Vector Calculus.

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Vector ftⁿ

If for each value of a scaler var t there corresponds a vector $\vec{F}$ then $\vec{F}$ is said to be vector ftⁿ of the scalar variable t.

For ex $\vec{F} = a\cos t\,\hat{i} + b\sin t\,\hat{j} + c\,\hat{k}$

in general,

$\vec{F} = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$

Const vector

A vector whose magnitude is const & whose direction is in fixed direction is a const vector.

Derivative of a vector ftⁿ

$$\frac{d\vec{F}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$$

⇒ Theorem.

1) $\frac{d}{dt}(c\vec{F}) = c\,\frac{d\vec{F}}{dt}$; c is scalar const.

2) $\frac{d}{dt}(\phi\vec{a}) = \frac{d\phi}{dt}\,\vec{a}$, $\vec{a}$ = const vector, ϕ = scalar ftⁿ

3) $\frac{d}{dt}(\phi\vec{F}) = \frac{d\phi}{dt}\vec{F} + \phi\frac{d\vec{F}}{dt}$; $\phi \to$ scalar ftⁿ, $\vec{F} \to$ vector ftⁿ

4) If $\vec{A}$ & $\vec{B}$ are vector ftⁿ

(i) $\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$

(ii) $\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \frac{d\vec{B}}{dt}\cdot\vec{A}$

(iii) $\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$

Velocity of a particle

Let a particle moves along a curve whose vector eqⁿ is $\vec{r} = \vec{r}(t)$ where $\vec{r}$ is a ftⁿ of t.
Let P be its position at time t & Q be its position at $t+\Delta t$, then the velocity vector is given by

$$\vec{V} = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

Scalar point ftⁿ

If for every point there corresponds a scalar $\phi(x,y,z)$ then $\phi(x,y,z)$ is called a scalar pt. ftⁿ

For example, at each & every pt in space, the temp. is scalar pt. ft^n

Vector Point Function.
If for every pt. there corresponds a vector $\vec{F} = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$ then $\vec{F}$ is a vector pt. ft^n

Level surface.
If $\phi(x,y,z) = c$ where c is a constant and $\phi(x,y,z)$ is a scalar pt. ft^n is called a level surface of the ft^n ϕ

Gradient of a Scalar Pt. ft^n.
Let $f(x,y,z)$ be a scalar pt. ft^n defined at each point (x,y,z) in a certain region of space & is differentiable then gradient of f is defined by

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Geometrical Significance of Gradient.

Let $\phi(x,y,z) = C$ represents a surface then $\nabla\phi$ is the normal vector (which is ⊥ to the tangent plane) to the surface at the point (x,y,z)

Angle b/w two surfaces $\phi(x,y,z) = c_1$ & $\psi(x,y,z) = c_2$ is given by

$$\theta = \cos^{-1}\left[\frac{\nabla\phi\cdot\nabla\psi}{|\nabla\phi|\,|\nabla\psi|}\right]$$

Q If $\phi(x,y,z) = x^2yz$, find $\nabla\phi$ at $(1,-2,1)$

→ $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}$

$= -4\hat{i} + \hat{j} - 2\hat{k}$

Q. Find the unit normal to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$

→ we have $xy^3z^2 = 4$

∴ $\phi(x,y,z) = xy^3z^2 - 4$

$\nabla\phi = y^3z^2\,\hat{i} + 3xy^2z^2\,\hat{j} + 2xy^3z\,\hat{k}$

$= -4\hat{i} - 12\hat{j} + 4\hat{k}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4}{\sqrt{176}}\hat{i} - \frac{12}{\sqrt{176}}\hat{j} + \frac{4}{\sqrt{176}}\hat{k}$

144
16
16
176

Q. Find the angle at pt $(2,-1,2)$ of the surface $x^2+y^2+z^2 = 9$ & $z = x^2+y^2-3$

→ Given 2 surfaces $\phi = x^2+y^2+z^2-9$

$\psi = -x^2-y^2+3+z$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$
$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla\phi\Big|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\nabla\psi = -2x\hat{i} + (-2y)\hat{j} + \hat{k}$$
$$\nabla\psi\Big|_{(2,-1,2)} = -4\hat{i} + 2\hat{j} + \hat{k}$$

$$\theta = \cos^{-1}\left[\frac{\nabla\phi \cdot \nabla\psi}{|\nabla\phi|\,|\nabla\psi|}\right]$$
$$= \cos^{-1}\left[\frac{-16 - 4 + 4}{6(\sqrt{21})}\right]$$
$$= \cos^{-1}\left[\frac{-16}{6\sqrt{21}}\right]$$
$$= \cos^{-1}\left[\frac{-8}{3\sqrt{21}}\right]$$

Divergence of Vector f^n

Let $f = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a given vector f^n differentiable at each pt. (x, y, z) in a certain region of space then the divergence of f is denoted by $\text{div}\,\vec{f}$ & is defined as

∇ with scalar $\rightarrow$ gradient
∇ with vector & dot product $\rightarrow$ divergence
∇ with vector & cross product $\rightarrow$ curl

$$\text{div}\,\vec{f} = \nabla\cdot\vec{f}$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$
$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Physical significance of divergence

If $\vec{f}$ represents the velocity of a fluid in the fluid flow, $\text{div}\,\vec{f}$ represent the rate of fluid flow through unit volume

Solenoidal vector

A vector $\vec{f}$ is said to be solenoidal if divergence of $\vec{f} = 0$

Curl of a vector f^n

Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector f^n differentiable at each point (x, y, z) in a certain region of space, then curl of $\vec{f}$ is defined as

$$\text{curl}\,\vec{f} = \nabla\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrotational vector.

A vector $\vec{f}$ is said to be irrotational if curl of $\vec{f} = 0$

Q. Prove that $\vec{f} = (2x + yz)\hat{i} + (4y + xz)\hat{j} - (6z - xy)\hat{k}$ is solenoidal as well as irrotational.

$\rightarrow$ A vector is solenoidal if $\operatorname{div} \vec{f} = 0$

$$\vec{f} = (2x+yz)\hat{i} + (4y+xz)\hat{j} + (6z+xy)\hat{k}$$

$$\operatorname{div} \vec{f} = 2 + 4 - 6$$
$$= 0 \quad \therefore \text{solenoidal}$$

$$\text{curl of } \vec{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$
$$= (x - x)\hat{i} - (y - y)\hat{j} + (z - z)\hat{k}$$
$$= 0$$

$\therefore$ vector is irrotational as curl of $\vec{f} = 0$

Q. Find const c if $\vec{F} = (cxy - z^3)\hat{i} + (c-2)y^2\hat{j} + c(1-c)xyz\hat{k}$ is solenoidal.

$\rightarrow$ A vector is solenoidal if $\operatorname{div} \vec{f} = 0$

$$\vec{f} = (cxy - z^3)\hat{i} + (c-2)y^2\hat{j} + c(1-c)yz\hat{k}$$

$$\operatorname{div} \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$
$$= cy + 2(c-2)y + c(1-c)y$$

as it is solenoidal,
$$cy + 2(c-2)y + cy(1-c) = 0$$
$$y(c + 2c - 4 + c - c^2) = 0$$
$$y(4c - 4 - c^2) = 0$$
$$-y(c^2 - 4c + 4) = 0$$
$$c^2 - 2(2)c + (2)^2 = 0$$
$$(c-2)^2 = 0$$
$$c = 2$$

Q. Find the const a & b so that $\vec{F} = (axy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (bxz^2 - y)\hat{k}$ is irrotational.

$\rightarrow$ vector is irrotational when curl of $\vec{f} = 0$.

$$\text{curl of } \vec{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$
$$= (-1+1)\hat{i} - (bz^2 - 3z^2)\hat{j} + (6x - ax)\hat{k}$$

for curl = 0,

$bz^2 - 3z^2 = 0$ $\quad\quad$ $6x - ax = 0$

$b = 3$ $\quad\quad$ $a = 6$

Q. If $\vec{A} = x^2 y\,\hat{i} + yz^3\,\hat{j} - zx^3\,\hat{k}$ find grad (div $\vec{A}$).

$\rightarrow$ $\text{div}\,\vec{A} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

$= 2xy + z^3 - x^3$

$\text{grad}\,(\text{div}\,\vec{A}) =$

$\nabla f = (2y - 3x^2)\hat{i} + (2x)\hat{j} + (3z^2)\hat{k}$

Q. If $\phi = xyz - 2y^2 z + x^2 z^2$, determine ~~div~~ div (grad ϕ) at pt. $(1, 2, 4)$

$\nabla f = (yz + 2xz^2)\hat{i} + (xz - 4yz)\hat{j} + (xy - 2y^2 + 2x^2 z)\hat{k}$

$= (8 + 32)\hat{i} + (4 - 32)\hat{j} + (2 - 8 + 8)\hat{k}$

$= 40\hat{i} - 28\hat{j} + 2\hat{k}$

$\cancel{\text{div}(\nabla f) = 40 - 28 - 5}$

$\cancel{= 12 + 2}$

$\cancel{= 14}$

$\text{div}(\nabla f) = 2z^2 + (-4z) + (2x^2)$

$= 32 + (-16) + 2$

$= 16 + 2$

$= 18$

$\rightarrow$ Result

1. Prove that curl grad $\phi = 0$

2. $\rightarrow$ Result

Prove that div (curl $\vec{F}$) = 0

1. $\text{grad}\,\phi = \nabla\phi = \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k}$

$$\text{curl}(\text{grad}\,\phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\hat{i} - \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)\right]\hat{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\hat{k}$$

$$= \left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right]\hat{i} - \left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right]\hat{j} + \left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right]$$

as $\dfrac{\partial^2\phi}{\partial y\partial z} = \dfrac{\partial^2\phi}{\partial z\partial y}$ & $\dfrac{\partial^2\phi}{\partial x\partial z} = \dfrac{\partial^2\phi}{\partial z\partial x}$ & $\dfrac{\partial^2\phi}{\partial x\partial y} = \dfrac{\partial^2\phi}{\partial y\partial x}$

$= 0$

Hence proved.

2. curl of $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

$$\text{div}(\text{curl of } \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial f_3}{\partial x} + \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y}$$

$= 0$

Hence proved