Notes on Preskill

States and Ensembles.

Hilbert space - vector space over complex numbers ... having an inner product defining:

① $\langle\phi|\phi\rangle > 0$ for $|\phi\rangle \neq 0$ Square of distance from origin

② $|\phi\rangle^{\dagger}(a|\psi_1\rangle + b|\psi_2\rangle) = \langle\phi|(a|\psi_1\rangle + b|\psi_2\rangle)$

$= a\langle\phi|\psi_1\rangle + b\langle\phi|\psi_2\rangle$ Linearity

③ $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$ Skew symmetry

- Complete in the norm - $||\psi|| = \langle\psi|\psi\rangle^{1/2}$
  - an important point in infinite dimensional function spaces (ensures convergence of certain eigenfunction expansions, ex Fourier analysis).
- $|\psi\rangle$ and $e^{i\alpha}|\psi\rangle$ are same if $|e^{i\alpha}| = 1$.

Observable - a property of the system that can be measured. In quantum mechanics, an ~~adjoint operator is a self~~ observable is a self-adjoint operator given by: ~~$|\phi\rangle$~~ $\langle\phi|A|\psi\rangle = \langle\phi|A^{\dagger}|\psi\rangle$

- Given A and B are self-adjoint,
  ① $A+B$ is self adjoint $(A+B)^{\dagger} = A^{\dagger} + B^{\dagger}$
  ② $(AB)^{\dagger} = B^{\dagger}A^{\dagger} = BA$ is ~~adj~~ self adjoint iff A and B commute.
- self-adjoint operator has spectral representation it's eigenstates form a complete orthonormal basis in the Hilbert space: $\boxed{A = \sum_n a_n P_n}$

a_n = eigenvalue and P_n = ~~$\sum$~~ $|n\rangle\langle n|$ where $|n\rangle$ is the corresponding eigenvector if a_n is nondegenerate

$P_n = |n\rangle\langle n| \rightarrow$ projection onto the corresponding eigenvector - so the outer product is like a projection. $\rightarrow$ ① $P_n = P_n^{\dagger}$ ② $P_n . P_m = \delta_{n,m} P_n$

Measurement - numerical value of an observable A after measurement is the eigenvalue i; and the state after measurement is the eigenstate A_i corresponding to the eigenvalue i.

i ~~an~~ is measured with probability: $||A_i|\psi\rangle||^2$

83) square of the norm of the state after application of the eigenstate $= \langle \psi | A_i | \psi \rangle$

Post-measurement state : $\dfrac{A_i |\psi\rangle}{\sqrt{\langle \psi | A_i | \psi \rangle}}$

– if measurement is repeated again, the state eigenstate comes up with probability 1.

Dynamics – $\dfrac{d}{dt} |\psi(t)\rangle = -i H |\psi(t)\rangle$

$H \equiv$ self-adjoint operator called Hamiltonian.

$\rightarrow |\psi(t+dt)\rangle = (1 - iH\,dt) |\psi(t)\rangle$

↳ unitary since H is self-adjoint $\equiv U(dt)$.

Then $U^\dagger U = I$

$$\boxed{U = e^{-itH}}$$

So what is the difference between a qubit and a classical bit that is probabilistic?

So what is the probabilistic classical bit – if a bit has some definite value 0 or 1 but that is unknown, so now the probability is also $\frac{1}{2}$ in this case.

Spin-$\frac{1}{2}$: symmetry is a transformation on the system that leaves all observables unchanged.

– if an observable A is measured in the state $|\psi\rangle$, it occurs with the probability $|\langle a|\psi\rangle|^2$

$|a\rangle \equiv$ an outcome / an eigenvector of A

Symmetry : mapping of vectors in the Hilbert space $|\psi\rangle \rightarrow |\psi'\rangle$ that preserves absolute values of the inner products – $|\langle \psi|\phi\rangle| = |\langle \psi'|\phi'\rangle|$ for all states $|\phi\rangle$ and $|\psi\rangle$. A famous theorem by Wigner ~~also~~ can always be chosen that such a mapping to be unitary or anti-unitary, such that : $|\psi'\rangle = U|\psi\rangle$ where U can be unitary (and linear) or anti-unitary.

– Symmetries form a group.

$$\begin{array}{ccc} \text{initial} & \xrightarrow{\text{dynamics}} & \text{final} \\ \downarrow \text{rotation} & & \downarrow \text{rotation.} \\ \text{new initial} & \xrightarrow[\text{dynamics}]{} & \text{new final} \end{array}$$

The time evolution operator commutes with symmetry

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83 $U(R)\, e^{-itH} = e^{-itH} \cdot U(R)$ – where $U(R)$ is th symmetry transformation.

$\Rightarrow$ $U(R) H = H\, U(R)$ — linear order in time t

– For continuous symmetry, R is very close to I

$U = 1 - i\varepsilon Q + O(\varepsilon^2)$ since $R = I + \varepsilon T$

U is unitary $\Rightarrow$ Q is an observable. or $Q = Q^{+}$

Expanding to linear order in ε, $[Q, H] = 0$
somewhat like Poisson brackets. ↳ conservation la

$[Q, H] = 0 \Rightarrow$ if we prepare an eigenstate of Q, then time evolution by Schrondinger's equation pre that eigenstate.

– Some abstract math here, but the sense that comes out is –

To implement rotations on a quantum system self-joint operator J_x, J_y, J_z are found in the Hilbert space satisfying the symmetry relations

$$J_x = \frac{1}{2}(\sigma_x);\ J_y = \frac{1}{2}(\sigma_y);\ J_z = \frac{1}{2}(\sigma_z)$$

where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices.

Eigenvalues of $J_k = \pm\frac{1}{2}$

Return to this section some time later

Measuring the spin along $\hat{n}$ is equivalent to rotating the system and measuring along $\hat{z}$. if rotation allowed

For instance, $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)$

and $|\downarrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - |\downarrow_z\rangle\right)$. Measuring along z axis we get $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ with equal probability.

Now $\frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle + |\downarrow_x\rangle\right)$. Measurement ~~on~~ along z-axis giv

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right) + \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - |\downarrow_z\rangle\right)\right)$$

$$= \frac{1}{2} \cdot 2 \cdot |\uparrow_z\rangle = |\uparrow_z\rangle$$ (quantum interference is at the root of all this)

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(84) Photon polarization – discussions on Poincaré groups

Density matrix – bipartite quantum system

The postulates of quantum mechanics give the rendering of the entire universe. If a part of it is considered, they tend to fail:

① states are no longer vectors.

② transformation (evolution) is no longer unitary

③ measurements are NOT orthogonal projections

ex observing one qubit in a system of 2 qubits.

Let the orthonormal basis be $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ — that means the Hilbert spaces of individual qubits are still separate. The quantum system is then a composite of these:

$$|\psi\rangle_{AB} = a|0\rangle_A \otimes |0\rangle_B + b|1\rangle_A \otimes |1\rangle_B$$

$$\text{or } |\psi\rangle_{AB} = \boxed{a(|0\rangle_A \otimes |0\rangle_B) + b(|1\rangle_A \otimes |1\rangle_B)}$$ – Page 76-

$$\text{or } |\psi\rangle_{AB} = |0\rangle_A \otimes a|0\rangle_B + |1\rangle_A \otimes b|1\rangle_B$$

A and B are correlated in this state.

Measuring A —

① $|0\rangle_A \otimes |0\rangle_B$ with probability $|a|^2$

② $|1\rangle_A \otimes |1\rangle_B$ with probability $|b|^2$

So outcomes are perfectly correlated in this case.

Considering a more complex observable – $M \otimes I$ such that M is a self-adjoint operator on A and I is identity operator on I given by

$$M \otimes I = \begin{bmatrix} m_{11}I & m_{12}I & \cdots & m_{1n}I \\ \vdots & & & \\ m_{n1}I & m_{n2}I & \cdots & m_{nn}I \end{bmatrix}$$

Expectation of an observable (or probability) M on $\psi = \langle\psi|M|\psi\rangle$

Thus $\langle\psi| M\otimes I |\psi\rangle = \left[a^* \langle 0|_A \otimes \langle 0|_B + b^* \langle 1|_A \otimes \langle 1|_B\right]$.

$$(M\otimes I)\left[a|0\rangle_A \otimes |0\rangle_B + b|1\rangle_A \otimes |1\rangle_B\right]$$

$$(M\otimes I)(|w\rangle \otimes |u\rangle) = M|w\rangle \otimes I|u\rangle$$

$$= [\ldots]\left[a\, M|0\rangle_A \otimes I|0\rangle_B + b\, M|1\rangle_A \otimes I|1\rangle_B\right]$$

$$= \left[a^* a \langle 0|_A M|0\rangle_A \otimes \langle 0|_B I|0\rangle_B + b^* b \langle 1|_A M|1\rangle_A \otimes \langle 1|_B I|1\rangle_B\right.$$

$$= |a|^2 \langle 0|_A M|0\rangle_A + |b|^2 \langle 1|_A M|1\rangle_A$$

(85) $\boxed{\langle M_A \rangle = \underset{\text{Trace}}{tr}(M_A \rho_A)}$ $\rho_A = |a|^2 |0\rangle_A \langle 0|_A + |b|^2 |1\rangle_A \langle 1|_A$

density matrix formalism

– ρ_A :: density operator – self-adjoint, positive (eigen values are nonnegative) and has unit trace.

In general, considering two Hilbert spaces H_a and H_b and the composite system $H_a \otimes H_b$, the expection of any observable M is given by:

$\langle M \rangle = tr(M\rho)$ where ρ is the density operator given by:

$\rho = \sum_a p_a |\psi_a\rangle\langle\psi_a|$ where $|\psi_a\rangle$ is a pure state acting one one subsystem and comes with probability p_a where $p_a = aa^*$ where a is the amplitude;

– If ρ is combined of only one state (it is pure and constitutes a vector), $\rho^2 = \rho$.

$|\psi_a\rangle\langle\psi_a|$ will then be a projection on the 1-D subspace spanned by $|\psi_a\rangle$ and $p_a = 1$

– More than 2 terms in the summation $\Rightarrow$ ρ is an incoherent superposition of states $|\psi_a\rangle$

Relative phases of $|\psi_A\rangle$ are inaccessible, or they are entangled (if some other subsystem exists).

– entanglement destroys the coherence of the superposition of states of A, such that some phases are inaccessible looking at A alone.

Bloch sphere – Note that σ_x, σ_y, and σ_z are traceless. The most general self-adjoint matrix can be expanded in the basis $\{I, \sigma_x, \sigma_y, \sigma_z\}$

$$\rho(\vec{P}) = \frac{1}{2}(I + \vec{P}\cdot\vec{\sigma}) = \frac{1}{2}(I + P_1\sigma_x + P_2\sigma_y + P_3\sigma_z)$$

$$\rho(\vec{P}) = \frac{1}{2}\begin{pmatrix} 1+P_3 & P_1 - iP_2 \\ P_1 + iP_2 & 1-P_3 \end{pmatrix} \Rightarrow \det(\rho) = \frac{1}{4}(1-\vec{P}^2)$$

$|\vec{P}| = 1$ for pure state
< 1 for mixed state.

(86) Necessary and sufficient condition for ρ to have nonnegative eigenvalues — $\vec{P}^2 \leq 1$ or $\det \rho \geq 0$

Thus there is a 1-1 correspondence between the possible density matrices of a single qubit and a unit 3-ball called the Bloch sphere. Such that $0 \leq |\vec{P}| \leq 1$.

The boundary of the sphere contains density matrices of vanishing determinant (since at boundary $|\vec{P}| = 1$ and $\det \rho = 0$)

Skipped parts beyond my mathematical intuition.

Gleason's theorem — density matrix is a very general feature of a broader framework.

— starts with the understanding that quantum mechanics must assign probabilities to each orthogonal projection in the Hilbert space (i.e. the state of a quantum system is a mapping from an orthogonal projection to a non-negative real number such that):

$$E \longrightarrow p(E); \quad 0 \leq p(E) \leq 1$$

① $p(0) = 0$ ② $p(1) = 1$

③ if $E_1 E_2 = 0$, then $p(E_1 + E_2) = p(E_1) + \cancel{E_2)}\ p(E_2)$

① - there is always an outcome ; ② all probs. add up to 1. ③ → if E_1 and E_2 are orthogonal, probabilities are additive.

Theorem: $p(E) = tr(\rho E)$ ρ is a Hermitian positive ρ.

as long as dimension > 2

— in 2 dimensions, things are simpler since there are not enough orthogonal projections. All non-trivial projections can be represented as:

$$\boxed{E(\hat{n}) = \frac{1}{2}(I + \hat{n}.\vec{\sigma})}$$

87 Evolution of density operator

Assumption – the subsystems are not coupled by the Hamiltonians, so they still evolve independently. Given as $H_A \otimes H_B = H_A \otimes I_B + I_A \otimes H_B$. Time evolution operator – $U_A(t) \otimes U_B(t)$

A state can be represented as:

$$|\psi(t)\rangle_{AB} = \sum_{i\mu} a_{i\mu} |i(t)\rangle_A \otimes |\mu(t)\rangle_B$$

Such that $|i(t)\rangle_A = U_A(t)\,|i(0)\rangle_A$ and
$|\mu(t)\rangle_B = U_B(t)\,|\mu(0)\rangle_B$

independent evolution as before.

Partial trace: $\rho_A(t) = \sum_{i,j,\mu} a_{i\mu} a^*_{j\mu} |i(t)\rangle_A \langle j(t)|_A$

They are diff...

But it doesn't matter.
$|i(t)\rangle_A \langle j(t)|_A$ is a projection and shall be 0 when $i \neq j$. ← There is no such assumption of this sort.

$$\boxed{\rho_A(t) = U_A(t)\,\rho_A(0)\,U_A(t)^\dagger}$$

, Similar partial trace for $\rho_B(t)$.

– Case: $\rho_A(0)$ is diagonal, it is possible to represent $\sum_a p_a |\psi_A\rangle\langle\psi_A|$ or the fact that the pure states are orthogonal at least (and may be orthonormal ; i don't know).

Schmidt decomposition:

Concept of Partial trace – any density matrix can be decomposed as (on $H_A \otimes H_B$)

$$\rho_{AB} = \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \otimes |b_k\rangle\langle b_l| \text{ where}$$

$\{|a_i\rangle\}$ is the orthonormal basis of H_A and $\{|b_i\rangle\}$ is the orthonormal basis of H_B.

$\mathrm{tr}_B(\rho_{AB})$ = a density matrix in H_A given by

$$= \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \langle b_l | b_k\rangle$$

(88) Back to Preskill — Arbitrary vector in $H_A \otimes H_B$ can be written as $\sum_{ij} a_{ij} |i\rangle_A \otimes |j\rangle_B = \sum_i |i\rangle_A \otimes |\tilde{i}\rangle_B$

where $|\tilde{i}\rangle_B = \sum_{j} a_{ij} |j\rangle_B$ need not be mutually orthogonal after all.

where a_{ij} are amplitudes.

— If $\{|i\rangle_A\}$ is the orthogonal and orthonormal basis such that ρ_A is diagonal:

$$\rho_A = \sum_i p_i |i\rangle_A \langle i|_A \quad \text{where } p_i \text{ is the probability.}$$

Computing ρ_A by a partial trace: $\rho_A = tr_B(|\psi\rangle_{AB}\langle\psi|_{AB})$

where $|\psi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A |j\rangle_B$

Thus $|\psi\rangle_{AB} \langle\psi|_{AB} = \sum_{ijkl} c_{ij} |i\rangle_A \langle k|_A \otimes |j\rangle_B \langle l|_B$

$tr_B(|\psi\rangle_{AB} \langle\psi|_{AB}) = \sum_{ij} c_{ijkl} \langle l|j\rangle_B \cdot |i\rangle_A \langle k|_A$ [Best to let these variables be all separate

Equating both — $\rho_A = \sum_{ijkl} c_{ijkl} \langle l|j\rangle_B |i\rangle_A \langle k|_A = p_i |i\rangle_A \langle i|_A$

OR $p_i |i\rangle_A \langle i|_A = p_i \delta_{ij} |i\rangle_A |j\rangle_B$

$\Rightarrow \boxed{c_{ijkl} \langle l|j\rangle_B = p_i \delta_{ij}}$ or as Preskill did — $\langle \tilde{j}|\tilde{i}\rangle = p_i \delta_{ij}$

$\Rightarrow |\tilde{i}\rangle_B$ ~~and iff~~ are orthogonal after all

Normalizing them: $|i'\rangle_B = p^{-1/2} |\tilde{i}\rangle_B$. Placing back;

any arbitrary vector in $H_A \otimes H_B$ can be written as $\sum_i |i\rangle_A \otimes p^{-1/2} |i'\rangle_B$ (Schmidt decomposition) of particular orthonormal basis.

A comparison of ~~traces from~~ ρ_A from definition and partial trace yielded the decomposition result.

— It is easy to see the orthonormal bases used for $H_A \otimes H_B$ will be different for different vector states $|\psi_A\rangle_{A\otimes B}$ and $|\phi_A\rangle_{A\otimes B}$ because of their different amplitudes..

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89 **Partial trace over H_A — or ρ_B.** The given arbitrary state: $|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B$.

Then $|\psi\rangle_{AB} = \sum_{i,j} |i\rangle_A \otimes |\tilde{i}\rangle_B$. ← Preskill's method.

Job is to normalize this new basis; for which we compare ρ_B from 2 sources — definition and partial trace.

Def: $\rho_B = \sum_{ij} p_{\tilde{ij}} |\tilde{i}\rangle_B \langle \tilde{j}|_B \cdot \delta_{ij}$

Trace — ~~State $|\psi\rangle_{AB}$~~ $|\psi\rangle_{AB}\langle\psi|_{AB} = \sum_{i,j} |i\rangle_A \langle j|_A \otimes |\tilde{i}\rangle_B \langle \tilde{j}|_B$

~~tr~~ $\rho_B = tr_A(|\psi\rangle_{AB}\langle\psi|_{AB})$

$\Rightarrow \rho_B = \sum_{ij} \langle j|i\rangle_A |\tilde{i}\rangle_B \langle \tilde{j}|_B$

Comparing — $\langle j|i\rangle_A |\tilde{i}_B\rangle\langle\tilde{j}|_B = p_{\tilde{ij}} \cdot \delta_{ij}$

or $|\tilde{i}\rangle_B$ is indeed orthonormal.

Final form — $|\psi\rangle_{AB} = \sum_{ij} \sqrt{p_{ij}}\ |i\rangle_A \otimes |i'\rangle_B$

such that $|i'\rangle_B = \{\sqrt{\tilde{p}_{ij}}\}^{-1} |\tilde{i}\rangle_B$

$\rho_A = \sum_i p_i |i\rangle_A \langle i|_A$ and $\rho_B = \sum_i p_i |\tilde{i}\rangle_B \langle\tilde{i}|_B$

— same non-zero eigenvalues. Number of zero eigenvalues may be different as H_A and H_B might have different dimensions.

Process — if there is no degenerate eigenvalue in ρ_A or ρ_B,

① find eigenstates of ρ_A and ρ_B

② diagonilize ρ_A and ρ_B to find $|i\rangle_A$ and $|\tilde{i}\rangle_B$.

③ Pair eigenstates with same eigenvalue to obtain the decomposition.

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Entanglement — Schmidt decomposition theorem —

Consider 2 Hilbert spaces H_1 and H_2, and the orthonormal bases $\{w_1, w_2, \ldots, w_m\} \in H_1$ and $\{x_1, x_2, x_3, \ldots, x_m\} \in H_2$, and dimension of $H_2 = n >$ dimension of $H_1 = m$. Then any vector y in $H_1 \otimes H_2$:

$$y = \sum_{i=1}^{m} \alpha_i w_i \otimes x_i$$ where α_i is uniquely determined by y.

So it is like expressing a vector in same number of orthonormal vectors, and refer to the point that says ρ_A and ρ_B have same non-zero eigenvalues. These m vectors must be the eigenvectors corresponding to those m eigenvalues.

The reason to do $\sum_{ij} c_{ij} |i\rangle_A \otimes |j\rangle_B \longrightarrow \sum_i c_i |i\rangle_A \otimes |\tilde{i}\rangle_B$ was to reduce the system to one variable, hence ensuring same number of orthonormal vectors are considered in both H_A and H_B.

Entanglement — entangled if the Schmidt no. is more than 1. → eigenvalue? (Yes)

— $|\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A \otimes |\tilde{i}\rangle_B$ is a non-entangled state if the Schmidt number is 1, or we can take a pure state in ~~H_A~~ H_A and an impure state in H_B and depict $|\psi\rangle_{AB}$ as a tensor product of the two.

Also if $|\psi\rangle_{AB}$ is non-entangled; $\rho_A = 1.|i\rangle_A \langle i|_A$ and $\rho_B = 1.|\tilde{i}\rangle_B \langle \tilde{i}|_B$

Entanglement can never be created within a subsystem. Note CORRELATIONS $\neq$ ENTANGLEMENT. Correlations can exist even in non-entangled states. $|00\rangle$ is correlated; but $|\uparrow\rangle_A |\uparrow\rangle_B$ are also correlated — not are entangled. $|\uparrow\rangle_A |\uparrow\rangle_B$ can be prepared without the subsystems ever interacting.

example of entangled state — $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\uparrow\rangle_B + |\downarrow\rangle_A |\downarrow\rangle_B\right)$

(91) Ambiguity of ensemble representation –

① Convexity – density operator ρ – self-adjoint, unit trace, and non-negative (or has non-negative eigenvalues)

Consider a convex linear combination of 2 density matrices ρ_1 and ρ_2:

$$\rho(\lambda) = \lambda \rho_1 + (1-\lambda)\rho_2 \quad \text{where } 0 \le \lambda \le 1.$$

To check if $\rho(\lambda)$ is non-negative – $\langle \psi | \rho(\lambda) | \psi \rangle$

$= \langle \psi | (\lambda \rho_1 + (1-\lambda)\rho_2) | \psi \rangle = \lambda \langle \psi | \rho_1 | \psi \rangle + (1-\lambda) \langle \psi | \rho_2 | \psi \rangle$

and ≥ 0, since $\langle \psi | \rho_1 | \psi \rangle \ge 0$ and $\langle \psi | \rho_2 | \psi \rangle \ge 0$.

– A set of a vector space is convex if the set contains the straight line segment connecting any 2 points on the set.

– Most density operators can be expressed as the convex sum of other density operators, but $\boxed{\text{not pure states}}$. Consider ~~$\rho = |\psi\rangle\langle\psi|$~~ $\rho = |\phi\rangle\langle\phi|$ where $|\phi\rangle$ is a vector. Let ~~ϕ~~ $|\psi\rangle$ be orthogonal to $|\phi\rangle$.

Assuming ρ can be expressed as a convex sum:

$$\rho = \lambda \rho_1 + (1-\lambda)\rho_2$$

① self-adjoint ② unit trace follows immediately

③ is positive – $\underbrace{\langle \psi | \rho | \psi \rangle}_{=0} = \underbrace{\lambda \langle \psi | \rho_1 | \psi \rangle + (1-\lambda) \langle \psi | \rho_2 | \psi \rangle}_{\text{non-negative}}$

This is possible iff $\rho_1 = \rho_2$ or it is impossible to express a density operator formed from pure state as a convex sum of ρ_1 and ρ_2. Such vectors are called extremal points. Only the pure states are extremal, because all others can be represented as $\sum_i p_i |i\rangle\langle i|$ in the basis where ρ is diagonal, and so is a convex of pure states.

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(12) <u>Ensemble preparation</u> — Consider $\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2$ and interpret the following:

$$\langle M \rangle = \lambda \langle M \rangle_1 + (1-\lambda)\langle M \rangle_2$$ where ρ_1 is prep. with prob. λ and ρ_2 with prob. $(1-\lambda)$

$$= \lambda\, tr(M\rho_1) + (1-\lambda)\, tr(M\rho_2)$$

$$\langle M \rangle = tr(M\rho(\lambda))$$

Hence expectation value is indistinguishable whether $\rho(\lambda)$ was prepared first or ρ_1 and ρ_2 were.

There are infinite ways to express ~~ρ~~ ρ as a convex sum of ρ_1 and ρ_2, and thus the ambiguity. But pure states are non-ambiguous.

<u>How ambiguous is a mixed state?</u> or in how many ways can ρ written as a sum of the eigenstates/ <u>pure states/extremals</u>? → but eigenstates ≠ pure states

Maximally mixed state of a qubit: $\rho = \frac{1}{2} I$

can be $\rho = \frac{1}{2}\left(|\uparrow\rangle_z \langle\uparrow|_z + |\downarrow\rangle_z \langle\downarrow|_z\right)$ or $\frac{1}{2}\left(|\uparrow\rangle_x \langle\uparrow|_x + |\downarrow\rangle_x \langle\downarrow|_x\right)$

There is absolutely no way to know, just by measuring spin of the system.

Recall the Bloch ball equation $\rho = \frac{1}{2}(I + \vec{P}\cdot\vec{\sigma})$

the center yields $\rho = \frac{1}{2} I$ (Maximally mixed state of the qubit). This could be reached by several ways (2 enumerated above).

– This ambiguous nature contrasts with the classical probability distribution.

<u>Faster than light</u>

Consider a state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B\right)$. It is clearly an entangled state. Then:

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B\right)\left(\langle\uparrow|_A\langle\uparrow|_B + \langle\downarrow|_A\langle\downarrow|_B\right)$$

same Hilbert space goes with same

$$= \cancel{\frac{1}{2}\left(|\uparrow\rangle_A|\uparrow\rangle_B \langle\uparrow|_A\langle\uparrow|_B + \right.}$$

$$= \frac{1}{2}\left(|\uparrow\rangle_A\langle\uparrow|_A \otimes |\uparrow\rangle_B\langle\uparrow|_B + |\downarrow\rangle_A\langle\downarrow|_A \otimes |\downarrow\rangle_B\langle\downarrow|_B\right)$$

$$\rho = tr(|\psi\rangle\langle\psi|) = \frac{1}{2}\left(\underbrace{\langle\uparrow|\uparrow\rangle_B}_{\text{orthonormal, hence } = I} |\uparrow\rangle_A\langle\uparrow|_A + \langle\downarrow|\downarrow\rangle_B |\downarrow\rangle_A\langle\downarrow|_A\right)$$

(93) $\rho_A = \frac{1}{2}\left(|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A\right)$. This ρ_A is same whether for $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$ or $\{|\downarrow_z\rangle, |\uparrow_z\rangle\}$.

So if several copies of $|\psi\rangle_{AB}$ are prepared, and Alice and Bob sit with A and B respectively, if Bob measures his, he causes A to collapse same.

Now alice measures along z-axis...
I can't quite see through the math here, but Bob needs to send classical info to Alice so she could determine things, else there are 2 ensembles with exactly the same ρ_A.
The possibility is that I am confusing one qubit math with a 2 qubit ensemble. In one qubit, measuring along z gives ~~ato~~ always $|\uparrow_z\rangle$ when $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ is used for preparation. But not here. Hence things are entangled, and hence queer.

Suppose original ensemble is : $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_A |\uparrow_z\rangle_B + |\downarrow_z\rangle_A |\downarrow_z\rangle_B\right)$

$$|\psi\rangle_{AB}\langle\psi|_{AB} = \frac{1}{2}\left(|\uparrow_z\rangle_A|\uparrow_z\rangle_B + |\downarrow_z\rangle_A|\downarrow_z\rangle_B\right)\left(\langle\uparrow_z|_A\langle\uparrow_z|_B + \langle\downarrow_z|_A\langle\downarrow_z|_B\right)$$

$$= \frac{1}{2}\left(|\uparrow_z\rangle_A\langle\uparrow_z|_A \otimes |\uparrow_z\rangle_B\langle\uparrow_z|_B + |\downarrow_z\rangle_A\langle\downarrow_z|_A \otimes |\downarrow_z\rangle_B\langle\downarrow_z\right.$$

$$\rho_A = tr_B(|\psi\rangle_{AB}\langle\psi|_{AB}) = \frac{1}{2}\left(|\uparrow_z\rangle_A\langle\uparrow_z|_A + |\downarrow_z\rangle_A\langle\downarrow_z|_A\right)$$

Similarly $\rho_A' = \frac{1}{2}\left(|\uparrow_x\rangle_A\langle\uparrow_x|_A + |\downarrow_x\rangle_A\langle\downarrow_x|_A\right)$

Recall that $|\uparrow_x\rangle_A = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_A + |\downarrow\rangle_z\right)$ and

$$|\downarrow_x\rangle_A = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_A - |\downarrow_z\rangle_A\right)$$

$$\therefore |\uparrow_x\rangle_A\langle\uparrow_x|_A = \frac{1}{2}\left(|\uparrow_z\rangle_A\langle\uparrow_z|_A + |\downarrow\rangle_z\langle\downarrow|_z\right)$$

$$|\downarrow_x\rangle_A\langle\downarrow_x|_A = \frac{1}{2}\left(|\downarrow_z\rangle_A\langle\downarrow_z|_A + |\uparrow_z\rangle_A\langle\uparrow_z|_A\right)$$

Hence implied $\boxed{\rho_A' = \rho_A}$. Thus Alice can't distinguish

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(94) Quantum erasure — coherent and incoherent superpositions are concepts detailing if the spin/phase is observable or not.

$\rho = \frac{1}{2}(I) \leftarrow$ incoherent ; $|\uparrow_x, \downarrow_x\rangle = \frac{1}{2}(|\uparrow_z\rangle \pm |\downarrow_z\rangle)$

= coherent where the phase is observable.

Incoherence comes from entanglement with another spin, rendering that spin inaccessible. Two questions — why and how?

Why — coherence ~~&~~ has everything to do with interference, and interference is destroyed on measurement. Being entangled with B opens a possibility of measuring A through B, thereby destroying interference.

How — Consider a non-entangled state — $|\psi_A\rangle \otimes |\psi_B\rangle$ or their ~~Gram~~ Schmidt decomposition has just 1 term, then it is possible to recover information from ~~$M\otimes$~~ the system using something like $M \otimes I$ as $(M\otimes I)\{|\psi_A\rangle \otimes |\psi_B\rangle\} = \cancel{M\otimes|\psi_A\rangle} \; M|\psi_A\rangle \otimes |\psi_B\rangle$.

Now consider an entangled state — $\frac{1}{2}\left[|\psi_A\rangle \otimes |\psi_B\rangle + |\psi_A'\rangle \otimes |\psi_B'\rangle\right]$

Application of $M \otimes I = \frac{1}{2}\left[M|\psi_A\rangle \otimes |\psi_B\rangle + M|\psi_A'\rangle|\psi_B'\rangle\right]$

and hence no information.

Quantum erasure is the restoration of coherence through some procedure after decoherence by entanglement.

Alice's spin is entangled with Bob's. So it is in incoherent superposition with that of the pure states $|\uparrow_z\rangle_A$ and $|\downarrow_z\rangle_A$. Now Bob measures his state along x and doesn't look at it. So the information is erased, and Alice's state goes back to coherence.

Q. If Alice can recover pure states from mix, then how is ρ_A a complete description of the subsystem?

(95) they are simply different. The ensemble where Alice doesn't know is diff. from the ensemble where Alice receives information.

GHJW theorem — Generalization of quantum erasure

- a mixed state may be realized as an ensemble of pure states in an ∞ number of ways. For a density matrix, consider one such:

$$\rho_A = \sum_i p_i |\phi_i\rangle_A \langle\phi|_A \quad \text{such that } \sum_i p_i = 1$$

where $\{\phi_i\}$ are ~~non~~ normalized vectors. do not assume they are mutually orthogonal.

Then a $\boxed{\text{purification}}$ leads to a bipartite pure state $|\Psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |\phi_i\rangle_A |\alpha_i\rangle_B$

where $\{\phi_i\} \in H_a$ and $\{\alpha_i\} \in H_B$ are ~~or~~ $\boxed{\text{mutually orthonormal.}}$ $\longrightarrow$ pure state meaning.

Then $tr_B(|\Psi\rangle_{AB} \langle\Psi|_{AB}) = \rho_A$

A measurement in B projects the system to an eigenvector α_i causing ρ_A to be pure $= |\phi_i\rangle\langle\phi_i|$ since they were entangled.

Thus a pure state can be found in a mixed state.

Interesting. — purification leading to $\{\alpha_i\}$ may be such that $\{\alpha_i\}$ do not span $\mathcal{H}_a$, but measurement outcomes orthogonal to all $\langle\alpha_i$ never occur.

Now consider another realization of ρ_A

$$\rho_A = \sum_i q_i |\psi_i\rangle_A \langle\psi_i|_A$$

Purification: $|\Psi\rangle_{AB} = \sum_i \sqrt{q_i}\, |\psi_i\rangle_A |\beta_i\rangle_A$

$$\boxed{|\Psi\rangle_{AB} = (I_A \otimes U_B)\, |\Psi\rangle'_{AB}}$$

(96) the two states differ by a unitary change of basis in B alone.

Relation to Schmidt decomposition:

$$|\phi_1\rangle_{AB} = \sum_k \sqrt{\lambda_k}\, |k\rangle_A \, |k_1'\rangle_B$$

$$|\phi_2\rangle_{AB} = \sum_k \sqrt{\lambda_k}\, |k\rangle_A \, |k_2'\rangle_B$$

λ_k = eigenvalues of ρ_A and $|k\rangle_A$ = eigenvectors.

and $|k_1'\rangle$ and $|k_2'\rangle$ are 2 orthonormal bases in $\mathcal{H}_B$, and thus $|k_1'\rangle_B = U_B |k_2'\rangle_B$ or a unitary basis change transformation.

How things finally work –

in $\rho_A = \sum_i p_i |\psi_i\rangle_A \langle\psi_i|_A$, the relative phases are inaccessible as system A is entangled with system B, the state being:

$|\phi_1\rangle_{AB} = \sum_k \sqrt{\lambda_k}\, |k\rangle_A |k'\rangle_B$. So measuring B gives info about A, making A incoherent.

But: $|\phi_2\rangle_{AB} = \sum_k \sqrt{\lambda_k}\, |k\rangle_A |k_2'\rangle_B$ makes things interesting, as now info can be found by measuring in B and relaying information. Now we can extract one of the pure states from the ensemble.

Information is physical – info acquired by measuring B, when relayed to A changes the physical description of state of A.

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