Of Symmetry and Conservation Laws

Conservation laws are connected to symmetry.

- related to our study of variations in quantities when the coordinates are shifted by infitesimal quantities (say δ whose square is 0, and $\delta \to 0$).

- For a simple function $F(x, y)$

$$\delta F(x,y) = \frac{\partial F}{\partial x} \cdot \delta x + \frac{\partial F}{\partial y} \cdot \delta y$$

And thus we study the <u>first order change in F</u> (or variation in F when x, y change with infinitesimal amount).

<u>Notation</u> - It's better to denote the Langragian as $\mathcal{L}(x, \dot{x})$ because $A = \int dt \, \mathcal{L}(X, \dot{X})$ means $\dot{X}$ is the derivative of X wrt t, and t is not necessarily time, so $\dot{X}$ is not always the velocity v.

(10) So the principle of least action implies the first order change in $A = \int dt\, \mathcal{L}(x, \dot{x})$ is ~~minimum~~ zero, i.e. $\delta A = 0$

So change in A is from contributions ~~to~~ of two variables : X and $\dot{X}$.

Notational convention -

① X is replaced by q. And X contains a set of different coordinates, $X_i \equiv q_i$.

② $\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} \rightarrow$ canonical momentum conjugate to q_i (P_i).

ex $\mathcal{L}(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x)$

The Euler - Langragian equation then becomes

$$\boxed{\frac{d}{dt} P_i = \frac{\partial \mathcal{L}}{\partial q_i}}$$ and there is one such equation for each i

ex. $\mathcal{L} = \dfrac{\dot{q}_1^2 + \dot{q}_2^2}{2} - V(q_1 - q_2)$

If we think of q_1, q_2 as positions, we imply that the potential depends on the difference ~~in~~ or the distance between them.

Now. $\dfrac{d}{dt} \cdot \dfrac{\partial \mathcal{L}}{\dot{q}_1} = \dfrac{\partial \mathcal{L}}{\partial q_1} \Rightarrow \dfrac{d}{dt} P_1 = -V'(q_1 - q_2)$

$\Rightarrow \boxed{\dot{P}_1 = -V'}$

Similarly, $\dot{P}_2 = -(-V'(q_1 - q_2)) = V'(q_1 - q_2)$

And that simply means incremental change in q_2 causes the distance to decrease.

Now,

$\dfrac{d}{dt}(P_1 + P_2) = 0$ [and that's a conservation law]

or $\dfrac{\partial \mathcal{L}}{\partial q_1} + \dfrac{\partial \mathcal{L}}{\partial q_2} = \text{constant}.$

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(11) <u>ex.</u> $\mathcal{L} = \frac{\dot{q}_1^2 + \dot{q}_2^2}{2} - V(a q_1 + b q_2)$

(1) $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{\partial \mathcal{L}}{\partial q_1} \Rightarrow \frac{d}{dt} P_1 = -V'(aq_1 + bq_2).a$

where $P_1 = \cancel{\beta}.\ \dot{q}_1$

(2) $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \frac{\partial \mathcal{L}}{\partial q_2} \Rightarrow \frac{d}{dt} P_2 = -V'(aq_1 + bq_2).b$

$\frac{d}{dt}(P_1 + P_2) = -(a+b).V'(aq_1 + bq_2)$ [And this doesn't look like conserved]

and $\frac{d}{dt}(bP_1 - aP_2) = 0$ and that is <u>conserved.</u>

So there is some different conservation law for this Langragian~~s~~. And by intuition, $\frac{\partial \mathcal{L}}{\partial \dot{x}}$ yields $\frac{1}{2}.m.2\dot{x}$ $= m\dot{x} = mv$ (which is the momentum).

<u>And how does such a potential come up in the first place?</u>

Consider $\mathcal{L} = \frac{m\dot{x}^2}{2} + \frac{M\dot{y}^2}{2} - V(x-y).$

To get away with m and M, we do a change of variables such that

$m\dot{x}^2 = \dot{q}_1^2$ and $M\dot{y}^2 = \dot{q}_2^2$

$mx^2 = q_1^2$ and $M.y^2 = q_2^2$

So the equation becomes

$$\mathcal{L} = \frac{\dot{q}_1^2 + \dot{q}_2^2}{2} - V\left(\frac{q_1}{\sqrt{m}} - \frac{q_2}{\sqrt{M}}\right)$$

So simplification of the kinetic terms introduce those constants in the potential energy.

Of ~~Sur~~ Symmetry

$\mathcal{L} = \frac{\dot{q}^2}{2}$ [a change (coordinate change) that doesn't affect the langragian].

Coordinate change can be done by-

① shifting the coordinate axes center

② putting the thing in motion itself.

Change is depicted by $\delta q = \delta$ (for analysis, let δ be constant that is positive)

$\dot{q} = \frac{q \pm \delta q}{\Delta t} \Rightarrow \dot{q} = 0 \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$ or $\delta \mathcal{L} = 0$ or $\frac{\partial \mathcal{L}}{\partial q} = 0$

And that is symmetry (translation symmetry).

or $\dot{P}$ (canonical momentum conjugate to q) is conserved.

- Try fixing this example in earlier notions of the Langragian.

ex. $\mathcal{L} = \frac{\dot{q}_1^2 + \dot{q}_2^2}{2} - V(aq_1 + bq_2)$

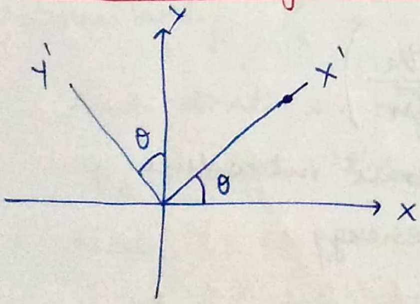
For any shift $q_1' = q_1 + b\delta$; $q_2' = q_2 - a\delta$

So no change in $\dot{q}_1, \dot{q}_2$ (or effectively no change in slope). Now $aq_1' + bq_2' = aq_1 + ab\delta + bq_2 - ab\delta$

$= aq_1 + bq_2$

Langragian remains invariant

Rotational symmetry -



$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2)$

(here cartesian coordinates are used)

depends on distance from origin

Symmetry - a change of coordinates that retains the same radius as the initial coordinate

(13) Suppose coordinates rotate to x' and y'. Then,

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= y\cos\theta - x\sin\theta \end{aligned}\right\}$$ For small angle $\theta = \delta$

Thus $\cos\theta = 1$, $\sin\theta = \delta$

Thus, $$\left.\begin{aligned} x' &= x + y\delta \\ y' &= y - x\delta \end{aligned}\right]$$ – change in coordinates is proportional to x, y.

$$\boxed{\begin{aligned}\delta x &= y\delta \\ \delta y &= -x\delta\end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned}\delta\dot{x} &= \dot{y}\delta \\ \delta\dot{y} &= -\dot{x}\delta\end{aligned}}$$ Transformations we need to make.

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2+y^2)$$

Since $$\delta(x^2+y^2) = 2x.\delta x + 2y.\delta y$$
$$= 2x.y\,\delta + 2y(-x).\delta = 0$$

and $$\delta\left(\frac{m}{2}(\dot{x}^2+\dot{y}^2)\right) = \frac{m}{2}\delta(\dot{x}^2+\dot{y}^2)$$
$$= \frac{m}{2}.(2\dot{x}.\delta\dot{x} + 2\dot{y}\,\delta\dot{y}) = \frac{m}{2}[2\dot{x}.\dot{y}\,\delta - 2\dot{y}\dot{x}\delta]$$
$$= 0$$

Hence, the Langragian remains ~~co~~ unchanged.

So generally $\delta q_i \propto \delta$, and the propotionality constant may be some q_j itself : $\boxed{\delta(q_i) = f_i(q).\,\delta}$

and $\boxed{\delta\dot{q}_i = \frac{d}{dt}(\delta q_i)}$ Rotational symmetry (generalized)

Now whether the Langragian does or does not change when the coordinates are changed depends on the type of function encoded in $f_i(q)$.

So $$\delta\mathcal{L}(q_i, \dot{q}_i) = \sum_i \frac{\partial\mathcal{L}}{\partial q_i}.\delta q_i + \sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}.\delta\dot{q}_i$$

Assumption –

(1) Assume that the motion of the system satisfies the laws of the Langragian (Euler-Langragian equation). So converting things to P_i

$$\delta\{\mathcal{L}(q_i, \dot{q}_i) = \sum_i(\dot{P}_i\,\delta q_i + P_i.\delta\dot{q}_i)$$
$$= \sum_i\left[\frac{d}{dt}(P_i.\delta q_i)\right] = \sum_i\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}.\delta q_i\right)\right]$$

$$= \frac{d}{dt} \sum_i \left(P_i \cdot f_i(q) \cdot \delta \right)$$ using initial condition

If this is a symmetry, then $\delta \mathcal{L}(q_i, \dot{q}_i) = 0$

Thus, $$\delta \cdot \frac{d}{dt} \left[\sum_i P_i \cdot f_i(q) \right] = 0$$

$$\Rightarrow \sum_i P_i \cdot f_i(q) = \text{constant} = Q$$ (some quantity that is conserved).

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ex $\delta q_1 = \delta$; $\delta q_2 = \delta$: $f_1(q) = 1 = f_2(q)$

$$Q = \frac{d}{dt} \left[P_1 f_1(q) + P_2 f_2(q) \right] = 0$$

$$= \frac{d}{dt} \left[P_1 + P_2 \right] = 0$$ ⇒ momentum conservation as a consequence of translation invariance

P_i = conjugate momentum to q_i (because $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is the velocity derivative of kinetic energy part of the Langragian, yielding momentum.

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ex $\delta q_1 = b\delta$; $\delta q_2 = -a\delta$, find the change in the Langragian.

$$\delta \mathcal{L}(q_i, \dot{q}_i) = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

From Euler-Langragian equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \Rightarrow \frac{d}{dt} P_i = \frac{\partial \mathcal{L}}{\partial q_i}$$ such that $P_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

Thus, $$\delta \mathcal{L}(q_i, \dot{q}_i) = \sum_i \left(\frac{d}{dt} P_i \cdot \delta q_i + P_i \delta \dot{q}_i \right)$$

$$= \sum_i \left(\dot{P}_i \delta q_i + P_i \delta \dot{q}_i \right) = \sum_i \left(\frac{d}{dt} P_i \cdot \delta q_i \right)$$

Now from initial conditions of symmetry:

$\boxed{\delta q_i = f_i(q) \cdot \delta}$ or change in a coordinate is proportional to δ, with the constant of proportionality being a function of q_i themselves.

(15) $\delta \mathcal{L}(q_i, \dot{q}_i) = \frac{d}{dt} \sum_i P_i \cdot f_i(q_i) \cdot \delta = 0$

$\Rightarrow \sum_i P_i \cdot f_i(q_i) \cdot \delta = Q = \text{constant}.$

Therefore, $\frac{d}{dt}\left(P_1 \cdot b \cdot \delta + P_2(-a\delta)\right) = 0$

where δ is a constant $\Rightarrow \frac{d}{dt}\left(P_1 \cdot b - a \cdot P_2\right) = 0$

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ex $\delta x = y\delta$ and $\delta y = -x\delta$ (Here $f_i(q_i)$ are NOT some constant functions. Hence one can deduce some kind of rotational symmetry from them).

From the same set of equations $\delta\mathcal{L}(q_i, \dot{q}_i)$

$$= \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \cdot q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \cdot \dot{q}_i \right) = \sum_i \left(P_i \cdot \dot{q}_i + \frac{d}{dt} P_i \cdot q_i \right)$$

$$= \sum_i \left(\frac{d}{dt}(P_i \cdot q_i)\right) = \frac{d}{dt}\left[\sum_i \left(P_i \cdot f_i(q_i) \cdot \delta\right)\right] = 0$$

Therefore:

$$\frac{d}{dt}\left[P_1(y\delta) + P_2(-x \cdot \delta)\right] = 0 \Rightarrow \frac{d}{dt}\Big(\underbrace{P_1 y - P_2 x}_{\text{Angular momentum}}\Big) = 0$$

So $\boxed{\text{Angular momentum is conserved as a consequence of rotational symmetry}}$

The above things were conservation of momentum is relating to lack of dependence ~~bet~~ where you put the origin of space (or invariant under translational / rotational symmetry)

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ex. Harmonic oscillator: $\mathcal{L} = \frac{m \cdot \dot{q}^2}{2} - \frac{k \cdot q^2}{2}$

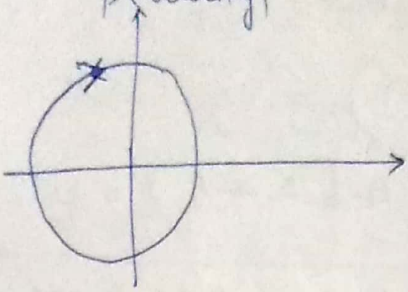
Simplicity, $m = 1 = k$

Euler-Langragian equation $- \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$

or $\boxed{\frac{d}{dt} P = \frac{\partial \mathcal{L}}{\partial q}}$: $\frac{d}{dt}(m \cdot \dot{q}) = -kq$

or $\frac{d}{dt} P = -kq$

In general, $m \cdot \frac{dq}{dt}$ = momentum = P.



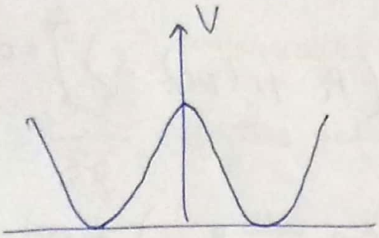
$$\boxed{\begin{aligned} \frac{dP}{dt} &= -kq \\ \frac{dq}{dt} &= \frac{P}{m} \end{aligned}}$$

→ motion in a circle with uniform angular velocity

Since $P = -kqt + c$ and $q = \frac{P}{m}t + c_2$

$$P = -kq \cdot \frac{q \cdot m}{P} \Rightarrow \boxed{P^2 + km \cdot q^2 = C}$$

The velocity - position space of a Harmonic oscillator in clockwise direction. So when q is maximally positive, $\frac{dP}{dt}$ is negative



If there is a potential such that : $q \to -q$ and $\dot{q} \to \dot{q}(-1)$ then the notions of infinitesimal q transformations do not apply.

Such symmetries do not have conservation laws, in classical mechanics (there is one in quantum mechanics though). So q to $-q$ can't be built up by incremental things δ, and thus do not have conservation laws.

If the Langragian is invariant under a small δ, how is it invariant under a summation of ∞ δ?

Prove that wherever q_i is, a small change in δ doesn't change the Langragian. So that allows to work on the above statement.

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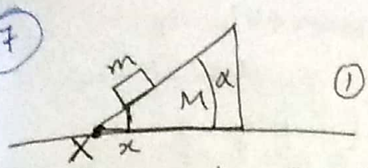


Motion of a block on a free wedge.

① How much coordinates are needed to describe the instantaneous configuration of the system? 2 coordinates - X and x (relative to X). ② Langragian = KE - PE. $\mathcal{L} = \frac{1}{2}M\dot{X}^2$ (wedge KE) $+ \frac{1}{2}m([\dot{X}+\dot{x}]^2 + [\dot{x}\tan\alpha]^2) - (mg\,x\tan\alpha)$.

For the block, $V_x = \dot{X} + \dot{x}$, $V_y = \dot{x}\tan\alpha$

For PE, there is no gravitation pulling it horizontally. Only the vertical movement for m is considered.

$$\mathcal{L}(X, x, \dot{X}, \dot{x}) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m([\dot{X}+\dot{x}]^2 + [\dot{x}\tan\alpha]^2) - (mg\,x\tan\alpha).$$

P_X (momentum conjugate to X) $= \frac{\partial\mathcal{L}}{\partial\dot{X}}$

$$= \frac{2}{2}\cdot M\cdot\dot{X} + \frac{1}{2}m\left(2(\dot{X}+\dot{x})\right) = \underline{M\dot{X} + m(\dot{X}+\dot{x})}$$

P_x (momentum conjugate to x) $= \frac{\partial\mathcal{L}}{\partial\dot{x}}$

$$= \frac{2}{2}m(\dot{X}+\dot{x}) + \frac{2}{2}m(\dot{x}\tan\alpha) \cancel{- mg\tan\alpha}$$

$$P_x = m(\dot{X}+\dot{x}) + m\dot{x}\tan\alpha$$

Symmetry: $\boxed{\delta X = \varepsilon \;;\; \delta x = 0}$ Since x is a relative coordinate

$$Q = \sum\left[P_i f_i(q_i)\right] = P_X\cdot 1 + P_x\cdot 0 = \boxed{P_X}$$

is conserved.

How is the symmetry deduced?

① $\delta X = \varepsilon$. There is no term containing X in $\mathcal{L}$. Hence $\frac{\partial\mathcal{L}}{\partial X} = 0$ or change in X keeps $\mathcal{L}$ invariant.

$f_X(q) = 1$ can be deduced by the diagram itself.

A small nudge ε to the right makes $\delta X = \varepsilon$

② x is rather relative to X. So $\delta x = 0$

Implies P_X = constant or $\dot{P}_X = 0$

(18) The wedge is NOT allowed to move in y direction; nor the block does not in y direction but the potential energy changes (implying an internal force on the system). Thus there is no symmetry there. Therefore,

$$\boxed{\dot{P}_X = 0 \;;\quad \dot{P}_x = mg \tan\alpha}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = \frac{\partial \mathcal{L}}{\partial X}$$

$$\frac{d}{dt}\left(M\dot{X} + m(\dot{X}+\dot{x})\right) = 0$$

$$\Rightarrow M\ddot{X} + m\ddot{X} + m\ddot{x} = 0$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt}\left(m(\dot{X}+\dot{x}) + m\dot{x}\tan\alpha\right) = mg\tan\alpha$$

$$\Rightarrow m\ddot{X} + m\ddot{x} + m\ddot{x}\tan\alpha = mg\tan\alpha.$$

Also V_X is the absolute x-component velocity of the block. $\dot{x}$ is the relative velocity. Therefore,

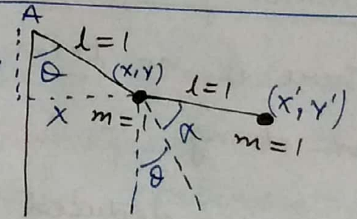
$$\dot{x} = V_X - \dot{X} \Rightarrow V_X = \dot{x} + \dot{X}$$

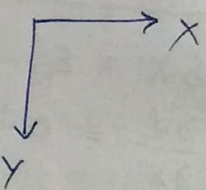
In this context, we did not need to talk about the forces between the wedge and the block.

① Simple Euler Langragian arrangement gave the 2 equations of motions required to describe the motion.

② Symmetry laws ($\delta X = \varepsilon$ and $\delta x = 0$) allow for the momentum conservation laws to hold.

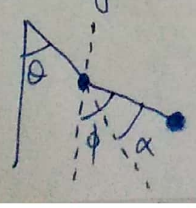
Double Pendulum –





Coordinates:

Some instinct about angular motion. The one obvious choice would be θ (angle of first pendulum to the wall).



Now φ or α? If θ is fixed, and assuming no gravitational field, φ^α would provide a better symmetry choice. We could find

(19) Some simple conservation law here.

Kinetic energy $\longrightarrow$ simpler to form in Cartesian coordinates and then transform into angles.

Considering Cartesian coordinates X and Y,

$X = 1.\sin\theta$; $Y = 1.\cos\theta$; $\dot{X} = \dot{\theta}.\cos\theta$; $\dot{Y} = (\dot{\theta}\sin\theta)(-1)$

Now for the second pendulum: (X', Y')

$X' = X + \sin(\theta+\alpha)$; $Y' = Y + \cos(\theta+\alpha)$

$\dot{X}' = \dot{X} + \cos(\theta+\alpha).(\dot{\theta}+\dot{\alpha})$; $\dot{Y}' = \dot{Y} - \sin(\theta+\alpha)(\dot{\theta}+\dot{\alpha})$

① formed in Cartesian coordinates : $X, Y, \dot{X}, \dot{Y}$

② then applied geometrical conditions.

KE of first pendulum: $\frac{1}{2}(\dot{X}^2+\dot{Y}^2) = \frac{1}{2}\dot{\theta}^2$

Second pendulum: $\frac{1}{2}(\dot{X}'^2 + \dot{Y}'^2)$

$$= \frac{1}{2}\Big(\cancel{X}\,\dot{\theta}^2\cos^2\theta + \cos^2(\theta+\alpha).(\dot{\theta}+\dot{\alpha})^2 + 2\dot{X}\cos(\theta+\alpha)(\dot{\theta}+\dot{\alpha})$$
$$+ \dot{\cancel{\theta}}^2\sin^2\theta + \sin^2(\theta+\alpha).(\dot{\theta}+\dot{\alpha})^2 - 2.\dot{Y}\sin(\theta+\alpha)(\dot{\theta}+\dot{\alpha})\Big)$$

$$= \frac{1}{2}\left(2.\dot{\theta}^2 + (\dot{\theta}+\dot{\alpha})^2 + 2.\dot{\theta}(\dot{\theta}-\dot{\alpha}).\cos\alpha\right)$$

$$KE = \boxed{\dot{\theta}^2 + \frac{(\dot{\theta}+\dot{\alpha})^2}{2} + \dot{\theta}(\dot{\theta}-\dot{\alpha}).\cos\alpha}$$

For the time being, if we assume no gravitational field, $\mathcal{L} = KE$ and we have a conserved quantity here, wrt θ. Thus $\frac{\partial\mathcal{L}}{\partial\dot{\theta}}$ shall be conserved as $\frac{\partial\mathcal{L}}{\partial\theta} = 0$ ($\frac{\partial\mathcal{L}}{\partial\dot{\theta}}$ = sum of angular momentum of both pendulum bobs). $\frac{\partial\mathcal{L}}{\partial\theta}$ is the symmetry we were looking for (Langragian invariance under small changes in θ such that $\delta\theta = f(\theta).\varepsilon$)

(20) Considering the potential field: considering point A as zero potential,

PE of first pendulum $= -Y = -\cos\theta \cdot g$ ← Y' is already relative to A

PE of second pendulum $= -(\cancel{Y}\, Y') \cdot g$

$= -(\cos\theta + \cos(\theta+\alpha))\, g$

$$V(X) = [-\cos\theta - \cos\theta - \cos(\theta+\alpha)] \cdot g$$

$\mathcal{L} = KE - V(X)$ now no longer independent of any coordinate. Hence $\frac{\partial \mathcal{L}}{\partial \theta} \neq 0$ and there is no symmetry

Also, energy is conserved. Thus, $T+V =$ fixed and $P_X =$ fixed

Equations of motion: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$

and $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\alpha}\right) = \frac{\partial \mathcal{L}}{\partial \alpha}$ Absence of ~~fi~~ $V(x)$

$$\boxed{\begin{aligned} &① \ (\dot\theta - \dot\alpha) \cdot \ddot\alpha \ \cancel{+}\ \dot\theta \cos\alpha\, \ddot\alpha = \dot\theta(\dot\theta - \dot\alpha)(-\sin\alpha) \cdot \dot\alpha + g\sin(\theta+\alpha) \cdot \dot\alpha \\ &\cancel{②} \\ &② \ 2 \cdot \dot\theta + (\dot\theta - \dot\alpha). + (2\dot\theta - \dot\alpha)\, co \end{aligned}}$$

Better find P_θ and P_α and then $\dot P_\theta$ and $\dot P_\alpha$.

①

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = 2\dot\theta + \cancel{2}(\dot\theta - \dot\alpha) + (2\dot\theta - \dot\alpha) \cdot \cos\alpha$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = 2\ddot\theta + \ddot\theta - \ddot\alpha + (2\dot\theta - \dot\alpha)(\sin\alpha) \cdot \dot\alpha(-1) + (2 \cdot \ddot\theta - \ddot\alpha)\cos\alpha$$

And $\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = 0}$

② Similarly $\frac{\partial \mathcal{L}}{\partial \dot\alpha}$ and $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\alpha}\right) = \frac{\partial \mathcal{L}}{\partial \alpha}$

(a) Sum of momenta is conserved, not individual motions.