21

Of ~~Hamiltonian~~ Hamiltonian / Hamiltonian / Hamilton's equations

- it is conserved
- Third formulation of mechanics:

① Newton ② Euler-Langrangian ③ ~~Hamiltonian~~

For the laws of physics to be reversible, neither convergence nor divergence is allowed.

Convergence - many states leading to the present state. Past is non-deterministic

Divergence - Future is non-deterministic

The flow of laws of physics should adhere to this.

$$\boxed{H = \sum_i \dot{q}_i P_i - \mathcal{L}}$$

From symmetry, $\sum_i \dot{q}_i P_i$ is constant or conserved.

$\dot{q}$ can be expressed as q and P.

Thus H is a function of P and q

ex Harmonic oscillator $\mathcal{L} = \frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2$

Restoring force.

$P = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\dot{q}}{\omega}$ or $\boxed{\dot{q} = P\omega}$

or I have expressed $\dot{q}$ as P

So Hamiltonian $= \dot{q}.P - \mathcal{L}$

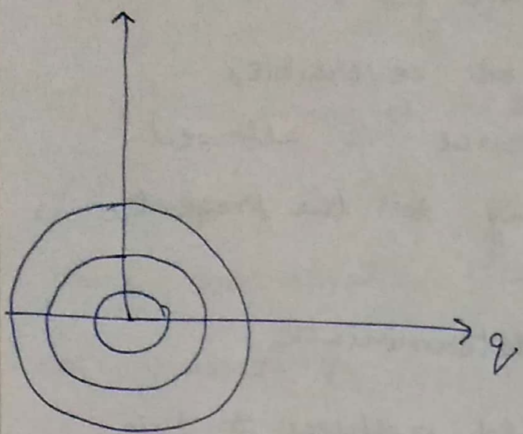
$$= P^2\omega - \frac{1}{2\omega}.(\omega^2 P^2) + \frac{\omega}{2}q^2$$

$$= P^2\omega - \frac{1}{2}P^2\omega + \frac{\omega}{2}q^2 = \boxed{\frac{1}{2}P^2\omega + \frac{\omega}{2}q^2}$$

$$H = \frac{1}{2}\omega(P^2 + q^2)$$

Any Langrangian that can't be expressed in ~~Hamiltonian~~ (or express $\dot{q}$ in P and q) is a bad system, or impossible physical system.

Phase space – 1 degree of freedom ⇒ 2 variables to solve. In general, n degrees of freedom ⇒ 2n ~~degrees of fr~~ variables



Phase space for 1 degree of freedom.

$\mathcal{H} = \frac{1}{2}\omega(p^2+q^2)$ are constants wrt time and are circles

Derivation of the Hamiltonian:

Consider a function $F(q,p)$ and make a small disturbance in the phase space

$$\delta F(q,p) = \frac{\partial F(q,p)}{\partial q}.\delta q + \frac{\partial F}{\partial p}.\delta p \quad - (1)$$

$$= A.\delta q + B.\delta p \quad - (2)$$

$$\delta H = \sum (\delta \dot{q}\, p + \dot{q}\,\delta p) - \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \frac{\partial \mathcal{L}}{\partial q}\delta q$$

cancel

$$= \sum \dot{q}\,\delta p - \frac{\partial \mathcal{L}}{\partial q}\delta q$$

From Euler Langrangian equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

$$\boxed{\delta H = \sum \dot{q}\,\delta p - \dot{p}\,\delta q} \quad - (3)$$

Comparing (3) with (2),

$$A = \frac{\partial F}{\partial q} = -\dot{p}\ ; \quad B = \frac{\partial F}{\partial p} = \dot{q}$$

23. Thus, ~~δH~~ = $\boxed{\frac{\partial H}{\partial P_i} = \dot{q}_i \; ; \; \frac{\partial H}{\partial q_i} = -\dot{P}_i}$

Harmonic ~~osclr~~ oscillator — equations of motion in Hamiltonian mechanics.

For n coordinates, n Langrangian equations and $2n$ Hamiltonian equations.

$$H = \frac{\omega}{2}\left(p^2 + q^2\right) \text{ for Harmonic oscillator}$$

$$\frac{\partial H}{\partial P} = \dot{q} \Rightarrow \frac{\partial H}{\partial P} = \omega P = \dot{q} \quad -①$$

$$\frac{\partial H}{\partial q} = -\dot{P} = \omega q \quad -②$$

① and ② : equations of motions

Diff wrt time, $\ddot{q} = \omega \dot{P}$; $\ddot{q} = \omega(-\omega q)$

Harmonic oscillator equation $\Rightarrow \boxed{\ddot{q} = -\omega^2 q}$

circular motion with constant angular velocity

Time derivative of H: Conservation of Energy using Hamilton equations of motion.

$$\frac{d}{dt} H = \sum_i \frac{\partial H}{\partial P_i} \cdot \left[\frac{d}{dt} P_i\right] + \frac{\partial H}{\partial q_i}\left[\frac{d}{dt} q_i\right]$$

$$= \sum_i \frac{\partial H}{\partial P_i} \cdot \dot{P}_i + \frac{\partial H}{\partial q_i} \cdot \dot{q}_i \quad \left[\text{Now Hamilton's equation is put up}\right]$$

$$= \sum_i \frac{\partial H}{\partial P_i} \cdot \left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial q_i} \cdot \frac{\partial H}{\partial P_i} = 0$$

$\Rightarrow \boxed{\frac{d}{dt} H = 0}$ or $\boxed{\dot{H} = 0}$ or $\boxed{\text{energy is conserved.}}$

If the Hamiltonian is time dependent, energy is <u>not</u> conserved. The trajectory of the body is such it remains on a surface of constant H in the phase space. For 2n dimensions, a surface of 2n-1 dimensions is given by a single equation.

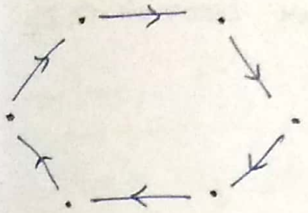
Fluid flow in phase space – Imagine a $2n$ phase space and spread dust all over it. This dust now resembles a starting point. The flow of this dust is governed by the Hamilton's. Fluid flow is like what the dust particles are doing in the phase space – where they go next depends where they are presently. Such a flow would be like velocity with components $\dot{p}$ and $\dot{q}$ (and they are given by known functions in the phase space $= \frac{\partial H}{\partial q}$ and $\frac{\partial H}{\partial p}$ which happen to be the Hamilton's equations of motion.) So every point has a fixed velocity, as H is fixed. So this phase space can be filled up with a vector field (every vector depicting the velocity in $\dot{p}$ and $\dot{q}$ at that point).

Is this fluid compressible? Compressibility implies the ability to change the density, or changing the velocity field of the phase space. A constant velocity field could imply incompressibility (or volume of fluid in a small segment coming into is getting out, otherwise compressibility). Or follow a segment and see if its volume is changing over movement. Velocity is uniform. For n dimensional space, $\frac{\partial V}{\partial q_i} = 0$. However, if the fluid is compressing in one direction and decompressing in other, fluid might still be incompressible. $\sum_i \frac{\partial V_i}{\partial q_i} = 0$ for incompressibility where V_i is the component of velocity in ith dimension, depending on where the particle is at that

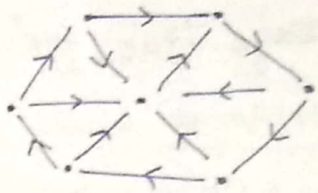
(25) time. $\sum_i \frac{\partial V_i}{\partial q_i} = \left(\sum_i \frac{\partial}{\partial q_i}\right) \cdot V = \vec{\nabla} \cdot \vec{V}$

where ∇ = divergence of the vector field

From this discussion and Pg 21, some good law of physics would require the past/future is fully deterministic.



Reversible law



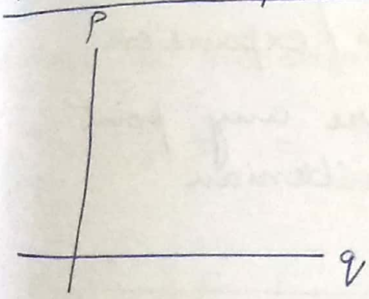
Some n state system

Non-reversible law

Stating the law to be reversible is like stating that the fluid is incompressible (arrows coming in = arrows going out or $\vec{\nabla} \cdot \vec{V} = 0$).

<u>Of Liouville's theorem</u>

<u>Flow in phase space is incompressible</u> and topology doesn't change.



Let the components be V_p and V_q.

$$V_p = \dot{p} \text{ and } V_q = \dot{q}$$

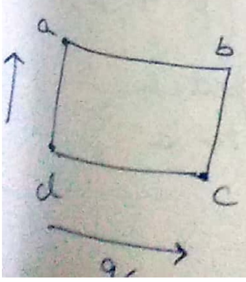
① The $\frac{\partial V_p}{\partial p} + \frac{\partial V_q}{\partial q} = 0$ (for incompressible)

② $V_p = \dot{p} = -\frac{\partial H}{\partial q}$

$$V_q = \dot{q} = \frac{\partial H}{\partial p}$$

Thus, $\frac{\partial}{\partial p}\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p}\right)$ and it is indeed equal to 0. The order of differentiation in partial form is independent of the order of differentiation

<u>Proof:</u>



$\frac{\partial F}{\partial p} = F(a) - F(d)$; $\frac{\partial F}{\partial q} = F(c) - F(d)$

$\frac{\partial}{\partial p} \cdot \frac{\partial F}{\partial q}$ = Vertical diff of 2 horizontal differences

$= [F(b) - F(a)] - [F(c) - F(d)]$

(26) $\frac{\partial}{\partial q} \cdot \frac{\partial F}{\partial P}$ = horizontal differences of 2 vertical differences. Gives the same expression.

Liouville's theorem: $\vec{\nabla} \cdot \vec{V} = 0$ in Hamilton's phase space. Consequence of Hamilton's equation.

The theorem stays true for any other coordinate system.

ex $\mathcal{H} = p \cdot q$; $V_p = \dot{p} = -\frac{\partial H}{\partial q} = -p$

$V_q = \dot{q} = \frac{\partial H}{\partial p} = q$

Liouville's condition of incompressibility: $\boxed{\vec{\nabla} \cdot \vec{V} = 0}$

$$\cancel{\partial H} \quad \frac{\partial V_p}{\partial p} + \frac{\partial V_q}{\partial q} = \frac{\partial}{\partial p}(q) + \frac{\partial}{\partial q}(-p) = 0$$

$\dot{p} = -p \Rightarrow$ exponential compression

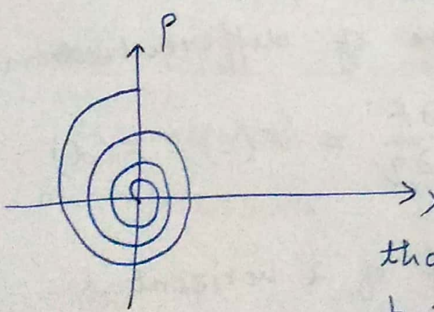
$\dot{q} = q \Rightarrow$ exponential decompression / expansion

Proof: Topology doesn't change. Where any point goes is deterministic. The Hamiltonian prevents the spilliting the topology.

Damped Harmonic Oscillator - after a long time one can't determine where it came from.

It is always 0 velocity and origin position at the end.

So $m\ddot{x} = -kx - c\dot{x}$ where $c\dot{x}$ = viscous force.



Constant energy - $\frac{\omega}{2}(p^2 + x^2)$

Now think about the patch that starts big and shrinks to a point in the end.

27) Now $P = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$; $\boxed{\cancel{\dot{P} = m\ddot{x}}}$ $\dot{p} = m\ddot{x}$

$= -kx - c\dot{x}$

$\Rightarrow \boxed{\dot{p} = -kx - c \cdot \frac{P}{m}}$ After substitutions.

Now $\vec{\nabla} \cdot \vec{V} = \frac{\partial V_P}{\partial P} + \frac{\partial V_x}{\partial x} = \frac{\partial \dot{p}}{\partial P} + \frac{\partial \dot{x}}{\partial x}$

$= \frac{\partial}{\partial P}\left(\overset{-kx - c\cdot\frac{P}{m}}{\cancel{m\ddot{x}}}\right) + \frac{\partial}{\partial x}\left(\frac{P}{m}\right) = -\frac{c}{m} \neq 0$. Now if c is positive, $\vec{\nabla} \cdot \vec{V} < 0 \Rightarrow$ fluid compresses.

Hamilton's equations are necessary but NOT sufficient for Liouville's. We might find an incompressible fluid NOT coming from Hamilton's equations. ex. a 3D incompressible fluid (Hamiltonian is not defined for odd dimensions). Incompressibility $\not\Rightarrow$ Hamiltonian existence but Hamiltonian existence $\Rightarrow$ Incompressibility.