Statistics Assignment 3

1. Let A_i be th event that there are no birthdays in the i^{th} season. The probability that all seasons occur atleast once is 1- $P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Also, $A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$. Using the inclusion – exclusion principle and the symmetry of the seasons,

$$P(A_1 \cup A_2 \ \cup A_3 \cup A_4)$$
$$= \sum_{i=1}^{4} P(A_i) - \sum_{i=1}^{3}\sum_{j>i} P(A_i \ \cap A_j) + \sum_{i=1}^{3}\sum_{j>i}\sum_{k>j} P(A_i \cap A_j \cap A_k)$$
$$= 4P(A_1) - 6P(A_1 \ \cap A_2) + 4P(A_1 \cap A_2 \cap A_3).$$

We have $P(A_1) = (3/4)^7$. Similarly,

$$P(A_1 \cap A_2) = \frac{1}{2^7}$$

And

$$P(A_1 \cap A_2 \cap A_3). = \frac{1}{4^7}$$

Therefore,

$$P(A_1 \cup A_2 \ \cup A_3 \cup A_4) = 4\left(\frac{3}{4}\right)^7 - \frac{6}{2^7} + \frac{4}{4^7}$$

So the probability that all 4 seasons occur atleast once is

$$1 - \left(4\left(\frac{3}{4}\right)^7 - \frac{6}{2^7} + \frac{4}{4^7}\right) \approx 0.513.$$

2. There are two general ways that Alice can have class every day: either she has 2 days with 2 classes and 3 days with 1 class, or she has 1 day with 3 classes, and has 1 class on each of the other 4 days. The number of possibilities for the former is $\binom{5}{2}\binom{6}{2}6^3$ (choose the 2 days when she has 2 classes, and then select 2 classes on those days and 1 class for the other days). The number of possibilities for the latter is$\binom{5}{1}\binom{6}{3}6^4$. So the probability is

$$\frac{\binom{5}{2}\binom{6}{2}6^3 + \binom{5}{1}\binom{6}{3}6^4}{\binom{30}{7}} = \frac{114}{377} \approx 0.302$$

we will use inclusion-exclusion to find the probability of the complement, which is the event that she has at least one day with no classes. Let $B_i = A_i^c$. Then

$$P(B_1 \cup B_2 \ldots \cup B_5) = \sum_{i} P(B_i) - \sum_{i<j} P(B_i \cap B_j) + \sum_{i<j<k} P(B_i \cap B_j \cap B_k)$$

(terms with the intersection of 4 or more B_i's are not needed since Alice must have classes on at least 2 days). We have

$$P(B_1) = \frac{\binom{24}{7}}{\binom{30}{7}}, P(B_1 \cap B_2) = \frac{\binom{18}{7}}{\binom{30}{7}}, P(B_1 \cap B_2 \cap B_3) = \frac{\binom{12}{7}}{\binom{30}{7}}$$

and similarly for the other intersections. So

$$P(\mathrm{B}_1 \cup B_2 \ldots \cup B_5) = 5\frac{\binom{24}{7}}{\binom{30}{7}} - \binom{5}{2}\frac{\binom{18}{7}}{\binom{30}{7}} + \binom{5}{3}\frac{\binom{12}{7}}{\binom{30}{7}} = \frac{263}{377}$$

Therefore,

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = 1 - \frac{263}{377} = \frac{114}{377} \approx 0.302.$$