曲线坐标公式推导

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1 预备公式

槽道上下壁面均认为可以任意波动，其运动方程为：

$$y_d = -1 + \eta_d(x, z, t), \qquad y_u = 1 + \eta_u(x, z, t) \tag{1}$$

其中 η_u, η_d 分别为上下壁面的位移大小。
定义：

$$\eta = \frac{1}{2}(\eta_u - \eta_d), \qquad \eta_0 = \frac{1}{2}(\eta_u + \eta_d) \tag{2}$$

定义变换：

$$\begin{cases} t = \tau \\ x_1 = \xi_1 \\ x_2 = \xi_2(1+\eta) + \eta_0 \\ x_3 = \xi_3 \end{cases} \tag{3}$$

则有微分变换：

$$\begin{cases} \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial \tau} + \phi_t \dfrac{\partial}{\partial \xi_2} \\ \dfrac{\partial}{\partial x_i} = \dfrac{\partial}{\partial \xi_i} + \phi_i \dfrac{\partial}{\partial \xi_2} \end{cases} \tag{4}$$

其中：

$$\phi_i = \varphi_i - \delta_{i2} \tag{5}$$

,

$$\varphi_i = \begin{cases} -\dfrac{1}{1+\eta}(\xi_2 \dfrac{\partial \eta}{\partial \xi_i} + \dfrac{\partial \eta_0}{\partial \xi_i}) & i = 1, 3 \\ \dfrac{1}{1+\eta} & i = 2 \end{cases} \tag{6}$$

$$\varphi_t = -\frac{1}{1+\eta}(\xi_2 \frac{\partial \eta}{\partial \tau} + \frac{\partial \eta_0}{\partial \tau}) \tag{7}$$

由以上公式可以推得以下常用的公式：

$$\frac{\partial \phi_i}{\partial \xi_2} = \begin{cases} -\dfrac{1}{1+\eta}\dfrac{\partial \eta}{\partial \xi_i} & i = 1, 3 \\ 0 & i = 2 \end{cases} \tag{8}$$

$$\frac{\partial \phi_i}{\partial \xi_i} = \begin{cases} -\dfrac{1}{(1+\eta)^2}((\xi_2 \dfrac{\partial^2 \eta}{\partial \xi_i^2} + \dfrac{\partial^2 \eta_0}{\partial \xi_i^2})(1+\eta) - \dfrac{\partial \eta}{\partial \xi_i}(\xi_2 \dfrac{\partial \eta}{\partial \xi_i} + \dfrac{\partial \eta_0}{\partial \xi_i})) & i = 1, 3 \\ 0 & i = 2 \end{cases} \tag{9}$$

2 动量方程

2.1 X 方向动量方程

$$\begin{aligned}
&\frac{u^{n+1}-u^n}{\Delta\tau}+\phi_t^n\frac{\partial u^n}{\partial\xi_2}+\frac{1}{2}(\frac{\partial u^n u^{n+1}}{\partial\xi_1}+\frac{\partial v^n u^{n+1}}{\partial\xi_2}+\frac{\partial w^n u^{n+1}}{\partial\xi_3})\\
&+\frac{1}{2}(\frac{\partial u^n u^{n+1}}{\partial\xi_1}+\frac{\partial u^n v^{n+1}}{\partial\xi_2}+\frac{\partial u^n w^{n+1}}{\partial\xi_3})\\
&+\frac{1}{2}(\phi_1^n\frac{\partial u^n u^{n+1}}{\partial\xi_2}+\phi_2^n\frac{\partial v^n u^{n+1}}{\partial\xi_2}+\phi_3^n\frac{\partial w^n u^{n+1}}{\partial\xi_2})\\
&+\frac{1}{2}(\phi_1^{n+1}\frac{\partial u^n u^{n+1}}{\partial\xi_2}+\phi_2^{n+1}\frac{\partial u^n v^{n+1}}{\partial\xi_2}+\phi_3^{n+1}\frac{\partial u^n w^{n+1}}{\partial\xi_3})\\
&=-\frac{\partial p}{\partial\xi_1}-\phi_1^{n+1/2}\frac{\partial p}{\partial\xi_2}+\frac{1}{2Re}(\Delta u^n+\Delta u^{n+1})
\end{aligned}\tag{10}$$

整理各项，得：

$$\begin{aligned}
I+M_{11}^1 \quad & (\frac{u^{n+1}}{\Delta\tau}+\frac{\partial u^n u^{n+1}}{\partial\xi_1}-\frac{1}{2Re}\frac{\partial^2 u^{n+1}}{\partial\xi_1^2})\\
M_{11}^2 \quad & +\frac{1}{2}(\frac{\partial v^n u^{n+1}}{\partial\xi_2}+(\phi_1^n+\phi_1^{n+1})\frac{\partial u^n u^{n+1}}{\partial\xi_2}+\phi_2^n\frac{\partial v^n u^{n+1}}{\partial\xi_2}+\phi_3^n\frac{\partial w^n u^{n+1}}{\partial\xi_2}-\frac{1}{Re}\Delta u^{n+1})\\
M_{11}^3 \quad & +\frac{1}{2}(\frac{\partial w^n u^{n+1}}{\partial\xi_3}-\frac{\partial^2 u^{n+1}}{\partial\xi_3^2})\\
Possion \quad & +(\frac{\partial p}{\partial\xi_1}+\phi_1^{n+1/2}\frac{\partial p}{\partial\xi_2})\\
M_{12} \quad & +\frac{1}{2}(\frac{\partial u^n v^{n+1}}{\partial\xi_2}+\phi_2^{n+1}\frac{\partial u^n v^{n+1}}{\partial\xi_2})\\
M_{13} \quad & +\frac{1}{2}(\frac{\partial u^n w^{n+1}}{\partial\xi_3}+\phi_3^{n+1}\frac{\partial u^n w^{n+1}}{\partial\xi_2})\\
R \quad & =\frac{u^n}{\Delta\tau}-\phi_t^n\frac{\partial u^n}{\partial\xi_2}+\frac{1}{2Re}\Delta u^n+\frac{1}{2Re}2\phi_i\frac{\partial^2 u^{n+1}}{\partial\xi_2\partial\xi_i}
\end{aligned}\tag{11}$$

为达到计算二阶精度，在算法中需做如下的分步变换：

$$\boldsymbol{u}^{n+1}=\boldsymbol{u}^*-\Delta tG\delta p \qquad \delta\boldsymbol{u}=\boldsymbol{u}^*-\boldsymbol{u}^n\tag{12}$$

之后，原方程可继续化简为：

$$\frac{\delta u}{\Delta\tau}+N(\delta u)-\frac{1}{2Re}\Delta\delta u=-Gp^{n+1/2}-\phi_t^n\frac{\partial u^n}{\partial\xi_2}-N(u^n)+\frac{1}{Re}\Delta u^n\tag{13}$$

其中 $\Delta\delta u$ 项中的交叉导数项 $2\phi_i\frac{\partial^2\delta u^{n+1}}{\partial\xi_2\partial\xi_i}$ 需提至右侧进行迭代处理。
此时相应的右端项为：

$$R \qquad -Gp^{n+1/2}-\phi_t^n\frac{\partial u^n}{\partial\xi_2}-N(u^n)+\frac{1}{Re}\Delta u^n+\frac{1}{2Re}2\phi_i\frac{\partial^2\delta u^{n+1}}{\partial\xi_2\partial\xi_i}+b.c.\tag{14}$$

其中边界条件项由以下各项在边界处的值组成：

$$
\begin{array}{ll}
M_{11}^2 & \frac{1}{2}(\frac{\partial v^n u^{n+1}}{\partial \xi_2} + (\phi_1^n + \phi_1^{n+1})\frac{\partial u^n u^{n+1}}{\partial \xi_2} + \phi_2^n \frac{\partial v^n u^{n+1}}{\partial \xi_2} + \phi_3^n \frac{\partial w^n u^{n+1}}{\partial \xi_2} - \frac{1}{Re}\Delta u^{n+1}) \\
M_{12} & \frac{1}{2}(\frac{\partial u^n v^{n+1}}{\partial \xi_2} + \phi_2^{n+1}\frac{\partial u^n v^{n+1}}{\partial \xi_2}) \\
M_{13} & \frac{1}{2}\phi_3^{n+1}\frac{\partial u^n w^{n+1}}{\partial \xi_2}
\end{array} \tag{15}
$$