GRAPH REALIZATION OF DISTANCE SETS

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Introduction: Network realization problem.

unweighted & weighted DISTANCE REALIZATION pr.

$D_{n\times n}$ $D_{ij} = d(v_i, v_j)$ $G=(V,E)$ $|V|=n$

• unweighted
- at most one graph realizes D
- $\exists$ poly time alg. star, path, cycle

• weighted
- D is a metric
- negative weights ⇒ symm. matrix with 0 diag

unique realization for trees

• unw. & weigh. RANGE DISTANCE REALIZ.

$D_{ij} = [a,b]$ $d(v_i,v_j) \in [a,b]$

• unw.

NP-hard

underlying graph tree ⇒ NP-hard

• weigh.
└ poly time alg ??

• unw. & weigh. SET DIST. REALIZ.

$D_{ij} = \{\text{set}\}$ $d_{ij} \in D_{ij}$

RANGE $\subseteq$ SET

$[a,b] \rightarrow \{a, a+1, \dots, b\}$

Graph family	Range-DR	Set-DR
General	2-RANGE-DR is NP-hard (Thm 1)	2-SET-DR is NP-hard (Thm 1)
	1-RANGE-DR is polynomial [12]	1-R~~ANGE~~-DR is polynomial [12]
Tree	3-RANGE-DR is NP-hard [4]	2-SET-DR is NP-hard (Thm 2)
	1-RANGE-DR is polynomial [12]	1-R~~ANGE~~-DR is polynomial [12]
Star	RANGE-DR is polynomial (Thm 4)	SET-DR is polynomial (Thm 4)
Path	2-RANGE-DR is polynomial (Thm 8)	2-SET-DR is polynomial (Thm 8)
	RANGE-DR is NP-hard (Thm 9)	5-SET-DR is NP-hard (Thm 10)
Cycle	2-RANGE-DR is polynomial (Thm 12)	2-SET-DR is polynomial (Thm 12)
	RANGE-DR is NP-hard (Thm 14)	5-SET-DR is NP-hard (Thm 15)

Table 1 Results for realization with unweighted graphs.

Graph family	Range-DR	Set-DR
General	RANGE-DR is polynomial [15]	Open problem
Tree	3-RANGE-DR is NP-hard [4]	2-SET-DR is NP-hard (Thm 2)
	1-RANGE-DR is polynomial [2]	1-RANGE-DR is polynomial [2]
Star	RANGE-DR is polynomial[1] (Thm 6)	2-SET-DR is polynomial (Thm 3)
		6-SET-DR is NP-hard (Thm 5)
Path	2-RANGE-DR is polynomial (Thm 7)	2-SET-DR is polynomial (Thm 7)
	RANGE-DR is NP-hard (Thm 9)	5-SET-DR is NP-hard (Thm 10)
Cycle	2-RANGE-DR is polynomial (Thm 13)	2-SET-DR is polynomial (Thm 13)
	RANGE-DR is NP-hard (Thm 14)	5-SET-DR is NP-hard (Thm 15)

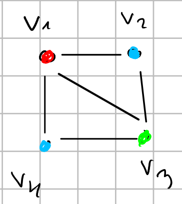
Table 2 Results for realization with weighted graphs.

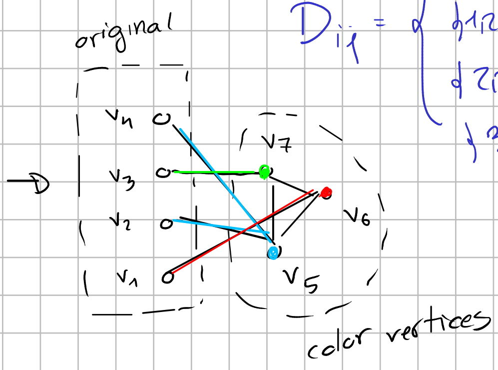
THEOREM 1: 2-RANGE DR is NP-hard on unweighted gr.

PF 3-COLORING $\longrightarrow$ 2-RANGE DR

$G=(V,E)$ $|V|=n$

$$V' = \underset{\text{orig}}{\overset{\{v_1 \dots v_n\}}{V}} \cup \{\underbrace{v_{n+1}, v_{n+2}, v_{n+3}}_{\text{color vert.}}\}$$

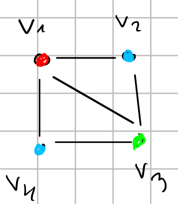


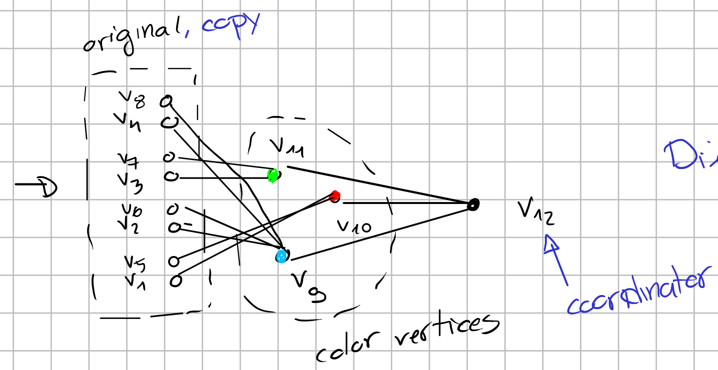


$$D_{ij} = \begin{cases} \{1\} ; & i,j \in \{n+1,n+2,n+3\} \\ \{1,2\} ; & i \in \{1..n\}, j \in \{n+1,..n+3\} \\ \{2,3\} ; & (v_i,v_j) \notin E(G) \quad i,j \in \{1 \dots n\} \\ \{3\} ; & (v_i,v_j) \in E(G) \quad i,j \in \{1 \dots n\} \end{cases}$$

THEOREM 2: 2-SET-DR is NP-hard for unw & weigh. trees.

PF 3-COLOR $\Longrightarrow$ 2-SET-DR





$$D_{ij} = \begin{cases} \{1\} & d(\text{color, coordinator}) \\ \{2\} & d(\text{color,color}),\ d(\text{original } i, \text{copy} i),\ d(\text{orig/copy, coordin.}) \\ \{4\} & v_i v_j \in E \quad d(v_i,v_j),\ d(\text{copy} i, \text{copy} j)\ d(\text{orig} i, \text{copy } j) \end{cases}$$

$\{2,4\}$ $(v_i\ v_j) \notin E$

$\{2,4\} \Rightarrow [2,4]$ 3-RANGE-DR is NP-hard.

THEOREM 3: Poly. time alg. for 2-SET-DR on stars.

PF $\{v_1 \dots v_n\}$

assume v_i center

$$D_{ij} = \{d_{ij}^0, d_{ij}^1\}$$

$x_j = \text{FALSE} \iff d(v_i,v_j) = d_{ij}^0$

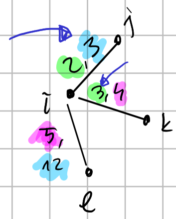
$x_j = \text{TRUE} \iff d(v_i,v_j) = d_{ij}^1$

D_{jk} $j,k \neq i$

(i) $\underset{\bar{x}_j}{d_{ij}^0} + \underset{\bar{x}_k}{d_{ik}^0}$ (ii) $\underset{\bar{x}_j}{d_{ij}^0} + \underset{x_k}{d_{ik}^1}$ (iii) $\underset{x_j}{d_{ij}^1} + \underset{\bar{x}_k}{d_{ik}^0}$ (iv) $\underset{x_j}{d_{ij}^1} + \underset{x_k}{d_{ik}^1}$

$\in D_{jk} = \{d_{jk}^0, d_{jk}^1\}$

(i) and (ii) $\rightarrow (\overline{x_j} \vee \overline{x_k}) \wedge (\overline{x_j} \vee x_k)$ $O(n^2)$



$(\overline{x_j} \wedge \overline{x_k}) \vee (\overline{x_j} \wedge x_k)$

$D_{ij} = \{2,3\}$ $D_{i,k} = \{3,4\}$ $D_{jk} = \{5,7\}$

$2 + 3 = 5$ $\overline{x_j} \vee \overline{x_k}$

$3 + 4 = 7$ $x_j \vee x_k$

$D_{k,l} = \{9, 100\} \rightarrow x_k \vee \overline{x_l}$

$D_{j,l} = \{15, 100\} \rightarrow x_j \vee x_l$

$(\overline{x_j} \vee \overline{x_k}) \wedge \underbrace{(x_j \vee x_k)}_{T} \wedge (x_k \vee \overline{x_l}) \wedge \underbrace{(x_j \vee x_l)}_{T}$

$x_j = T$ $x_k = F$ $x_l = F$

$(\overline{x_j} \wedge \overline{x_k}) \vee (x_j \wedge x_k) \vee (x_k \wedge \overline{x_l}) \vee (x_j \wedge x_l)$

MISTAKE :(

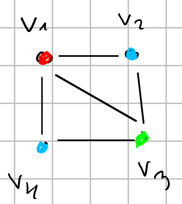
$n \cdot O(n^2) = O(n^3)$

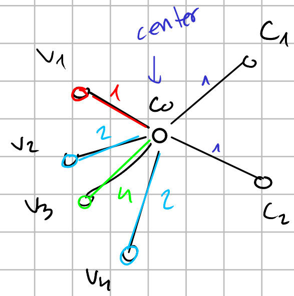
k-SET $\underline{\underline{k=2}}$

THEOREM 5: 6-SET-DR is NP-hard on weighted stars.

PF:

3-COLORING





colors $c \in \{0,1,2\}$

v_i on distance 2^c from the center

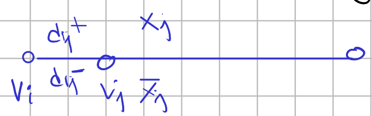
$\{1, 2, 4\}$

$d(v_i, v_j)$: $(v_i, v_j) \in E(G)$ $\in \{3, 5, 6\}$

$(v_i, v_j) \notin E(G)$ $\in \{2, 3, 4, 5, 6, 8\}$

THEOREM 7: Poly time alg. for $\underline{\underline{\text{2-SET-DR}}}$ on weighted paths.

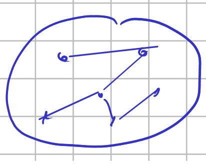
$\boxed{\text{PF}}$

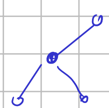


THEOREM 9: RANGE-DR is NP-hard on weig. and unw. paths

HAMILTONIAN PATH

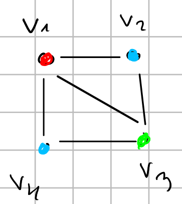
$G \rightarrow D_{ij} = \begin{cases} \{1 \ldots n-1\}; & (v_i, v_j) \in E(G) \\ \{2, \ldots n-1\}; & (v_i, v_j) \notin E(G) \end{cases}$

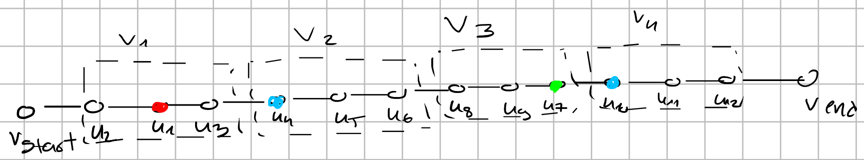




THEOREM 10: 5-SET-DR is NP-hard on unw. & weigh. paths.

$\boxed{\text{PF}}$ 3-COLORING





u_1, u_2, u_3 $\quad u_4, u_5, u_6$ $\quad u_7, u_8, u_9$ $\quad u_{10}, u_{11}, u_{12}$

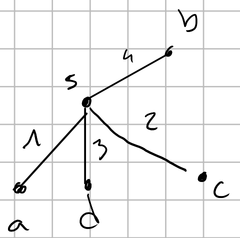
$v_i \longrightarrow u_{3i-2}$

$n \longrightarrow 3n+2 \qquad u_0 \dots u_{3n+1}$

points. More formally, define $\bar{k} \triangleq \lceil k/3 \rceil$. The matrix is defined as follows for any two indice $0 \le k < \ell \le 3n+1$:

$$D_{k,\ell} = \begin{cases} \{3n+1\} & k=0, \ell = 3n+1, \\ \{3\bar{\ell}-2, 3\bar{\ell}-1, 3\bar{\ell}\} & k=0, \bar{\ell} \in \{1,\dots,n\}, \\ \{3n-3\bar{k}+1, 3n-3\bar{k}+2, 3n-3\bar{k}+3\} & \bar{k} \in \{1,\dots,n\}, \ell = 3n+1, \\ \{1,2\} & \bar{k} = \bar{\ell}, \\ \{3(\bar{\ell}-\bar{k})+\Delta : \Delta \in \{-2,-1,0,1,2\}\} & \bar{k} < \bar{\ell}, \\ & k \bmod 3 \neq 1 \text{ or } \ell \bmod 3 \neq 1 \\ & \text{or } (v_{\bar{k}}, v_{\bar{\ell}}) \notin E(G), \\ \{3(\bar{\ell}-\bar{k})+\Delta : \Delta \in \{-2,-1,1,2\}\} & \bar{k} < \bar{\ell}, \\ & k, \ell \bmod 3 = 1, \underline{(v_{\bar{k}}, v_{\bar{\ell}}) \in E(G)}\,. \end{cases}$$

$8k$



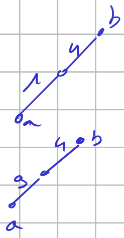
$dur(a,c) = 2$

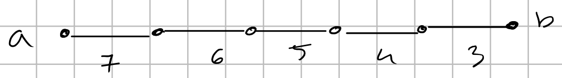
$$\boxed{\begin{aligned} d(a,b) &= x \leq 8 \\ d(b,a) &\geq 8 - x \end{aligned}}$$

in ~~general~~ ?? on stars

$d(a,b) = 3$

$d(b,a) = 5$





$d(a,b) + d(b,a) \leq \text{period} \cdot \#\,\text{edges}(a,b)$

PROBLEM 1

INPUT: $\underbrace{\text{Underlying graph } G}_{\text{given class}}$, period Δ_e $\forall e$, matrix D of $\underbrace{\text{shortest temporal paths}}_{\text{fastest}}$ among vertices

OUTPUT: Find labeling (if exists) s.t. $\forall$ edge gets exactly 1 label.

PROBLEM 2

INPUT: $\underbrace{\text{Underlying graph } G}_{\text{given class}}$, k-integer, matrix D of $\underbrace{\text{shortest temporal paths}}_{\text{fastest}}$ among vertices

OUTPUT: Find labeling (if exists) s.t. $\forall$ edge gets at most k-labels.