Statistics Assignment 6

1)

a) The joint distribution of X, Y, Z is

$$P(X = a, Y = b, Z = c) = \frac{n!}{a!\,b!\,c!}\left(\frac{1}{3}\right)^{a+b+c}$$

where a, b, c are any nonnegative integers with $a + b + c = n$, since $(1/3)^{a+b+c}$ is the probability of any specific configuration of choices for each player with the right numbers in each category, and the coefficient in front counts the number of distinct ways to permute such a configuration.

Alternatively, we can write the joint PMF as

$$P(X = a, Y = b, Z = c) = P(X = a)P(Y = b|X = a)P(Z = c|X = a, Y = b),$$

where for $a + b + c = n, P(X = a)$ can be found from the $Bin(n, 1/3)$ PMF $P(Y = b|X = a)$ can be found from the $Bin(n - a, 1/2)$ PMF, and $P(Z = c|X = a, Y = b) = 1$.

This is a $Multinomial(n, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ distribution.

b) The game is decisive if and only if exactly one of X, Y, Z is 0. These cases are disjoint so by symmetry, the probability is 3 times the probability that X is zero and Y and Z are nonzero. Note that if $X = 0$ and $Y = k$, then $Z = n - k$. This gives

$$\mathrm{P(decisive)} = 3\sum_{\mathrm{k=1}}^{\mathrm{n-1}} \frac{\mathrm{n!}}{\mathrm{0!\,k!\,(n-k)!}}\left(\frac{1}{3}\right)^{n} = 3\left(\frac{1}{3}\right)^{n}\sum_{k=1}^{n-1}\binom{n}{k} = \frac{2^n - 2}{3^{n-1}}$$

Since $\sum_{k=1}^{n-1}\binom{n}{k} = -1 - 1 + \sum_{k=0}^{n}\binom{n}{k} = 2^n - 2.$

As a check, when $n = 2$ this reduces to 2/3, which makes sense since for 2 players, the game is decisive if and only if the two players do not pick the same choice.

c) For $n = 5$, the probability is $(2^5 - 2)/3^4 = 30/81 \approx 0.37$. As $n \to \infty, (2^n - 2)/3^{n-1} \to 0$, which make sense since if the number of players is very large, it is very likely that there will be at least one of each of Rock, Paper, and Scissors.