方程组求根的迭代法

- Jacobi迭代法
- Gauss-Seide迭代法
- SOR迭代法
- 向量的范数

Jacobi迭代法

已知方程组：

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ ... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases}$$

我们依据第i个方程解出第i个未知量，可得到：

$$\begin{cases} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - ... - a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - ... - a_{1n}x_n) \\ ... \\ x_3 = \frac{1}{a_{nn}}(b_n - a_{n2}x_2 - a_{n3}x_3 - ... - a_{nn-1}x_{n-1}) \end{cases}$$

等式左边加上上标$k+1$,右边加上k,k表示迭代第k次。则可得到Jacobi迭代公式。

$$\begin{cases} x_1^{(k+1)} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - ... - a_{1n}x_n^{(k)}) \\ x_2^{(k+1)} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - ... - a_{1n}x_n^{(k)} \\ ... \\ x_3^{(k+1)} = \frac{1}{a_{nn}}(b_n - a_{n2}x_2^{(k)} - a_{n3}x_3^{(k)} - ... - a_{nn-1}x_{n-1}^{(k)} \end{cases}$$

Gauss-Seidel迭代法

在Jacobi公式的基础上，只需让等号右边下标小于i的变量的上标全部改成$k+1$即可得到Gauss-Seidel。

$$\begin{cases} x_1^{(k+1)} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - ... - a_{1n}x_n^{(k)}) \\ x_2^{(k+1)} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - ... - a_{2n}x_n^{(k)}) \\ ... \\ x_n^{(k+1)} = \frac{1}{a_{nn}}(b_3 - a_{n1}x_1^{(k+1)} - a_{12}x_3^{(k+1)} - ... - a_{nn-1}x_{n-1}^{(k+1)}) \end{cases}$$

SOR迭代

在Gauss-Seidel的基础上，将第i个迭代公式$a_{ij}x_j^{(k)}$加入到等号右边

$$\begin{cases} x_1^{(k+1)} = x_1^{(k)} + \frac{1}{a_{11}}(b_1 - a_{11}x_1^{(k)} - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \ldots - a_{1n}x_n^{(k)}) \\ x_2^{(k+1)} = x_2^{(k)} + \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k+1)} - a_{22}x_2^{(k)} - a_{23}x_3^{(k)} - \ldots - a_{2n}x_n^{(k)}) \\ \ldots \\ x_n^{(k+1)} = x_n^{(k)} + \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - a_{n3}x_3^{(k+1)} - \ldots - a_{nn-1}x_{n-1}^{k+1} - a_{nn}x_n^{(k)}) \end{cases}$$

为等号右边加入松弛因子ω.

$$\begin{cases} x_1^{(k+1)} = x_1^{(k)} + \frac{\omega}{a_{11}}(b_1 - a_{11}x_1^{(k)} - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \ldots - a_{1n}x_n^{(k)}) \\ x_2^{(k+1)} = x_2^{(k)} + \frac{\omega}{a_{22}}(b_2 - a_{21}x_1^{(k+1)} - a_{22}x_2^{(k)} - a_{23}x_3^{(k)} - \ldots - a_{2n}x_n^{(k)}) \\ \ldots \\ x_n^{(k+1)} = x_n^{(k)} + \frac{\omega}{a_{nn}}(b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - a_{n3}x_3^{(k+1)} - \ldots - a_{nn-1}x_{n-1}^{k+1} - a_{nn}x_n^{(k)}) \end{cases}$$

GS迭代就是$\omega = 1$的SOR迭代。

向量的范数

设$\vec{x} = (x_1, x_2, \ldots, x_n)^T$,则有:

$$||\vec{x}||_1 = |x_1| + |x_2| + \ldots + |x_n| = \sum_{i=1}^{n} |x_i| \text{称为向量的}\vec{x}\text{的}1\text{范数}$$

$$||\vec{x}||_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2} \text{称为向量的}\vec{x}\text{的}2\text{范数}$$

$$||\vec{x}||_\infty = \max|x_i|, 1 \le i \le n, \text{称为向量的}\vec{x}\text{的}\infty\text{范数}$$