Poisson Brackets

Given a function $F(q, p)$, its variation with time:-

$$\frac{d}{dt} F(q, p) = \sum_i \frac{\partial F}{\partial P} \cdot \dot{P} + \frac{\partial F}{\partial q} \cdot \dot{q}$$

$$= \sum_i \frac{\partial F}{\partial P_i} \cdot \left(- \frac{\partial H}{\partial \overset{*}{q}_i} \right) + \frac{\partial F}{\partial q_i} \cdot \frac{\partial H}{\partial P_i}$$ ← Resemblance to the time derivative of H.

Given any two functions, (may or may not be the Hamiltonian), the above quantity is called the Poisson Bracket

$$\{F, G\} = \sum_i \frac{\partial F}{\partial P_i}\left(\frac{-\partial G}{\partial q_i}\right) + \left(\frac{\partial F}{\partial q_i} \cdot \frac{\partial G}{\partial P_i}\right)$$

Time derivative of any function $= \{F, H\}$ most condensed form of mechanics.

$$\boxed{\dot{F}(q, P) = \{F, H\}}$$

(28) ① Poisson brackets are anti-symmetric:

$$\{A, B\} = -\{B, A\}$$

② $\{A+B, C\} = \{A, C\} + \{B, C\}$

③ $\{\lambda A, B\} = \lambda\{A, B\}$ ④ $\{AB, C\} = \{A, C\}B + \{B, C\}.A$

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① $\{q_i, q_j\} = 0$ since there is nothing here depending on p. Similarly $\{p_i, p_j\} = 0$ since there is nothing here depending on q.

② $\{q_i, p_j\} = \frac{\partial F}{\partial q_i}\cdot\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\cdot\frac{\partial G}{\partial q_i}$

$= 0$ if $i \neq j$

$\neq 1$ if $i = j$ or δ_{ij} $\dot{F} = \{F, H\}$

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ex Harmonic oscillator: $H = \frac{p^2}{2m} + \frac{k}{2}q^2$

$\dot{q} = \{q, H\} \Rightarrow \dot{q} = \{q, \frac{p^2}{2m} + \frac{k}{2}q^2\}$

$\Rightarrow \dot{q} = \{q, \frac{p^2}{2m}\} + \underbrace{\{q, \frac{k}{2}q^2\}}_{=0} \Rightarrow \dot{q} = \{q, \frac{p^2}{2m}\}$

$\Rightarrow \dot{q} = \frac{1}{2m}\{q, p^2\} \Rightarrow \dot{q} = \frac{1}{2m}\{q, p.p\}$ Product rule
(λ out)

$\Rightarrow \dot{q} = \frac{1}{2m}\left[\{q, p\}.p + \{q, p\}.p\right] \Rightarrow \boxed{\dot{q} = \frac{p}{m}\{q, p\}}$

$\Rightarrow \boxed{\dot{q} = \frac{p}{m}}$ first equation of motion

Second equation of motion $= \dot{p}$

$\dot{p} = \{p, H\} \Rightarrow \dot{p} = \{p, \frac{p^2}{2m} + \frac{k}{2}q^2\}$

$\Rightarrow \dot{p} = \{p, \frac{p^2}{2m}\} + \{\overset{\times}{p}, \frac{k}{2}q^2\} \Rightarrow \dot{p} = \{p, \frac{k}{2}q^2\}$

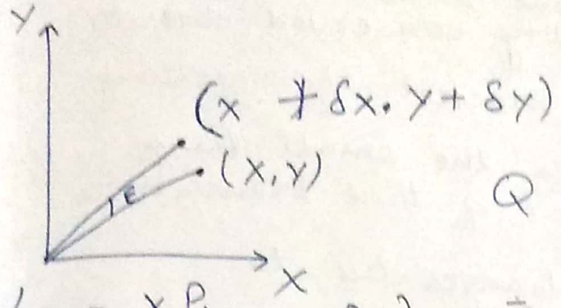
$\Rightarrow \dot{p} = \frac{k}{2}\{p, q^2\} \Rightarrow \dot{p} = \frac{k}{2}\{p, q.q\}$

$\Rightarrow \dot{p} = \frac{k}{2}\left[\{p, q\}.q + \{p, q\}.q\right] \Rightarrow \boxed{\dot{p} = kq\{p, q\}} \Rightarrow \boxed{\dot{p} = -kq}$

since $\{q, p\} = 1 \Rightarrow \{p, q\} = -1$

Of Angular momentum

Quantity that is conserved due to rotational invariance (or rotational symmetry discussed here).



rotational symmetry.

$$\boxed{\begin{aligned}\delta x &= -y\cdot\varepsilon \\ \delta y &= +x\cdot\varepsilon\end{aligned}} \begin{aligned}&\Rightarrow f_x(x) = -y \\ &\Rightarrow f(y) = +x\end{aligned}$$

$$\delta z = 0$$

$$Q = \sum_i P_i \cdot f(q_i) \text{ is conserved.}$$

$$\left.\begin{aligned} L_z &= xP_y - yP_x \\ L_x &= yP_z - zP_y \\ L_y &= zP_x - xP_z \end{aligned}\right\} \text{conserved for isolated systems.}$$

$= P_x\cdot(-y) + P_y(+x) = L_z$ angular momentum in z-axis.

$$\vec{L} = \vec{r} \times \vec{P} \quad \text{or} \quad L = L_x + L_y + L_z$$

$$\{x, L_z\} = \{x, xP_y - yP_x\} = \{x, xP_y\} - \{x, yP_x\}$$

$$= \underbrace{\{x, x\}}_{\substack{=0 \\ \text{No } P \text{ term}}} P_y + \{x, P_y\}\cdot x - \Big[\underbrace{\{x, y\}}_{\substack{=0 \\ \text{No } P \text{ term}}}\cdot P_x + \{x, P_x\}\, y\Big]$$

$$= \{x, P_y\}\cdot P_y * -\{x, P_x\}\, y$$

$$= 0 - 1\cdot y$$

All using Poisson properties described above.

$$\{y, L_z\} = \{y, xP_y - yP_x\} = x\{y, P_y\} = x$$

$$\{z, L_z\} = \{z, xP_y - yP_x\} = 0$$

<u>Note;</u> $\delta x = -y\varepsilon = \{x, L_x\}\cdot\varepsilon;\ \delta y = +x\cdot\varepsilon = \{y, L_z\}\cdot\varepsilon$

where z is the <u>axis of rotation</u>.

Similarly, $\boxed{\delta P_x = -P_y;\ \delta P_y = P_x;\ \delta P_z = 0}$

① $\{P_x, L_z\} = \{P_x, x\cdot P_y - y\cdot P_x\} = \{P_x, x\cdot P_y\} - \{P_x, y\cdot P_x\}$

$$= \underbrace{\{P_x, P_y\}}_{\substack{=0 \\ \text{No } q \text{ term}}}\cdot x + \{P_x, x\}P_y - \Big[\underbrace{\{P_x, P_x\}}_{\substack{=0 \\ \text{No } q \text{ term}}}\cdot y + \underbrace{\{P_x, y\}}_{\substack{=0 \text{ since } i\neq j \\ \text{in } \delta_{ij}}}\cdot P_x\Big]$$

$$= \{P_x, x\}\, P_y = -\{x, P_x\}\, P_y = -P_y \quad \text{since } \{x, P_x\} = 1$$

(30) Similar other calculations.

Poisson bracket with angular momentum gives the small change in the quantity by virtue of rotation. $\{x, L_z\}$ Generator of rotations about z axis is L_z (31)

H was the quantity conserved due to time invariance. So poisson bracket of something with the Hamiltonian gives the small change wrt time. $\{F, H\}$ Generator of time translations is H.

– Poisson brackets give changes due to certain symmmetry operations.

Translations: conserved quantity = sum of momenta.

To see what Poisson brackets give in such a case.

Symmetry: $\delta q = 1 \cdot \varepsilon$; Thus, $Q = \sum_i P_i \cdot f(q_i)$

$$\Rightarrow Q = \sum_i P_i \cdot 1 \Rightarrow \boxed{P = \sum_i P_i} \text{ (the momentum)}$$

Claim: change in some $F(q)$ is $\delta F_i = \frac{\partial F}{\partial q_i} \cdot \delta q_i$

Now do Poisson brackets give the same thing.

$$\{F(q_i), P\} = \{F(q_i), P_1 + P_2 + \dots + P_i + \dots + P_n\}$$

$$= \{F(q_i), P_i\} = \frac{\partial F}{\partial q_i} \cdot \frac{\partial P_i}{\partial P_i} - \underbrace{\frac{\partial F}{\partial P_i} \cdot \frac{\partial P_i}{\partial q_i}}_{=0}$$

$$= \boxed{\frac{\partial F}{\partial q_i}}$$ exactly what we were looking for.

Symmetry operations are connected to Poisson. So if Poisson brackets with $\boxed{\text{conserved}}$ quantity are taken, small transformations are obtained that retain the symmetry. iff conserved.

So P is the generator of translations that follow the conservation law implied by the conserved quantity P.

(31) Let G be a generator. Let $\boxed{G \text{ be conserved}}$ and thus $\frac{dG}{dt} = 0$ or $\{G, H\} = 0$ or $\{H, G\} = 0$

$\{G, H\} = 0 \Rightarrow$ G doesn't change when the system evolves. Conservation law.

$\{H, G\} = 0 \Rightarrow$ H doesn't change when G changes (or G involves a symmetry. Presence of symmetry

Poisson brackets involve giving small transformations of some quantity G. If G is conserved, we can derive some symmetry retained transformations.

$$\{L_x, L_z\} = \{yP_z - zP_y, xP_y - yP_x\}$$

$$= \{yP_z, xP_y - yP_x\} - \{zP_y, xP_y - yP_x\}$$

$$= \{yP_z, xP_y\} + \{zP_y, yP_x\} = P_z \cdot x - \cancel{P_z \cdot x}^{\,z \cdot P_x} = \cancel{0} - L_y$$

~~So~~ change in L_x as L_z changes is $\cancel{0}$. $- L_y$.

Similar to $\{x, L_z\} = -y$

Similarly: $\{L_y, L_z\} = \{zP_x - xP_z, xP_y - yP_x\}$

$$= \{zP_x, xP_y\} + \{xP_z, yP_x\} = yP_z - z \cdot P_y$$

$= L_x$ (same as $\{y, L_z\} = x$)