Attributed Graph Alignment

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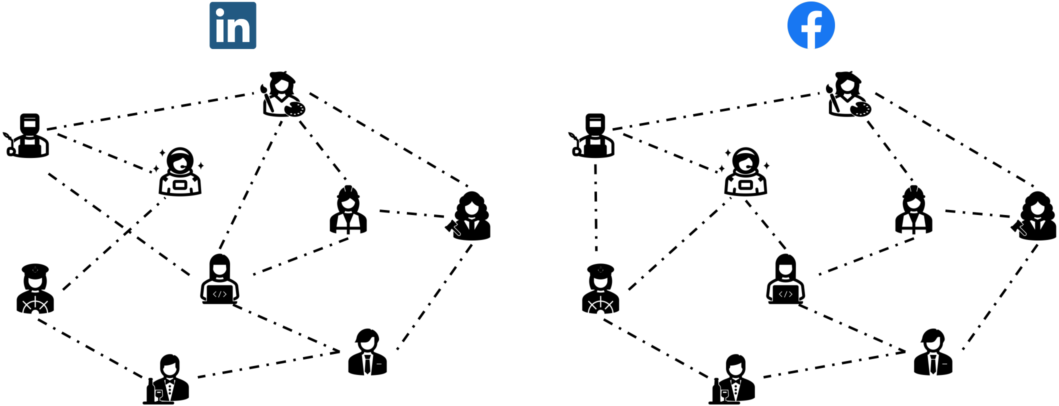
Social network de-anonymization



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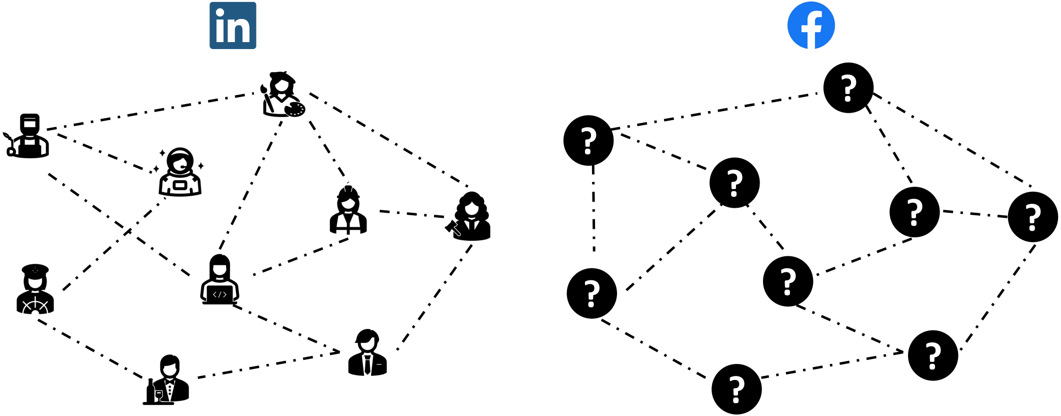

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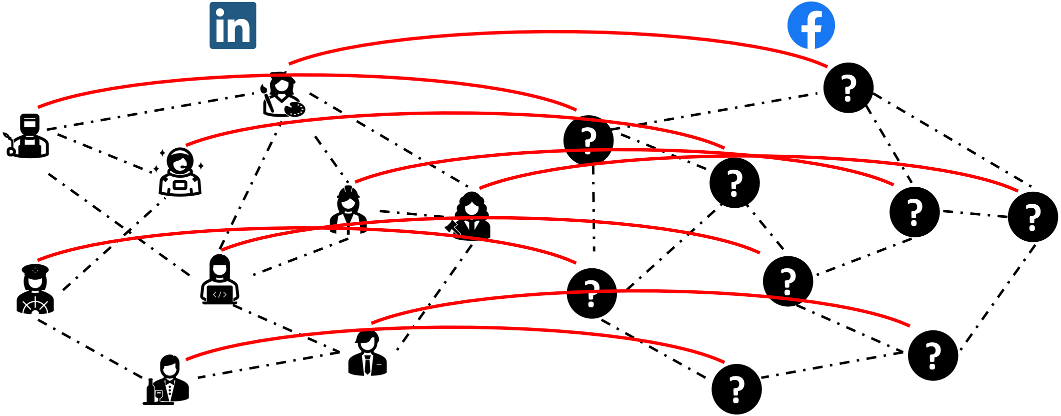

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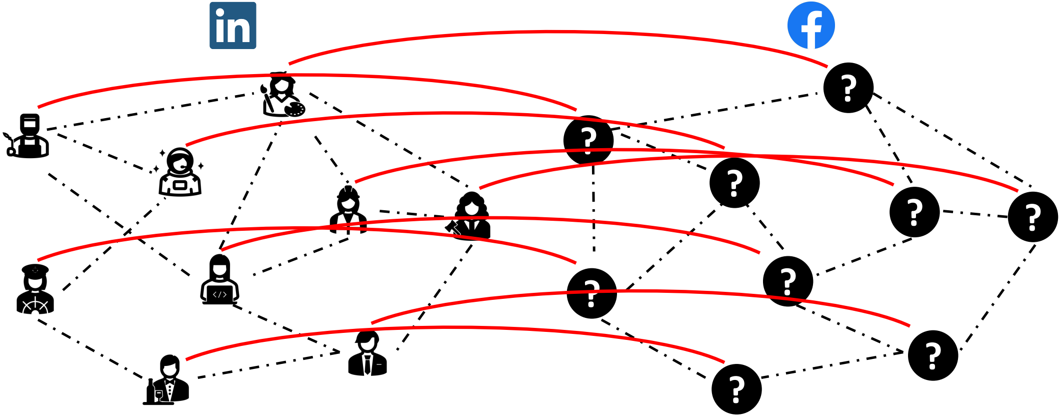


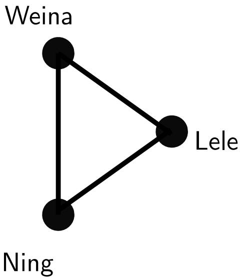
Other applications: computer vision, biological network analysis, natural language processing...

Related works

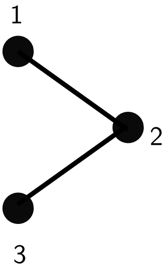
- **Information-theoretic limit:**
  Pedarsani and Grossglauser (2011), Cullina and Kiyavash (2016), Cullina and Kiyavash (2017), Wu et al. (2021)

- **Efficient algorithms:**
  - **Convex optimization:** Lyzinski et al. (2015);
  - **Spectral method:** Nassar et al. (2018), Feizi et al. (2019), Fan et al. (2020) etc. ;
  - **Belief propagation:** Onaran and Villar (2020)
  - **Others:** Barak et al. (2019)

What if graph structure is not enough?



G_1



G'_2

We do know more ...

Ning Zhang

Intro

Studies ECE at **University of British Columbia**

Studied Physics Major at Nankai University, China

Lives in **Vancouver, British Columbia**

From **Shijiazhuang Shi, Hebei, China**

Joined February 2017

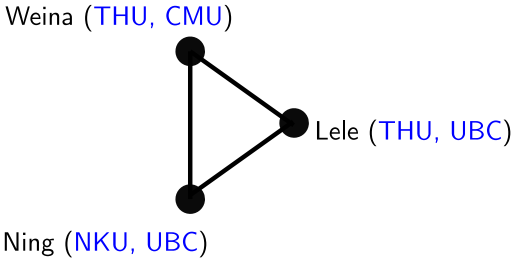
Edit Details

Hiking **Badminton**

Ning Zhang

Information theory; MASc@UBC; Vancouver

We do know more ...



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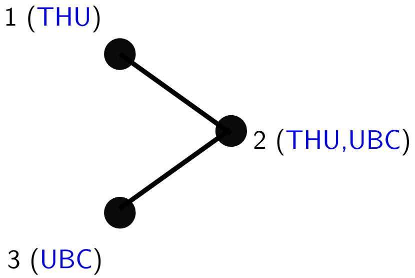
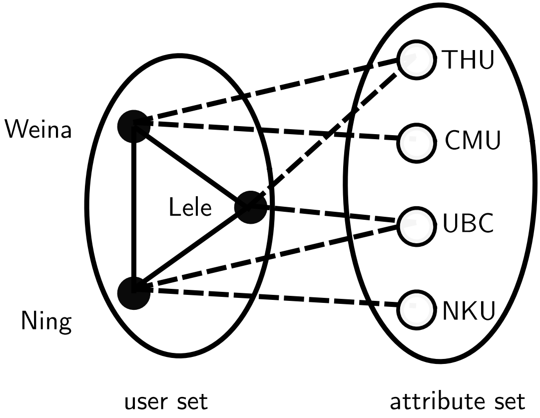


G_2'

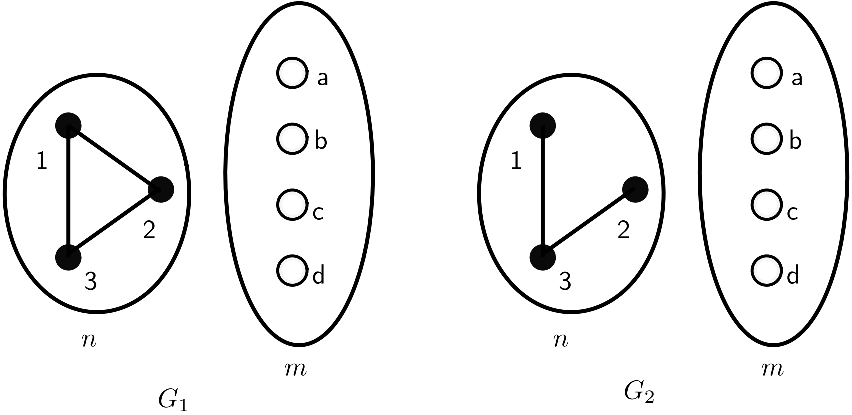
Attributes as vertices



How much benefit can vertex attributes bring?

Model: attributed Erdős–Rényi graph pair

- $(G_1, G_2) \sim \mathcal{G}(n, \boldsymbol{p}; m, \boldsymbol{q})$
  - For $e \in \mathcal{E}_u$, $(\mathbb{1}\{e \in E_1\}, \mathbb{1}\{e \in E_2\}) \overset{\text{i.i.d}}{\sim} \boldsymbol{p}$, where $\boldsymbol{p} = (p_{11}, p_{10}, p_{01}, p_{00})$.



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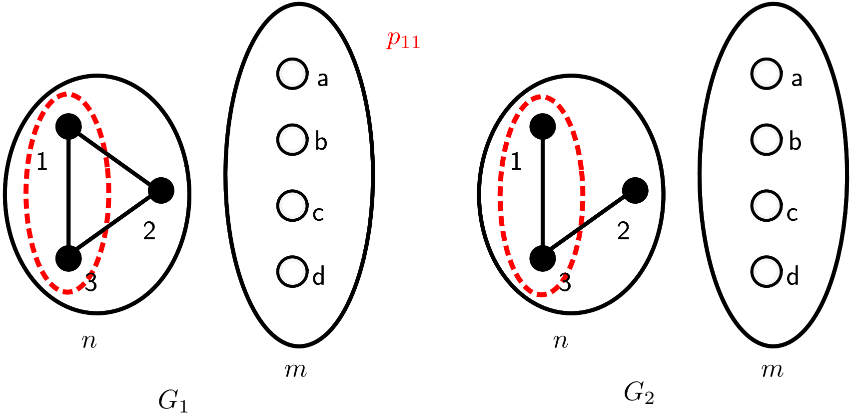

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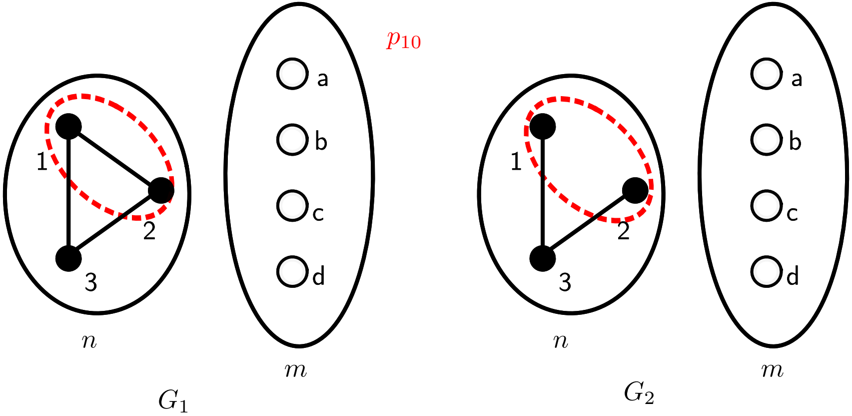

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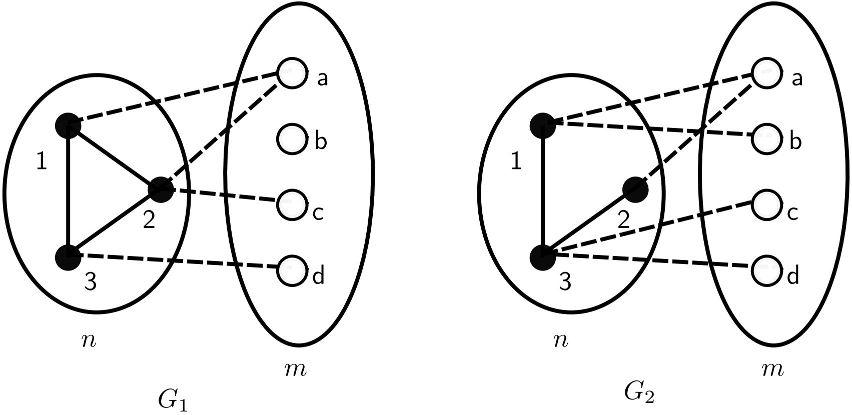

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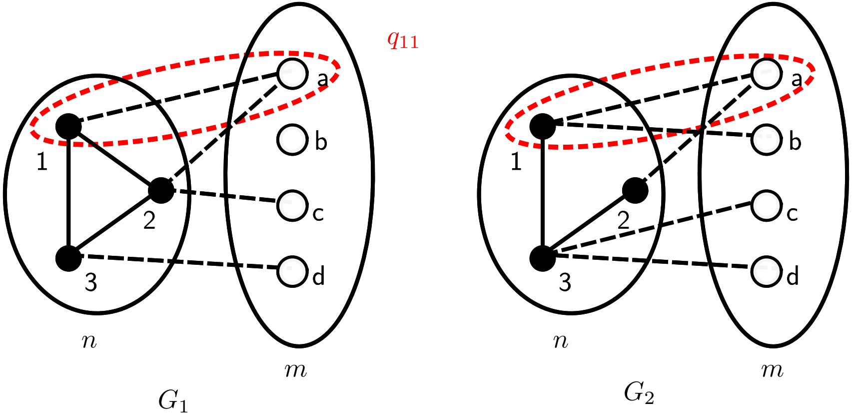

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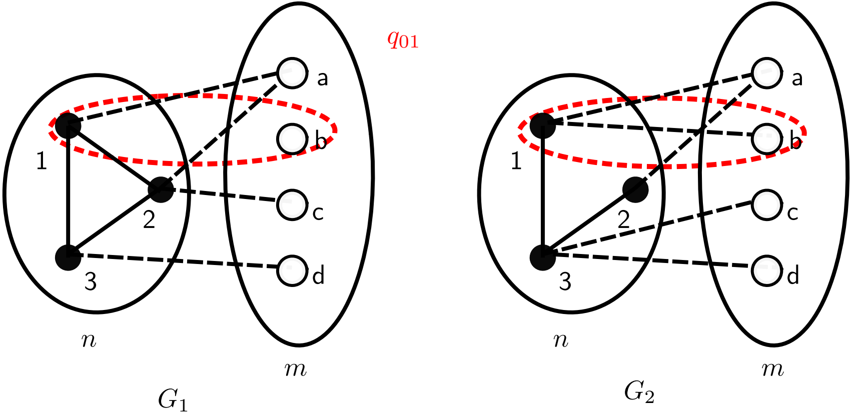

Model: attributed Erdős–Rényi graph pair

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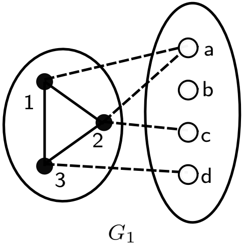
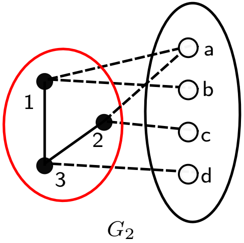
- **Anonymization:** Obtain G_2' by applying random Π^* on user vertex set





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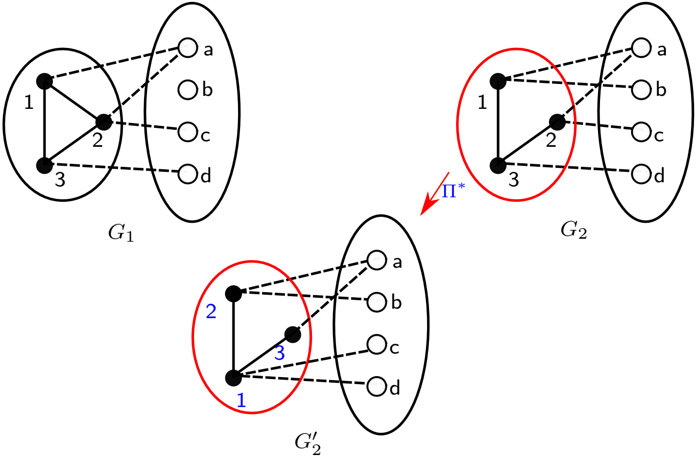

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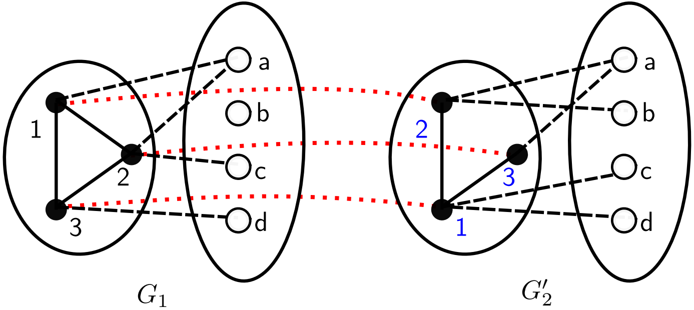
- **Exact alignment:** Recover $\hat{\pi}(G_1, G_2')$ s.t.

$$\lim_{n \to \infty} \mathsf{P}(\hat{\pi}(G_1, G_2') = \Pi^*) = 1$$



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$$\lim_{n \to \infty} \mathsf{P}(\hat{\pi}(G_1, G_2') = \Pi^*) = 1$$

- **Our goal:** characterize the information-theoretic limit

Achievability

Conditions on $(n, \boldsymbol{p}, m, \boldsymbol{q})$ to achieve exact alignment;

Converse

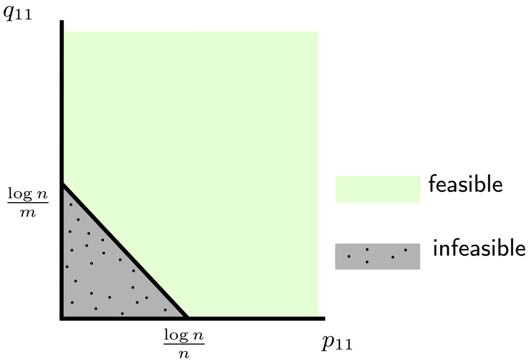
Conditions on $(n, \boldsymbol{p}, m, \boldsymbol{q})$ s.t. exact alignment is not possible.

Achievability (simplified)

- $m = \Omega((\log n)^3)$: $np_{11} + mq_{11} - \log n \to \infty$
- $m = o((\log n)^3)$:
  $np_{11} + mq_{11} - O(mq_{11}^{3/2}) - \log n \to \infty$

Converse

$np_{11} + mq_{11} - \log n \to -\infty$



Main results

Achievability (simplified)

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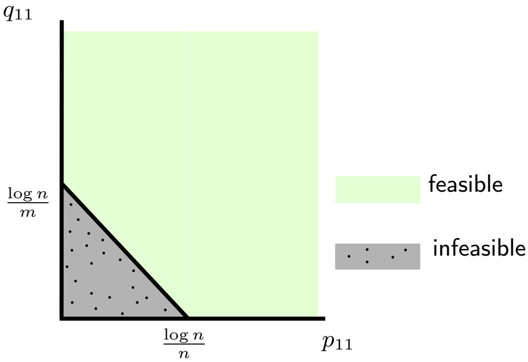
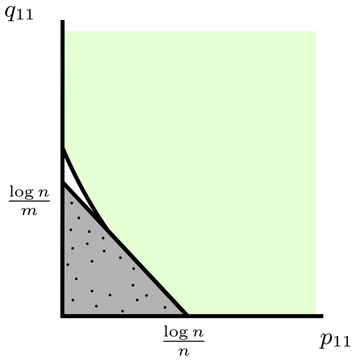
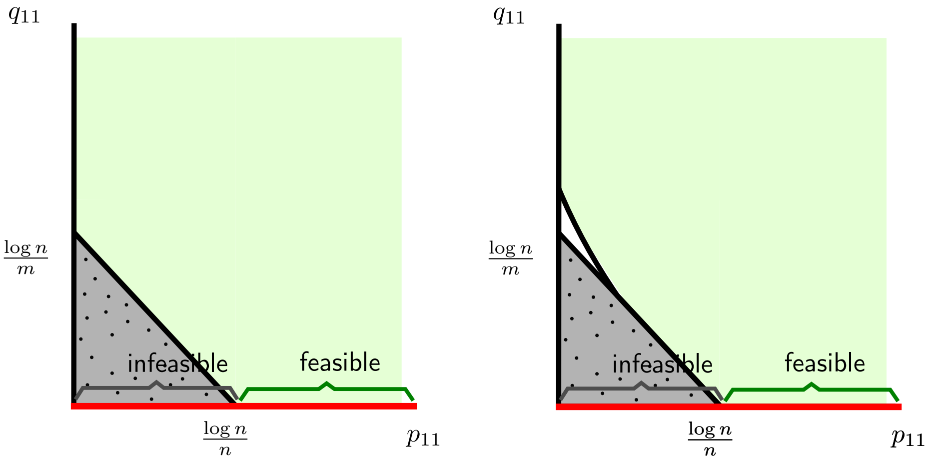




1. **Erdős–Rényi graph alignment** ($q_{00} = 1$) Cullina and Kiyavash (2017)



(a) $m = \Omega((\log n)^3)$ (b) $m = o((\log n)^3)$

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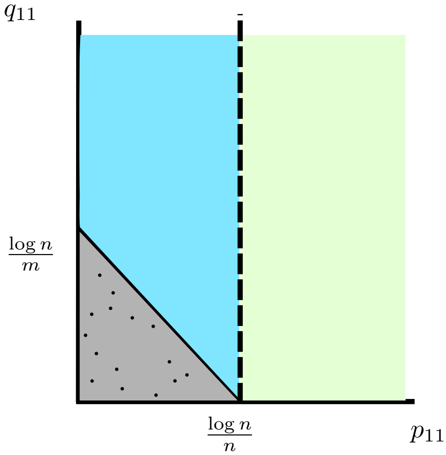

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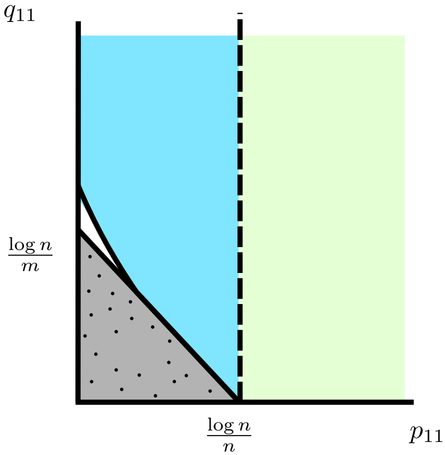
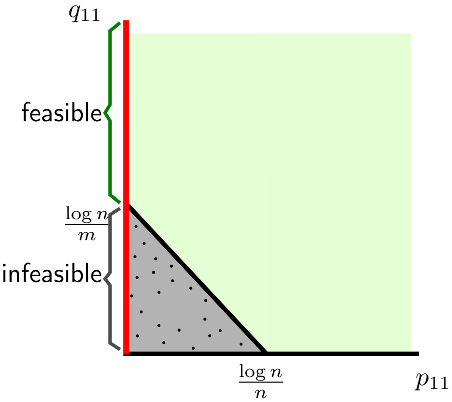


(b) $m = o((\log n)^3)$

2. **Bipartite graph alignment** $(p_{00} = 1)$ Dai et al. (2019)



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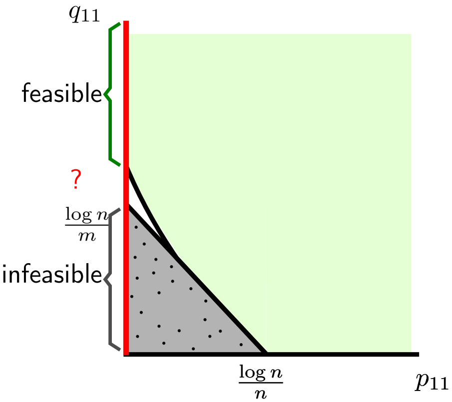
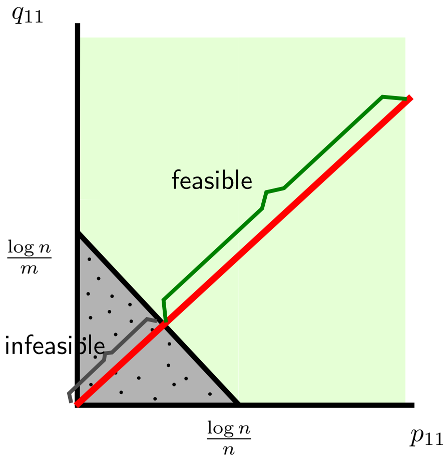


(b) $m = o((\log n)^3)$

Improve the seeded graph alignment

3. **Seeded graph alignment** $(p = q)$ Yartseva and Grossglauser (2013)



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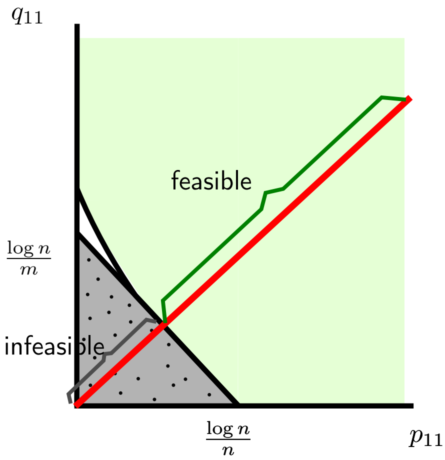

(b) $m = o((\log n)^3)$

Proof of achievability (sketch)

Achievability

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Key ideas:

- MAP estimator (see Lemma1)

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Key ideas:

- MAP estimator (see Lemma1)
- Upper bound error probability of MAP (Cullina and Kiyavash (2017)):
  - Orbit decomposition
  - Generating functions

Proof of converse (sketch)

Converse

$np_{11} + mq_{11} - \log n \to -\infty$

Key ideas:

- $\mathsf{P}(\hat{\pi}_{\text{MAP}}(G_1, G_2') = \Pi^*) \leq \frac{1}{|\text{Aut}(G_1 \wedge G_2)|}$

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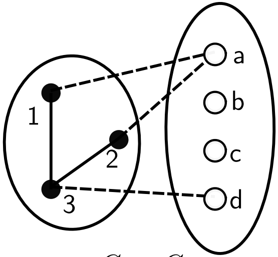
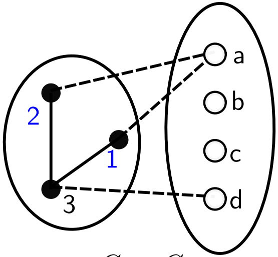


$G_1 \wedge G_2$



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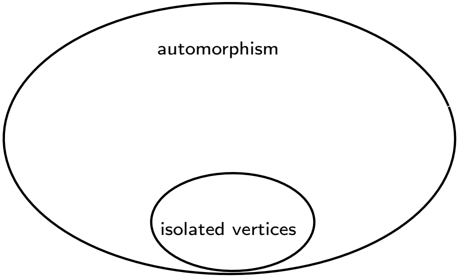
- $|\mathrm{Aut}| \geq |\mathrm{Aut}_{\mathrm{iso}}|$



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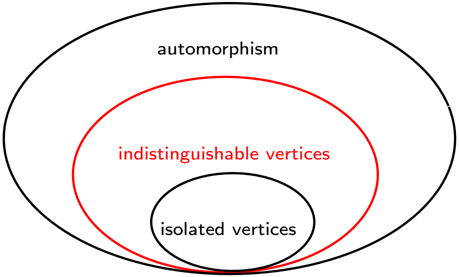

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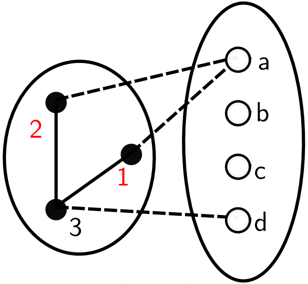

$G_1 \wedge G_2$

$|\mathrm{Aut}_{\mathrm{iso}}| = 0$
$|\mathrm{Aut}_{\mathrm{ind}}| = 1$

Proof of converse (sketch)

Converse

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Key ideas:

- $\mathsf{P}(\hat{\pi}_{\text{MAP}}(G_1, G_2') = \Pi^*) \leq \frac{1}{|\text{Aut}(G_1 \wedge G_2)|}$
- $|\text{Aut}| \geq |\text{Aut}_{\text{ind}}| \geq |\text{Aut}_{\text{iso}}|$
- Use the second moment method to show that w.h.p. $|\text{Aut}(G_1 \wedge G_2)| \geq 2$
  In particular, we show # indistinguishable vertex pair >0.

Contributions

- Propose attributed Erdős–Rényi graph pair model
- Characterize the information-theoretic limits
- Unify existing models for graph alignment

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