1) Of the postulates of quantum mechanics –

Quantum mechanics provides a mathematical framework to develop laws for physical systems. The postulates of quantum mechanics were derived after a long process of trial and error.

1) state space – Associated to any isolated physical system is a complex vector space with inner product known as the state space of the system. The system is completely described by its $\boxed{\text{state vector}}$, which is a unit vector in the system's state space.

The isolated physical system we are concerned with is the qubit. It has a two dimensional state space (with the arbitrary state vector $|\psi\rangle = a|0\rangle + b|1\rangle$ (where $|0\rangle$ and $|1\rangle$ are vectors in the orthonormal basis of the state space).

Normalization condition :: $|\psi\rangle = 1$ or $a^2 + b^2 = 1$.

~~2) Evolution~~ – Why Hilbert space? extends the notion of linear algebra and calculus from 2D spaces to infinite dimensional spaces. They are abstract inner product spaces that allow length and angle to be measured.

2) Evolution – evolution of a $\boxed{\text{closed}}$ quantum system is described with unitary transformation ($UU^\dagger = U^\dagger U = I$ where I is the identity element). That is, the state $|\psi\rangle$ of the system at time t_1 is related to state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U

$|\psi'\rangle = U|\psi\rangle$. Quantum mechanics doesn't tell us which unitary operators to consider. Turns out for the particular quantum system (qubit), any unitary operator can be realized in ~~physi~~ realistic systems.

Some unitary operators – Pauli matrices ::

1) $X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (NOT gate or the bit flip). $\xrightarrow{}$ changes the bit state.

$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

2) $Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \equiv$ phase flip (leaves $|0\rangle$ invariant and transforms $|1\rangle$ to $-|1\rangle$ with $-1 \equiv$ phase factor)

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \end{bmatrix}$ is another such

Hadamard gate — $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

unitary operator.

To verify $H.H^{+} = I$ where $H^{+} = \text{adjoint}(H)$. Now,

$H^{-1} = \frac{(\text{adj } H)}{|H|} \Rightarrow H^{+} = (H^{-1}).|H| \equiv H^{+} \begin{bmatrix} -0.707 & -0.707 \\ -0.707 & 0.7071 \end{bmatrix}$

exercise 2.51

2.52 $H^2 = I$ ⇒ Hadamard is a Hermitian operator

$H^{+} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. And $H.H^{+} = I$ (thus H is unitary).

Action $-H|0\rangle \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}}$; $H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

ex. 2.53 – eigenvalues and eigenvectors of $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\det(H - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & -(1+\lambda) \end{bmatrix}\right) = 0$

$\Rightarrow -(1-\lambda)(1+\lambda) - 1 = 0 \Rightarrow 1 - \lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm\sqrt{2}$

For eigenvectors,

$H.v = \lambda . v \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} v = \sqrt{2}\, v$

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \sqrt{2}\alpha \\ \sqrt{2}\beta \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha + \beta \\ \alpha - \beta \end{bmatrix} = \begin{bmatrix} \sqrt{2}\alpha \\ \sqrt{2}\beta \end{bmatrix}$

$\Rightarrow \beta = (\sqrt{2} - 1)\alpha$ and $\alpha = (\sqrt{2} + 1).\beta$

$\Rightarrow \cancel{\beta = (\sqrt{2}-1).(\sqrt{2}+1).\beta \Rightarrow \beta = (2-1).\beta} = \boxed{\text{Wow}}$

Let $\alpha = 1$, $\beta = \sqrt{2} - 1$ is one eigenvector.

Intuition — make understanding of linear transformation easier. They are the axes/directions along which the operator acts by simply stretching/compressing/flipping the direction vector itself. It all boils down to the fact that decoupling the ways into which an operator acts on a set of variables into independent actions along separate directions, that can be dealt independently.

7 Postulate 2 requires the system to be closed. In reality, this is not always possible. However, one can think of an open system as a part of another larger closed system that is undergoing unitary evolution.

Postulate 2 for continuous time – $i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$

where H is an unitary operator but NOT the Hadamard operator. It is the Hamiltonian operator (another fixed Hermitian operator).

In principle, if someone knows the Hamiltonian of a system and the experimental value of ~~H~~ $\hbar$, the dynamics of the system can be completely defined.

Spectral decomposition of the Hamiltonian since it is a Hermitian operator – $H = \sum_E E|E\rangle\langle E|$. Eigenvalues = E
Normalized eigenvectors = $|E\rangle$ (energy eigenstates)
E is the energy of the state E

For a single qubit →

$H = \hbar\omega X$ (ω is experimentally determined)

Energy eigenstates of H same as those eigenvectors of X $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ with corresponding energies $\hbar\omega$ and $-\hbar\omega$.

Ground state – lowest energy $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ (energy $-\hbar\omega$).

Solution to the Schrodinger equation – $i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$
or the unitary evolution of states in continuous time.

$$|\psi(t_2)\rangle = \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right]|\psi(t_1)\rangle$$

Since H is a Hermitian operator, this particular combination is also a Hermitian
And $\exp(iK)$ is Unitary

Any ~~uno~~ unitary operator U can be expressed in terms of Hermitian operator K such that $\boxed{U = \exp(iK)}$. for some Hermitian operator K.

↑ spectral decomposition proofs.

Postulate 2 describes a $\boxed{\text{closed}}$ quantum system, with no outer interaction at all. This seems to contradict the statement that we apply a unitary gate to a particular quantum system. So, we should NOT be able to write down the Hamiltonian because there is an external 'we' that is interacting with the system. However, it is possible to write ~~the H~~ a time varying Hamiltonian operator for the system (despite it not being closed). This is time varying because it encompasses the time-varying effects of the external factor in it. ex. consider an atom-laser system. The evolution of the state vector of the atom system is given by a time evolving ~~Hatn~~ Hamiltonian operator that contains terms related to laser intensity and so on. Thus, the interactions of the LASER is very well captured in the atom system and the evolution of state vector is reasonably approximated.

(3) Quantum measurement – Postulate 2 describes the unitary evolution in case of closed quantum systems. However, there are sometimes the experimentalist and the experimentation equipment (something that is external to the closed quantum system being discussed). Thus,

Postulate 3: Quantum measurements are described by a collection of $\{M_m\}$ of measurement operators. The index m refers to the measurement outcomes possible in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement, the probability that result m occurs is

$$p(m) = \langle\psi| M_m^{\dagger} M_m |\psi\rangle$$

and the state of the system after the measurement –

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi| M_m^{\dagger} M_m|\psi\rangle}}$$

(1) Completeness of measurement operators = $\boxed{\sum_m M^{\dagger}M = I}$

(2) $\sum_m p(m) = 1$. Now to check out some examples:

Measurement of qubit in computational basis –

Measurement operators – $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$ (In this case, measurement operator is a 2×2 matrix)

Restriction – each measurement operator is Hermitian
– $M_0^2 = M_0$, $M_1^2 = M_1$

ex $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$; $p(0) = \langle\psi| M_0^{\dagger}M_0 |\psi\rangle$

$|0\rangle\langle 0|$

$= \langle\psi| M_0^2 |\psi\rangle$ [Hermitian] $= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$= \langle\psi| M_0 |\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle$

$= [\alpha^* \ \beta^*]\begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \ 0]\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha.\alpha = |\alpha|^2$ [Measurement done]

9) $p(1)$: The measurement operator $|1\rangle\langle 1| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ (M_1)

Not soluable

Probability $p(1) = |\psi\rangle M_1^{+} M_1 \langle\psi| = |\psi\rangle M_1 \langle\psi|$ [M_1 is Hermitian and $M_1^2 = M$]

$= \cancel{\begin{bmatrix}\alpha\\\beta\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}\alpha & \beta\end{bmatrix}} = \begin{bmatrix}\alpha & \beta\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}$

$= \begin{bmatrix}0 & \beta\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = |\beta|^2$. Now to find the state.

State $s(m)$ — $\dfrac{M_m|\psi\rangle}{\sqrt{p(m)}}$ $\Rightarrow$ $s(0) = \dfrac{M_m|\psi\rangle}{\sqrt{p(0)}} = \dfrac{a}{|a|}|0\rangle$

$= \dfrac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}}{|\alpha|} = \dfrac{1}{|\alpha|}\cdot\begin{bmatrix}\alpha\\0\end{bmatrix}$ $\boxed{= \dfrac{\alpha}{|\alpha|}|0\rangle}$

Inferences —

with coefficient.

1. $M_m|\psi\rangle$ gives the $\boxed{\text{state}}$ we are measuring
2. $\langle\psi| M^{+} M |\psi\rangle$ gives the probability of measurement
3. state $= \dfrac{M_m|\psi\rangle}{\sqrt{p(m)}} = \dfrac{a}{|a|}\cdot|m\rangle$

— Considering the system and adding to it the measurement system (which are themselves quantum mechanical systems), we can obtain a completely isolated system whose evolution is defined by a unitary transformation.

$\boxed{\text{ex 2.57}}$ → first measurement $= L_t|\psi\rangle$. Second measurement $= M_m(L_t|\psi\rangle)$. Now both L_t and M_m are measurement operators (Hermitian and $L_t^2 = L_t$ and $M_m^2 = M_m$).

They follow the basic rules of linear algebra and are associative. $= (M_m \cdot L_t)|\psi\rangle$ Better solution

Measurement operator — ① actions on state $|\psi\rangle$ ② completeness ③ probability

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Distinguishing quantum states

From a given set of orthonormal states, it is easy to identity by measurement which state has been chosen. This can be found with certainty. But this is NOT true for not orthonormal states. Why? In non-orthonormal cases, there is some component of $|\psi_2\rangle$ along $|\psi_1\rangle$. It is hence impossible to tell that if we have measured some value i, does that map to $|\psi_1\rangle$ or $|\psi_2\rangle$.

Projective measurements

[Measurements using eigenvectors and eigenvalues]

— in many applications of QC and QI

— [projective measurements are equivalent to the general measurement postulate, when combined / augmented with the ability to perform unitary transformation] projective measurement

— described by an $\boxed{\text{observable}}$ M (a Hermitian operator on the state space being observed). 10

1. $M = \sum_m m P_m$ (P_m is the ~~projection~~ ~~or~~ ~~of the~~ projector onto the eigenspace of M with eigenvalue m).

Probability

2. $p(m) = \langle\psi| P_m |\psi\rangle$.

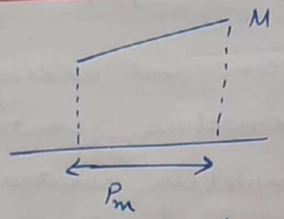
3. state $= \dfrac{P_m|\psi\rangle}{\sqrt{p(m)}}$

4. completeness $\sum_m P_m = 1$

The projective measurement can be derived from Postulate 3 by imposing one additional restriction —

(1) M_m are $\boxed{\text{orthogonal projectors}}$ (M_m are Hermitian and $\boxed{M_m M_{m'} = \delta_{m,m'} M_m}$)

orthogonal projection —



— easy to calculate average values (expected values) of projective measurements —

$$E(M) = \sum_m m\, p(m) = \sum_m m \langle\psi| P_m |\psi\rangle$$

$$= \langle\psi| \sum_m m \cdot P_m \cdot |\psi\rangle = \boxed{\langle\psi| M |\psi\rangle}$$

Variance — $\langle M^2\rangle - [\langle M\rangle]^2$

$\boxed{\text{ex 2.58}}$: same as above solution

$\boxed{\text{ex 2.57}}$ $|\psi'\rangle = \dfrac{\alpha_t|\psi\rangle}{\sqrt{\langle\psi|\alpha_t^{\dagger}\alpha_t|\psi\rangle}} = \dfrac{\alpha_t|\psi\rangle}{\sqrt{\langle\psi|\alpha_t|\psi\rangle}}$

Nomenclature —

(1). Rather than giving an observable M, people sometimes straightaway give P_m such that $\boxed{\sum_m P_m = I}$ and $P_m P_{m'} = \delta_{m,m'} P_m$). Corresponding observable is fixed

$$M = \sum_m m \cdot P_m$$

(2) measure in basis $|m\rangle$, where $|m\rangle$ forms an orthonormal basis simply means to perform projective measurements with projectors $P_m =$ ~~$\langle m||m\rangle$~~ $|m\rangle\langle m|$

1) ex projective measurements on single qubits –

Measurement of observable Z $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. On solving for eigenvalues = 1 ($|0\rangle$) and -1 ($|1\rangle$).

Let $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$. Then measurement of Z on the state $|\psi\rangle$: can give two of the following results:

+1

corresponding eigenvector = $|0\rangle$

$P_0 = |0\rangle\langle 0|$

$p(0) = \langle\psi|\,|0\rangle\langle 0|\,|\psi\rangle$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}$$

−1

Similarly, the probability that measurement of Z on $|\psi\rangle$ yields −1 is given by $\langle\psi|\,\underbrace{|1\rangle\langle 1|}\,|\psi\rangle$

eignvector projector for −1

General – suppose $\vec{v}$ is any 3d unit vector, then we can define an observable $\boxed{\vec{v}\cdot\vec{\sigma}} \equiv v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$.

Measurement of this observable is $\boxed{\text{measurement of spin along } \vec{v} \text{ axis.}}$

ex 2.59: ? Given state $|\psi\rangle = |0\rangle$, and observable X is measured. $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: Eigenvalues ≡ −1 and +1

Eigenvectors: $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$.

Average $E(x) = \sum_m m \cdot P_m = \sum_m m \cdot \langle\psi| \cdot P_m \cdot |\psi\rangle$

↳ orthogonal projector.

$E(X) = \langle\psi| . X\,|\psi\rangle$

$$= [1 \quad 0] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [0 \quad 1] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Wrong

Denote observable X as the sum of projectors $\sum_m m\,P_m$

where $P_m = |m\rangle\langle m|$

$$X = +1\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).(\langle 0| + \langle 1|)\right) + (-1).\left(\frac{1}{2}\,(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)\right)$$

$$E(x) = \frac{1}{2}\langle 0|0\rangle\langle 0|0\rangle - \frac{1}{2}\langle 0|0\rangle\langle 0|0\rangle = 1$$

Basic — The observable needs to be written as a linear combination of projections (where each projection is the project $|m\rangle\langle m|$ where m is the eigenvector).

Probably now you see why eigenvectors (or spectral decomposition) are so important.

$\boxed{\text{ex 2.60}}$ $\vec{v}\cdot\vec{\sigma}$ has eigenvalues ± 1 and that the projectors are given by $P_\pm = \frac{(I \pm \vec{v}\cdot\vec{\sigma})}{2}$.

For every unit vector (3D) v, an observable can be defined $\vec{v}\cdot\vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$. Now, the spectral decomposition has to be studied —

$$\vec{v}\cdot\vec{\sigma} = \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}.$$

characteristic polynomial $\det(\vec{v}\cdot\vec{\sigma} - \lambda I) = 0$

$\Rightarrow \lambda^2 - (v_1^2 + v_2^2 + v_3^2) = 0$. (implicit — the vector $\vec{v}$ is normalized. Only then $v_1^2 + v_2^2 + v_3^2 = 1$)

$$\boxed{\lambda = \pm 1}$$

Alternative — when given observable

① $\vec{v}\cdot\vec{\sigma} = +1 \cdot P_{+1} - 1 \cdot P_{-1}$ (From ④)

② Also $P_{+1} + P_{-1} = I$ (From ④)

Rearranging — $P_\pm = \frac{(I \pm \vec{v}\cdot\vec{\sigma})}{2}$

$\boxed{\text{ex 2.61}}$ Given state prior to measurement $= |\psi\rangle = |0\rangle$.

Measurement of $\vec{v}\cdot\vec{\sigma}$ to yield $+1$. $\left(\frac{I + \vec{v}\cdot\vec{\sigma}}{2}\right) = P_{+1}$

Probability $p(1) =$ $\cancel{\langle\psi|0\rangle\langle 0|}$ $\langle\psi| P_{+1} \cdot |\psi\rangle$

In general, probability $p(m) = \langle\psi| P_m^{+} \cdot P_m |\psi\rangle$ → Hermitian and $P_m^2 = P_m$.

Is this even true, coz here spectral decomposition is taking place.

$$p(1) = \langle 0| \left[\frac{I + \vec{v}\cdot\vec{\sigma}}{2}\right] |0\rangle = \cancel{[1 \quad 0]}$$

$$= \frac{1}{2}\langle 0| I |0\rangle + \frac{1}{2}\langle 0| \vec{v}\cdot\vec{\sigma} |0\rangle$$

$$= \frac{1}{2}\langle 0|0\rangle + \frac{1}{2}\langle 0| (v_1 X + v_2 Y + v_3 Z) |0\rangle$$

$$= \frac{1}{2} + \frac{1}{2}[1 \quad 0] \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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③ $= \frac{1}{2} + \frac{1}{2}\begin{bmatrix} v_3 & v_1 - iv_2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}(1 + v_3)$

State after measurement — action on state (previous) / $\sqrt{\text{probability}}$

$$= \frac{P_{+1}|0\rangle}{\sqrt{p(P_{+1})}} = \frac{\frac{I + \vec{v}\cdot\vec{\sigma}}{2}|0\rangle}{\sqrt{\frac{1}{2}(1+v_3)}}$$

$$\rightarrow \frac{1}{2}\underbrace{I.|0\rangle}_{\text{No effect}} + \frac{1}{2}\underbrace{\begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}}_{2\times2}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{2\times1} = \frac{1}{2}|0\rangle + \frac{1}{2}\begin{bmatrix} v_3 & v_1 + iv_2 \end{bmatrix}$$

Can't calculate this way

→ Currently stateless. We need to convert into some known state.

$$\begin{bmatrix} v_3 \\ v_1 + iv_2 \end{bmatrix} = c_1|0\rangle + c_2|1\rangle$$

$$\Rightarrow c_1 = v_3 \; ; \; c_2 = v_1 + iv_2$$

Thus,

$$\text{After state} = \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} \cdot \left[\left(\frac{1}{2} + v_3\right)|0\rangle + (v_1 + iv_2)|1\rangle\right]$$

Of POVM measurements

- Means to study the general measurement statistics without having to deal with post measurement states.

POVM formalism $\equiv$ $\boxed{\text{Positive Operator-Valued measure}}$

$p(m) = \langle\psi| M_m^{\dagger} M_m |\psi\rangle$ where M is/are measurement operators of the measurement.

Let $E_m = M_m^{\dagger} M$. Now,

① E_m is a positive operator such that ~~E~~ $\sum_m E_m = I$

② $p_m = \langle\psi| E_m |\psi\rangle$

$\{E_m\} \equiv$ POVM, sufficient to determine the probabilities of different measurement outcomes.

In case of projective measurement, $P_m P_{m'} = \delta_{m,m'} P_m$ and $\sum_m P_m = I$, all POVM are same as measurement operators $E_m = P_m$.

Clarification on exercise on Pg 8 — The measurement of qubit in computational basis is done using the projective measurement as can be deduced how ~~$M^{\dagger}M = \langle 0|0\rangle = E_m = M$~~ $M^{\dagger}M = M^2 = \boxed{M = E_m}$. Thus, projective measurements from POVM.

(14) Why general measurements are preferred to projective ones?

— ① simpler as less restrictions on the operators

② problems in QCQI require general measurements

③ projective measurements involve repeatability.

But many quantum mechanics measurements are not repeated. ex measuring photon's position destroys the photon itself.

— POVMs provide the simplest means to study general measurement statistics.

Ex 2.63: According to POVM : $E_m = M_m^\dagger M_m$

$\Rightarrow \sqrt{E_m} = \sqrt{M_m^\dagger M_m}$. Now, $M_m = V D V^{-1}$ (diagonalization)

$M_m^\dagger = V D^\dagger V^{-1}$. Thus, $\sqrt{E_m} = V \sqrt{D^\dagger D}\, V^{-1}$

Now, $M_m = V P \sqrt{D^\dagger D}\, V^{-1} \Rightarrow \underline{M_m = U_M \sqrt{E_m}}$

---

ex 2.64 :

Intuition — in real physical systems, the quantum state is destroyed after measurement (so the evolution of the system can't be well defined by projection on the subspace). Also, the evolution is not Hamiltonian and not unitary (because energy might enter and leave the system). Because only in closed systems, energy is conserved and time evolution is unitary.

In the above scenario, the POVM formalism is better as it does NOT require us to actually know about the environment.

However, if one starts with a POVM, one has no idea how to measure the post-measurement state

Replacing the old closed system postulate with open system one —

Measurements are described by collection of operators $\{M_m\}$ that are NOT necessarily projections but fulfil $\sum_m M_m^\dagger M_m = I$

(15) My thoughts about this – The quantum mechanics postulates are essentially applicable on closed systems only. In closed systems, one may have a unitary time evolution by a Hamiltonian, and measurements may be described by projective measurements (or using spectral decomposition of an observable such that there are m eigenvalues and m corresponding eigenvectors $|A_m\rangle$. The probability of the measurement yielding m is given by $\langle\psi||A_m\rangle\langle A_m||\psi\rangle$). Thus measurements are $\boxed{\text{non-demolition}}$ measurements.

In practice, however, the systems are not closed. Evolution of a system is NOT unitary (or a time varyeng Hamiltonian is present that takes into consideration the effects of the environment as well): $H_S \otimes H_E$ ($S \equiv$ system, $E \equiv$ environ.)

Now, we face 2 choices –

1. consider the system as a part of a larger closed system and apply time evolution on it (cumbursome as complete knowledge about the environment is required).
2. consider the system as $\boxed{\text{open}}$ (which will then modify the means of doing the measurement). no environment. And since by definition, POVM is restriction on projection that doesn't care about the environment.

Also, POVM $\equiv$ projections (if we factor in the environment + some additional unitary evolution). Working only with POVM allows one to work without the environment.

Considering a POVM : E, ~~one~~ $= M^{\dagger}.M$, one can simply find the required probability $\langle\psi|E|\psi\rangle$ and post-measurement state $\dfrac{M.|\psi\rangle}{\sqrt{\langle\psi|E|\psi\rangle}}$.

But, we do NOT start our analysis with POVM. Given a E, it is impossible to determine M. This means we have essentially lost the ability to calculate a post-measurement state. If one starts with general measurement operators M, one retains the ability to calculate anything using POVM, as well as the post-measurement state.

Of Phase

Global phase factor — Consider any two states, $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$, where $|\psi\rangle$ is a state vector and θ is a real number. It is interesting to note that the statistics of these 2 states the same

Relative prob: $\langle\psi|M^{+}M|\psi\rangle$ and $\langle\psi|e^{-i\theta}M^{+}Me^{i\theta}|\psi\rangle$
$= \langle\psi|M^{+}M|\psi\rangle$ where M is any measurement operator.

Hence, global phase factors are essentially irrelevant to the observed properties of the physical system

Relative phase — Two amplitudes, a and b, are differing by a relative phase if there exists a real θ such that $a = \exp(i\theta)\, b$

Composite systems —

Postulate 4 : The state space of a composite physical system is the tensor product of the state spaces of the component physical systems.

Understanding Postulate 4 by proving projective measurements with unitary dynamics are sufficient to implement a general measurement.
→ Proofs —

$\boxed{2.68}$ Entanglement — Consider a quantum state $|\psi\rangle$ and single qubit states $|a\rangle$ and $|b\rangle$. Then, there no way to decompose $|\psi\rangle$ into product of $|a\rangle$ and $|b\rangle$. (or $|\psi\rangle \neq |a\rangle|b\rangle$).

~~Proof~~

Given $\psi = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Let $|\psi\rangle = |a\rangle \otimes |b\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}$

Now

$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$|11\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Adding → $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$a_0 b_1 = 0$
$a_1 b_0 = 0$

Atleast ~~one~~ two of $a_0, a_1, b_1, b_2 = 0$. But, then $a_0 b_0, a_1 b_1$ (at least 1 would be 0). He...