2

Of Langragian Mechanics

Newtonian mechanics suffers from 2 limitations – ① deals with particles ② describes motion in special Cartesian coordinate system. Since any extended body can be broken into particles, and because the chain rule can transform equations of motions to any reference frame, Newtonian mechanics does just fine.

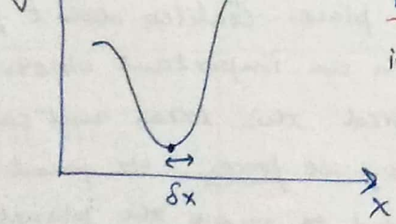
Let there be N generalized coordinates ~~to~~ required to completely define a system. These are the number of degrees of freedom of the system.

Configuration space – an imaginary N-dimensional space for which q_i constitute Cartesian coordinates

Of Principle Of Least Action – Kind of misnomer (the action isn't minimum), it tells the action is stationary.

Consider any potential of a system $V(x)$, where x is the coordinates of the system. $\frac{d\,V(x)}{dx} = 0$ signifies the condition of equilibrium (or no force on the system or the system is stationary).

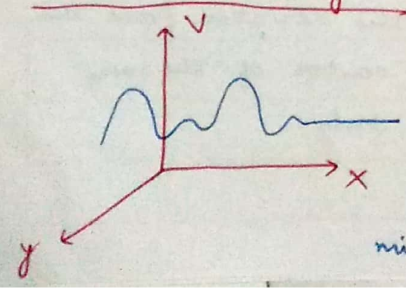
Formally, you are in equilibrium when $\frac{dV}{dx} \cdot \delta x = \delta V$ (or nudging the x in a direction on the following graph:



So on nudging, a δU potential builds up in the system. A system is in equilibrium if $\boxed{\delta V = 0}$

Equilibrium is not about the potential energy be minimum, it simply means the PE is stationary.

Now extending to higher dimensions –



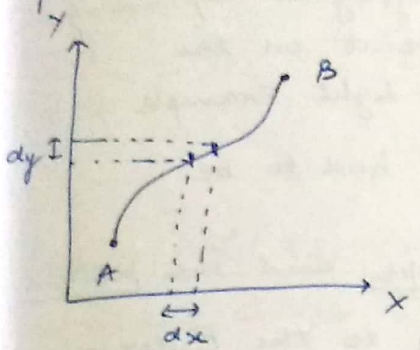
The ~~PE~~ system is in equilibrium when $\frac{\partial V}{\partial x_i} = 0$ for all i

so at ~~the~~ some points, the system remains stationary (it might be minima, maxima, saddle etc).

(3) So $\delta V = \frac{\partial V}{\partial x} \cdot \delta x + \frac{\partial V}{\partial y} \cdot \delta y$. If $\delta V = 0$ for every possible choice of δx or δy, the system is stationary (or the system is in equilibrium.

Principle of least action – For a particle of fixed energy traveling from point A to point B, the trajectory is such that the corresponding action has a minimum. So basically, we need to find a function that minimizes some quantity. And this can be found by principles of variational calculus.

ex shortest line between 2 points – so which curve on a 2D plane minimizes the distance between 2 points.



Length of the segment $= ds$

$= \sqrt{(dx)^2 + (dy)^2}$

$= dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ a function of position and the slope, something that the $\mathcal{L}$ is full of.

Total distance – $\int dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

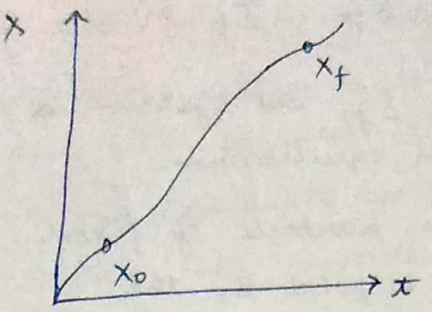
You can think this like – if for a given function dS is minimium, so any change in the function increases the length dS. Thus, $\delta(dS) = 0$. So the solution is a straight line. (just because the surface is a plane On a curve, the shortest distance function isn't so smooth).

ex light travel direction – tries to minimize time, and if moving with a constant velocity, this is equivalent to minimizing distance.

A light ray in a material depends on the index of refraction. So the velocity of the light would be variable. So using the previous example, the time taken to cross ds would be $\frac{ds}{dv} = \frac{dx\sqrt{1 + (dy/dx)^2}}{c(x,y)}$

Now, according to the principle of least action:

$$\delta \int \frac{dx \sqrt{1 + (dy/dx)^2}}{c(x,y)} = 0$$



Given initial conditions, x_0, t_0, $\left(\frac{dx}{dt}\right)_{x=x_0}$ That's all what we need to know

Newton's equations allow the entire trajectory to be constructed.

The idea is to minimize (or make stationary) the $\boxed{\text{action}}$. Also, in classical mechanics, there are forms where the start and end points and conditions are known and the interpolation between them has to be found out. Consider a quantity A (action) that is an integral of something

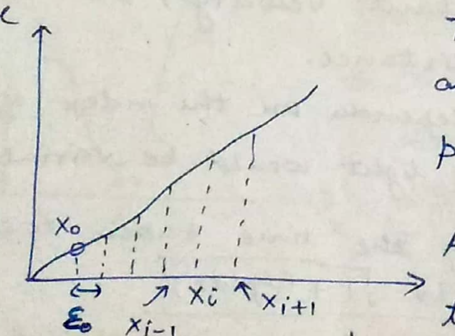
$$A = \int_{t_0}^{t_1} dt\, \mathcal{L}(\overset{x(t)}{x}, v)$$

that depends on the trajectory — a function of time — $\mathcal{L}$ (the Langragian depending on position x and velocity v at every point on the trajectory). This is analogous to the light example before where something $\frac{dx\sqrt{1+(dy/dx)^2}}{c(x,y)}$ had to be minimized and depended on the slope and the position — such problems can always to reduced to the Euler–Langragian equation (a differential equation).

Problem — $\delta A = 0$ (make "action" stationary). or

$$\delta \int_{t_0}^{t_1} dt\, \mathcal{L} = 0.$$

Try to reduce it to a normal problem of minimizing equations, by approximating this problem



(discretizing the time axis)

discrete

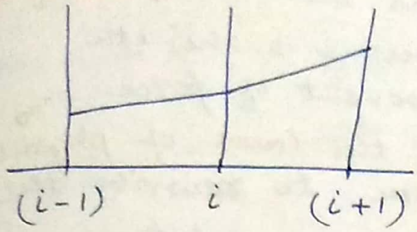
The integral then collapses to an approximate sum. The problem then reduces to:

$$A = \sum \varepsilon \cdot \mathcal{L}\left(x_i, \frac{x_{i+1}-x_i}{\varepsilon}\right)$$

So that $x(t)$ in the Langragian collapses to a discrete function of position wrt time and a normal $\frac{\Delta x}{\Delta t}$ for velocity

5 Now this quantity can be made stationary by $\frac{\partial A}{\partial X_i} = 0$ or $\delta A = \left(\frac{\partial A}{\partial X_i}\right) \cdot \delta X_i$ and $\delta A = 0 \Rightarrow \frac{\partial A}{\partial X_i} = 0$

Question – is this equally valid in discrete case. Probably not with same notation. So any alternative notation for discrete things should work – $(A_{i+1} - A_i) = \left(\frac{\Delta A}{\Delta X_i}\right) \cdot (X_{i+1} - X_i)$ and here $\left(\frac{\Delta A}{\Delta X_i}\right) = 0$ since $X_{i+1} - X_i \neq 0$ and $A_{i+1} - A_i = 0$ for stationary 'action'. Let's expand on our region of interest:



Then, A (action) $= \varepsilon . \sum \mathcal{L}\left(x_i, \frac{x_{i+1} - x_i}{\varepsilon}\right)$

$$A = \varepsilon . \mathcal{L}\left(x_i, \overbrace{\frac{x_{i+1} - x_i}{\varepsilon}}^{V_{i+1}}\right) + \varepsilon \mathcal{L}\left(x_{i-1}, \underbrace{\frac{x_i - x_{i-1}}{\varepsilon}}_{V_i}\right)$$

By the principle of stationary action,

$$\frac{\partial A}{\partial X_i} = 0 \Rightarrow \varepsilon \left[\frac{\partial \mathcal{L}(x_i, V_{i+1})}{\partial x_i} + \frac{\partial \mathcal{L}(x_i, V_{i+1})}{\partial V_{i+1}} \cdot \left(-\frac{1}{\varepsilon}\right) + \frac{\partial \mathcal{L}(x_{i-1}, V_i)}{\partial V_i} \cdot \frac{1}{\varepsilon} \right]$$

Now, $\left[\frac{\partial \mathcal{L}(x_i, V_{i+1})}{\partial V_{i+1}} - \frac{\partial \mathcal{L}(x_{i-1}, V_i)}{\partial V_i}\right] \frac{1}{\varepsilon}$ is simply the derivative wrt time. $\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial V_i}\right]$

~~Then~~, Thus, for least action

$$\delta A = \left(\frac{\partial A}{\partial X_i}\right) \cdot \delta x_i = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial V} + \frac{\partial \mathcal{L}}{\partial x} = 0$$

Given a Langragian, the Euler–Langragian differential to ensure the principle of least action – $\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial V} = \frac{\partial \mathcal{L}}{\partial x}}$

It is possible to write down a $\mathcal{L}$ such that this corresponds to the Newton's equations

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ex Given $PE = V(x)$; $KE = \frac{1}{2} m v^2$. So the Langragian is just $\mathcal{L} = \frac{1}{2} m v^2 - V(x)$.

Thus, $\frac{d}{dt}\left(\underbrace{\frac{\partial \mathcal{L}}{\partial v}}_{\text{momentum}}\right) = \frac{\partial \mathcal{L}}{\partial x} \Rightarrow \frac{d}{dt}(mv) = -\frac{dV}{dx}$

$$m . \frac{\partial^2 x}{\partial t^2} = -\frac{dV}{dx}$$

$\Rightarrow \boxed{\underbrace{ma}_{a \,\equiv\, \text{acceleration}} = \text{Force}}$

⑥ In many coordinated system,

$$A = \int_{t_0}^{t_1} dt\, \mathcal{L}(x_i, v_i)$$

Here i stands for all of the coordinates and all of the velocities

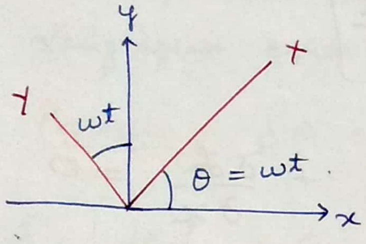
By applying the same principle of least action and approximating the integral using discretized lengths of time

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_i} = \frac{\partial \mathcal{L}}{\partial x_i}.$$

So the previous example turns out $m.a_i = F_i$ where i is the ith component of force.

One Langragian packages all of the laws of physics for a system. It is a lot easier to rewrite the Langragian ~~principle~~ equation in any arbitrary coordinate system, since the least action outputting the curve is independent of the coordinate system. So it's an extremely convenient tool to change coordinate systems.

ex How does laws of physics look like standing on a turntable.



So the red coordinate system is rotating wrt time.

$\boxed{\text{So what is the transformation from one system to the other?}}$

So,
$$\boxed{\begin{aligned} x &= X\cos\omega t + Y\sin\omega t \\ y &= -X\sin\omega t + Y\cos\omega t \end{aligned}}$$

For this problem, consider the Langragian

$$\mathcal{L} = \frac{m}{2}(v_x^2 + v_y^2) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)$$

To write this in terms of the ~~to~~ red system,

$$\dot{x} = \dot{X}\cos\omega t + X.(-\sin\omega t).\omega + \dot{Y}\sin\omega t$$

Look how these sum to 1.

$$+ Y.(\cos\omega t).\omega.$$

$$\dot{y} = -\dot{X}\sin\omega t + \ldots$$

~~Corori~~ Coriollis force

$$\mathcal{L} = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + \frac{m}{2}\omega(\dot{X}Y - \dot{Y}X)$$

(the rotating coordinates due to ω and ω^2).

(f) The second term gives a feel of a centrifugal force : $\boxed{-\omega^2 \frac{1}{2} m (x^2 + y^2)}$ → a radially outward pointing force. the centrifugal force. The coriollus force is the velocity dependent force. Now the Euler-Langragian :

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + \frac{m\omega}{2} . y \; : \; \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x} + \frac{m\omega}{2} . \dot{y}$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{m\omega^2}{2} . 2 . x + \frac{m}{2}\omega . (-\dot{y})$$

Now, $m\ddot{x} + \frac{m\omega}{2} . \dot{y} = m\omega^2 x - \frac{m\omega}{2}\dot{y}$

$$\Rightarrow \underbrace{m a_x}_{\cancel{\text{centrifugal force}}} = \cancel{\text{shift}} \; \underbrace{\omega^2 m x}_{\text{centrifugal force}} - \underbrace{m\omega \dot{y}}_{\text{a velocity dependent force, where the } a_x \text{ depends on } \dot{y}}$$

$\boxed{\text{coriolis force}}$

Similarly,

$$m a_y = \omega^2 m y + \underbrace{m\omega \dot{x}}$$

→ velocity dependent force, the $\boxed{\text{coriolis force}}$

Some rethinking –

what is the Langragian $\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$ tell.

Let's see. $\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{m}{2} . 2\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m . \ddot{x}$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{m}{2} . \frac{d(\dot{x}^2)}{dx} = \frac{m}{2} . \frac{d}{dx} . \left(\frac{dx}{dt}\right)^2$$

I don't know for others, but I find this tough.

Notes on derivation of Euler-Langragian differential –

$$A = \sum \varepsilon \, \mathcal{L}\left(x_i, \frac{x_{i+1} - x_i}{\varepsilon}\right)$$

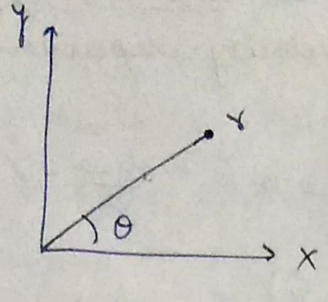
$$A = \varepsilon \, \mathcal{L}\left(x_i, \frac{x_{i+1} - x_i}{\varepsilon}\right) + \varepsilon . \mathcal{L}\left(x_{i-1}, \frac{x_i - x_{i-1}}{\varepsilon}\right)$$

$$\frac{\partial A}{\partial x_i} = \varepsilon . \frac{\partial \mathcal{L}}{\partial x_i} + \varepsilon . \frac{\partial \mathcal{L}}{\partial v_{i+1}} . \frac{\partial v_{i+1}}{\partial x_i} + \frac{\varepsilon \partial \mathcal{L}}{\partial v_i} . \frac{\partial v_i}{\partial x_i}$$

and $\frac{\partial v_{i+1}}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{x_{i+1} - x_i}{\varepsilon}\right) = -\frac{1}{\varepsilon}$; $\frac{\partial v_i}{\partial x_i} = \frac{1}{\varepsilon}$

Substituting – $\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v}\right) = \frac{\partial \mathcal{L}}{\partial x}}$

⑧ Try conversion of $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ to polar coordinates r and θ.



such that $X = r\cos\theta$ and

$Y = r\sin\theta$

$\dot{x} = r.(-\sin\theta).\dot{\theta} + \dot{r}\cos\theta$

$\dot{y} = r(\cos\theta).\dot{\theta} + \sin\theta.\dot{r}$

$\dot{x}^2 = r^2.\sin^2\theta.\dot{\theta}^2 + \dot{r}^2\cos^2\theta - 2.\dot{\theta}.r.\dot{r}\sin\theta\cos\theta$

$\dot{y}^2 = r^2\cos^2\theta.\dot{\theta}^2 + \dot{r}^2\sin^2\theta + 2r.\dot{r}.\dot{\theta}.\sin\theta.\cos\theta$

$\dot{x}^2 + \dot{y}^2 = r^2.\dot{\theta}^2 + \dot{r}^2$

So, $\mathcal{L} = \frac{1}{2}m(r^2.\dot{\theta}^2 + \dot{r}^2) - V(r)$

Euler - Langragian equation -

① $\frac{d}{dt}.\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\partial\mathcal{L}}{\partial r}$ [r kind of replaces the x]

$\frac{d}{dt}\left(\frac{1}{2}.m.2\dot{r}\right) = \frac{1}{2}.m.\dot{\theta}^2.2r - \frac{dV}{dr}$

$$\boxed{m.\ddot{r} = m.r.\dot{\theta}^2 - \frac{dV}{dr}}$$ where $m\ddot{r}$ = radial acceleration

$-\frac{dV}{dr}$ = r component of the force.

$mr\dot{\theta}^2$ = centrifugal force

② $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta}$

$\frac{d}{dt}\left[\frac{1}{2}.m.r^2.2.\dot{\theta}\right] = 0 \Rightarrow$ Something is conserved, and that is angular momentum

$\Rightarrow \boxed{m.r^2\ddot{\theta} = 0}$

Had V depended on θ, it would not have been conserved

$\frac{d}{dt}(I\omega) = 0 \Rightarrow I\omega = \text{constant}$

So absence of a coordinate in the Langragian means something is conserved.

(9) Now $\dot{\theta} = \frac{L}{mr^2}$ can be plugged into (1) to get an equation solely in r.

Key - points -

(1) Langragian packages problems in simple ways. One function of all coordinates and velocities determines all the equations of motion. And every system existent can be written in Langragian form.

(2) Euler - Langragian equation is invariant of the coordinate system, so conversion of coordinates becomes easy.

(3) it's easier to see how some coordinate is conserved, an ~~ob~~. absence from the Langragian converts to a conservation principle on application of the Euler - Langragian equation.