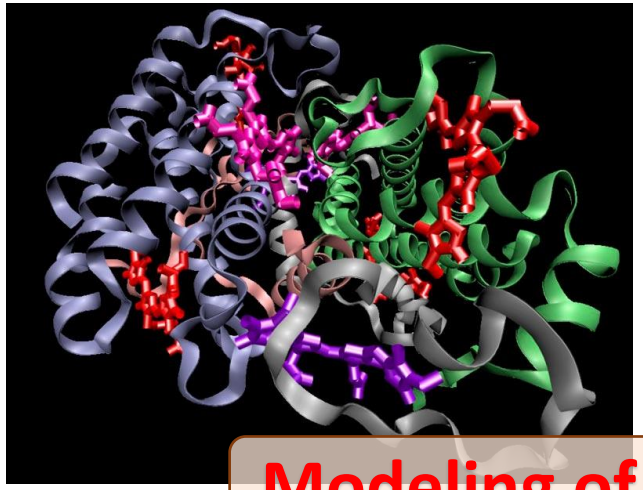
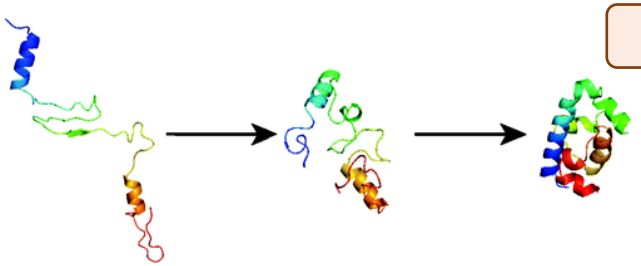
Widely dominated by number theoretic cryptography.



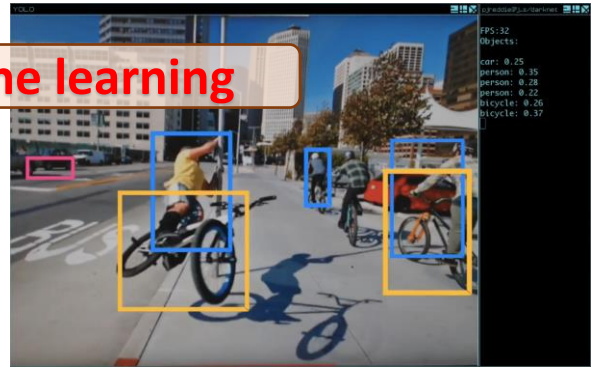
Motivation: the quantum menace

Quantum computing

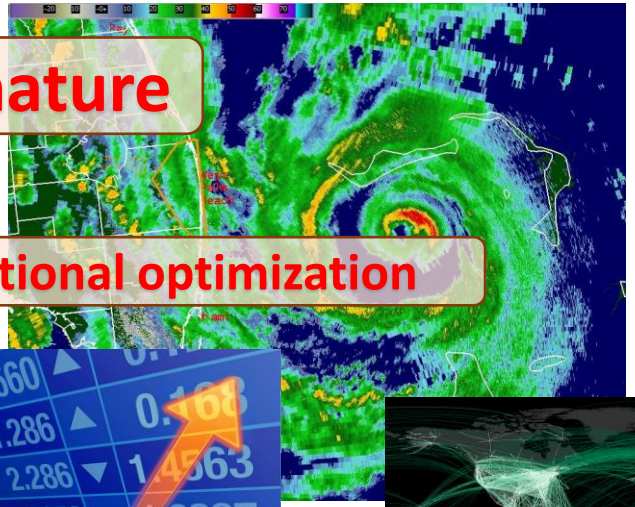




Machine learning

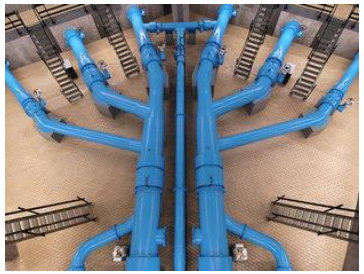


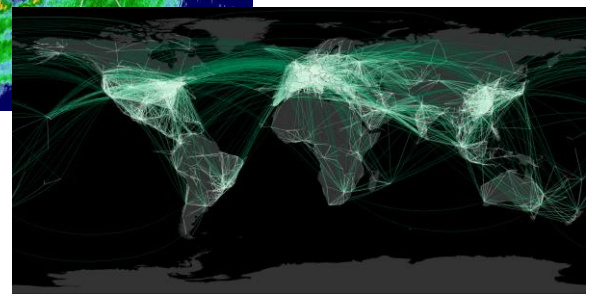
Modeling of nature



Computational optimization

Database search





Quantum computing

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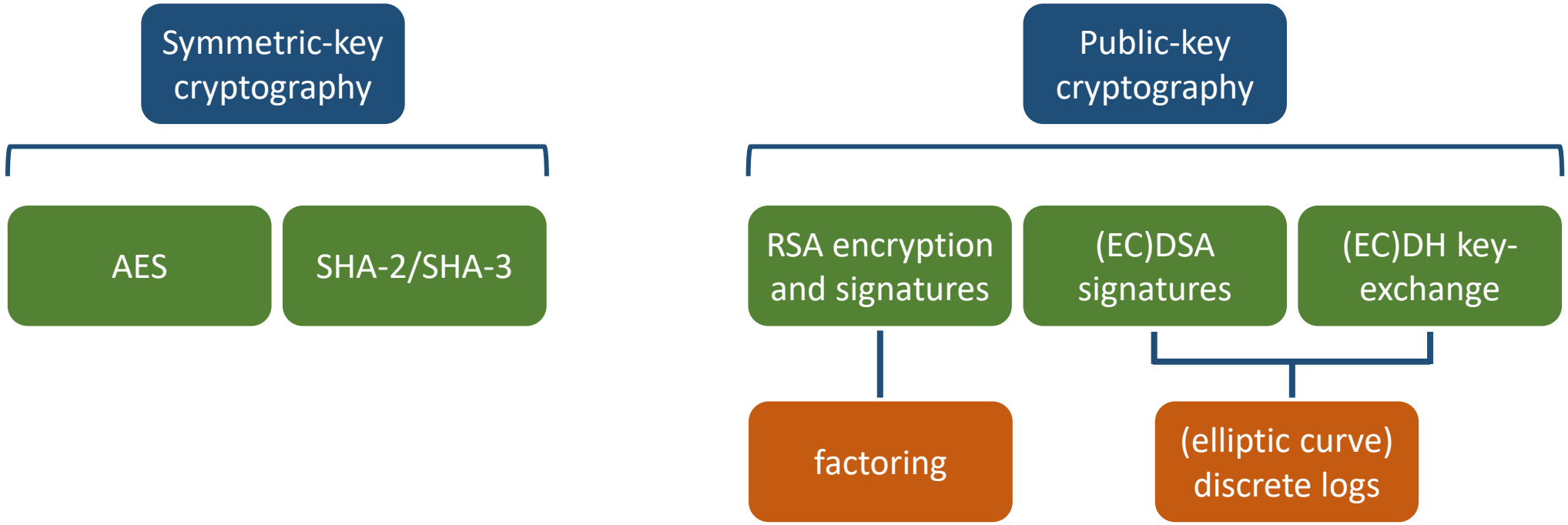

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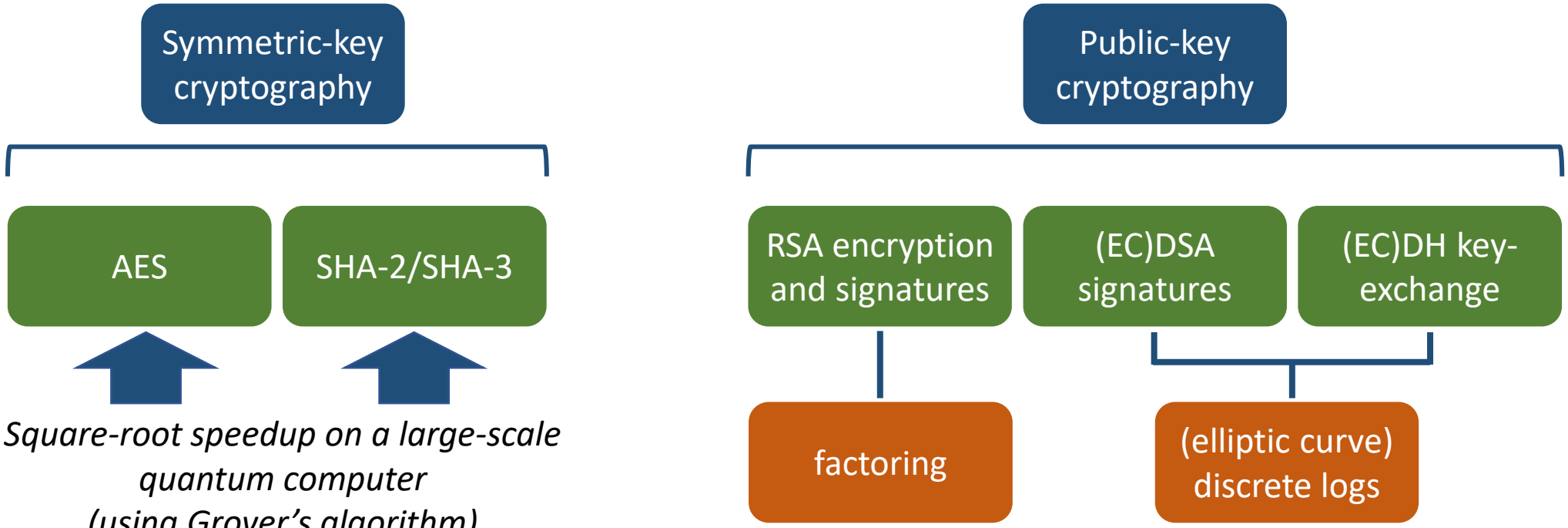

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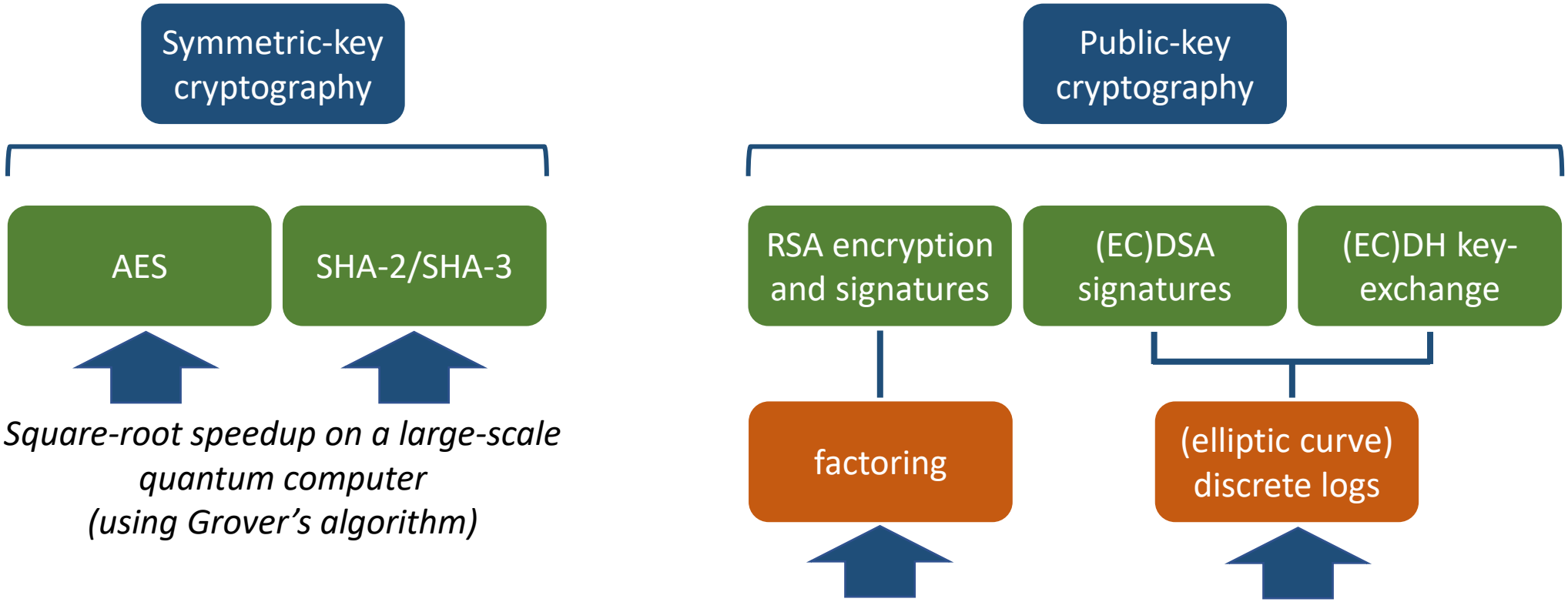
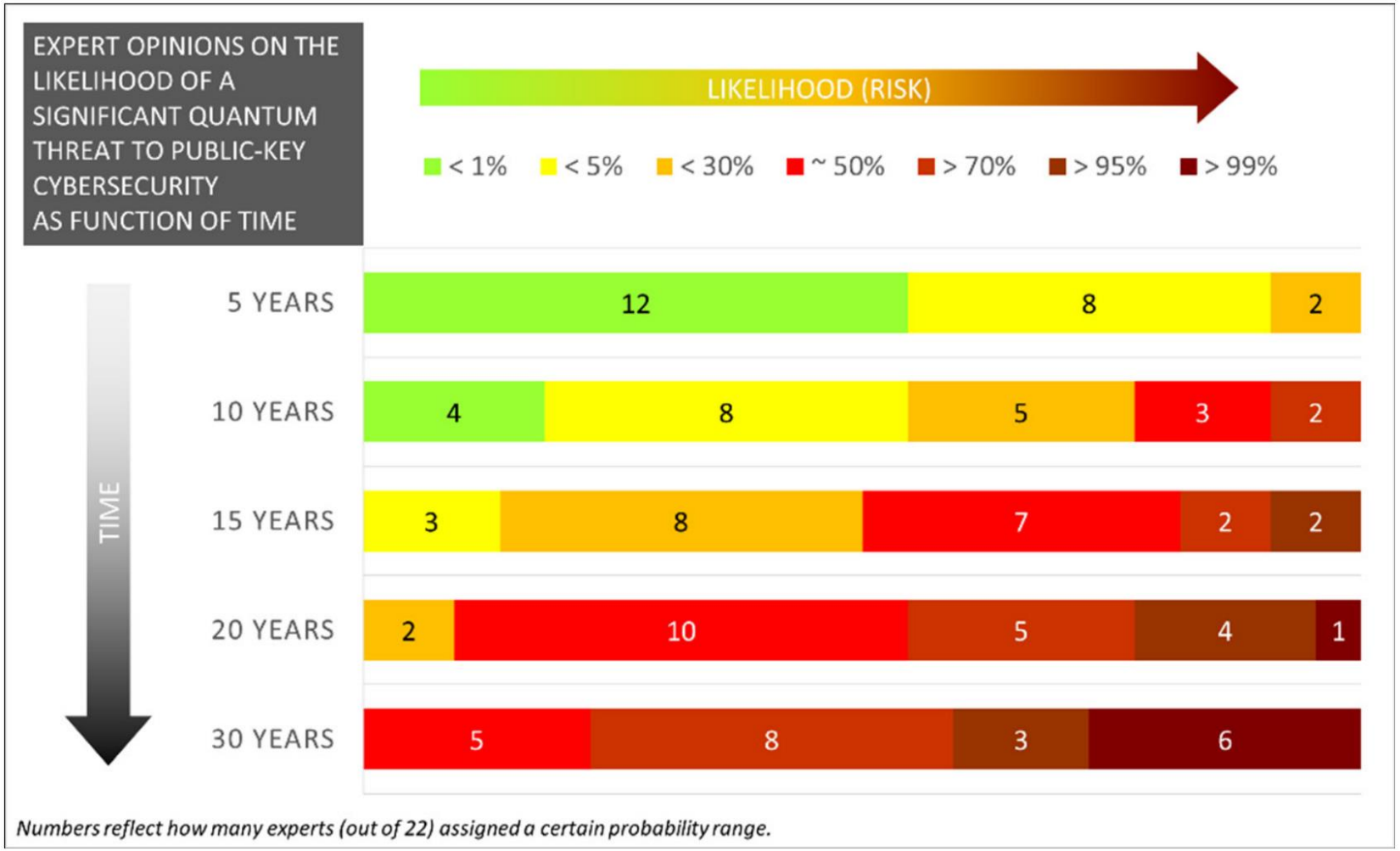


When will a large-scale, fault-tolerant quantum computer be built?

Expert opinions on the likelihood of the quantum threat to current public-key cryptosystems



(*) M. Mosca, M. Piani, “Quantum Threat Timeline”, Global Risk Institute (2019)

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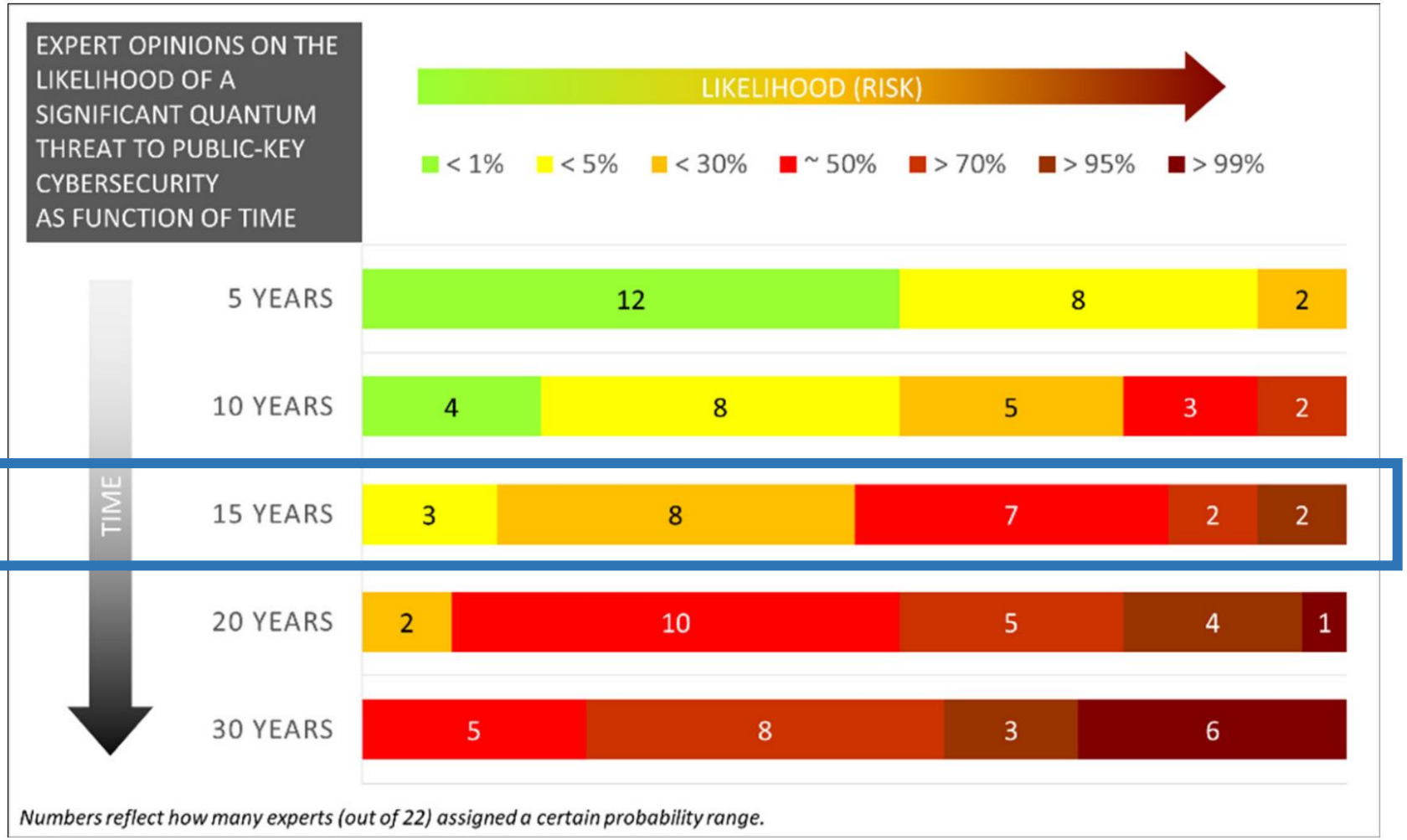

50% respondents say chances are $\geq 50\%$

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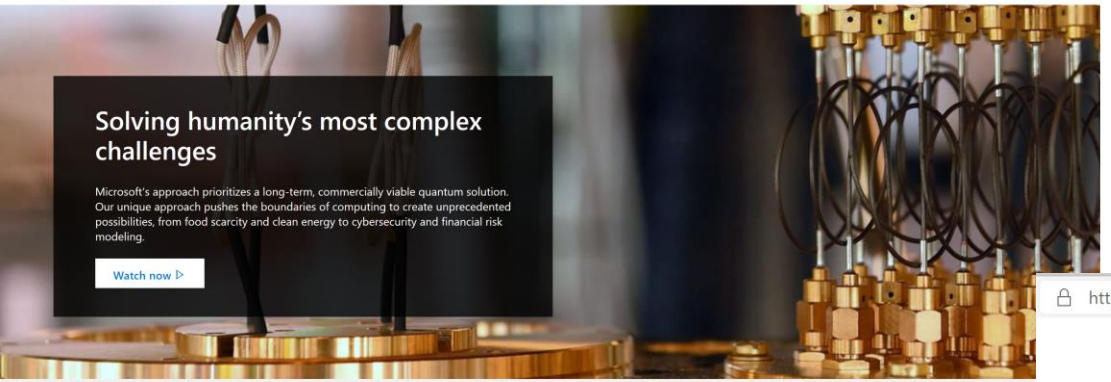
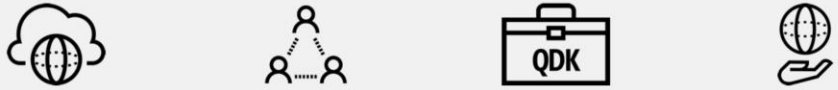
https://www.microsoft.com/en-us/quantum

Microsoft | Quantum Vision Development kit Quantum network Resources All Microsoft Search

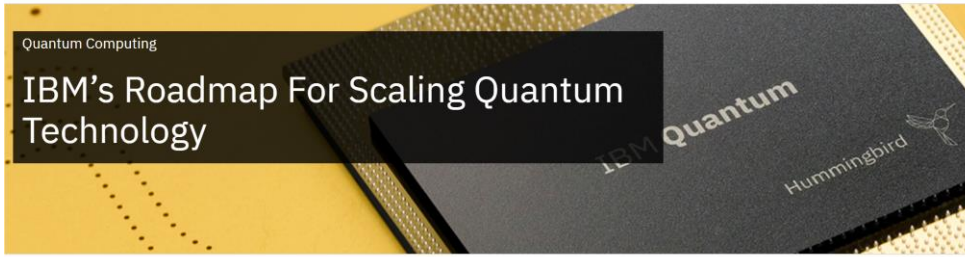


A scalable, open approach to quantum computing



https://www.ibm.com/blogs/research/2020/09/ibm-quantum-roadmap/

IBM Research Blog Topics Labs About



September 15, 2020 | Written by: Jay Gambetta

Categorized: Quantum Computing

Back in 1969, humans overcame unprecedented technological hurdles to make history: we put two of our own on the Moon and returned them safely. Today's computers are capable, but assuredly earthbound when it comes to accurately capturing the finest details of our universe. Building a device that truly captures the behavior of atoms—and can harness these behaviors to solve some of the most challenging problems of our time—might seem impossible if you limit your thinking to the computational world you know. But like the Moon landing, we have an ultimate objective to access a realm beyond what's possible on classical computers: we want to build a large-scale quantum computer. The future's quantum computer will pick up the slack where classical computers falter, controlling the behavior of atoms in order to run revolutionary applications across industries, generating world-changing materials or transforming the way we do business.

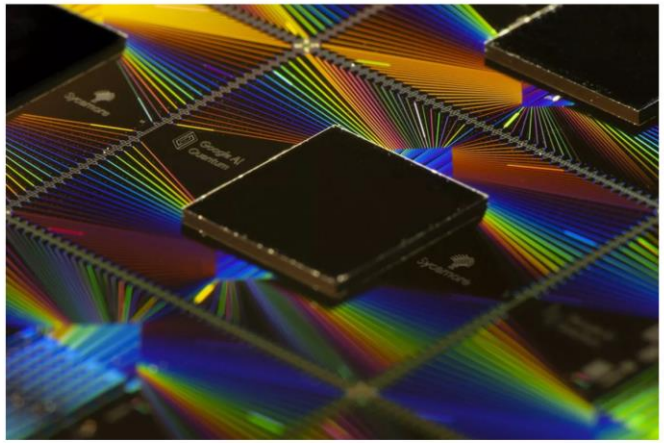
Today, we are releasing the roadmap that we think will take us from the noisy, small-scale devices of today to the million-plus qubit

https://www.theverge.com/2019/10/23/20928294/google-quantum-supremacy-sycamore-computer-c

Google confirms 'quantum supremacy' breakthrough

Its research paper is now available to read in its entirety

By Jon Porter | @JonPorty | Oct 23, 2019, 6:31am EDT



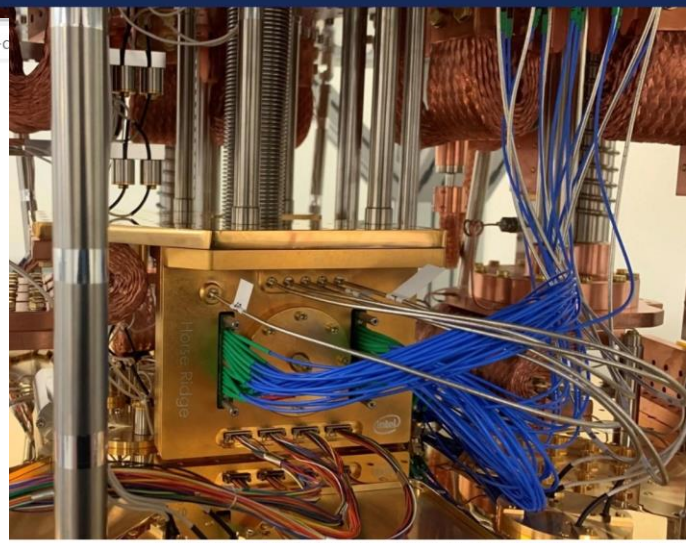
Google's Sycamore quantum processor, which was behind the breakthrough. | Credit: Google

Google has officially announced that it's achieved quantum supremacy in a new article published in the scientific journal *Nature*. The announcement comes exactly one month after it initially leaked, when Google's paper was accidentally published early. Now, however, it's official, meaning the full details of the research are public, and the broader scientific community can fully scrutinize what Google says it's achieved.

https://venturebeat.com/2020/12/03/intel-unveils-its-second-generation-quantum-computing-control-chip/

Intel unveils its second-generation quantum computing control chip

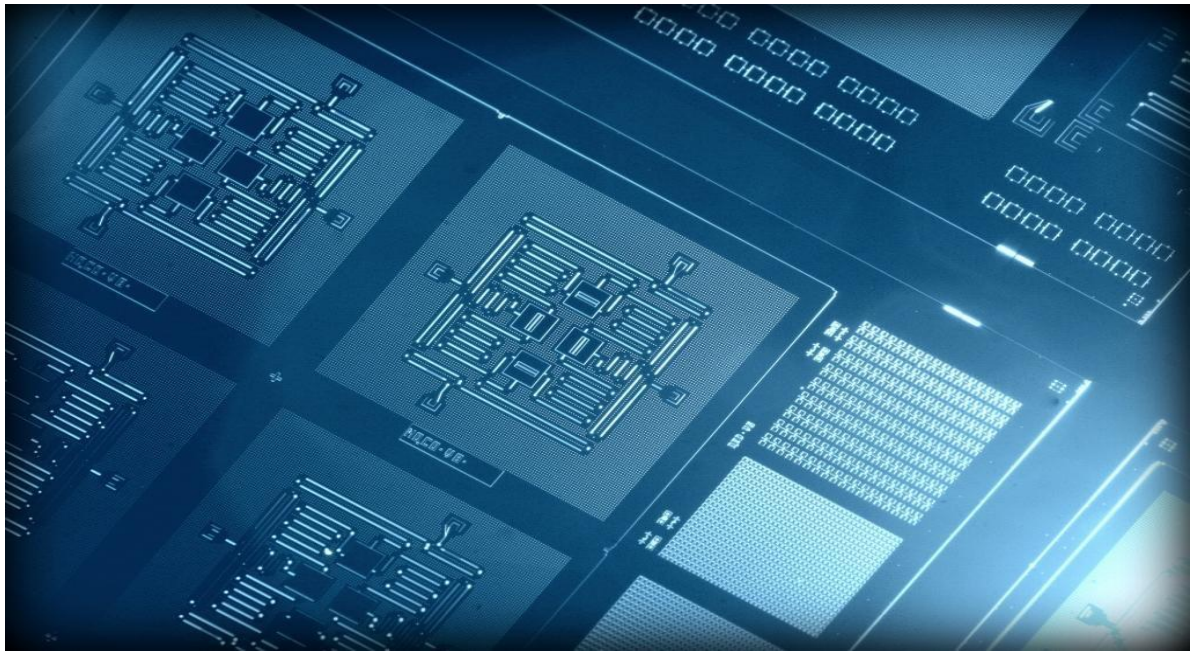
Dean Takahashi @deantak December 3, 2020 12:00 PM **Entrepreneur**



Intel's Horse Ridge II quantum computing chip is hiding under all that refrigeration, packaging, and wiring. Image Credit: Int

Intel unveiled its second-generation quantum computing control chip during its Intel Labs virtual event today. The chip, codenamed Horse Ridge II, is another milestone toward making quantum computing — one of the holy grails of computing — more practical. The new prototype builds on the first-generation Horse Ridge controller introduced in 2019. Horse Ridge II has more capability and higher levels of integration to control a quantum computer, which remains a long-term goal for the company.

Cryptopocalypse and the NIST response



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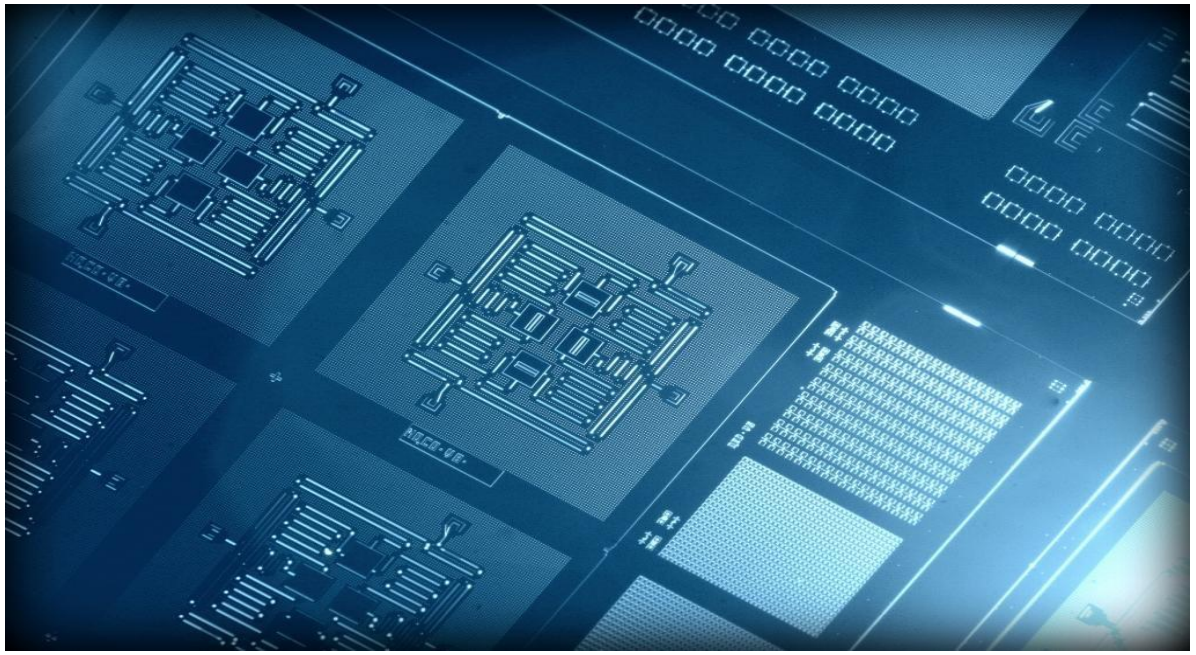

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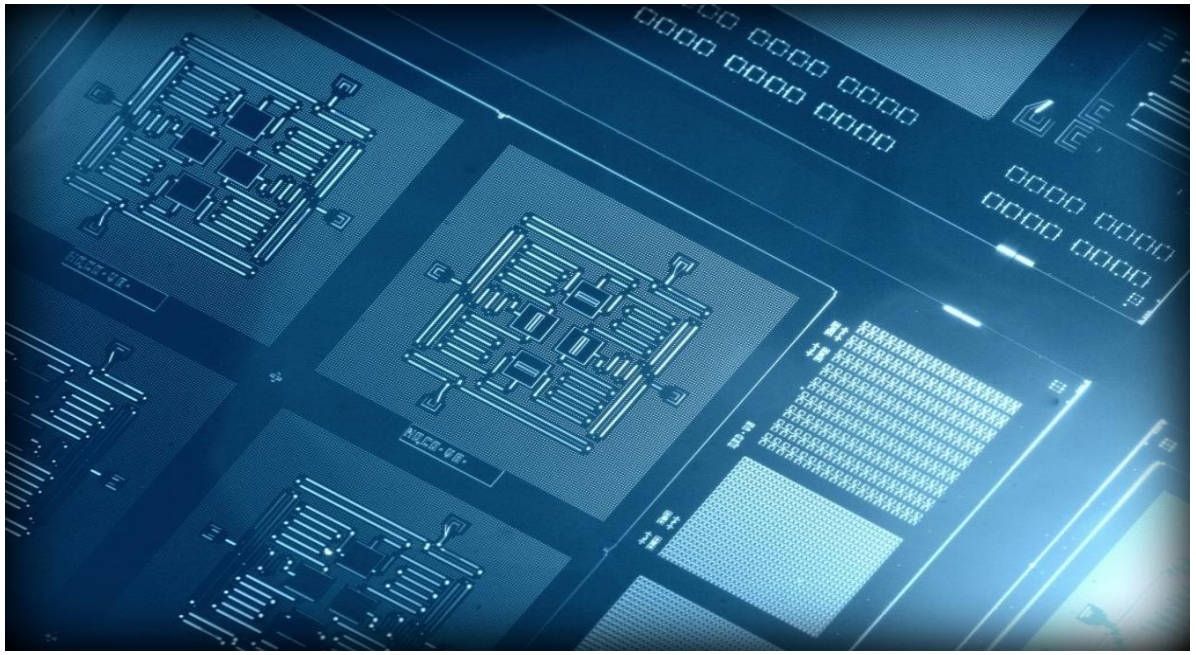

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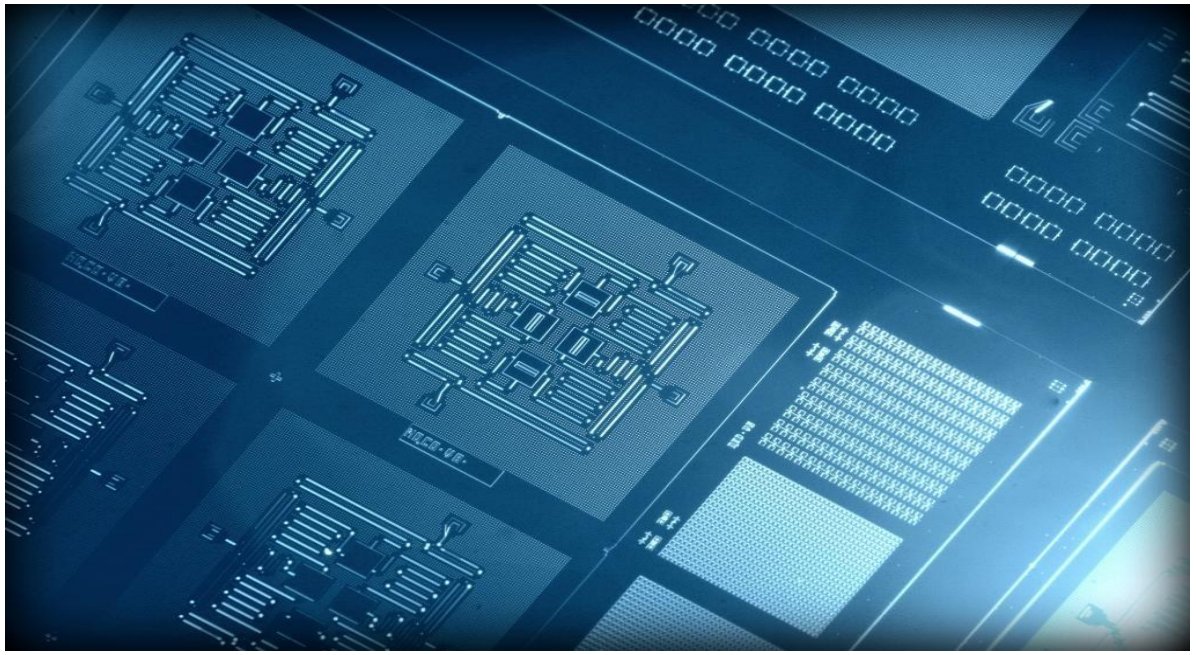

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NIST: "The goal of this process is to select a number of acceptable candidate cryptosystems for standardization."
(This includes key encapsulation, encryption and digital signatures).

What do we need to protect *now*?

Assuming a *large-scale, fault tolerant* quantum computer will be developed in, say, 15 years:

- Attacker records encrypted data *today*…

  … uses quantum computer to access secret data *in 15 years* from now.

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Need quantum-resistant key agreement and encryption for long-term security

Post-quantum (KEM) families

Code-based

Lattice-based

Isogeny-based

Post-quantum (KEM) families

Code-based	**Classic McEliece, BIKE, HQC**
Lattice-based	**NTRU, FrodoKEM (LWE), Kyber (M-LWE), Saber (M-LWR)**
Isogeny-based	**SIKE**

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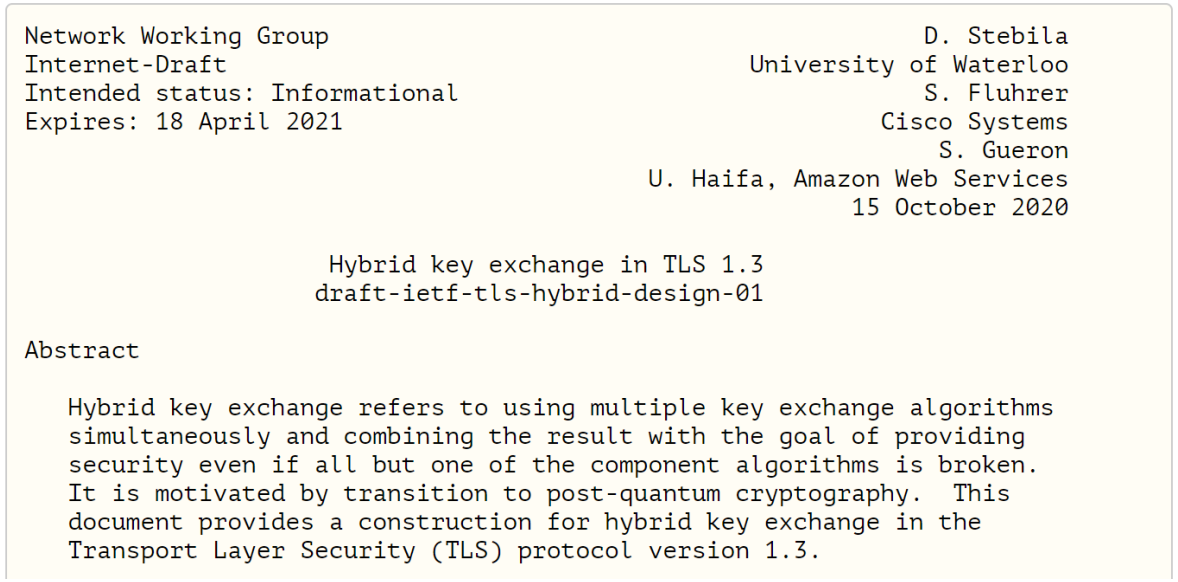
- Post-quantum crypto is still maturing
  - Our confidence on its security and efficiency is growing rapidly but there is still a gap
- The current recommendation is to use a hybrid approach
  - Combine classical schemes (e.g., ECC) with post-quantum (e.g., SIKE)



Network Working Group — D. Stebila
Internet-Draft — University of Waterloo
Intended status: Informational — S. Fluhrer
Expires: 18 April 2021 — Cisco Systems
S. Gueron
U. Haifa, Amazon Web Services
15 October 2020

Hybrid key exchange in TLS 1.3
draft-ietf-tls-hybrid-design-01

Abstract

Hybrid key exchange refers to using multiple key exchange algorithms simultaneously and combining the result with the goal of providing security even if all but one of the component algorithms is broken. It is motivated by transition to post-quantum cryptography. This document provides a construction for hybrid key exchange in the Transport Layer Security (TLS) protocol version 1.3.

PKC: implementation aspects

Preliminaries

Computer architectures: CISC versus RISC

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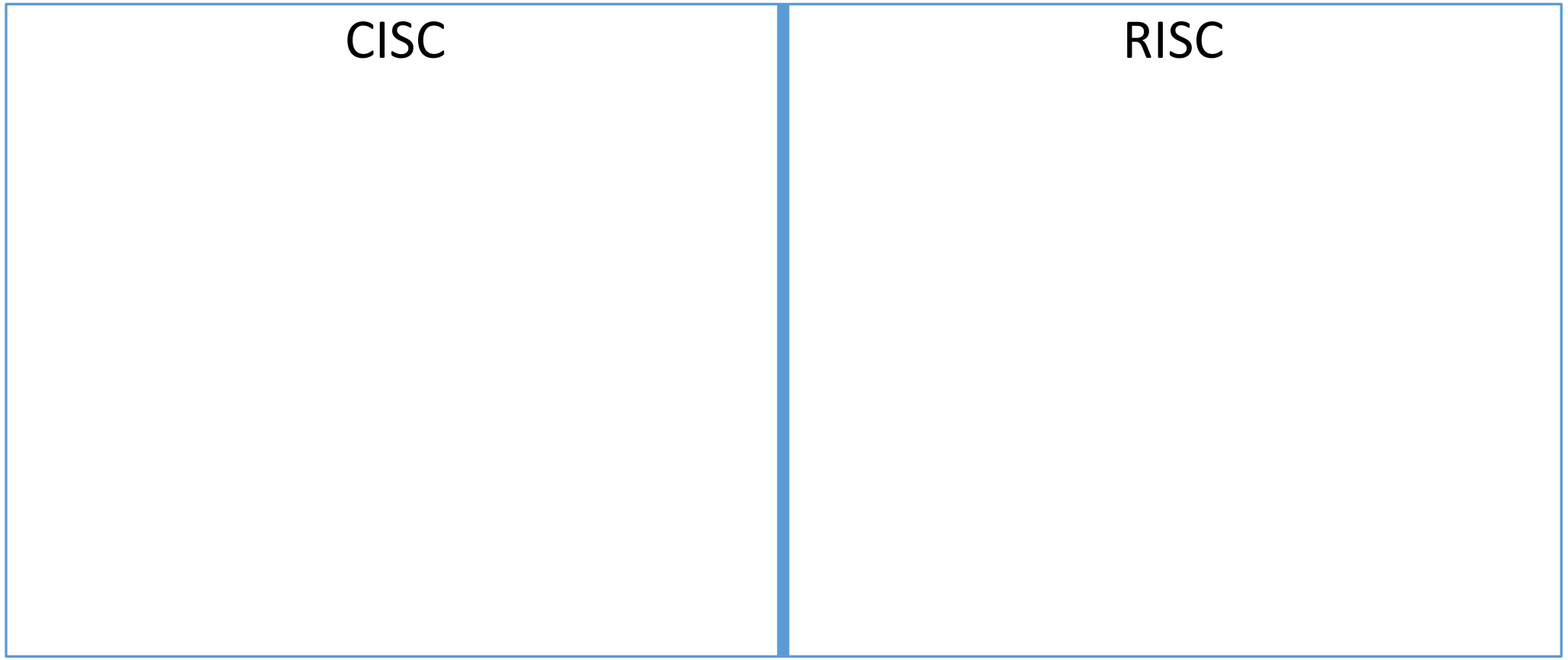
Before …



Computer architectures: CISC versus RISC

Before …

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• **Example:** x64 processors	• **Example:** ARM processors

Computer architectures: CISC versus RISC

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CISC*ish*	RISC*ish*
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- Software implementations can still take advantage of coarse-grained CISC/RISC differences, but **without losing track of specific architectural characteristics on a CPU-by-CPU basis**

x64 Instruction Set Architecture

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- Sixteen 64-bit general purpose registers (`rax`, `rbx`, `rcx`, `rdx`, `rbp`, `rsp`, `rsi`, `rdi`, `r8-r15`)
- Arithmetic, logical, memory-to-register, and register-to-memory instructions operate over 64-bit integers
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- Some relevant instructions:

  `add`, `adc`, `sub`, `sbb`, `shl`, `shr`, `shld`, `shrd`, `btr`, `bts`, `xor`, `and`

  `mov`, `push`, `pop`

  `mul` $(128 \leftarrow 64 \times 64)$

  `mulx` $(128 \leftarrow 64 \times 64)$, does not affect carry flag

PKC: implementation aspects

Side-channel attacks

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  - This blocks timing and cache attacks
- (Depending on application) software implementations on small devices should also include protection against other attacks (e.g., power and EM attacks)
- We will mainly discuss about the **first line of defense** here

Timing and cache attacks

There are *two* main sources of timing leakage in a computer program:

- Conditional statements (e.g., in if-then-else, do-while, etc.)
- Memory accesses through table lookups

Represent a risk if **execution conditions** or **table lookup indices** are secret-dependent.

Timing and cache attacks

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```
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```

- Even when instr(`do_this`) ≡ instr(`do_that`), it can hold that time(`do_this`) ≠ time(`do_that`)
- This timing difference is mainly due to the **branch predictor**

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Example: assume that `condition=1` (TRUE) or `0` (FALSE)

```
mask ← condition-1     (if condition=1 then mask is all 0s, else mask is all 1s)
xx ← do_this
yy ← do_that
x ← (mask & (x XOR xx)) XOR xx
y ← (mask & (y XOR yy)) XOR y
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- In some cases, constant-time version can be *faster* than if-then-else version
  - Branch mispredictions can be expensive, especially in architectures with deep pipelines

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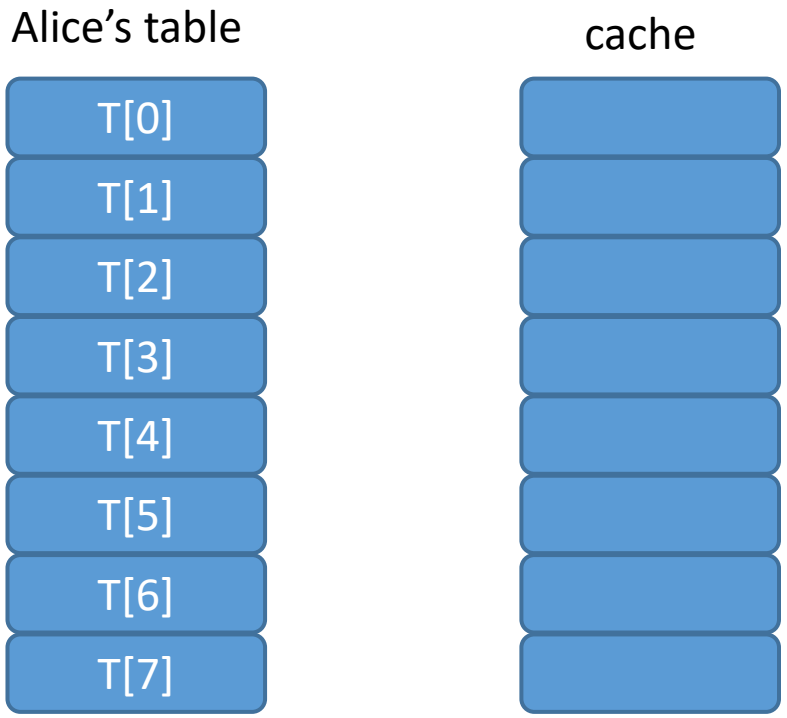

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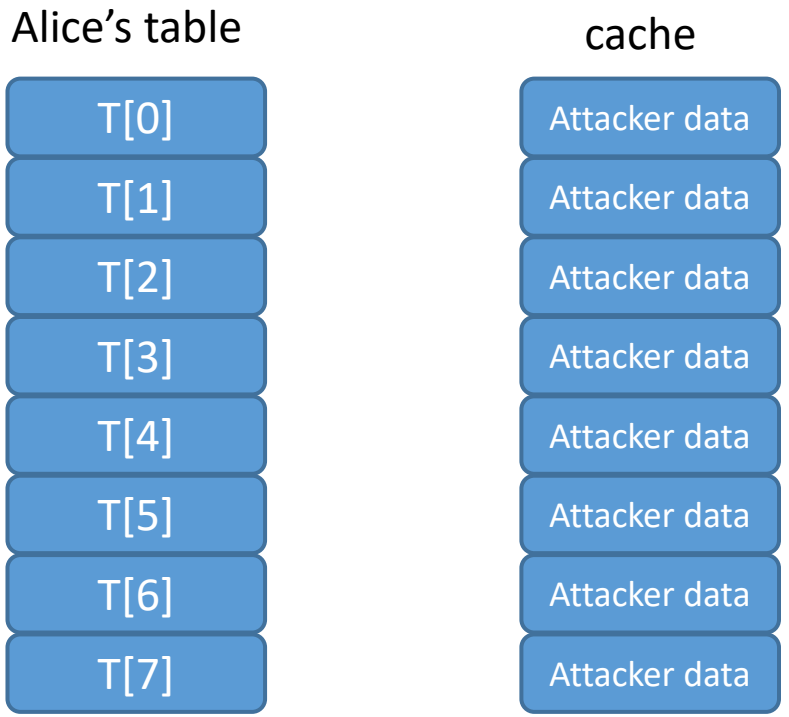


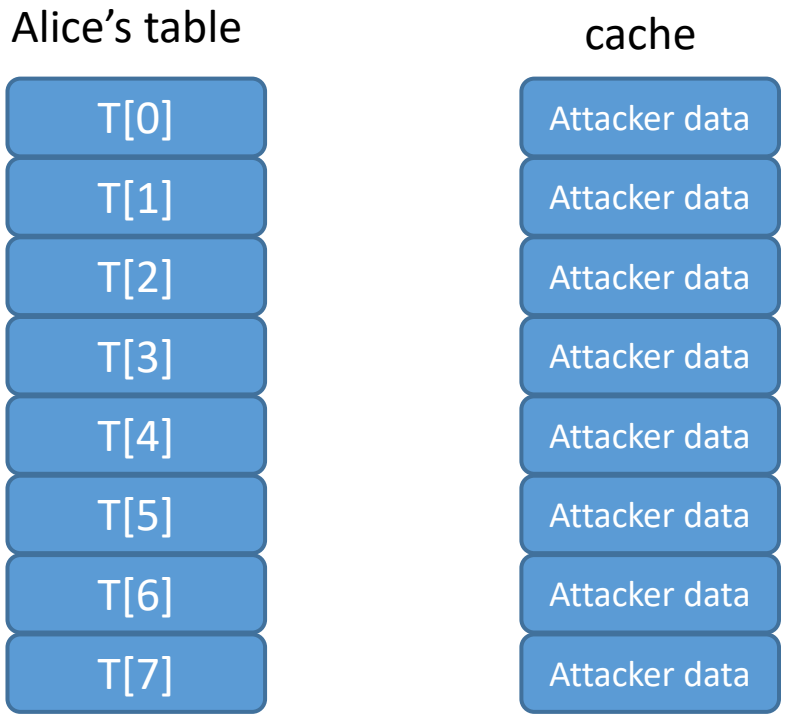
Attacker:

- Flushes and then populates cache memory

Timing and cache attacks

- Memory accesses through table lookups

- Alice needs first entry T[0]. She retrieves it



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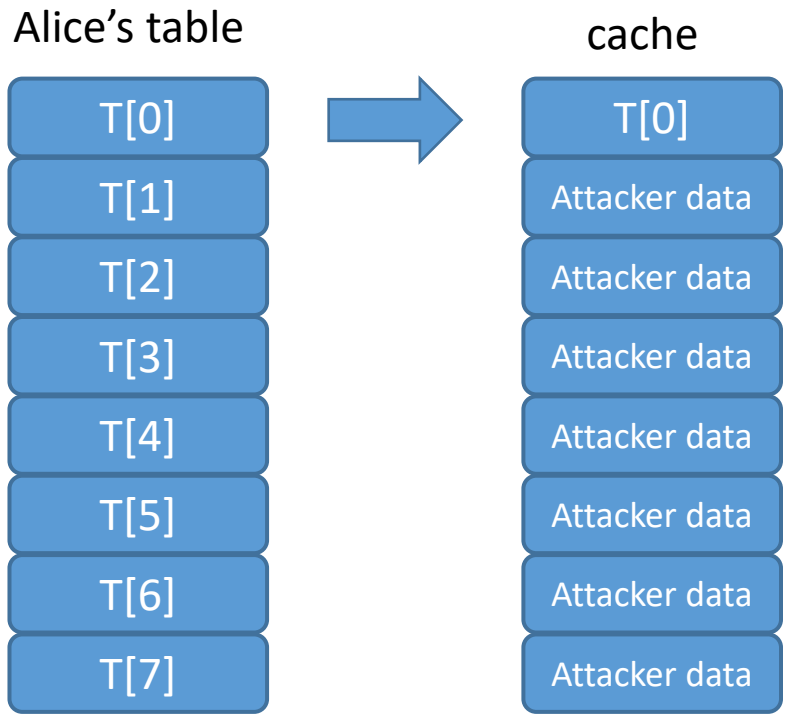

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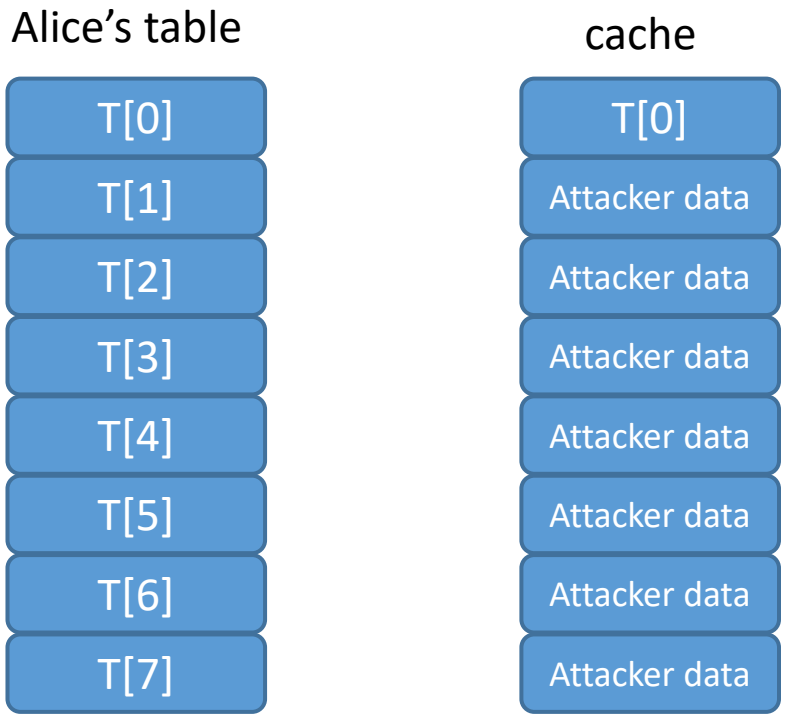

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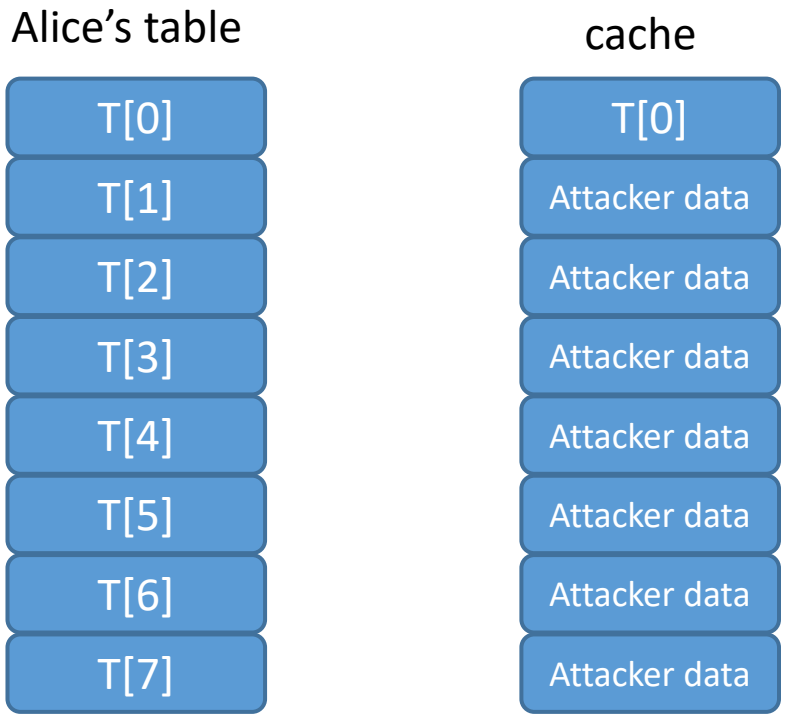

Attacker:
- Flushes and then populates cache memory
- Makes pass over data

Reading first entry takes longer ➔ learns that Alice retrieved T[0]

Timing and cache attacks

- **Countermeasure:** do linear pass over the full table and retrieve value through masking

Timing and cache attacks

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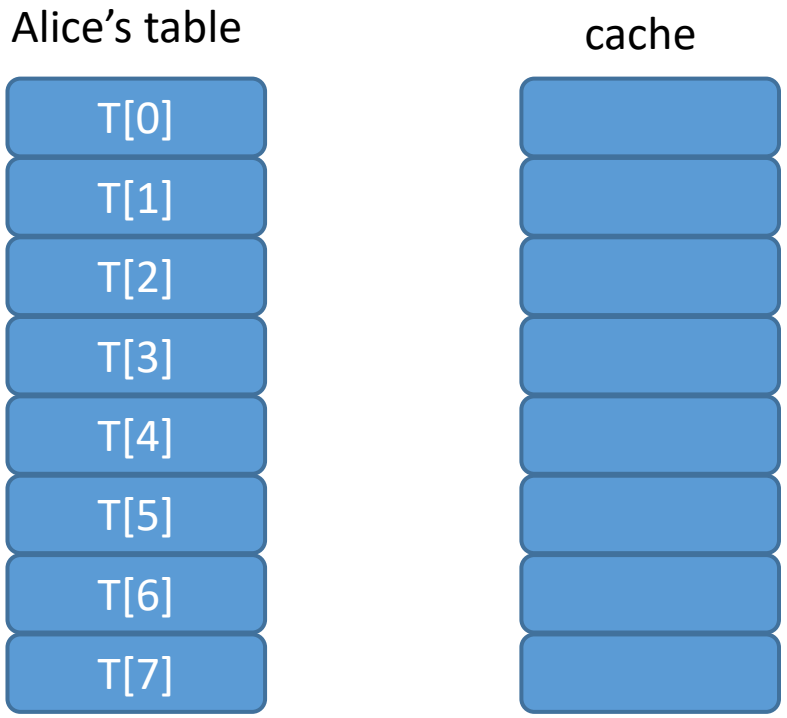

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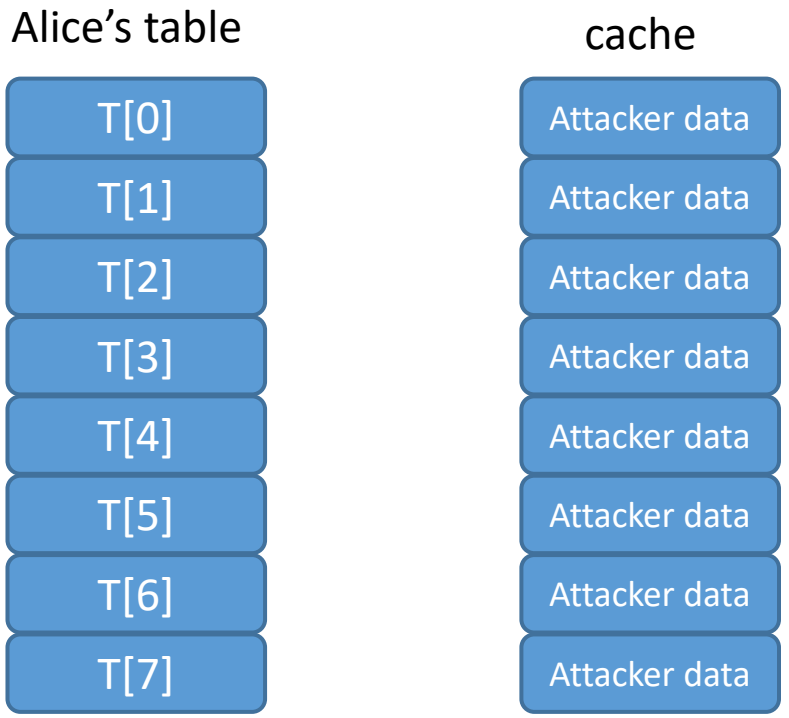

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Timing and cache attacks

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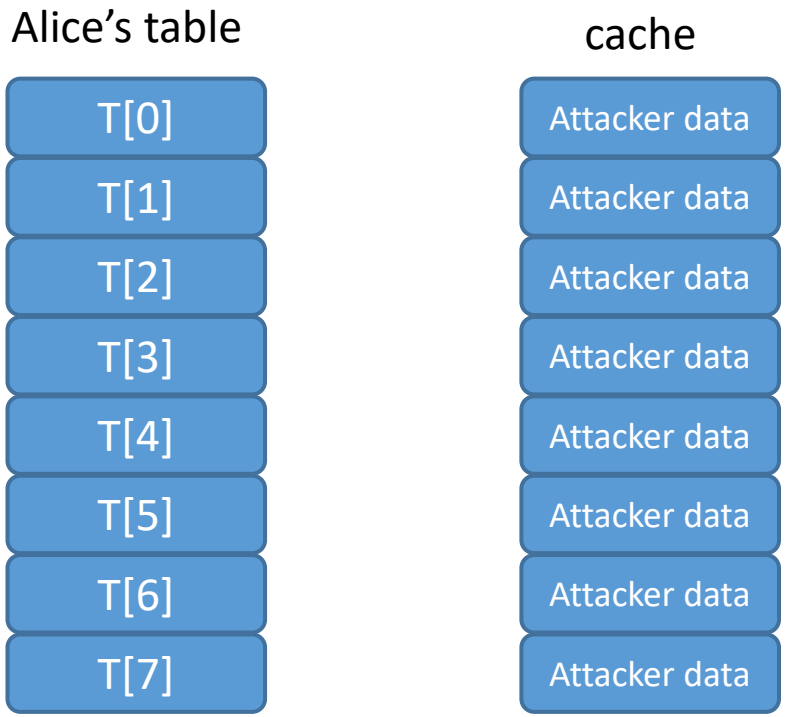
- Reads T[0]. Applies masking
- Reads T[1]. Applies masking
  ⋮
- Reads T[7]. Applies masking



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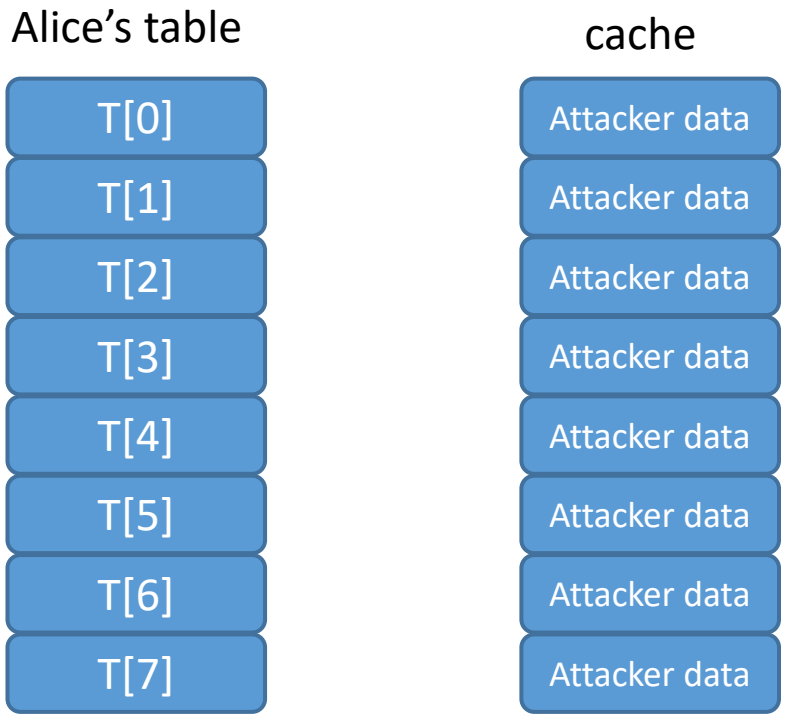

Attacker:
- Flushes and then populates cache memory

- Whatever the cache contains, it **does not leak information** about the exact value that Alice needed

PKC: implementation aspects

Computer representation

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- A t-bit integer a is represented in radix 2^r by

$$a = \sum_{i=0}^{s-1} a_i 2^{ir}$$

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- Let w be the computer wordsize (i.e., 8, 16, 32 or 64)
  - When $r = w$, we call it *full-radix* representation
  - When $r < w$, we call it *reduced-radix* representation

Representation: full-radix vs. reduced-radix

Full-radix: e.g., view of $t = 128$-bit integer in radix 2^{32}

- #limbs $= \lceil t/w \rceil$
- No room for accumulating intermediate values without word spilling

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- No room for accumulating intermediate values without word spilling

Reduced-radix:

- #limbs $\geq \lceil (t + \delta)/w \rceil$, for some $\delta > 0$
- Extra room for accumulating intermediate values without word spilling
- Requires extra words if $\lceil (t + \delta)/w \rceil > \lceil t/w \rceil$

Representation: full-radix vs. reduced-radix

Full-radix representation:

- Example: a 127-bit integer a is represented by (a_0, a_1, a_2, a_3) in radix 2^{32}
- More amenable for “generic” libraries
- Efficient when instructions with carries are efficient, multiplication is relatively expensive
- Example: non-vector implementations on 64-bit AMD, Intel Atom, Intel Quark, 32-bit ARM

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Reduced-radix representation:

- Example: a 127-bit integer a is represented by $(a_0, a_1, a_2, a_3, a_4)$ in radix 2^{26}
- More amenable for "special purpose" libraries
- Efficient when instructions with carries are relatively expensive, multiplication is efficient
- Example: vector implementations on 64-bit Intel desktop/server and 32-bit ARM,
  (in some cases) non-vector implementations on 64-bit Intel desktop/server
- Convenient when there is no direct access to instructions with carries (e.g., generic C)

Representation: full-radix vs. reduced-radix

- Relative performance between Curve25519-based implementations:
  - **amd64-51** (reduced-radix 2^{51}) and **amd64-64** (full-radix 2^{64})
  - 255-bit finite field defined over $p = 2^{255} - 19$

(*) Source: SUPERCOP (2015)

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- Results in **BLACK** indicate amd64-51 is better, results in **RED** indicate amd64-64 is better

Intel Haswell (“wintermute”):	**10%**
Intel Ivy Bridge (“hydra8”):	**6%**
Intel Sandy Bridge (“hydra7”):	**5%**
Intel Atom (“h8atom”):	**-36%**
AMD Piledriver (“hydra9”):	**-39%**
AMD Bulldozer (“hydra6”):	**-38%**
AMD Bobcat (“h4e450”):	**-47%**

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PKC: implementation aspects

Computing arithmetic

PKC: computational aspects

We will focus on two cases:

- **Classical side:** ECC
- **Post-quantum side:** SIKE

PKC: computational aspects

Computation tree:

- **Underlying arithmetic level**
- **Primitive level**
- **Protocol level**

PKC: computational aspects

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PKC: computational aspects

Computation tree:

- **Underlying arithmetic level:** e.g., modular arithmetic
- **Primitive level:** e.g., exponentiation, curve and isogeny arithmetic
- **Protocol level:** e.g., key-exchange, signature, encryption scheme

PKC: computational aspects

Computation tree: ECC

- **Field arithmetic**
- **Curve arithmetic**
- **Protocol level**

PKC: computational aspects

Computation tree: ECC

- **Field arithmetic:** multiplication, squaring, addition, inversion over $\mathbb{F}_p$ for p prime
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PKC: computational aspects

Computation tree: ECC

- **Field arithmetic:** multiplication, squaring, addition, inversion over $\mathbb{F}_p$ for p prime
- **Curve arithmetic:** point addition/doubling, scalar multiplication
- **Protocol level**

PKC: computational aspects

Computation tree: ECC

- **Field arithmetic:** multiplication, squaring, addition, inversion over $\mathbb{F}_p$ for p prime
- **Curve arithmetic:** point addition/doubling, scalar multiplication
- **Protocol level:** ECDH key exchange, EC-based signatures (ECDSA, EdDSA)

PKC: computational aspects

Computation tree: SIKE

- **Field arithmetic**
- **Curve and isogeny arithmetic**
- **Protocol level**

PKC: computational aspects

Computation tree: SIKE

- **Field arithmetic:** multiplication, squaring, addition, inversion over $\mathbb{F}_{p^2}$ for p prime
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PKC: computational aspects

Computation tree: SIKE

- **Field arithmetic:** multiplication, squaring, addition, inversion over $\mathbb{F}_{p^2}$ for p prime
- **Curve and isogeny arithmetic:** point addition/doubling/tripling, scalar multiplication, isogeny evaluation and computation, isogeny-tree traversal
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PKC: computational aspects

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- **Protocol level:** key exchange (SIDH), IND-CCA2-secure key encapsulation (SIKE)

PKC: implementation aspects

Modular arithmetic

Modular arithmetic

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Modular multiplication

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  Toom-Cook multiplication: asymptotic complexity $O(n^{\log_2 5})$
- **Karatsuba becomes attractive for a relatively large number of limbs:** in some cases, useful for high-security ECC (e.g., NIST P-521)

Modular multiplication

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- *Two* generic flavours:
  - **Interleaved multiplication/reduction:** usually faster when standalone
  - **Non-interleaved multiplication/reduction:** usually faster when lazy reduction applies

Modular reduction

- Two big categories for modular reduction:
  General-form modulus (RSA, ECC over general primes: e.g., Brainpool curves)
  Special-form modulus (high-performance ECC: e.g., NIST P-256, Curve25519, Four$\mathbb{Q}$)
- Two main reduction methods: Barrett reduction and Montgomery reduction

Modular reduction: special form primes

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    E.g., Four$\mathbb{Q}$'s prime $p = 2^{127} - 1$,
    NIST P-521's prime $p = 2^{521} - 1$
- NIST prime choices targeted 32-bit platforms (popular at that time). Except for P-521, these primes are relatively expensive in constant-time and vectorized implementations

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- t holds 2×64 bits.
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integer mult

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\begin{aligned}
[t_0, t] &= a_0 \times b_0 \\
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integer mult result $\{t_0 - t_7\}$

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Reduction (using $2^{256} \equiv 38$)

- Set output in the range $[0, 2^{256} - 38)$

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Reduction (using $2^{256} \equiv 38$)

1st pass

$$[t_0, t] = t_0 + 38 \times t_4$$
$$[t_1, t] = t_1 + 38 \times t_5 + t$$
$$[t_2, t] = t_2 + 38 \times t_6 + t$$
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[t_0, c] &= t_0 + 38 \times (t + 1) \\
[t_1, c] &= t_1 + c \\
[t_2, c] &= t_2 + c \\
[t_3, c] &= t_3 + c
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1st pass

2nd pass
Add extra 38

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[t_4, t] &= a_1 \times b_3 + a_3 \times b_1 + a_2 \times b_2 + t \\
[t_5, t] &= a_2 \times b_3 + a_3 \times b_2 + t \\
[t_6, t_7] &= a_3 \times b_3 + t
\end{aligned}
$$

integer mult result $\{t_0 - t_7\}$

Reduction (using $2^{256} \equiv 38$)

$$
\begin{aligned}
[t_0, t] &= t_0 + 38 \times t_4 \\
[t_1, t] &= t_1 + 38 \times t_5 + t \\
[t_2, t] &= t_2 + 38 \times t_6 + t \\
[t_3, t] &= t_3 + 38 \times t_7 + t \\
[t_0, c] &= t_0 + 38 \times (t + 1) \\
[t_1, c] &= t_1 + c \\
[t_2, c] &= t_2 + c \\
[t_3, c] &= t_3 + c \\
[t_0, c] &= t_0 - 38 \times c \text{ if } c = 1 \\
[t_1, c] &= t_1 - c \\
[t_2, c] &= t_2 - c \\
[t_3, c] &= t_3 - c
\end{aligned}
$$

Output $[t_0, t_1, t_2, t_3]$

1st pass

2nd pass
Add extra 38

3rd pass with condition

- t holds 2×64 bits.
- t_x hold 64 bits.
- c is a carry or borrow bit.

- Set output in the range $[0, 2^{256} - 38)$

Modular reduction: special form primes

- *Constant-time* field multiplication with $p = 2^{255} - 19$ (x64 assembly)

```
mov  rax, [reg_p1]
mul qword ptr [reg_p2]     // a0*b0
mov  r13, rax              // t0
mov  r12, rdx

mov  rbx, [reg_p2+8]
mov  rax, [reg_p1]
mul  rbx                   // a0*b1
add  r12, rax
mov  r10, rdx
adc  r10, 0

xor  r11, r11
mov  rax, [reg_p1+8]       // a1*b0
mul qword ptr [reg_p2]
add  r12, rax
mov  [reg_p3+8], r12       // t1
adc  r10, rdx
adc  r11, 0

mov  rax, [reg_p2]
mul qword ptr [reg_p1+16] // a2*b0
add  r10, rax
adc  r11, rdx

xor  rcx, rcx
mov  rax, [reg_p1+8]
mul  rbx                   // a1*b1
add  r10, rax
adc  r11, rdx
adc  rcx, 0

mov  rax, [reg_p2+16]
mul qword ptr [reg_p1]     // a0*b2
add  r10, rax
mov  [reg_p3+16], r10      // t2
adc  r11, rdx
adc  rcx, 0
```

```
xor  r10, r10
mov  rax, [reg_p2]
mul qword ptr [reg_p1+24] // a3*b0
add  r11, rax
adc  rcx, rdx
adc  r10, 0

mov  rax, [reg_p2+8]
mul qword ptr [reg_p1+16] // a2*b1
add  r11, rax
adc  rcx, rdx
adc  r10, 0

mov  rax,[reg_p2+16]
mul  rbx                   // a1*b2
add  r11, rax
adc  rcx, rdx
adc  r10, 0

mov  rax, [reg_p2+24]
mul qword ptr [reg_p1]     // a0*b3
add  r11, rax
mov  [reg_p3+24], r11      // t3
adc  rcx, rdx
adc  r10, 0

xor  r11, r11
mov  rax, [reg_p2+8]
mul qword ptr [reg_p1+24] // a3*b1
add  rcx, rax
adc  r10, rdx
adc  r11, 0

mov  rax, [reg_p2+16]
mul qword ptr [reg_p1+16] // a2*b2
add  rcx, rax
adc  r10, rdx
adc  r11, 0
```

```
mov  rax, [reg_p2+24]
mul  rbx                   // a1*b3
add  rcx, rax              // t4
adc  r10, rdx
adc  r11, 0

xor  r8, r8
mov  rax, [reg_p2+16]
mul qword ptr [reg_p1+24] // a3*b2
add  r10, rax
adc  r11, rdx
adc  r8, 0

mov  rax, [reg_p2+24]
mul qword ptr [reg_p1+16] // a2*b3
add  r10, rax              // t5
adc  r11, rdx
adc  r8, 0

mov  rax, [reg_p2+24]
mul qword ptr [reg_p1+24] // a3*b3
add  r11, rax              // t6
adc  r8, rdx               // t7

// Reduction

mov  rax, constant_38
mul  rcx
add  r13, rax              // r13 <- t0
adc  rdx, 0
mov  rcx, rdx

xor  r12, r12
mov  rax, constant_38
mul  r10
add  rax, rcx
adc  r12, 0
mov  r10, [reg_p3+8]
add  r10, rax              // r10 <- t1
adc  r12, rdx
```

```
xor  rcx, rcx
mov  rax, constant_38
mul  r11
add  rax, r12
adc  rcx, 0
mov  r11, [reg_p3+16]
add  r11, rax              // r11 <- t2
adc  rcx, rdx

xor  r12, r12
mov  rax, constant_38
mul  r8
add  rax, rcx
adc  r12, 1
mov  r8, [reg_p3+24]
add  r8, rax               // r8 <- t3
adc  rdx, r12              // rdx <- t4 + 1

xor  r12, r12
mov  rax, constant_38
mul  rdx
add  r13, rax              // t0
adc  r10, 0                // t1
adc  r11, 0                // t2
adc  r8, 0                 // t3

mov  rax, constant_38      // final correction
cmovc rax, r12
sub  r13, rax              // t0
mov  [reg_p3], r13
sbb  r10, 0                // t1
mov  [reg_p3+8], r10
sbb  r11, 0                // t2
mov  [reg_p3+16], r11
sbb  r8, 0                 // t3
mov  [reg_p3+24], r8
```

Modular addition

- *Constant-time* field addition $a + b \bmod p$
  - If $r = a + b \geq p$ then $r = r - p$. Output r
  - Eliminating **if-then-else statement** is usually straightforward

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Example: field addition with $p = 2^{127} - 1$ (64-bit)

$[t_0, c] = a_0 + b_0$
$t_1 = a_1 + b_1 + c$

Write $c = t_1 \gg 63,\ t_1 = t_1 \,\&\, \text{mask\_63}$ (using $2^{127} \equiv 1$)
$[t_0, c] = t_0 + c$
$t_1 = t_1 + c$
Output $[t_0, t_1]$

Modular addition

- **Example:** *constant-time* field addition with $p = 2^{127} - 1$ (x64 assembly)

```
mov     r8, [reg_p1]
mov     r9, [reg_p1+8]
add     r8, [reg_p2]
adc     r9, [reg_p2+8]

btr     r9, 63
adc     r8, 0
mov     [reg_p3], r8
adc     r9, 0
mov     [reg_p3+8], r9
```

Modular addition

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adc     r9, [reg_p2+8]

btr     r9, 63
adc     r8, 0
mov     [reg_p3], r8
adc     r9, 0
mov     [reg_p3+8], r9
```

- **Example:** *constant-time* field subtraction with $p = 2^{127} - 1$ (x64 assembly)

```
mov     r8, [reg_p1]
mov     r9, [reg_p1+8]
sub     r8, [reg_p2]
sbb     r9, [reg_p2+8]

btr     r9, 63
sbb     r8, 0
mov     [reg_p3], r8
sbb     r9, 0
mov     [reg_p3+8], r9
```

Modular arithmetic: case with reduced-radix

- So far, we have explored operations using full-radix. What about **reduced-radix**?
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Example: Set $w = 64$. An element in $\mathbb{F}_{2^{127}-1}$ can be represented as (a_0, a_1, a_2) in radix 2^{43}. When fully reduced, element values are [43, 43, 41]-bit long

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With full-radix:

$$[t_0, c] = a_0 + b_0$$
$$t_1 = a_1 + b_1 + c$$
$$c = t_1 \gg 63,\ t_1 \,\&=\, \text{mask\_63}$$
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127-bit field subtraction

$$t_0 = a_0 - b_0$$
$$t_1 = a_1 - b_1$$
$$t_2 = a_2 - b_2$$

With full-radix:

$$[t_0, c] = a_0 + b_0$$
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After result of additions/subtractions in $\mathbb{F}_{2^{127}-1}$:

$t_0 = t_0 + (t_2 \gg 41)$
$t_2 = t_2 \,\&\, \text{mask\_}41$
$t_1 = t_1 + (t_0 \gg 43)$
$t_0 = t_0 \,\&\, \text{mask\_}43$
$t_2 = t_2 + (t_1 \gg 43)$
$t_1 = t_1 \,\&\, \text{mask\_}43$

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After result of additions/subtractions in $\mathbb{F}_{2^{127}-1}$:

$t_0 = t_0 + (t_2 \gg 41)$
$t_2 = t_2 \,\&\, \text{mask\_41}$
$t_1 = t_1 + (t_0 \gg 43)$
$t_0 = t_0 \,\&\, \text{mask\_43}$
$t_2 = t_2 + (t_1 \gg 43)$
$t_1 = t_1 \,\&\, \text{mask\_43}$

- A reduced representation [43, 43, 41]-bit is obtained by running **two rounds** of the carry correction routine
- In some cases, it might be enough to run **one round** (after multiplications and squarings)

Modular arithmetic: case with reduced-radix

- For our example, if there is access to `adc` then **full-radix is more efficient than reduced-radix** (e.g., when using assembly or intrinsics)

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- For our example, if there is access to `adc` then **full-radix is more efficient than reduced-radix** (e.g., when using assembly or intrinsics)
- Otherwise, reduced radix can be more efficient (e.g., when using high-level languages)
- When using a high-level language, **reduced-radix can be simpler to implement in some cases**

Modular arithmetic: case with reduced-radix

- **Example:** C implementation of field multiplication with $p = 2^{127} - 1$ on Linux
- GNU GCC and clang support 128-bit datatypes (`int128_t`, `uint128_t`), not in ANSI C

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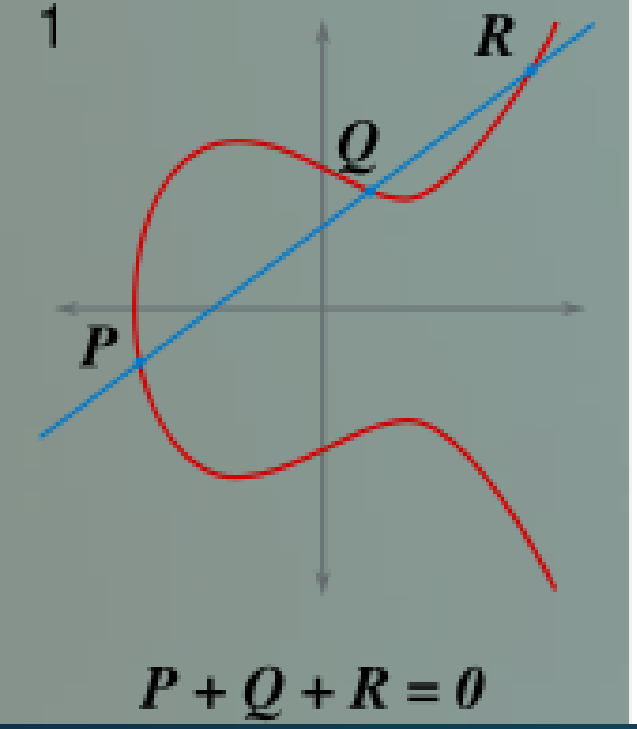
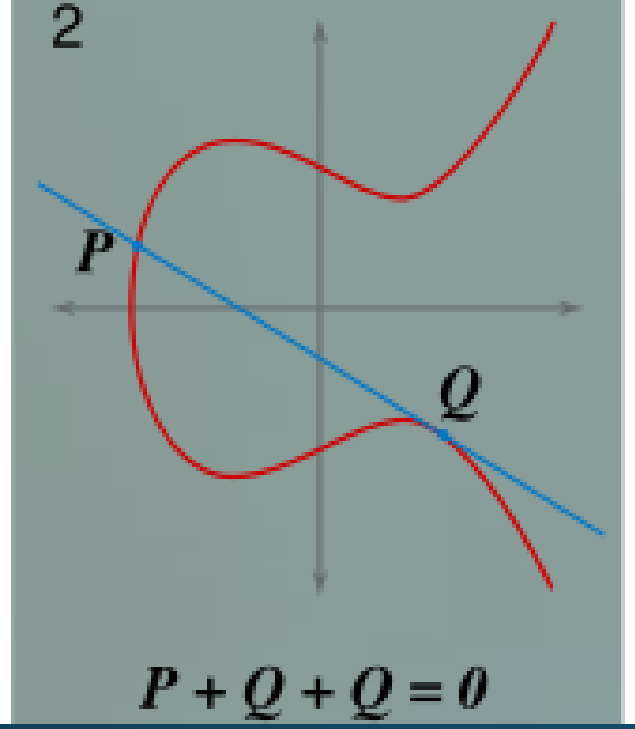
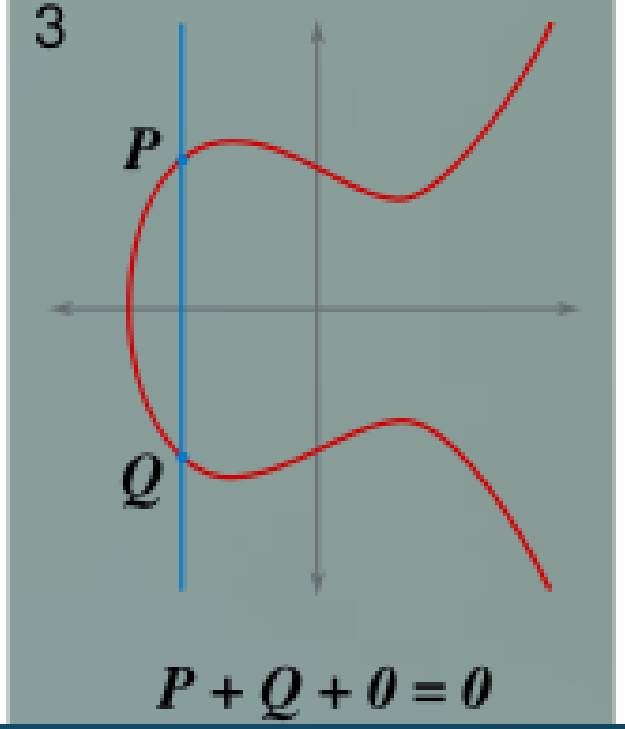
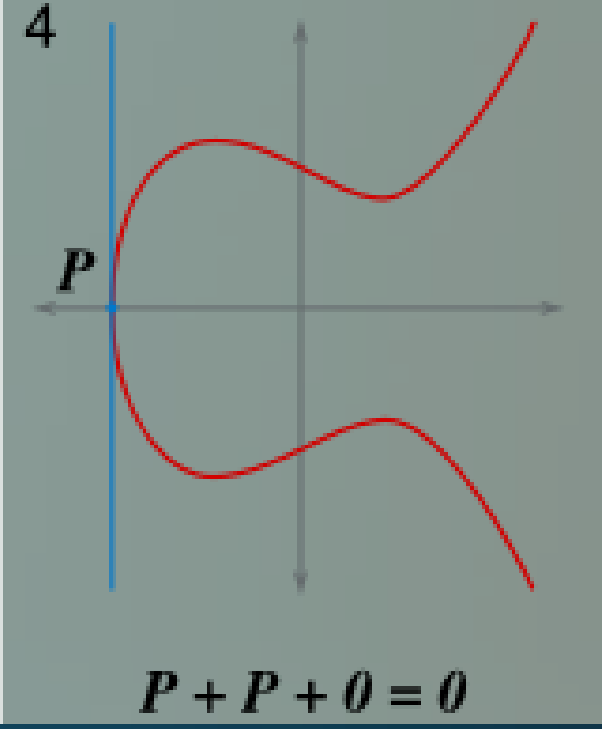
- **Example:** C implementation of field multiplication with $p = 2^{127} - 1$ on Linux
- GNU GCC and clang support 128-bit datatypes (`int128_t`, `uint128_t`), not in ANSI C

```
void fpmul1271(const int64_t* a, const int64_t* b, int64_t* c)
{
   int128_t t[3];
   int64_t  d1, d2;

   a0 = a[0]; a1 = a[1]; b0 = b[0]; b1 = b[1];

   d1 = a1 << 2;
   d2 = a2 << 2;
   t[0] = (int128_t)a0*b0 + (int128_t)d1*b2 + (int128_t)d2*b1;
   t[1] = (int128_t)a0*b1 + (int128_t)a1*b0 + (int128_t)d2*b2;
   t[2] = (int128_t)a0*b2 + (int128_t)a2*b0 + (int128_t)a1*b1;

   t[0] += (int64_t)(t[2] >> 41); c[2] = (int64_t)t[2] & mask_41;
   t[1] += (int64_t)(t[0] >> 43); c[0] = (int64_t)t[0] & mask_43;
   c[2] += (int64_t)(t[1] >> 43); c[1] = (int64_t)t[1] & mask_43;
   c[0] += (int64_t)(c[2] >> 41); c[2] &= mask_41;
}
```









The case of ECC

ECC basics

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- Elliptic curve-based protocols require different variants of scalar multiplication:
  - **Variable-base scalar multiplication:** $[k]P$ with variable point P
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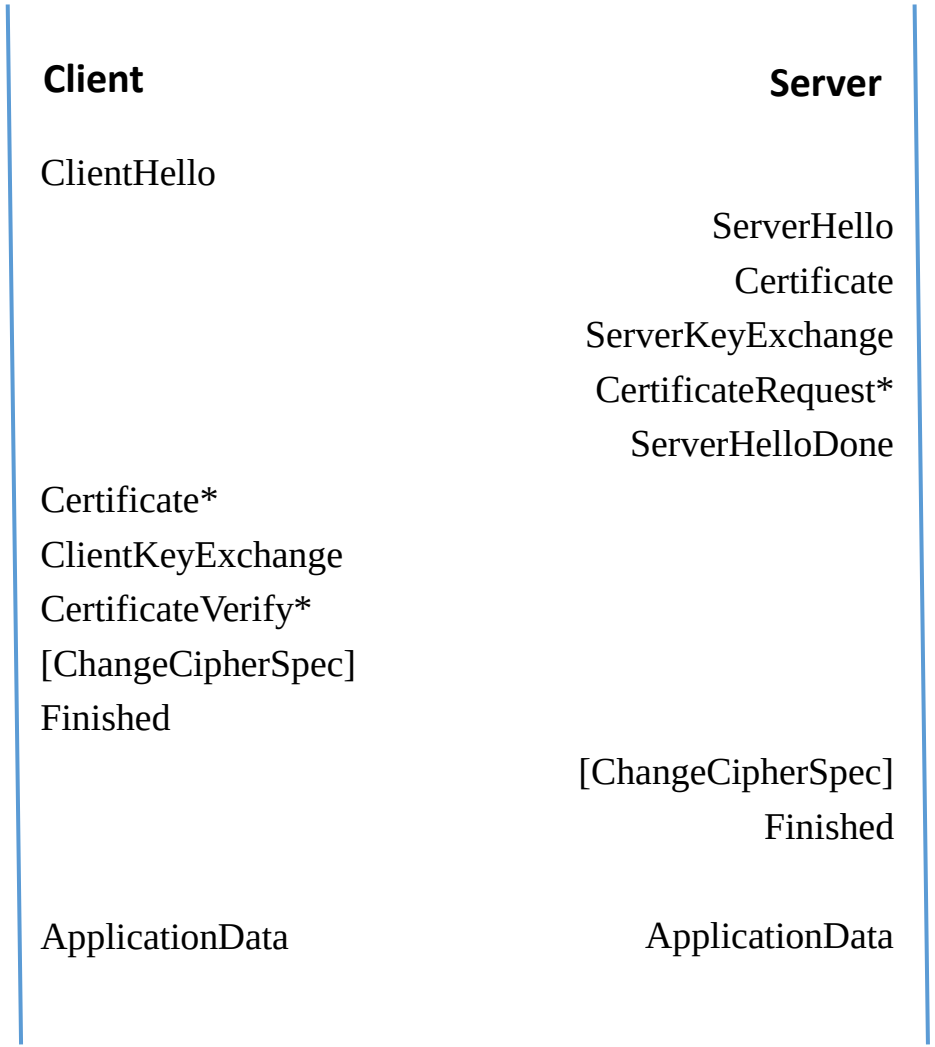
- We will focus on **finite fields with large prime characteristic** ($\mathbb{F}_p$ for p prime)

TLS Handshake with ECDHE_ECDSA: an example



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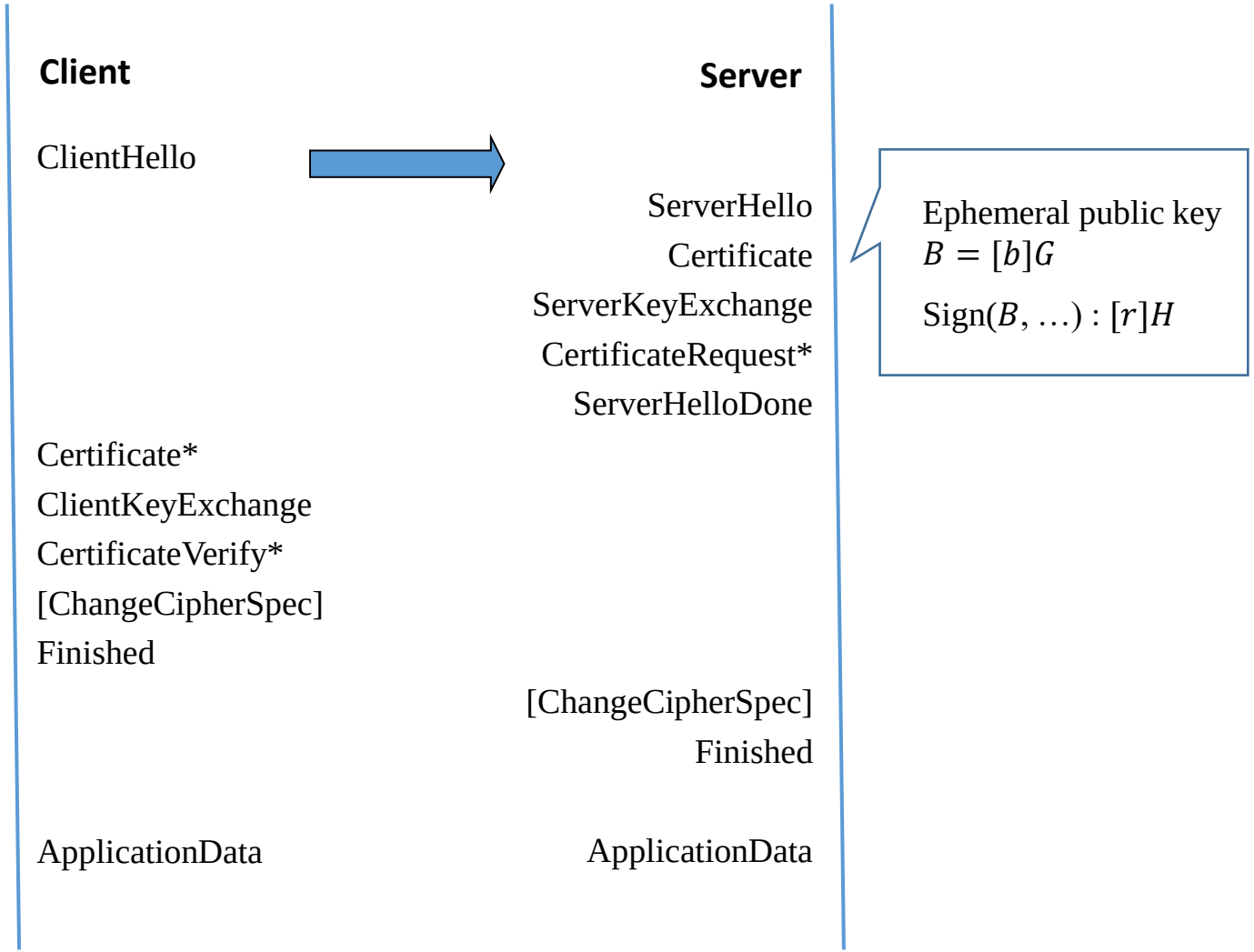

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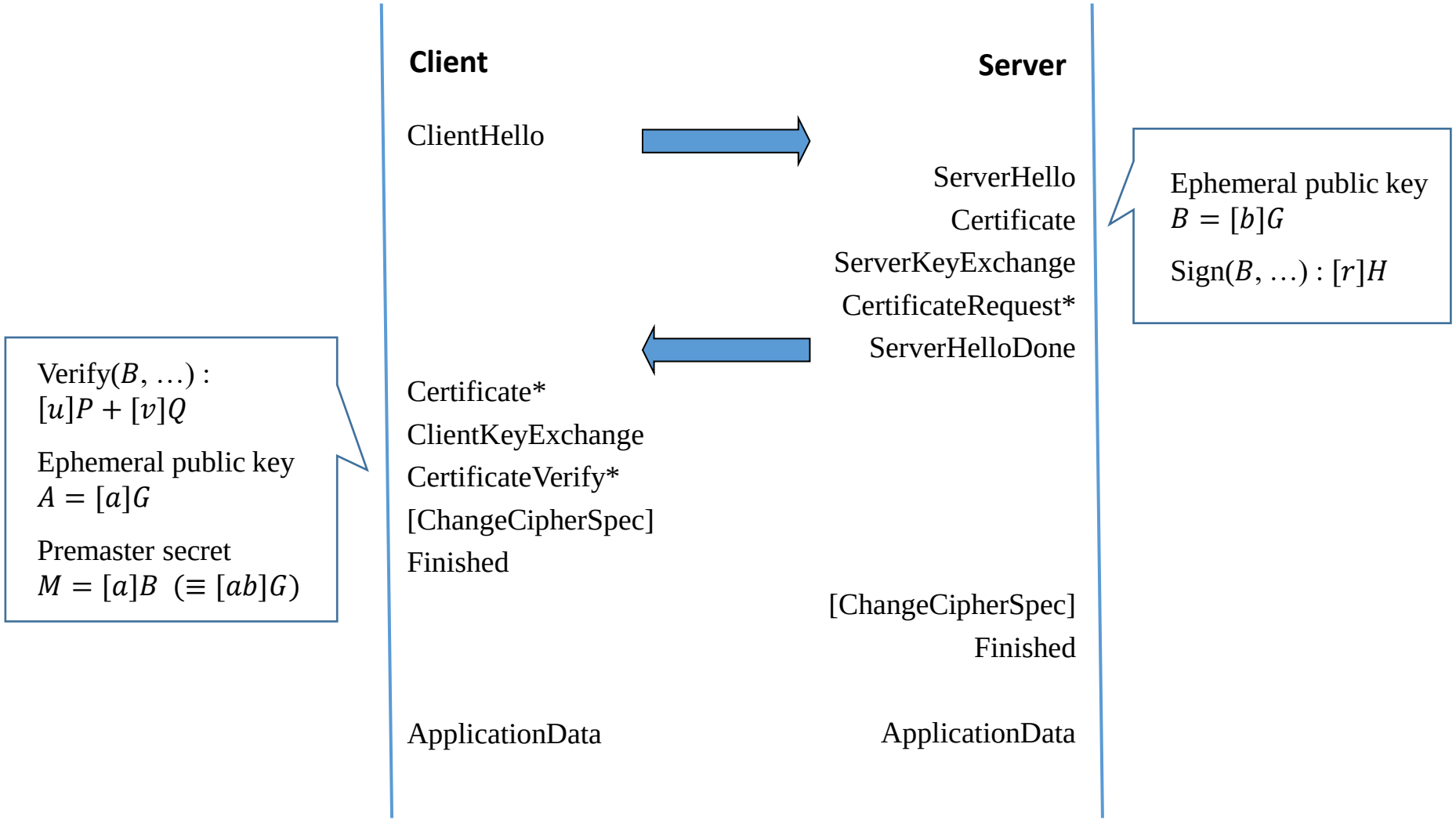

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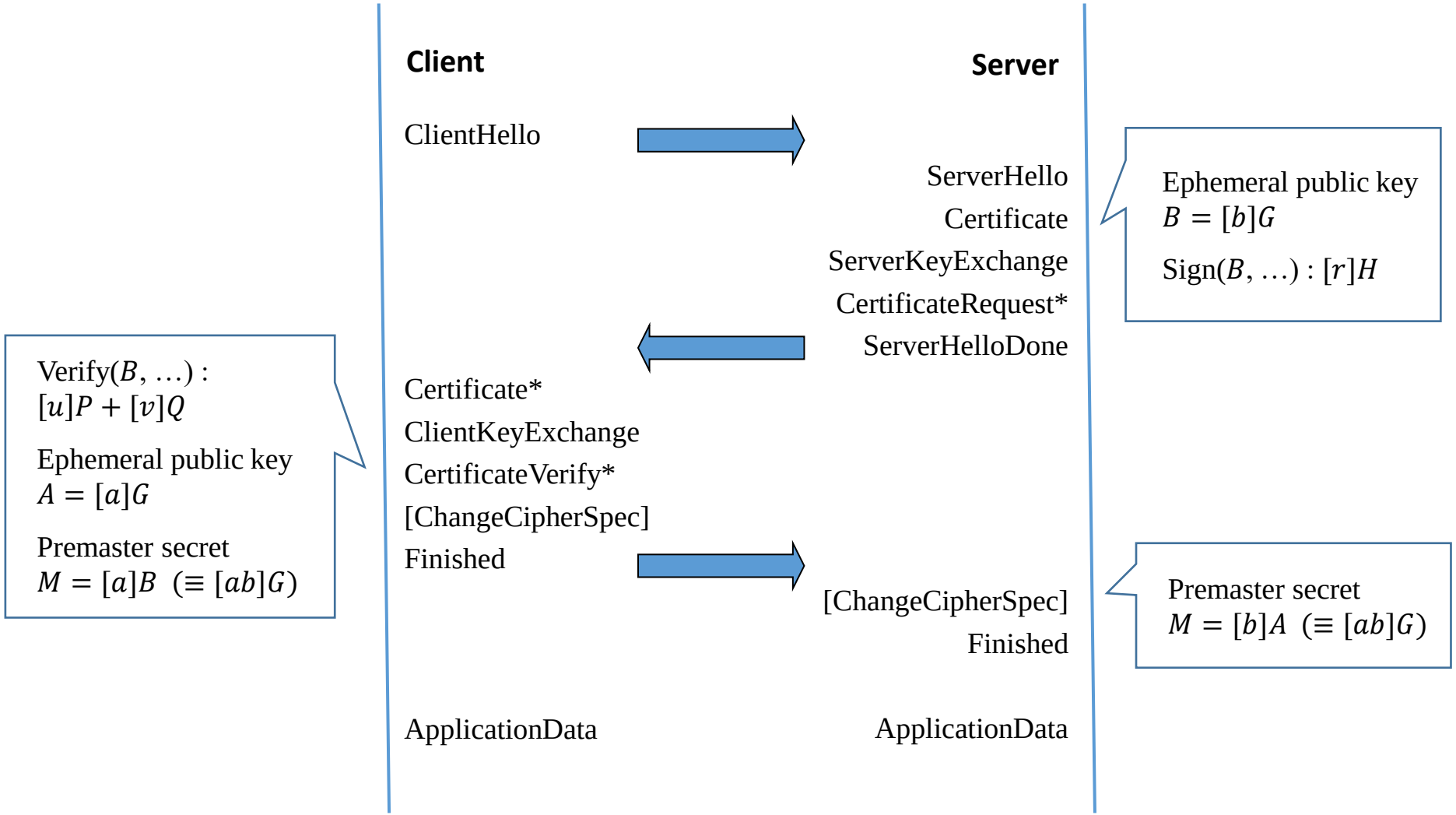

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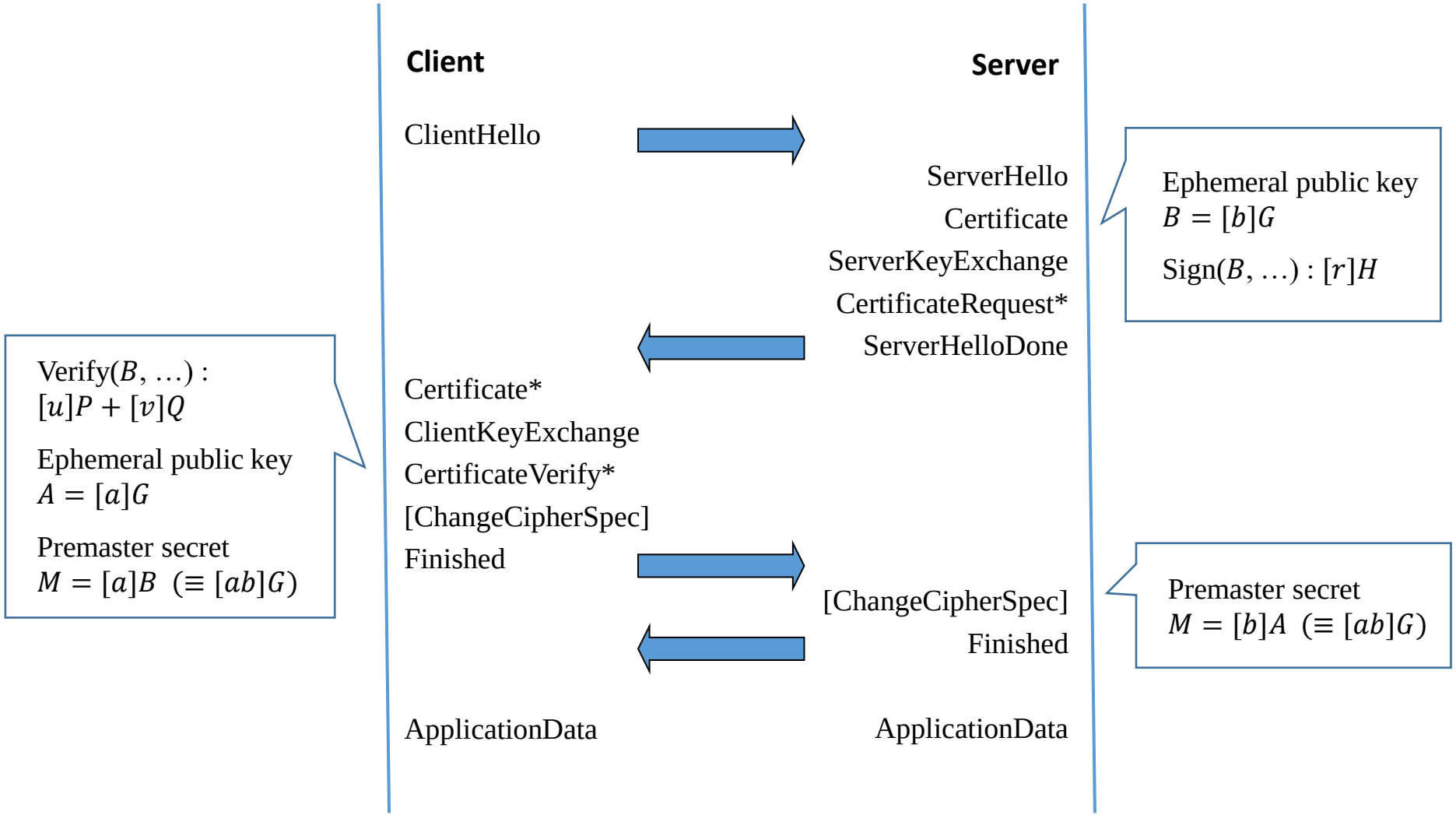

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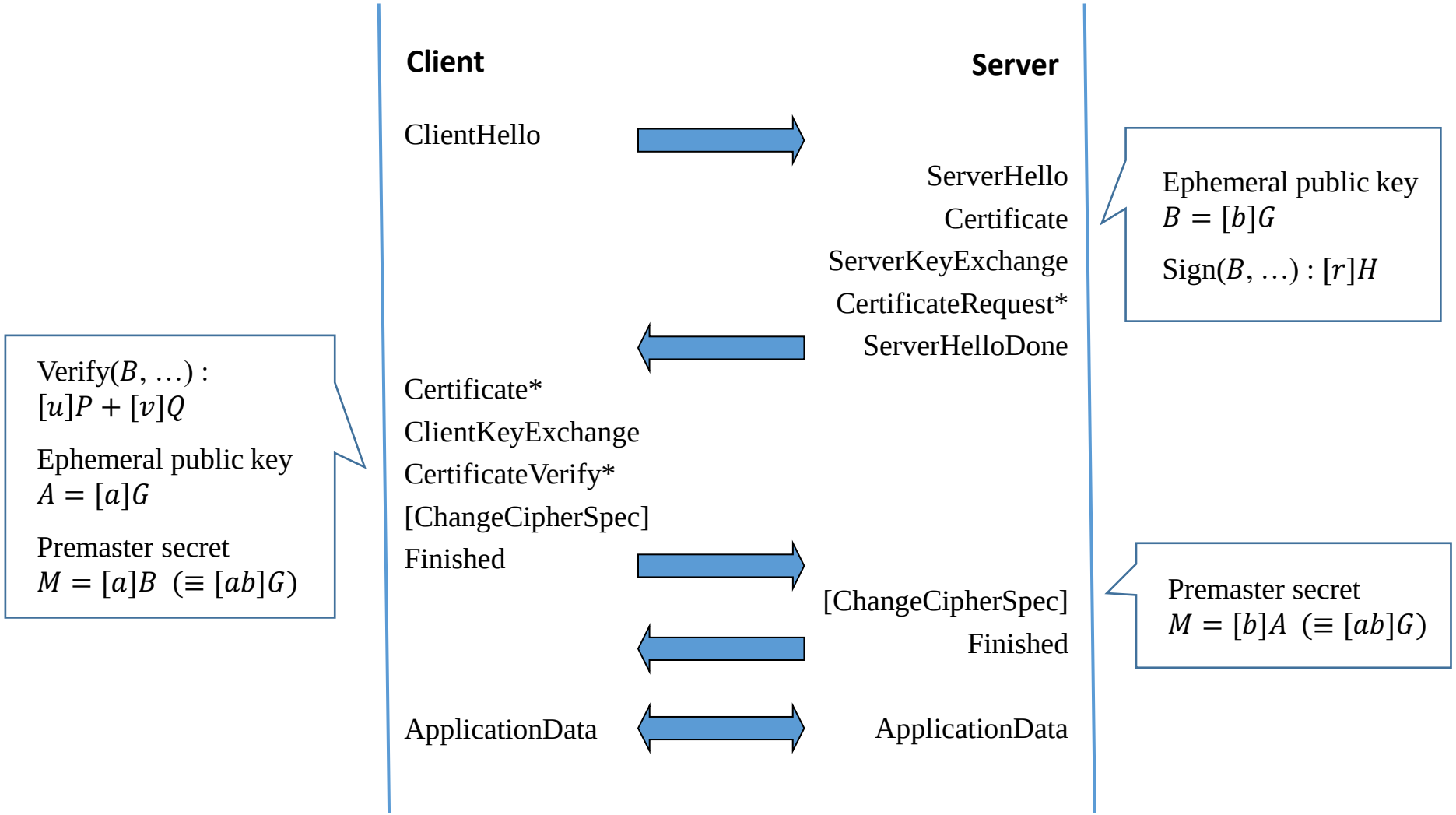

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A (short Weierstrass) elliptic curve over a prime field $\mathbb{F}_p$ is given by:

$$E/\mathbb{F}_p\colon\ y^2 = x^3 + ax + b$$

where $a, b \in \mathbb{F}_p$ and $\Delta = -16(4a^3 + 27b^2) \neq 0$.

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- $\{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : y^2 - x^3 - ax - b = 0\} \cup \{\infty\}$ is an abelian group with group operation (+)
- Given points P and Q, any of the following cases can arise for $R = P + Q$:

$P = \infty \Rightarrow R = Q$
$Q = \infty \Rightarrow R = P$
$P = -Q \Rightarrow R = \infty$
$P = Q \Rightarrow R = 2P$ (point doubling)
otherwise $R = P + Q$ (point addition)

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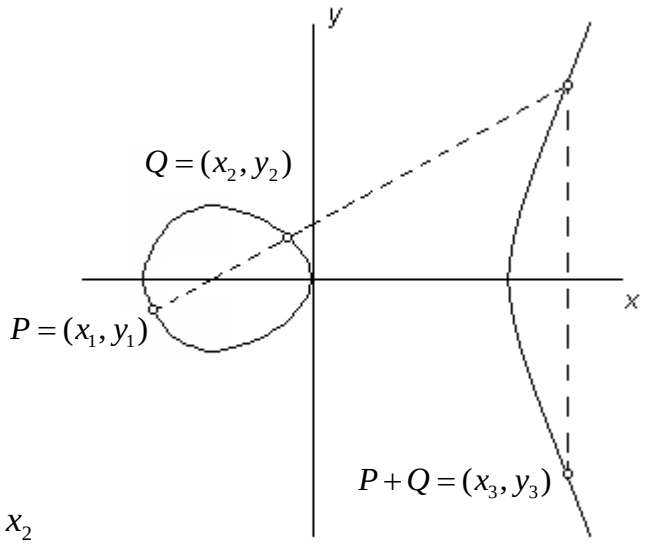

$$\begin{cases}
\lambda = \dfrac{y_2 - y_1}{x_2 - x_1} \\
x_3 = \lambda^2 - x_1 - x_2 \\
y_3 = \lambda(x_1 - x_3) - y_1
\end{cases}$$

point addition

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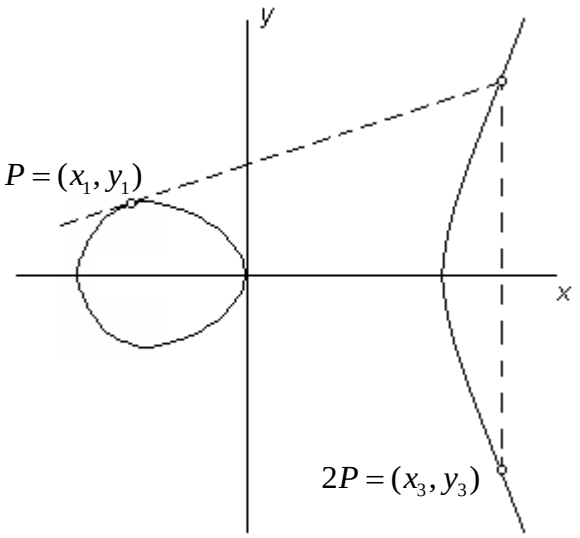
where $a, b \in \mathbb{F}_p$ and $\Delta = -16(4a^3 + 27b^2) \neq 0$.

- $\{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : y^2 - x^3 - ax - b = 0\} \cup \{\infty\}$ is an abelian group with group operation (+)
- Given points P and Q, any of the following cases can arise for $R = P + Q$:

$$\begin{aligned}
&P = \infty && \Rightarrow R = Q \\
&Q = \infty && \Rightarrow R = P \\
&P = -Q && \Rightarrow R = \infty \\
&P = Q && \Rightarrow R = 2P && \text{(point doubling)} \\
&\text{otherwise} && R = P + Q && \text{(point addition)}
\end{aligned}$$

$$\begin{cases}
\lambda = \dfrac{3x_1^2 + a}{2y_1} \\
x_3 = \lambda^2 - 2x_1 \\
y_3 = \lambda(x_1 - x_3) - y_1
\end{cases}$$



point doubling

Elliptic curve forms

Weierstrass curves

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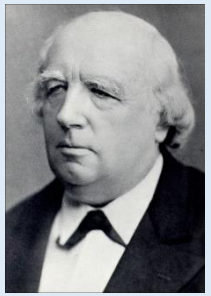

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- Exceptions in group law (but complete formulas exist)
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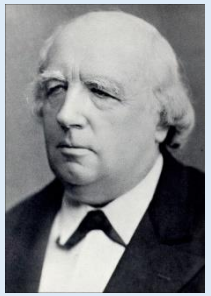

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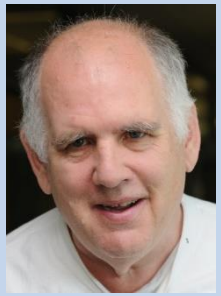
Montgomery curves

$$By^2 = x^3 + Ax^2 + x$$

- Subset of curves
- Not prime order
- Montgomery ladder

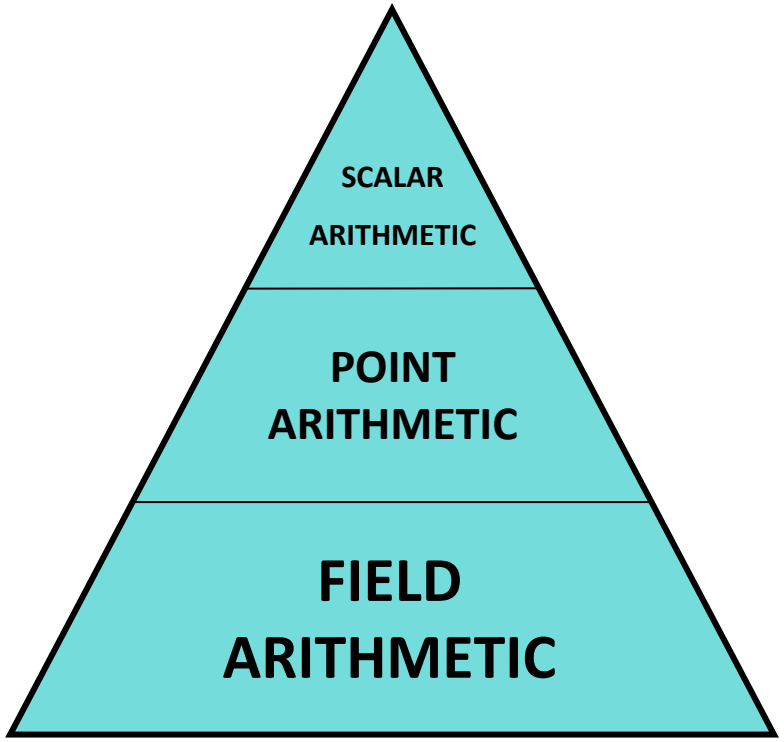


(Twisted) Edwards curves

$$ax^2 + y^2 = 1 + dx^4y^4$$

- Subset of curves
- Not prime order
- Fastest arithmetic
- Some have complete group law

ECC arithmetic layers



Scalar multiplication operations
E.g., $[k]P$, $[u]P+[v]Q$, *etc.*

Doubling and addition of points:
$2P$, $P+Q$

Field addition, subtraction, multiplication, squaring, inversion

Implementation Security

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- Not a problem for twist-secure curves

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Unified addition:

- Simple and compact, but expensive

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Today, it runs in…

46,000 cc	**Costello-Longa 2015 (Four$\mathbb{Q}$)**

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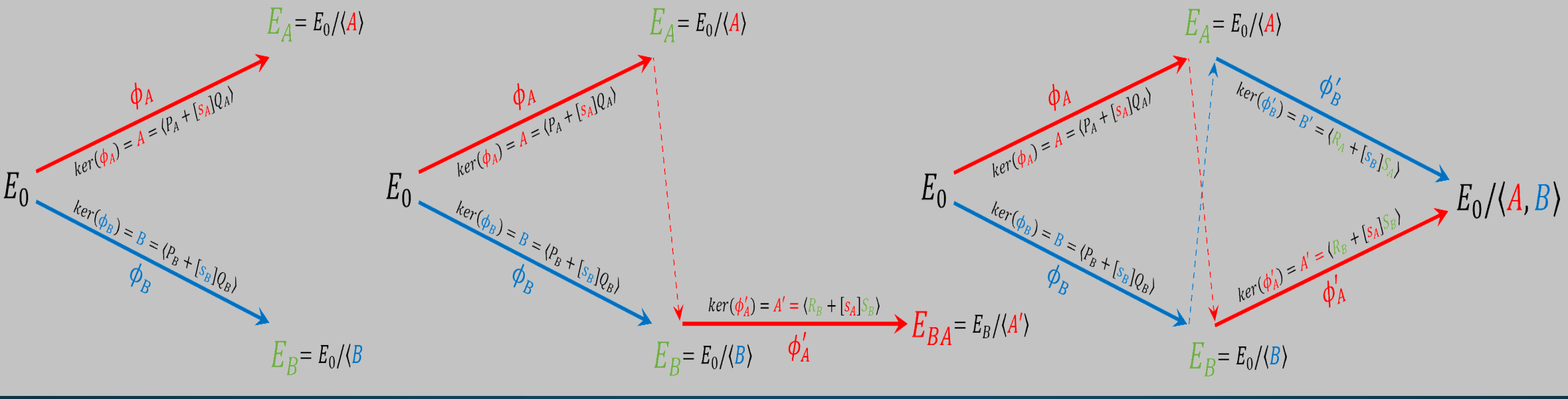

The case of SIKE

Supersingular isogeny key exchange

Supersingular isogeny Diffie-Hellman key exchange (SIDH)

- Proposed by Jao and De Feo in 2011.
- Compared to “predecessors” based on *ordinary isogenies*, SIDH has:
  - Much better performance
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Supersingular isogeny key encapsulation (SIKE)

- Designed by Costello–De Feo–Jao–L–Naehrig–Renes in 2017.
- IND-CCA secure key encapsulation protocol based on SIDH.
- Submitted to the NIST PQC standardization process.

Elliptic curves and isogenies

- Every elliptic curve over a field K with $\mathrm{char}(K) > 3$ can be defined in (short) Weierstrass form by

$$E: y^2 = x^3 + ax + b,$$

where $a, b \in K$ and $\Delta = -16(4a^3 + 27b^2) \neq 0$.

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- For an extension field L of K, the set of L-rational points on E

$$E(L) = \{(x, y) \in L \times L\colon y^2 - x^3 - ax - b = 0\} \cup \{\mathcal{O}\},$$

together with the *group addition law*, forms an abelian group with identity $\mathcal{O}$.

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- **Isomorphism classes** are determined by the ***j*-invariant**: $j(E) = 1728 \cdot \frac{4a^3}{4a^3 + 27b^2}$

Elliptic curves and isogenies

- Let E_1 and E_2 be elliptic curves defined over an extension field L.
- An isogeny is a (non-constant) rational map

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Relevant properties:

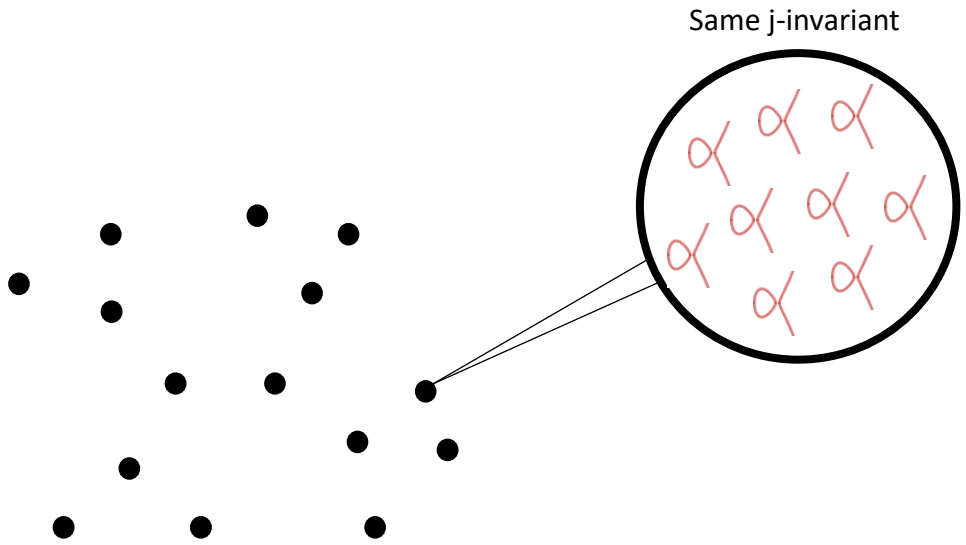
- Isogenies are group homomorphisms.
- For every finite subgroup $\langle G \rangle \subseteq E_1$, there is a unique curve E_2 (up to isomorphism) and isogeny $\phi: E_1 \to E_2$ with kernel $\langle G \rangle$. Write $E_2 = \phi(E_1) = E_1/\langle G \rangle$.
- (Separable) isogenies have $\deg(\phi) = \#\ker(\phi)$.

Supersingular curves

- An elliptic curve E/L is supersingular if $\#E(L) \equiv 1 (\mathrm{mod}\ p)$.
- All supersingular curves can be defined over $\mathbb{F}_{p^2}$.
- There are $\sim\lfloor \boldsymbol{p/12} \rfloor$ **isomorphism classes** of supersingular curves.

Supersingular isogeny graphs

- Vertices: the $\sim\lfloor p/12 \rfloor$ isomorphism classes of supersingular curves over $\mathbb{F}_{p^2}$.



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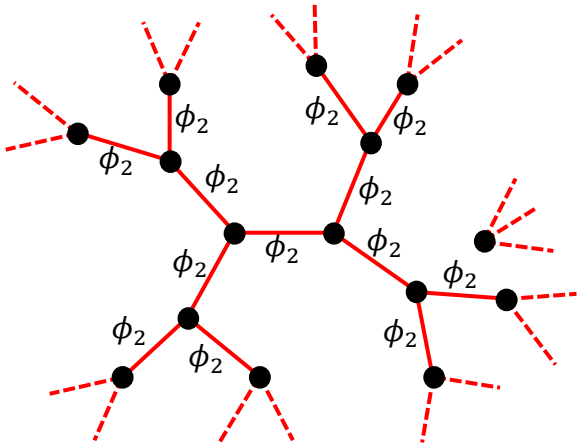
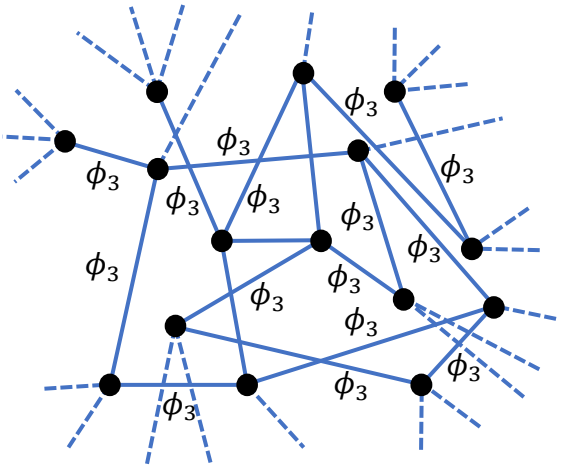


$\ell = 2$



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Supersingular isogeny graphs

- Vertices: the $\sim\lfloor p/12 \rfloor$ isomorphism classes of supersingular curves over $\mathbb{F}_{p^2}$.
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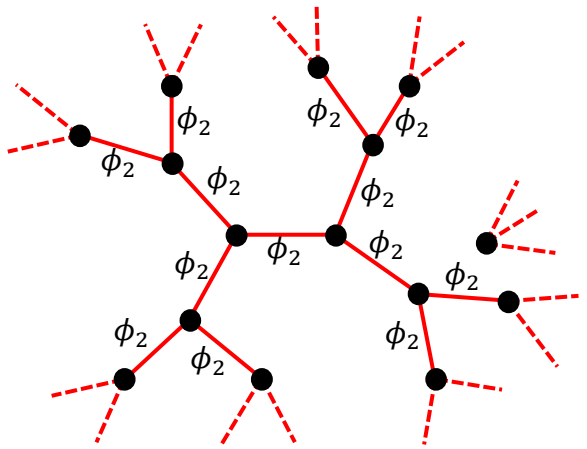
For any prime $\ell \nmid p$, there exist $(\ell + 1)$ isogenies of degree ℓ originating from every supersingular curve.



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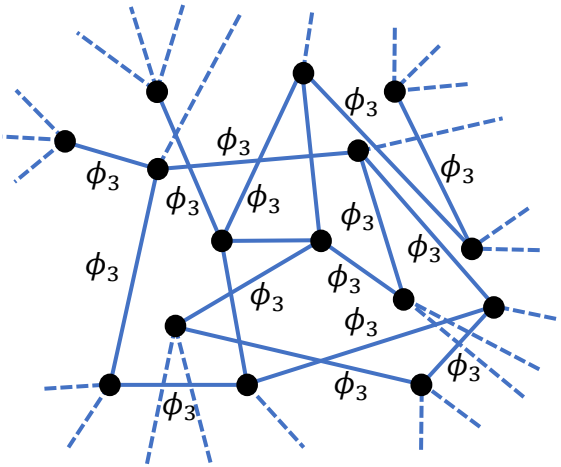

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SIDH in a nutshell

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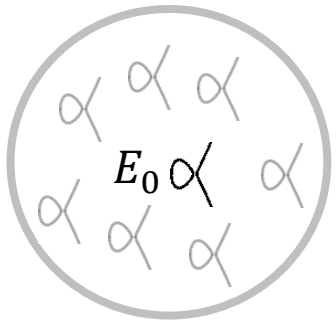

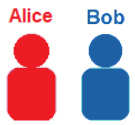

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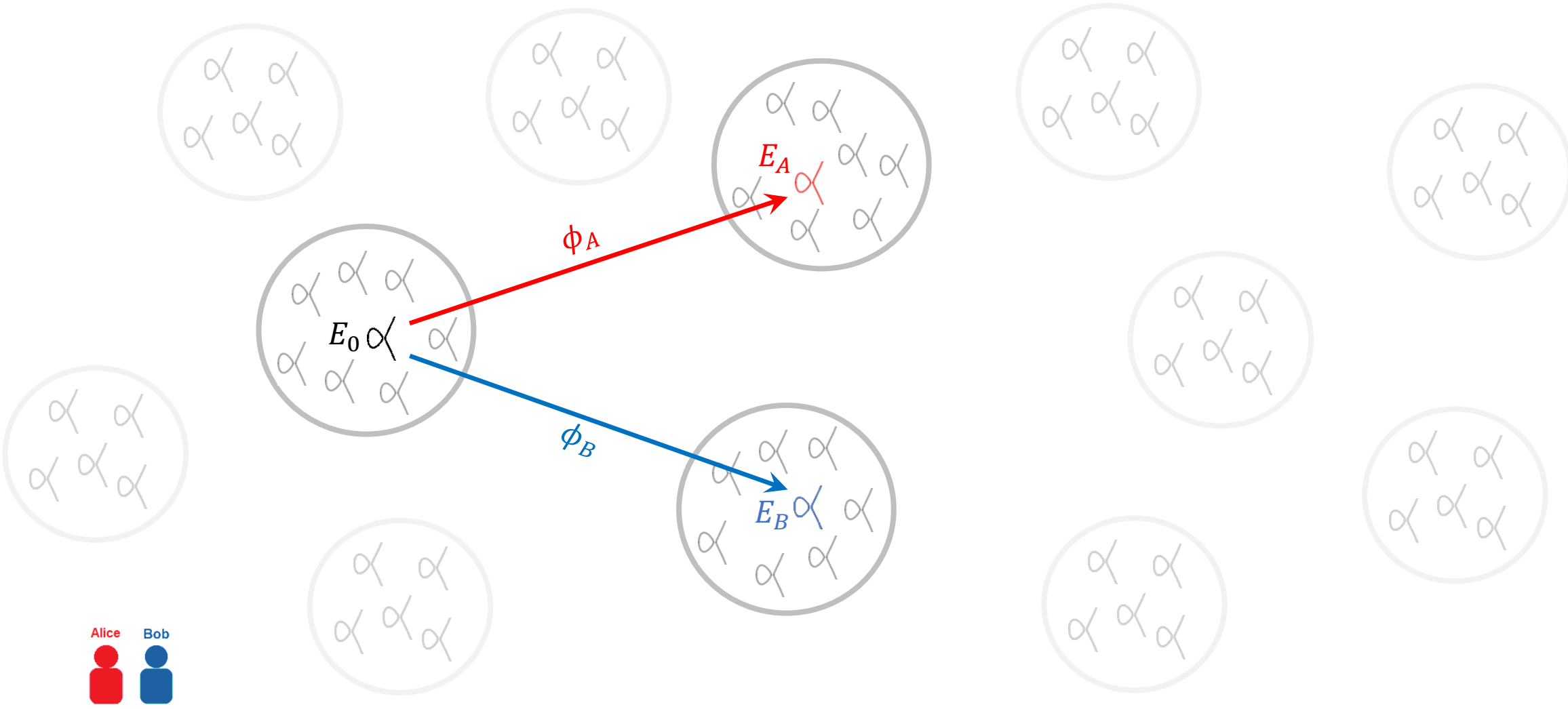

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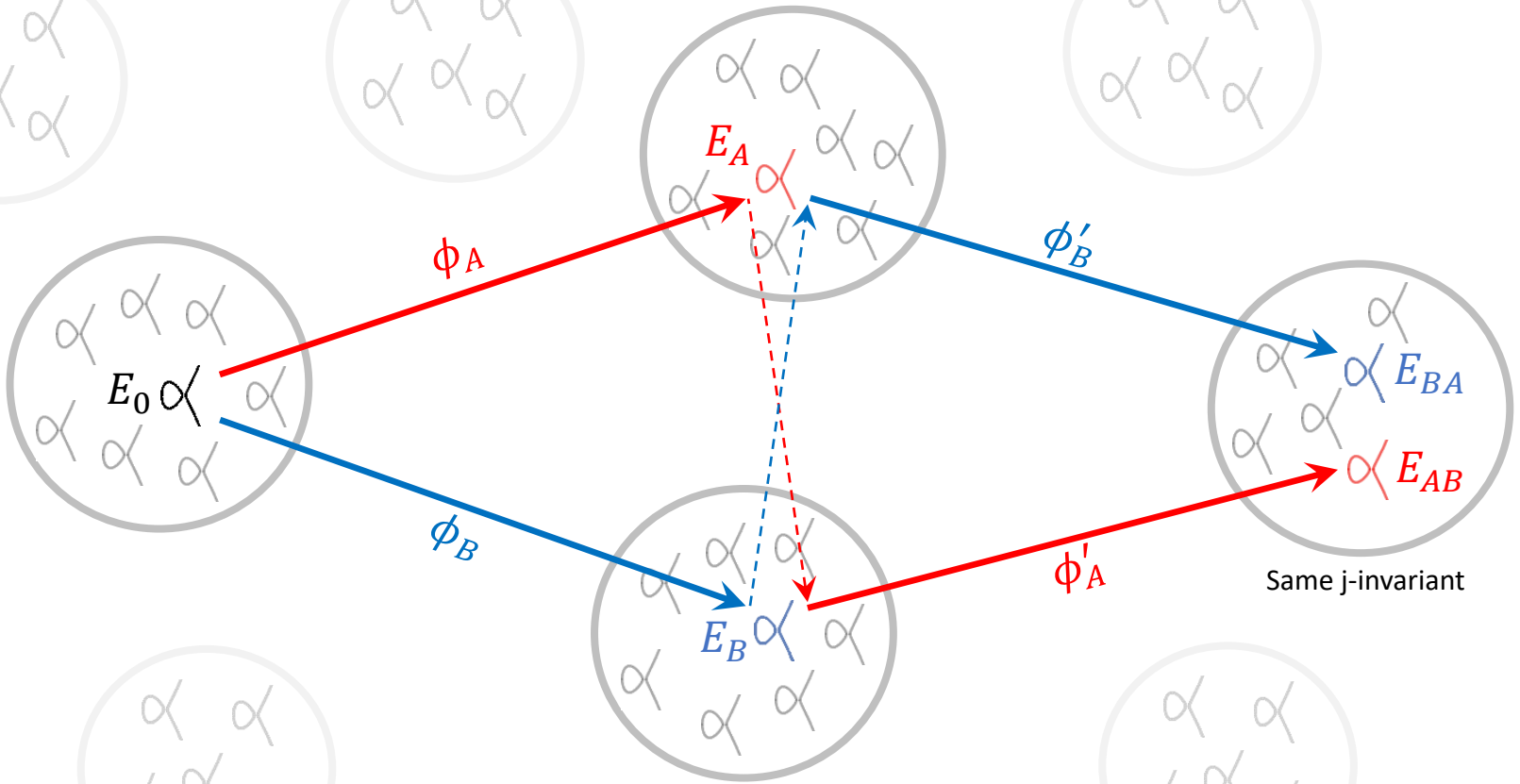
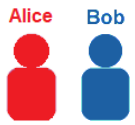




SIDH: setup

Set $\ell \in \{2,3\}$, supersingular curve $E_0/\mathbb{F}_{p^2}$ with a prime $p = f \cdot 2^{e_A}3^{e_B} - 1$ such that $2^{e_A} \approx 3^{e_B}$ and f small.

- Then: $E[2^{e_A}], E[3^{e_B}] \subset E_0(\mathbb{F}_{p^2})$

Alice

works over $E[2^{e_A}]$ using **2-isogenies** and linearly independent points P_A, Q_A.

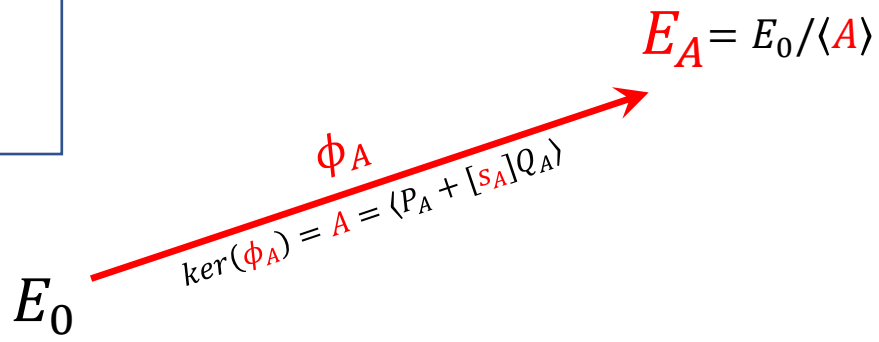
Bob

works over $E[3^{e_B}]$ using **3-isogenies** and linearly independent points P_B, Q_B.

SIDH protocol

private Alice private Bob
public params

E's are isogenous curves
P's, Q's, R's, S's are points



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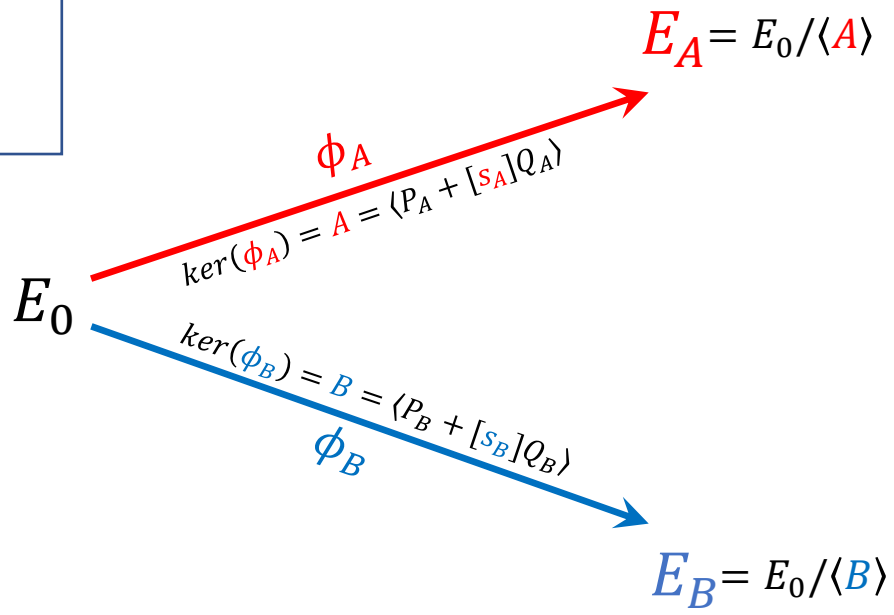

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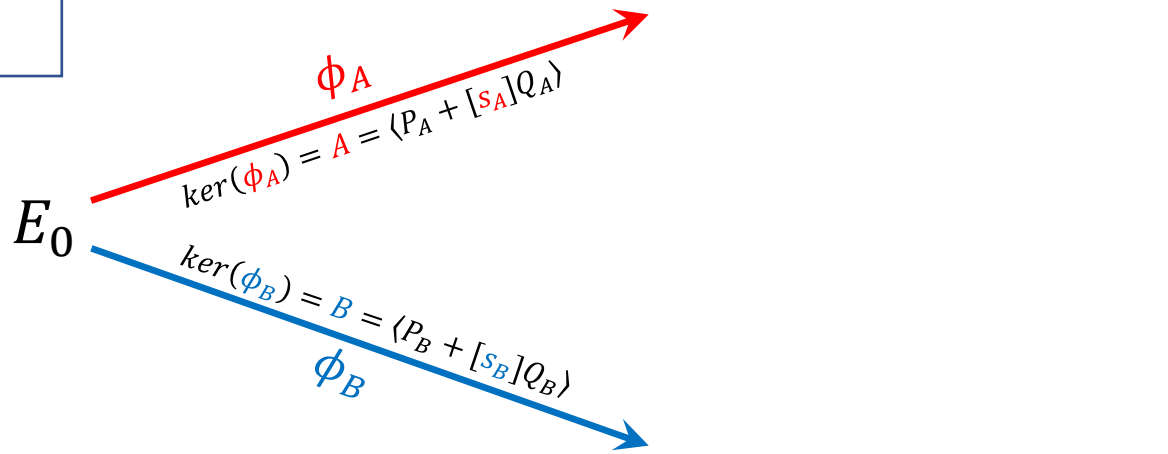

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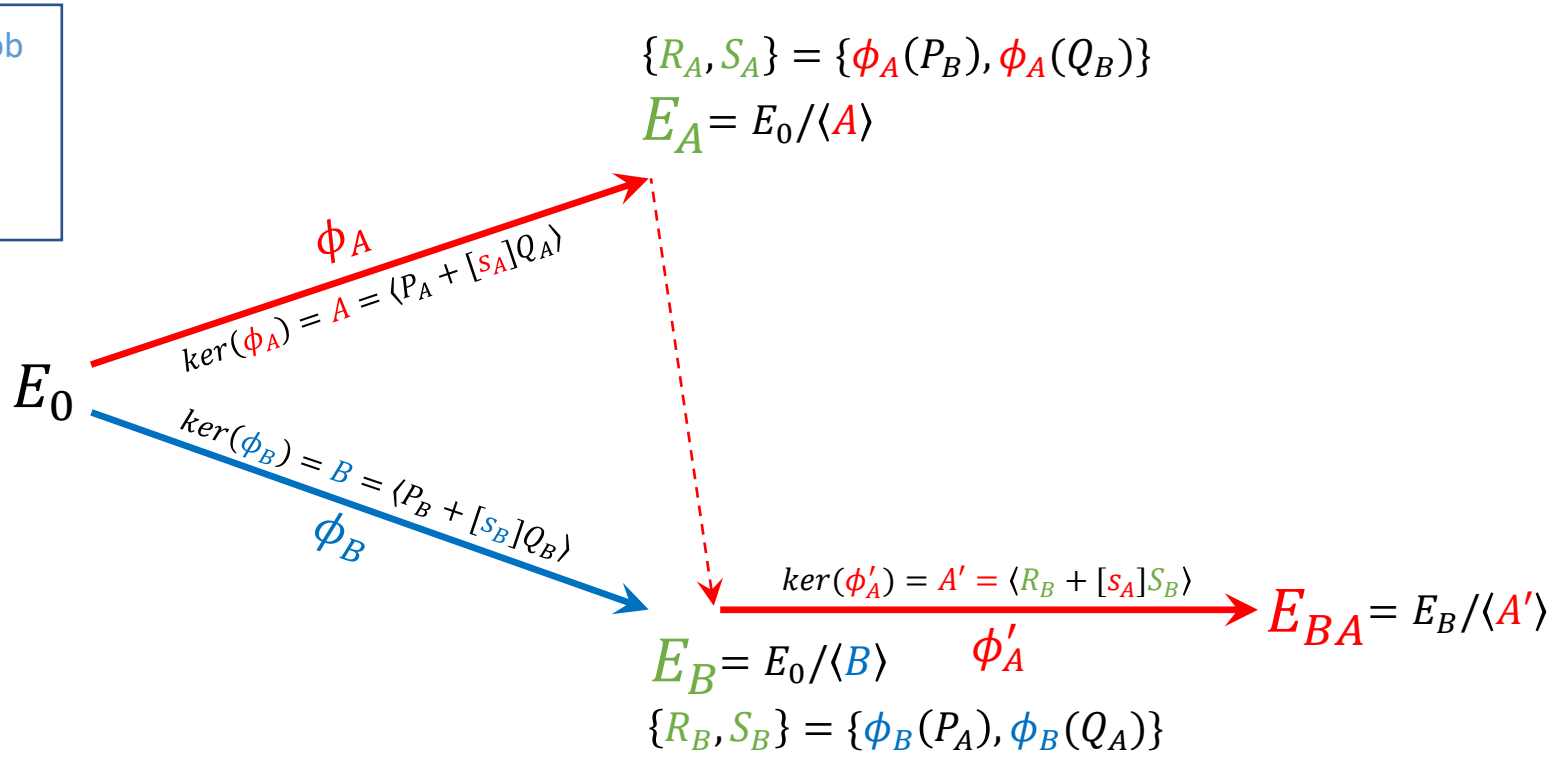

$$A' = \langle \phi_B(P_A) + [s_A]\phi_B(Q_A) \rangle = \langle \phi_B(P_A + [s_A]Q_A) \rangle = \langle \phi_B(A) \rangle$$

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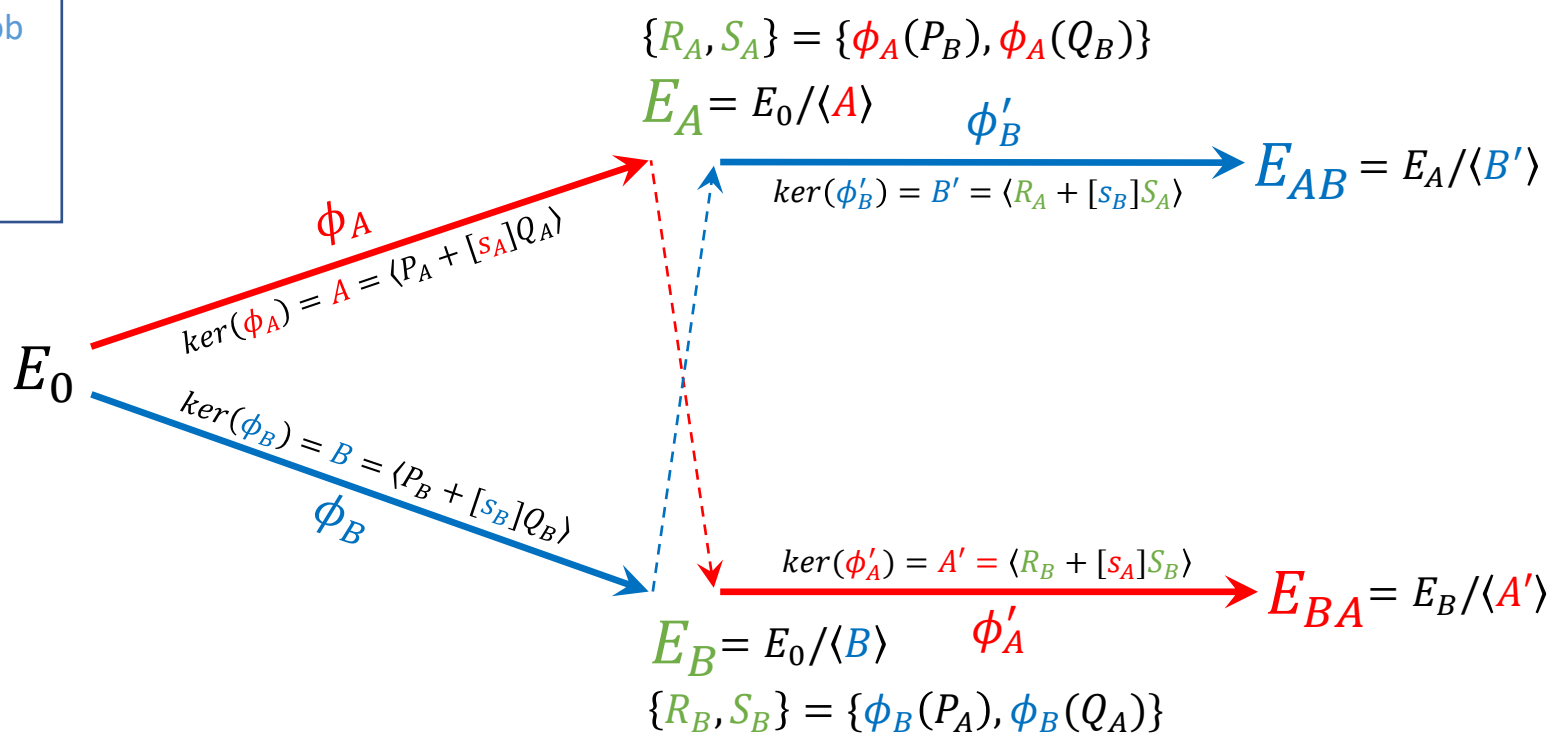

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$$E_{AB} = \phi'_B(\phi_A(E_0)) \cong E_0/\langle P_A + [s_A]Q_A, P_B + [s_B]Q_B \rangle \cong E_{BA} = \phi'_A\ (\phi_B(E_0))$$

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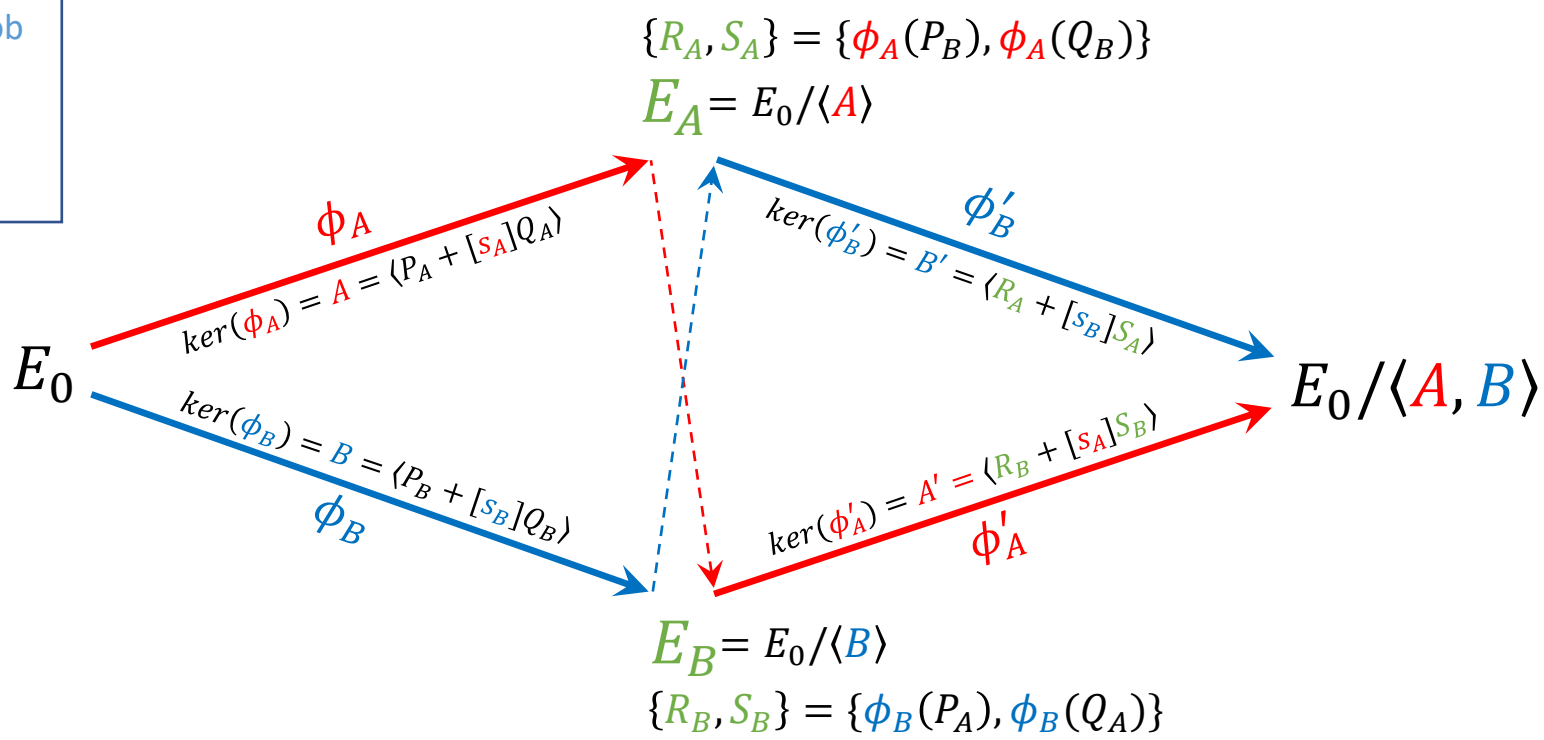
E's are isogenous curves
P's, Q's, R's, S's are points



$$A' = \langle \phi_B(P_A) + [s_A]\phi_B(Q_A) \rangle = \langle \phi_B(P_A + [s_A]Q_A) \rangle = \langle \phi_B(A) \rangle$$
$$B' = \langle \phi_A(P_B) + [s_B]\phi_A(Q_B) \rangle = \langle \phi_A(P_B + [s_B]Q_B) \rangle = \langle \phi_A(B) \rangle$$

$$E_{AB} = \phi'_B(\phi_A(E_0)) \cong E_0/\langle P_A + [s_A]Q_A, P_B + [s_B]Q_B \rangle \cong E_{BA} = \phi'_A\ (\phi_B(E_0))$$

SIDH protocol

Drawback:

- SIDH is not secure when keys are reused (Galbraith-Petit-Shani-Ti 2016)
  - Only recommended in **ephemeral mode**

Supersingular isogeny key encapsulation (SIKE)

- IND-CCA secure key encapsulation: no problem reusing keys!
- Uses a variant of Hofheinz–Hövelmanns–Kiltz (HHK) transform:

IND-CPA PKE → IND-CCA KEM

- HHK transform is secure in **both the classical and quantum ROM models**
- Offline key generation gives performance boost (no perf loss SIDH → SIKE)

Supersingular isogeny key encapsulation (SIKE)

KeyGen

1. $s_B \in_R [0, 2^{\lfloor \log_2 3^{e_B} \rfloor})$
2. Set $ker(\phi_B) = \langle P_B + [s_B]Q_B \rangle$
3. $\mathrm{pk}_B = \{\phi_B(E_0), \phi_B(P_A), \phi_B(Q_A)\}$
4. $s \in_R \{0,1\}^n$
5. **keypair:** $\{\mathrm{sk}_B = (s, s_B), \mathrm{pk}_B\}$

F, G, H instantiated with cSHAKE256.

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pk_B →

Encaps

1. message $m \in_R \{0,1\}^n$
2. $r = G(m, \mathrm{pk}_B) \bmod 2^{e_A}$
3. Set $ker(\phi_A) = \langle P_A + [r]Q_A \rangle$
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Decaps

1. $j' = j(E_{BA}) = j(\phi'_B(\phi_A(E_0)))$
2. $m' = F(j') \oplus c[2]$
3. $r' = G(m', \mathrm{pk}_B) \bmod 2^{e_A}$
4. Set $ker(\phi_A) = \langle P_A + [r']Q_A \rangle$
5. $\mathrm{pk}'_A = \{\phi_A(E_0), \phi_A(P_B), \phi_A(Q_B)\}$
6. If $\mathrm{pk}'_A = c[1]$ then
   Shared key: $ss = H(m', c)$
7. Else $ss = H(s, c)$

pk_B →

← $c = (\mathrm{pk}_A, F(j) \oplus m)$

Encaps

1. message $m \in_R \{0,1\}^n$
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pk_B →

Encaps

1. message $m \in_R \{0,1\}^n$
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2. $m' = F(j') \oplus c[2]$
3. $r' = G(m', \mathrm{pk}_B) \bmod 2^{e_A}$ *decryption*
4. Set $ker(\phi_A) = \langle P_A + [r']Q_A \rangle$
5. $\mathrm{pk}'_A = \{\phi_A(E_0), \phi_A(P_B), \phi_A(Q_B)\}$
6. If $\mathrm{pk}'_A = c[1]$ then *partial re-encryption*
   Shared key: $ss = H(m', c)$
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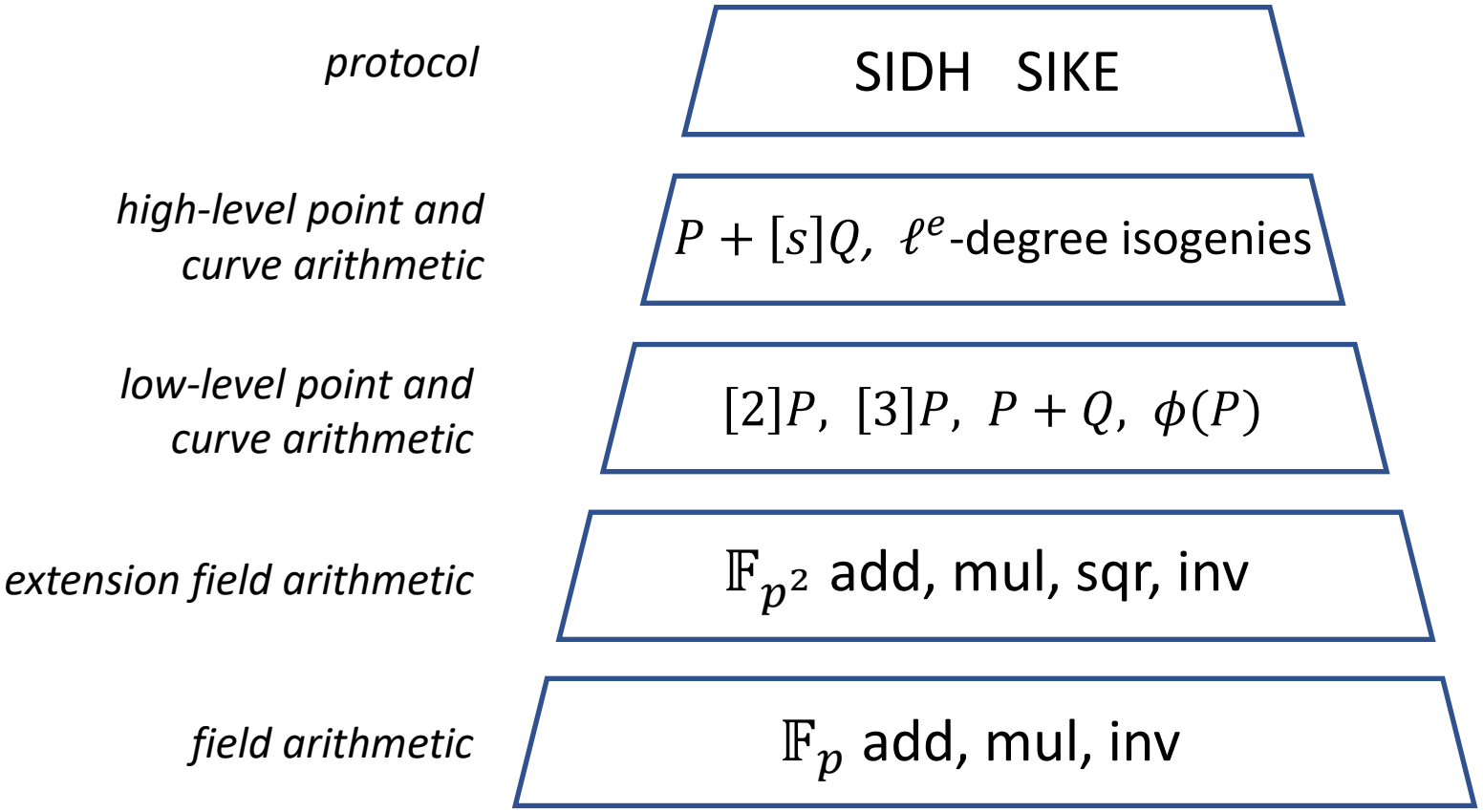
F, G, H instantiated with cSHAKE256.

Computational aspects

Computation layers



Computational aspects:
High-level point and curve arithmetic

High-level point and curve arithmetic

Two main internal computations:

- **Double-scalar multiplications** to construct kernels $\langle P + [s]Q \rangle$
- **Smooth, ℓ^e-degree isogeny** computations ϕ: $E_0 \rightarrow E'$

Computing $P + [s]Q$

Three-point differential ladder (x-only, variable point)

- De Feo-Jao-Plût (2014), step cost = **1DBL + 2ADD**
- Faz-Hernández et al. (2018), step cost = **1DBL + 1ADD**

Computing $P + [s]Q$

[Faz-Hernández–López–Ochoa-Jiménez–Rodríguez-Henríquez 2018]

$\boldsymbol{s = (01100)_2}$ $\quad$ $\boldsymbol{R_1 = P}$ $\quad$ $\boldsymbol{R_0 = Q}$ $\quad$ $\boldsymbol{R_2 = Q - P}$

Computing $P + [s]Q$

[Faz-Hernández–López–Ochoa-Jiménez–Rodríguez-Henríquez 2018]

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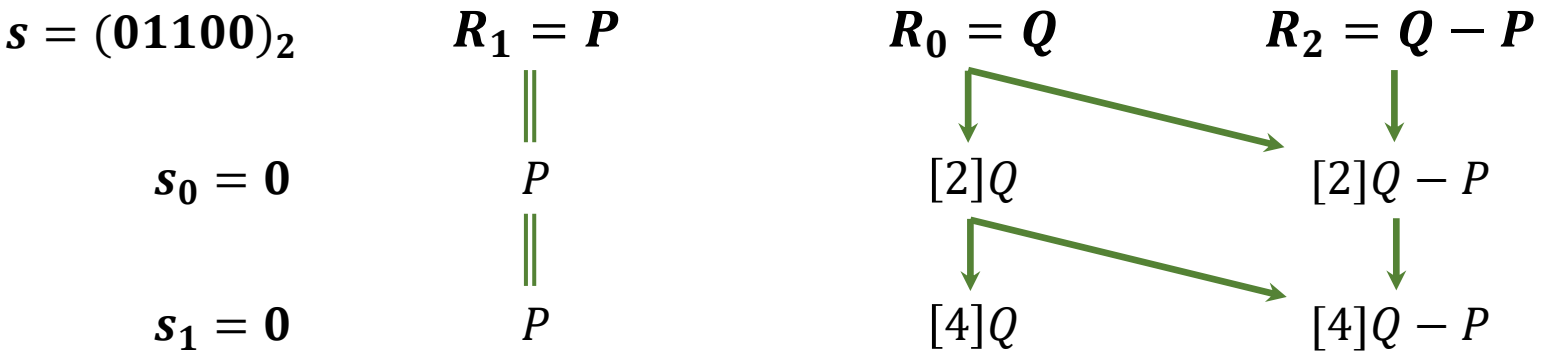

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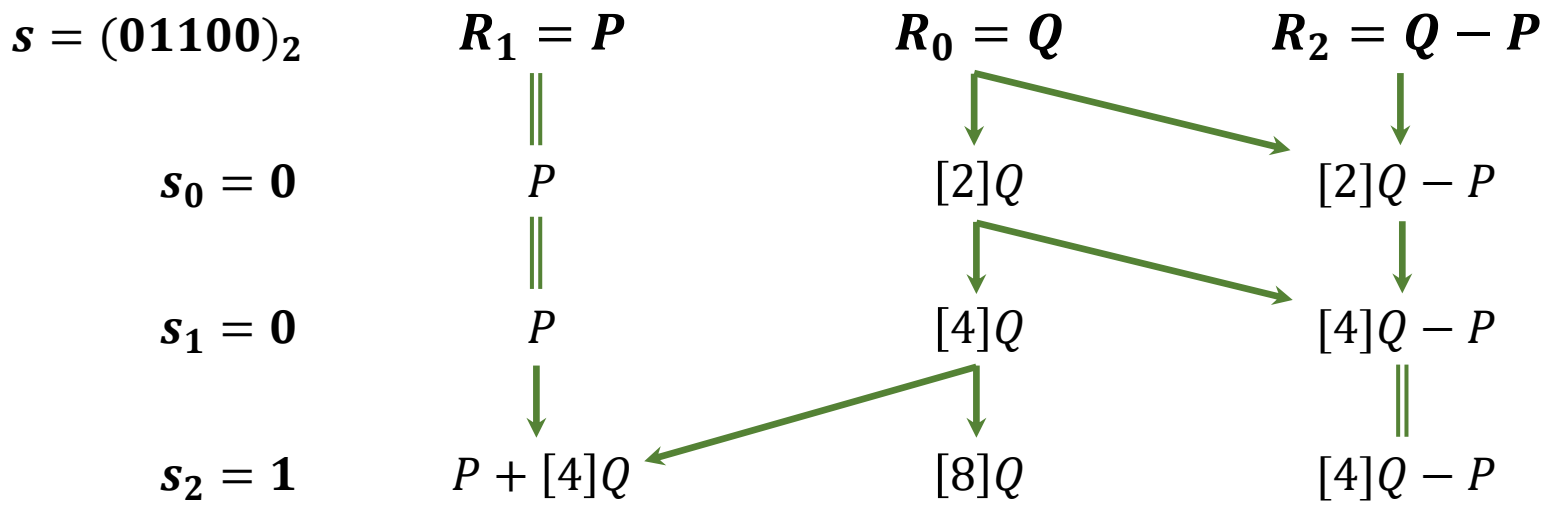

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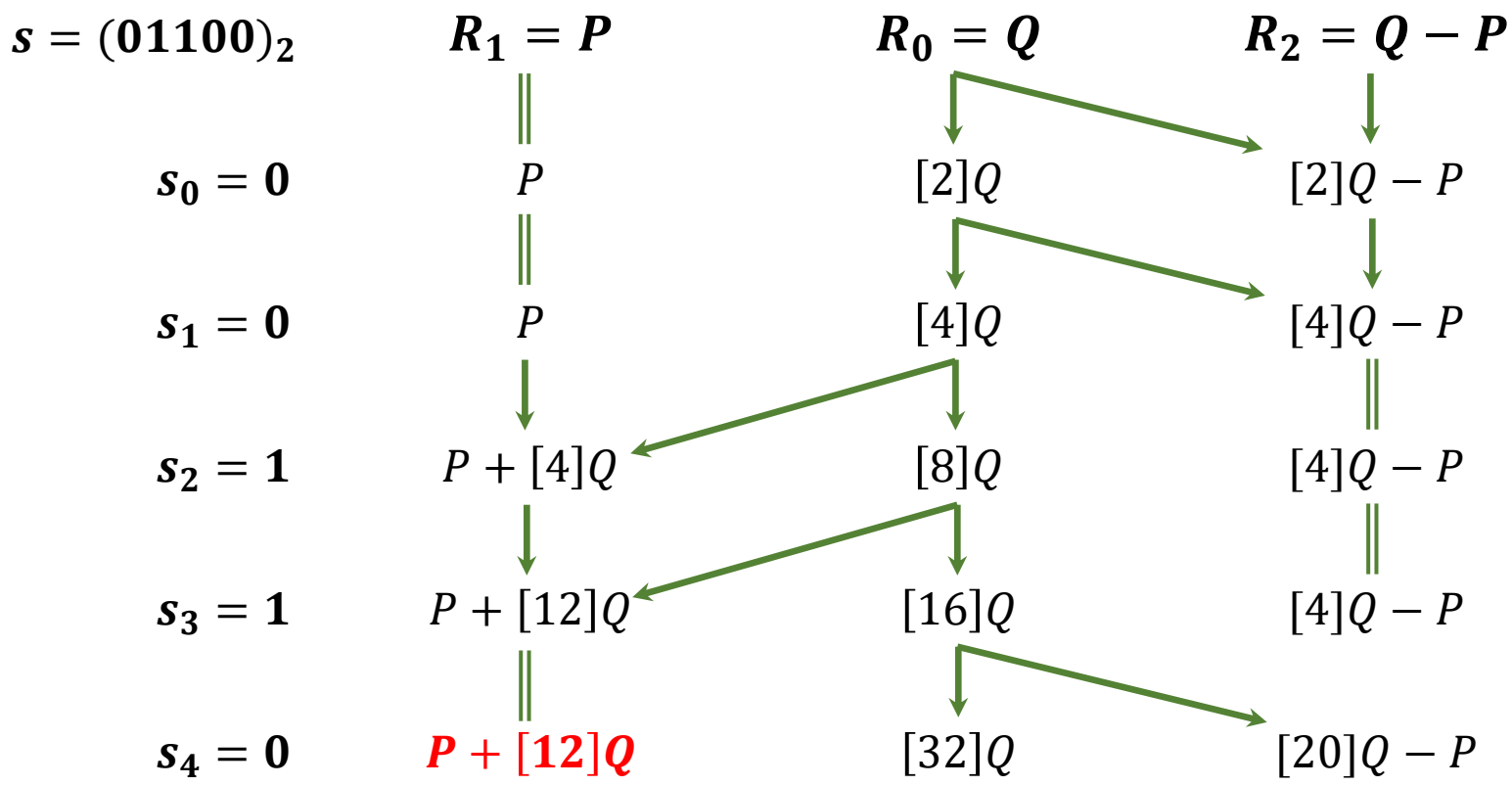
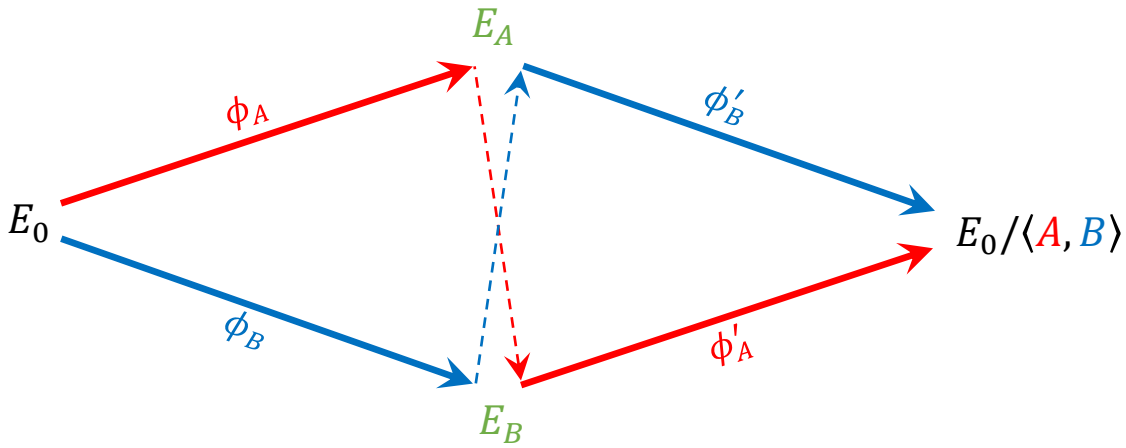


Computing ℓ^e-degree isogenies

- Construct it as a composition of multiple (small, prime-degree) isogenies



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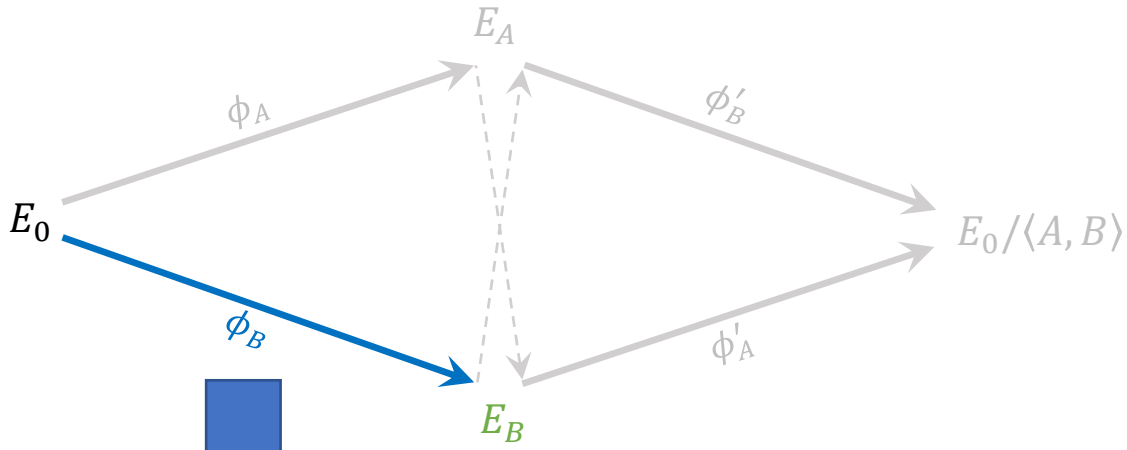
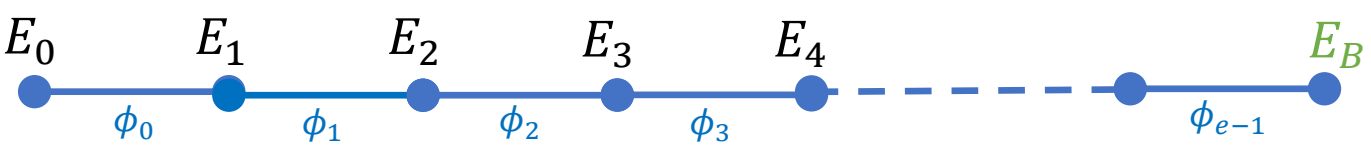


$$\phi_B \colon E_0 \to E_B$$

$$\phi_B = \phi_0 \cdot \phi_1 \cdot \phi_2 \cdot \cdots \cdot \phi_{e-1}$$



Computing ℓ^e-degree isogenies

- **Example:** Bob ($\ell = 3$) computes $E_B = \phi_B(E_0)$

  Let base point $P_0 \in E_0$. Assume $e = 4$

  Compute 3^4-degree isogeny:

$$\phi_B \colon E_0 \to E_4$$

$$\phi_B = \phi_0 \cdot \phi_1 \cdot \phi_2 \cdot \phi_3$$

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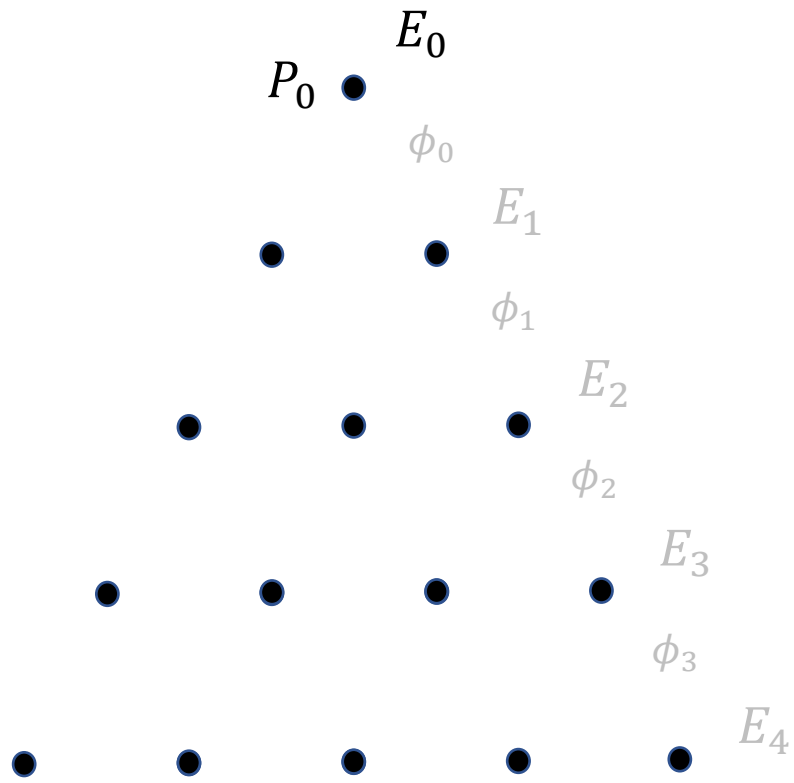

- Iteratively compute:

$$E_{i+1} = E_i/\langle [\ell^{e-i-1}]P_i \rangle$$

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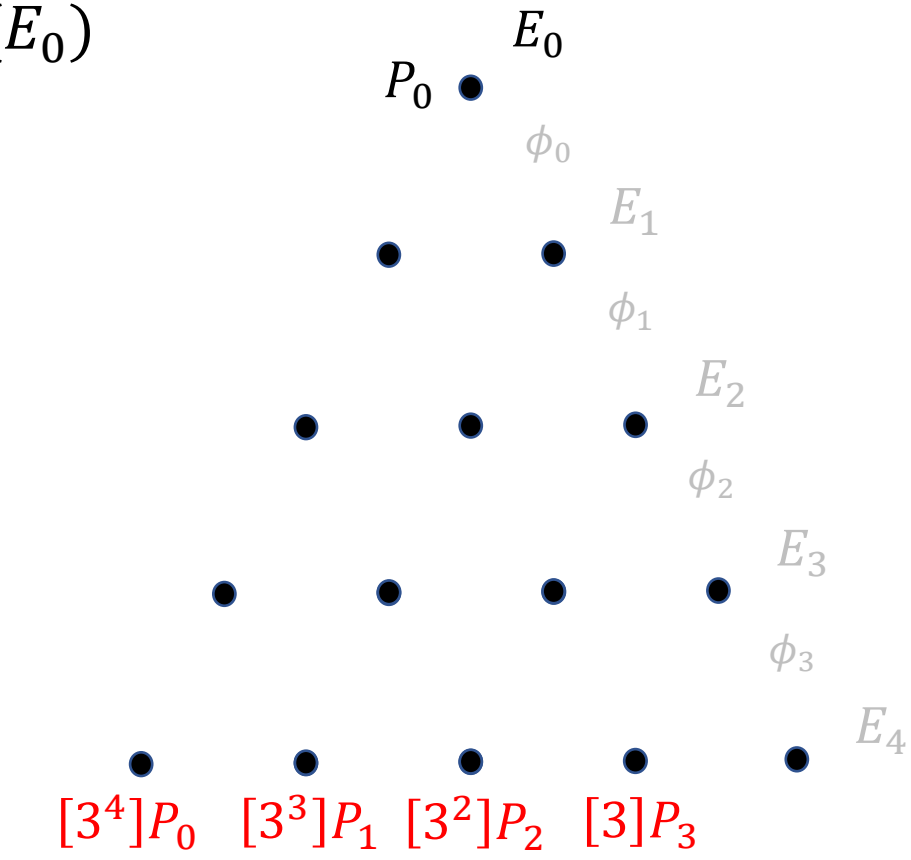

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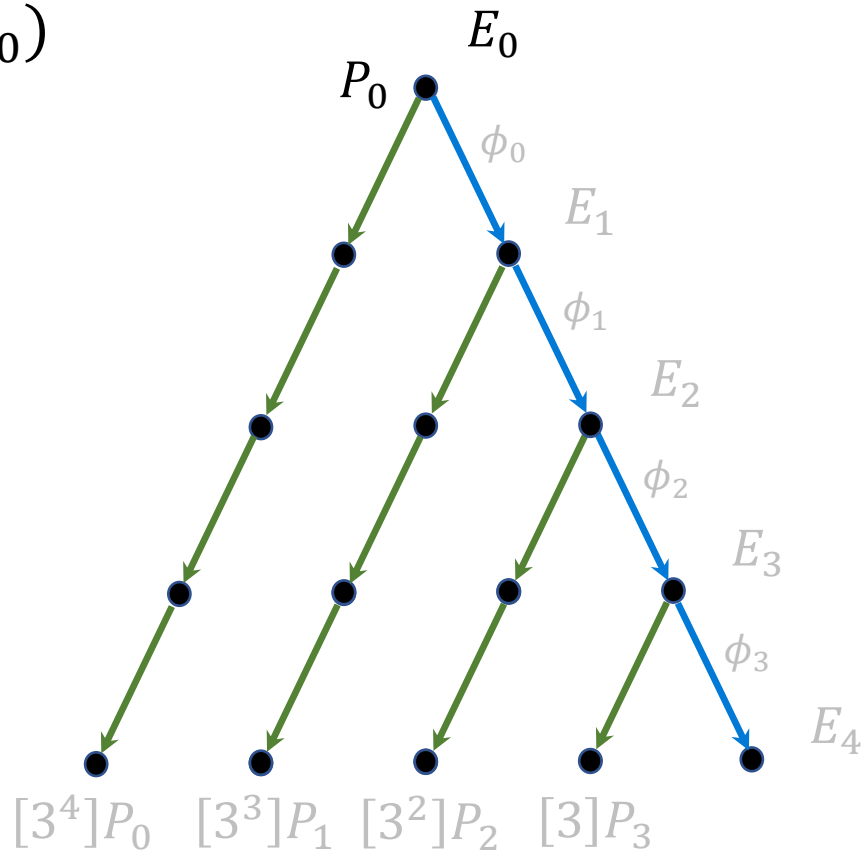

(+) slope: point operations

(−) slope: isogeny operations

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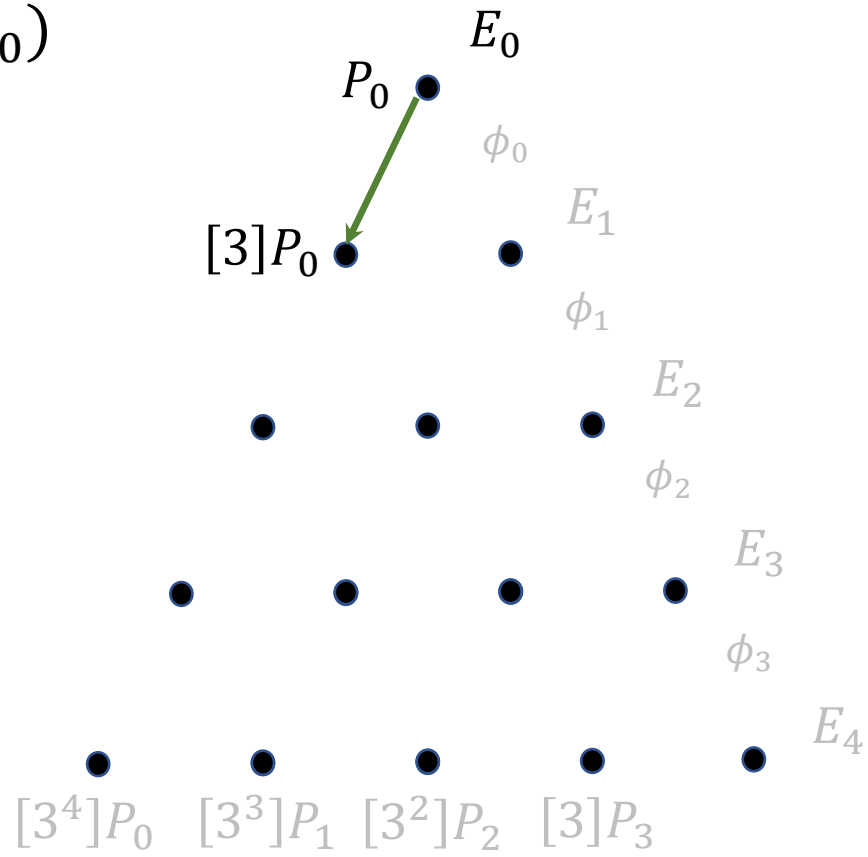

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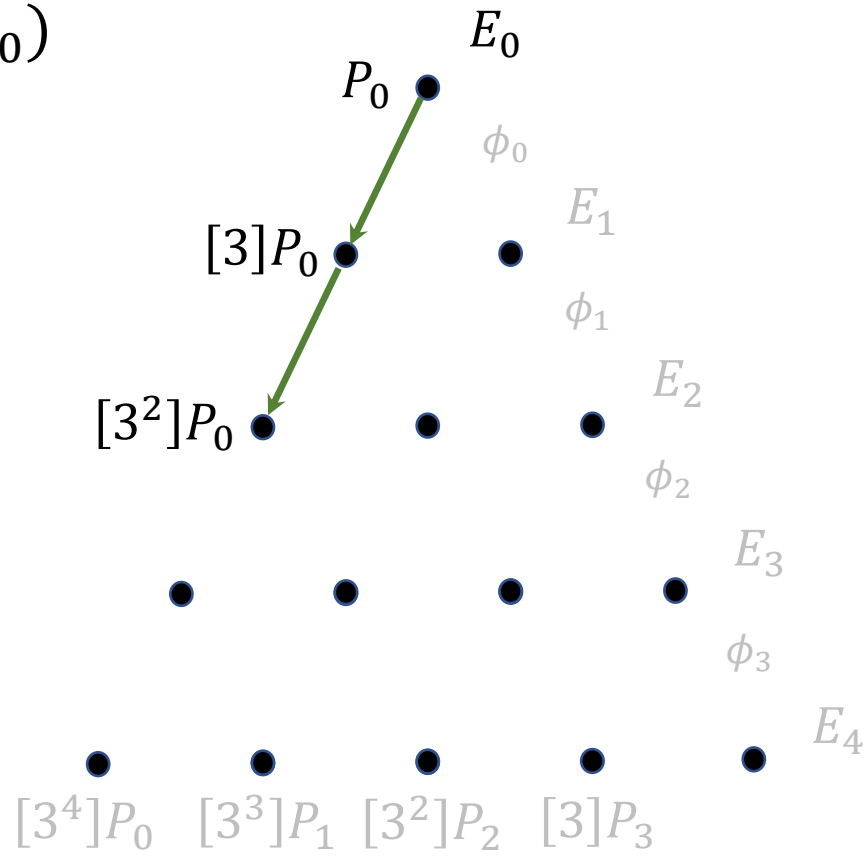

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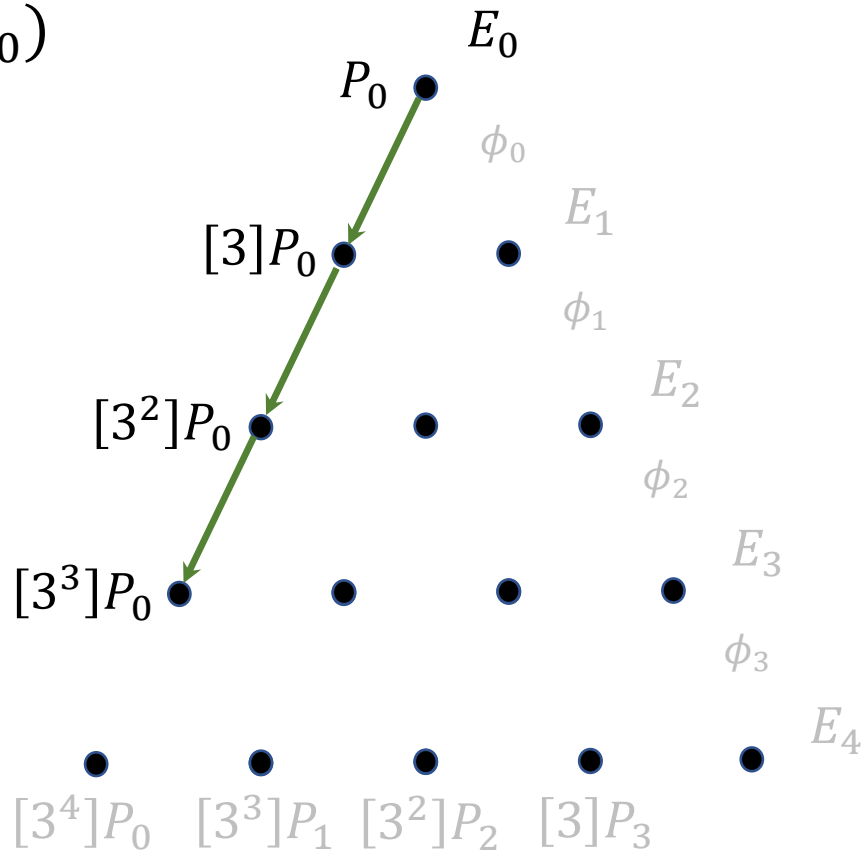

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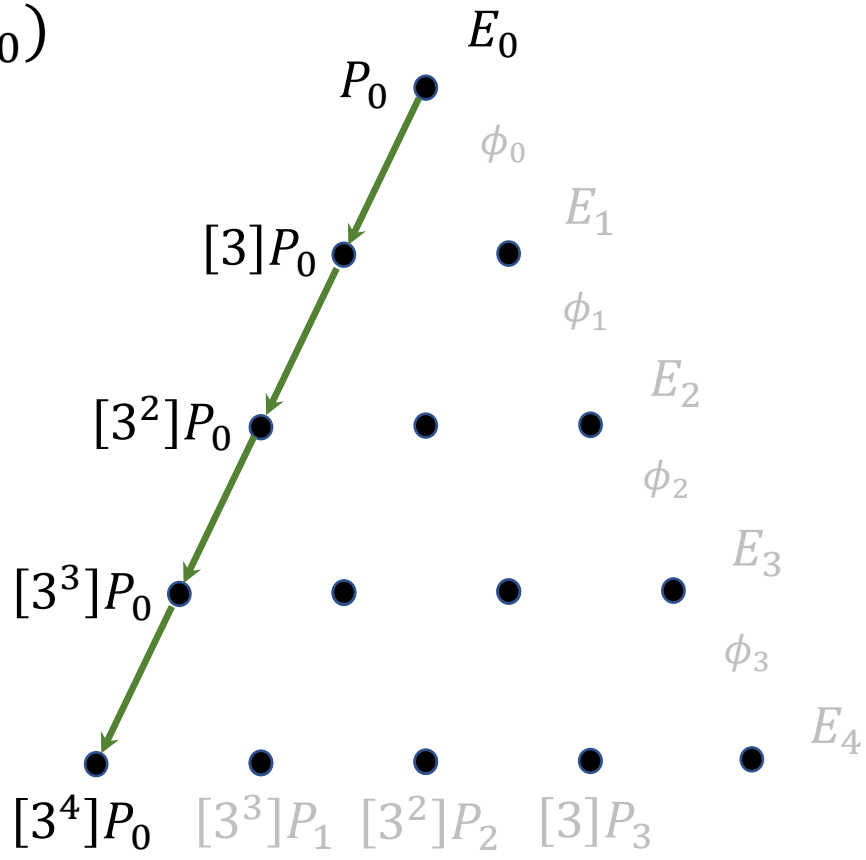

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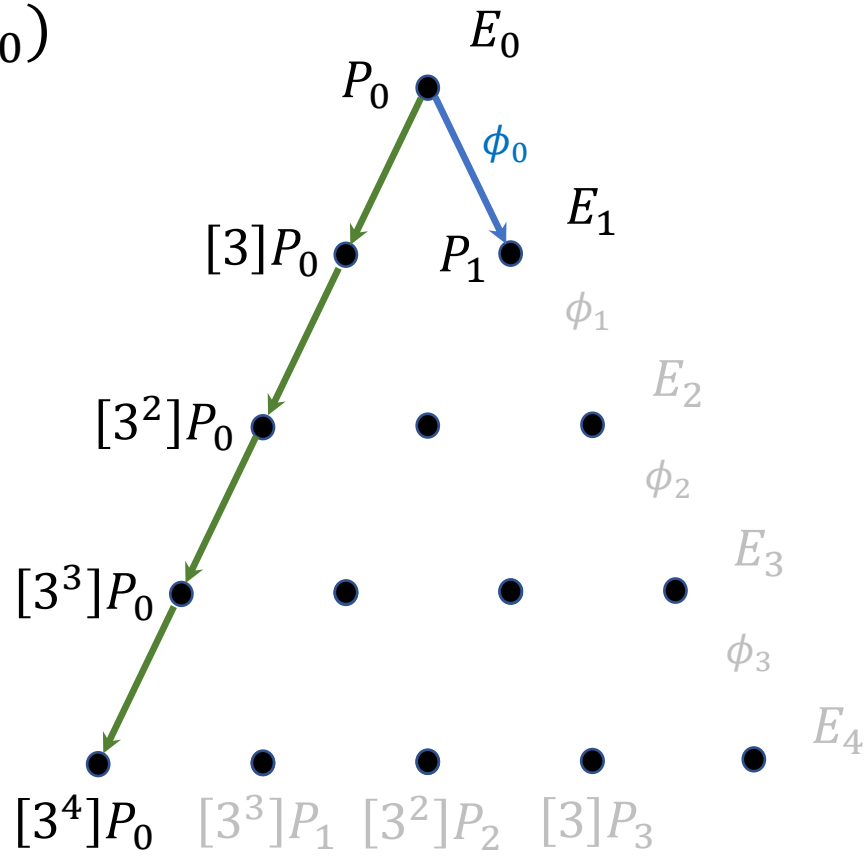

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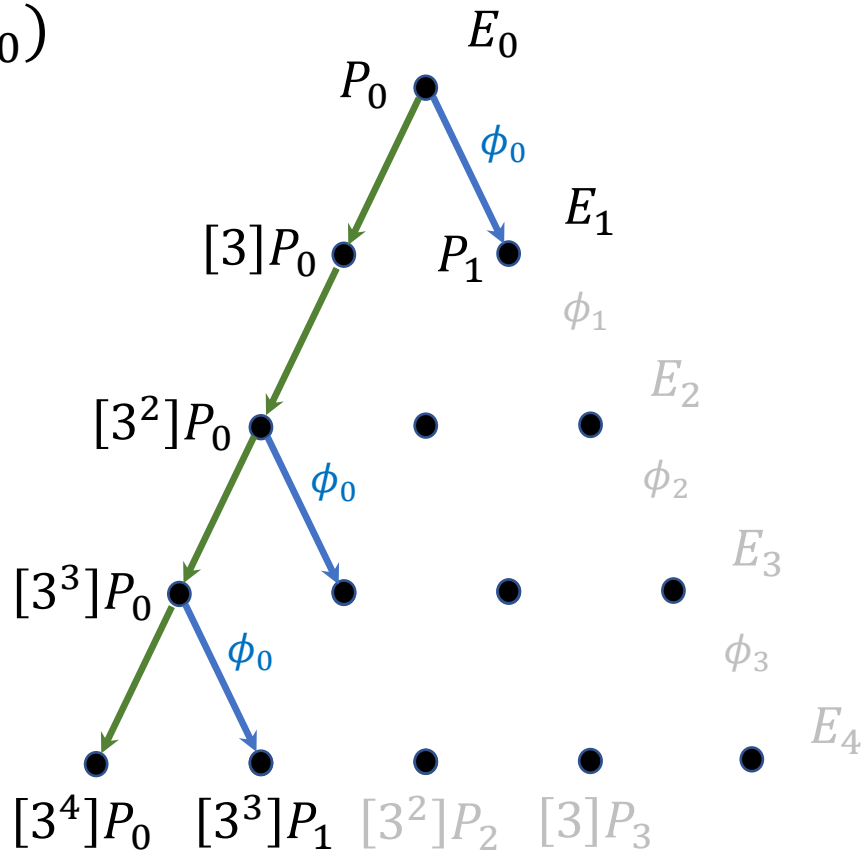

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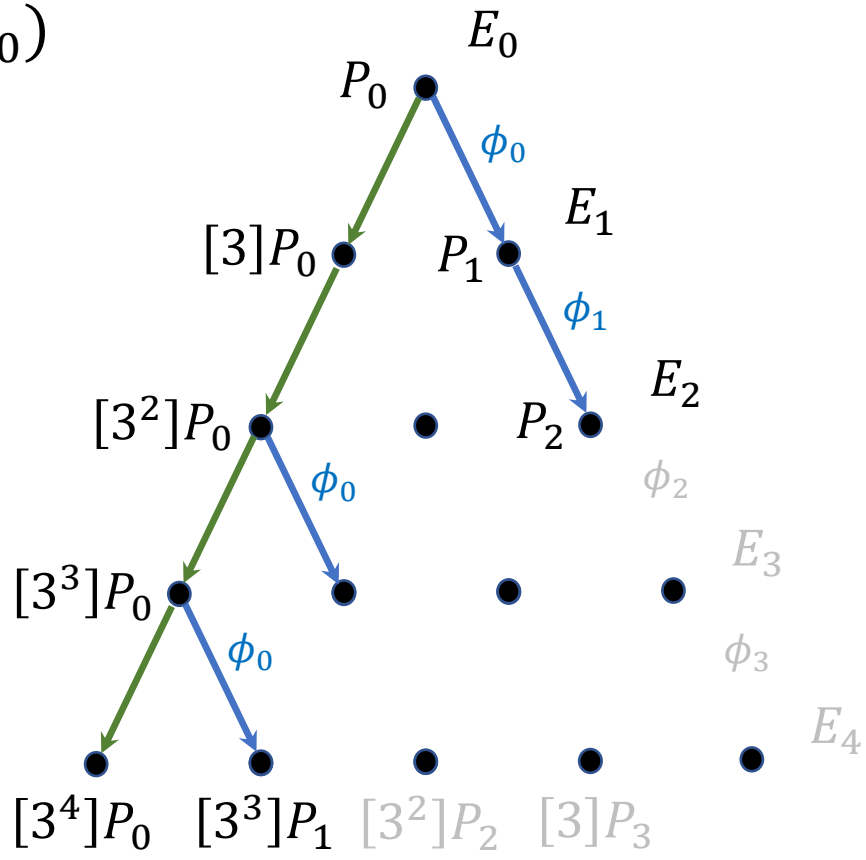

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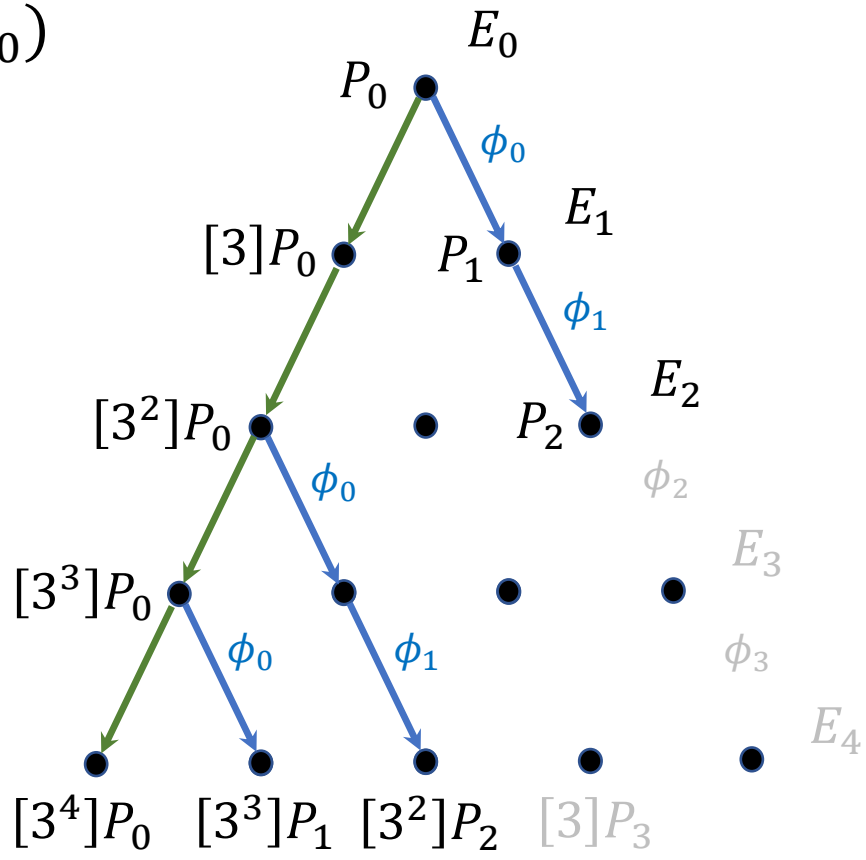

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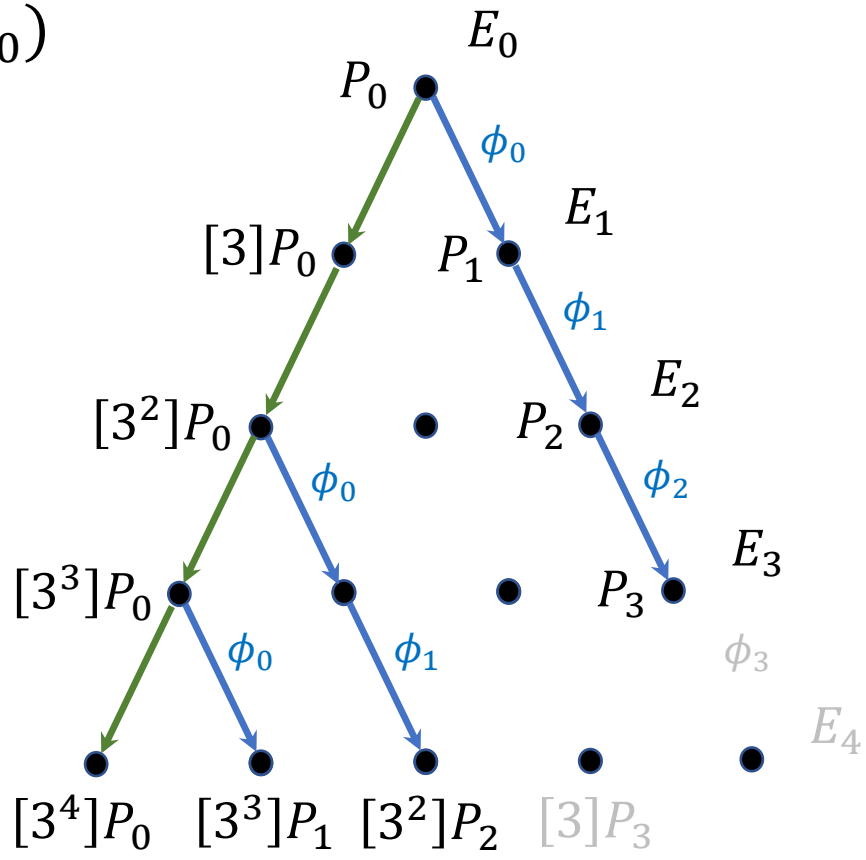
$$\phi_2 = E_2/\langle 9P_2 \rangle$$
$$E_3 = \phi_2(E_2)$$
$$P_3 = \phi_2(P_2)$$



Computing ℓ^e-degree isogenies

- **Example:** Bob ($\ell = 3$) computes $E_B = \phi_B(E_0)$

  Let base point $P_0 \in E_0$. Assume $e = 4$

  Compute 3^4-degree isogeny:

$$\phi_B \colon E_0 \to E_4$$

$$\phi_B = \phi_0 \cdot \phi_1 \cdot \phi_2 \cdot \phi_3$$

$$E_4 = E_0/\langle P_0 \rangle$$

$$\phi_2 = E_2/\langle 9P_2 \rangle$$
$$E_3 = \phi_2(E_2)$$
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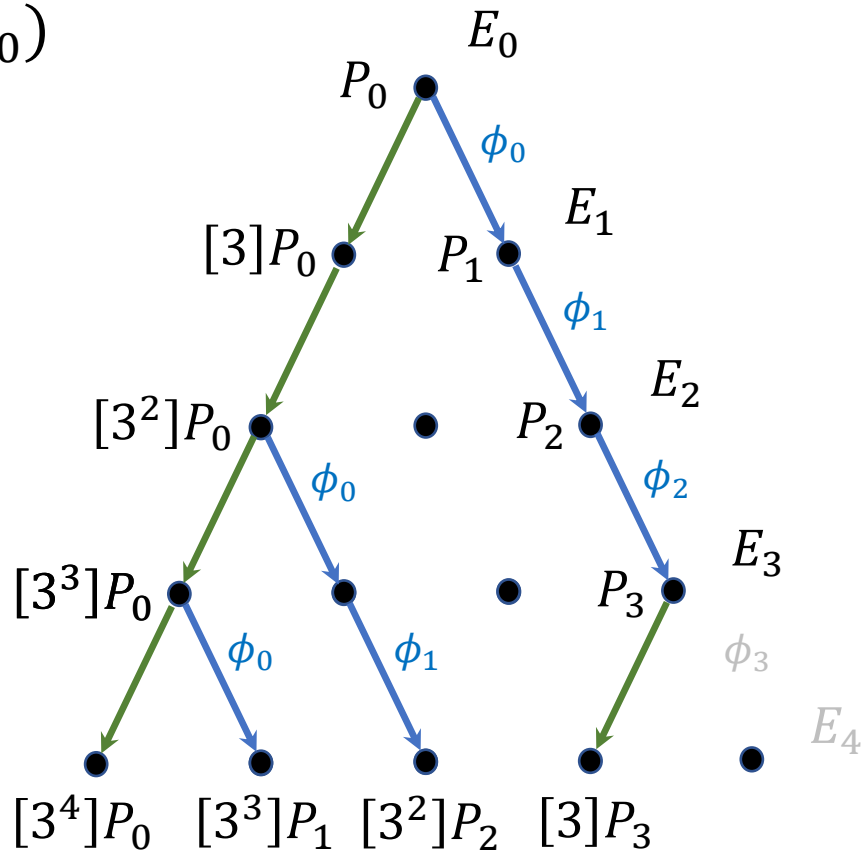

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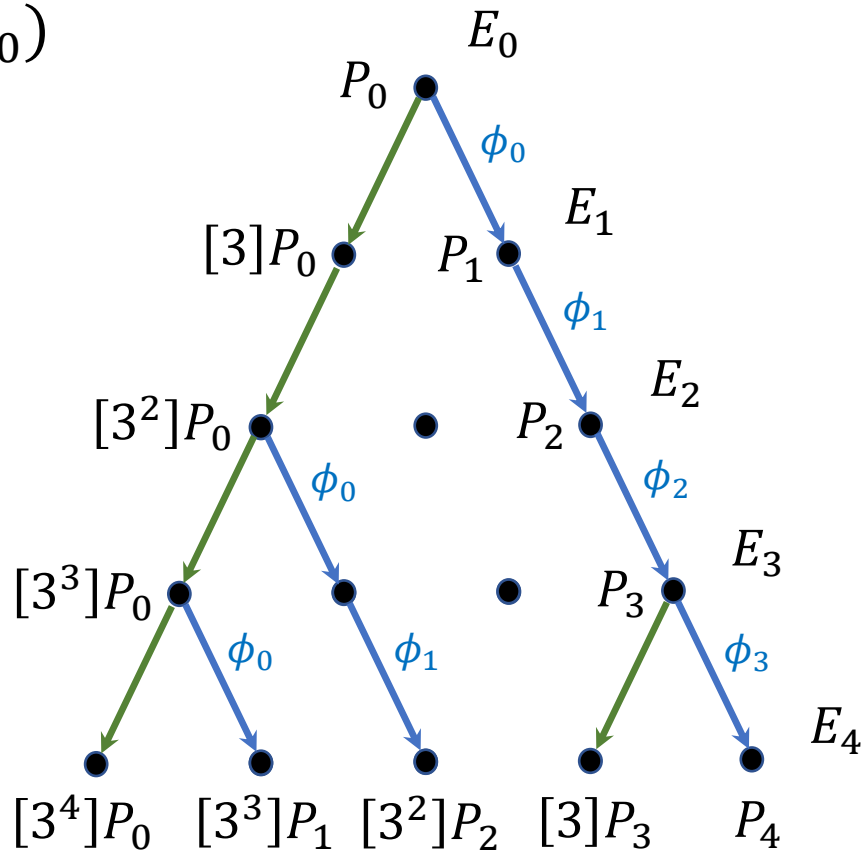

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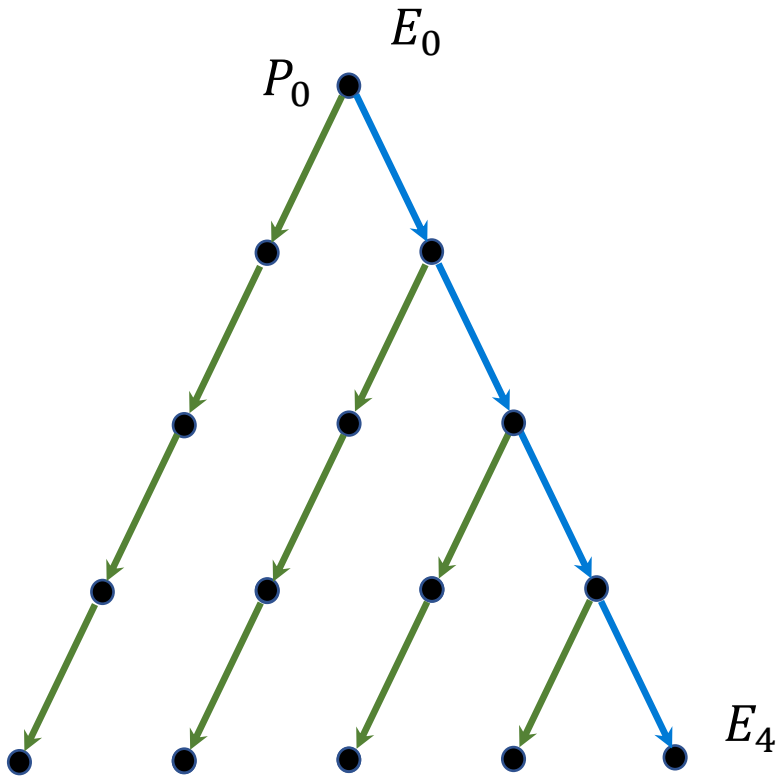
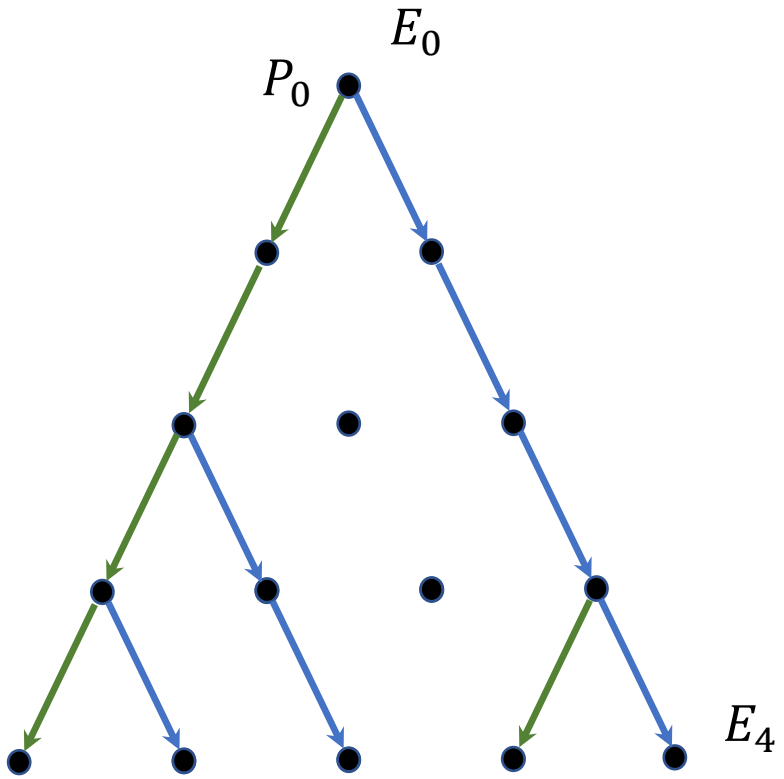


naïve



optimal

Optimal strategy: reduction from $\mathcal{O}(e^2)$ operations to $\mathcal{O}(e \log e)$

Computational aspects: Field arithmetic

Extension field arithmetic

Constructing degree-2 extension field $\mathbb{F}_{p^2}$ of a finite field $\mathbb{F}_p$:

Fix $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$, with degree-2 irreducible polynomial $f(x)$ in $\mathbb{F}_p[x]$ s.t. $f(\alpha) = 0$

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In our case: for a prime $p \equiv 3 \bmod 4$, take $\mathbb{F}_{p^2} = \mathbb{F}_p(i)$, with $i^2 + 1 = 0$

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Let $a = (a_0, a_1), b = (b_0, b_1) \in \mathbb{F}_{p^2}$, then:

- $a \pm b$
- $a \times b$
- a^2
- a^{-1}

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- $a^{-1} = \left(a_0 \times \left(a_0^2 + a_1^2\right)^{-1}, -a_1 \times \left(a_0^2 + a_1^2\right)^{-1}\right)$

  cost: 2 $\mathbb{F}_p$ mul + 2 $\mathbb{F}_p$ sqr + 1 $\mathbb{F}_p$ add + 1 $\mathbb{F}_p$ neg + 1 $\mathbb{F}_p$ inv

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- Two main approaches to implement integer multiplication + reduction:

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- Separated (integer multiplication and reduction) approach is preferred in most software platforms
  - Enables lazy reduction at $\mathbb{F}_{p^2}$ level

Integer multiplication

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- x64: limited number of registers, availability of carry-preserving instructions (e.g., `mulx`)
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Also: $p' \bmod 2^w \equiv 1$ for $w = 32, 64$

Security and parameters

SIDH security

Setting: supersingular curves $E_1/\mathbb{F}_{p^2}$ and $E_2/\mathbb{F}_{p^2}$, a large prime p, and isogeny $\phi: E_1 \to E_2$ with fixed, smooth, public degree.

> ***Supersingular isogeny problem:*** given $P, Q \in E_1$ and $\phi(P), \phi(Q) \in E_2$, compute ϕ.

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  For SIDH/SIKE: number of order-$\ell^{e/2}$ subgroups of $E[\ell^e] = |S| \approx p^{1/4}$

  Assume storage $w \approx 2^{80}$

$$\mathcal{O}\left(\frac{|S|^{\frac{3}{2}}}{\sqrt{w}}\right)$$

SIKE parameters (round 2 and 3)

- Eight parameter sets submitted to NIST:
  - SIKEp434, SIKEp503, SIKEp610, SIKEp751, and their corresponding compressed variants

Scheme (SIKEp + $\lceil \log_2 p \rceil$)	(e_A, e_B)	Security level	Standard (bytes)		Compressed (bytes)	
			pk	ct	pk	ct
SIKEp434	(216,137)	AES-128 (level 1)	330	346	196	209
SIKEp503	(250,159)	SHA3-256 (level 2)	378	402	224	248
SIKEp610	(305,192)	AES-192 (level 3)	462	486	273	297
SIKEp751	(372,239)	AES-256 (level 5)	564	596	331	363

Starting curve $E_0/\mathbb{F}_{p^2}$: $y^2 = x^3 + 6x^2 + x$, where $p = 2^{e_A}3^{e_B} - 1$.

Implementation results

SIDH Library

- Current release: **version 3.3**

  https://github.com/Microsoft/PQCrypto-SIDH

- Implements **SIDH** and **SIKE** with the *four standard parameter sets*: SIDH/SIKE{p434, p503, p610, p751} and their *four compressed* counterparts

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    - With high-speed x64 assembly for the field arithmetic
    - With high-speed ARMv8-A assembly for the field arithmetic
- No secret branches, no secret memory accesses: protected against cache and timing attacks

Performance on x64 (SIDH v3.3)

- Total time: Encapsulation + Decapsulation

Primitive	Standard		Compressed	
	Cycles ($\times 10^6$)	Time	Cycles ($\times 10^6$)	Time
SIKEp434	20.0	5.9 ms	26.2	7.7 ms
SIKEp503	27.9	8.2 ms	36.3	10.7 ms
SIKEp610	54.7	16.1 ms	66.9	19.7 ms
SIKEp751	84.6	24.9 ms	110.1	32.4 ms

(*) Obtained on a 3.4GHz Intel Core i7-6700 (Skylake) processor.

Performance on 64-bit ARM (SIDH v3.3)

Primitive	NIST sec level	Cycles ($\times 10^6$)	Time @1.992GHz
Passively secure key-exchange			
SIDHp434	1	60.8	30.5 ms
SIDHp503	2	88.4	44.4 ms
IND-CCA secure KEMs			
SIKEp434	1	59.4	29.8 ms
SIKEp503	2	82.7	41.5 ms

(*) Obtained on a 1.992GHz ARM Cortex-A72 (ARMv8-A) processor.

Performance on x64

Primitive	PQ security	Problem	Speed	Comm.
Classical				
RSA 3072	~0 bits	factoring	4.6 ms	0.8 KB
ECDH NIST P-256	~0 bits	EC dlog	1.4 ms	0.1 KB
Passively secure key-exchange				
SIDHp434	128 bits	isogenies	6.0 ms	0.6 KB
IND-CCA secure KEMs				
Kyber	100 bits	M-LWE	0.03 ms	0.7 KB
FrodoKEM	108 bits	LWE	1.1 ms	9.5 KB
SIKEp434	128 bits	isogenies	5.9–7.7 ms	0.2–0.3 KB

(*) Obtained on 3.4GHz Intel Haswell (Kyber) or Skylake (FrodoKEM and SIKE).

Recent research

The Cost to Break SIKE: A Comparative Hardware-Based Analysis with AES and SHA-3, L-Wang-Szefer, 2020. https://eprint.iacr.org/2020/1457

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- Uses a budget-based cost model to estimate cryptanalysis cost of vOW versus brute forcing AES.
- E.g., suggests SIKEp377 to match AES128 security (level 1).
  - ~1.4x faster than SIKEp434, ~13% smaller public keys

SIKE in the NIST post-quantum “competition”

- SIKE website: http://sike.org/
- SIKE specification: http://sike.org/files/SIDH-spec.pdf
- SIDH Library: https://github.com/Microsoft/PQCrypto-SIDH
- Package (protocol specification and implementations) submitted to NIST:

https://csrc.nist.gov/CSRC/media/Projects/post-quantum-cryptography/documents/round-3/submissions/SIKE-Round3.zip

Software Implementation of (Post-Quantum) Public-Key Cryptography

Patrick Longa

Microsoft Research

https://microsoft.com/en-us/research/people/plonga
http://patricklonga.com
Twitter: @PatrickLonga

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