Statistics Assignment 1

1.

a) Let X be the number of students that take the same seat in both classes and S_j be the j[th] student who has the same seat, then we have

$$P(X=0) = 1 - P(X \geq 1) = 1 - P\left(\bigcup_j S_j\right)$$

Using the inclusion- exclusion formula and the symmetry, we have

$$P\left(\bigcup_j S_j\right) = \sum_j (-1)^{j-1}\binom{100}{j} P\left(\bigcap_{k=1}^{j} S_k\right)$$

The probability that first j students sit on their seats is simply $\frac{(100-j)!}{100!}$. Thus we have

$$P\left(\bigcup_j S_j\right) = \sum_j (-1)^{j-1}\binom{100}{j}\frac{(100-j)!}{100!} = \sum_{j=1}^{100}\frac{(-1)^{j-1}}{j!}$$

Finally,

$$P(X=0) = 1 - \sum_{j=1}^{100}\frac{(-1)^{j-1}}{j!} = \sum_{j=1}^{100}\frac{(-1)^{j}}{j!}$$

b) If I_j is the indicator random variable that indicates if S_j has occurred. Then we have,

$$X = \sum_{j=1}^{100} I_j$$

We know that $P(I_j) = \frac{1}{100}$ and that we can approximate

$$P\left((I_j = 1) \cap (I_k = 1)\right) = \frac{1}{100}\cdot\frac{1}{99} \approx \left(\frac{1}{100}\right)^2 = P(I_j = 1)P(I_k = 1)$$

So, I_j and I_k are independent random variables. Now we can approximate X with Poisson distribution with parameter λ = E(X) = 100E(I_1). So, we have

$$P(X=0) \approx \frac{1^0}{0!}e^{-1} \approx 0.37$$

c) Using Poisson approximation to finally obtain that

$$P(X \geq 2) = 1 - P(X=0) - P(X=1) \approx 1 - e^{-1} - e^{-1} = 1 - 2e^{-1} \approx 0.26$$

2. If there are only 2 passengers A & B and 2 seats, then
P(B sits in his or her assigned seat) = P(A sits in his or her seat) = ½

If there are 3 passengers A, B, C and 3 seats, then

P(C sits in his or her assigned seat) = P(A sits in his or her assigned seat) + P(A sits in B's assigned seat)* P(B sits in A's assigned seat) =1/3 + (1/3)*(1/2) = 1/3 +1/6 = 1/2

If there are 4 passengers A, B, C, D and 4 seats, then
P(D sits in his or her assigned seat)= P(A sits in his or her assigned seat) + P(A sits in B's assigned seat)* P(B sits in A's assigned seat) + P(A sits in C's assigned seat)* P(C sits in A's assigned seat) + P(A sits in B's assigned seat)* P(B sits in C's assigned seat)* P(C sits in B's assigned seat)+ P(A sits in C's assigned seat)* P(C sits in B's assigned seat)* P(B sits in C's assigned seat)* =1/4+(1/4)*(1/3) +(1/4)*(1/3) +(1/4)*(1/3)*(1/2) +(1/4)*(1/3) *(1/2)= 1/4 + 1/12+1/12+1/24 +1/24 = 1/4 + 2/12+1/12 = ½
Similarly,
P(the last passenger in line gets to sit in his or her assigned seat) =1/2