Introduction to the Yoneda's Lemma

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Category

Definition (Category)

A category $\mathcal{C}$ consist of the following datas:

(1) a collection of objects :$X, Y, Z \in obj(\mathcal{C})$;

(2) a collection of morphisms :f, g, h such that

  (a) For any morphism f, there exists specific domain and codomain;

  (b) For any $X \in obj(\mathcal{C})$, there exists identity morphism $1_X : X \to X$;

  (c) Morphisms have composition.

which satisfies the following conditions:

- $\forall f : X \to Y, \ f 1_X = 1_Y f = f$;
- $f(gh) = f(gh)$.

We denote all morphisms $f : X \to Y$ by $Hom_{\mathcal{C}}(X, Y)$, and it should be a set here(locally small).

We define a subcategory $\mathcal{D} \subset \mathcal{C}$ whose objects contained in $obj(\mathcal{C})$ and $Hom_{\mathcal{D}}(X, Y) \subset Hom_{\mathcal{C}}(X, Y)$ for all $X \in obj(\mathcal{D})$. If in this case, $Hom_{\mathcal{D}}(X, Y) = Hom_{\mathcal{C}}(X, Y)$, we call it full subcategory.

We will omit the subscript of $Hom_{\mathcal{C}}(X, Y)$, if it is clear from the context.

Example

(1) **Sets,Group**. They belong to the concrete categroy in which every object has a underlying set structure.

(2) a group G indeuces a category BG, which has only one object denoted by $*$, and the elements in G are morphisms in BG.

(3)

$$\omega : \quad obj : 0, 1, \cdots$$

$$Mor : Hom(i, j) = \begin{cases} \xi_j^i & i \leqslant j \\ \varnothing & others \end{cases}$$

Functor

We can regard a functor as a morphism between categories.

Definition (Functor)

A functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following datas:

(1) An object $Fc \in \mathcal{D}$ for each $c \in \mathcal{C}$;

(2) A morphism $Ff : Fc \to Fc^{'} \in \mathcal{D}$for each $f : c \to c^{'} \in \mathcal{C}$

and satisfies the following conditions:

(a) For each composable pair $f, g \in Mor(\mathcal{C})$, $Fg \cdot Ff = F(g \cdot f)$;

(b) For any $c \in \mathcal{C}$, $F(1_C) = 1_{F(C)}$.

Example

(1) Forgetful functor.

(2) $X : BG \to \mathbf{Sets}$. It has a unique image object in $\mathcal{C}$ denoted by X, and morphisms $g_* : X \to X$. It defines a left action of the group G on X.

(3) $\mathcal{C}$ is a locally small category, and for any $c \in \mathcal{C}$. We have a functor represented by c :

$$\mathcal{C} \xrightarrow{Hom_{\mathcal{C}}(c,-)} \mathbf{Sets}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, "f"'] & Hom_{\mathcal{C}}(c,x) \arrow[d, "f_*"] \\
{} \arrow[r, mapsto] & {} \\
y \arrow[r, mapsto] & Hom_{\mathcal{C}}(c,y)
\end{tikzcd}$$

Natural Transformation

Definition (Natural Transformation)

Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two parallel functor, a natural transformation $\alpha : F \Rightarrow G$ consist of :

- a morphism $\alpha_c : Fc \to Gc$ in $\mathcal{D}$ for each $c \in \mathcal{C}$ which calls the component of α.

and for any $f : c \to c'$ in $\mathcal{C}$, the following diagram commute:

$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \end{tikzcd}$$

Now we know all the concepts in Yoneda's Lemma, so we present it here first.

Theorem (Yoneda's Lemma)

For any functor $F : \mathcal{C} \to \textbf{Sets}$, $\mathcal{C}$ is locally small, then for each $c \in \mathcal{C}$, there exists a bijection:

$$\begin{aligned}\Phi_{F,c} : \ Nat(Hom_{\mathcal{C}}(c,-), F) &\longrightarrow Fc \\ \alpha &\mapsto \alpha_c(1_c)\end{aligned}$$

and $\Phi_{F,c}$ natural in F and c.

We probe into some examples that will be used to explain the Yoneda's Lemma.

Example (sequence functor)

Define a functor:

$$F : \omega \to \textbf{Sets}$$
$$obj : n \mapsto F_n$$
$$Mor : \xi^n_{n+1} \mapsto f_{n,n+1}(: F_n \to F_{n+1})$$

for some $k \in \mathbb{N}$,

$$Hom_\omega(k, -) = \begin{cases} \varnothing & n < k \\ \xi^k_n & n \geqslant k \end{cases}$$

Then let us check out what conditions $\alpha : Hom_\omega(k, -) \Rightarrow F$ a natual transformation should satisfied:

Example (sequence functor)

First of all, its naturality:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, "\alpha_0"] & \cdots \arrow[r] & \varnothing \arrow[r] \arrow[d, "\alpha_{k-1}"] & * \arrow[r] \arrow[d, "\alpha_k"] & * \arrow[r] \arrow[d, "\alpha_{k+1}"] & \cdots \arrow[r] & * \arrow[r] \arrow[d, "\alpha_n"] & \cdots \\
F_0 \arrow[r] & \cdots \arrow[r] & F_{k-1} \arrow[r, "f_{k-1,k}"'] & F_k \arrow[r, "f_{k,k+1}"'] & F_{k+1} \arrow[r] & \cdots \arrow[r] & F_n \arrow[r] & \cdots
\end{tikzcd}$$

$*$ denotes a singleton $Hom(k, n)$. If $n < k$, α_n is trivial.

If $n \geqslant k$, since $*$ is a singleton, so α_n uniquely determines a element in F_n denoted by $\bar{\alpha}_n$. By the commutativity of the diagram, we have:

$$\bar{\alpha}_{k+1} = f_{k,k+1}(\bar{\alpha}_k)$$

So $\bar{\alpha}_{k+1}$ is determined by $\bar{\alpha}_k$. Continue recursion as above we get:

$$\bar{\alpha}_{n+1} = f_{k,n+1}(\bar{\alpha}_k) = f_{k,n+1}(\alpha_k(1_k))$$

Example (sequence functor)

so this natural transformation is uniquely determined by the representing object k.

Example (functor on category BG)

Assume that $G, X : BG \to \mathbf{Sets}$ be two functors, and $G = Hom_{BG}(*, -)$, so $G(*)$ is exactly the group G, and X define a left action on $X(*)$.

Consider the natural transformation $\eta : G \Rightarrow X$, it has only one component $\eta(*) = \eta$, for any $h, g \in G$:

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[d, "\eta"'] & G \arrow[r, "g"] \arrow[d, "\eta"'] & G \arrow[d, "\eta"'] \\ X \arrow[r, "h_*"'] & X \arrow[r, "g_*"'] & X \end{tikzcd}$$

Example (functor on category BG)

Namely, we have $\eta(gh) = g_* \cdot \eta(h)$. If we choose $h = e$ (the identity element in G):

$$\eta(g) = g_* \cdot \eta(e)$$

so the natrual transformation η is uniquely determined by the identity morphism of the component at representing object $*$ (there is only one object), and $\eta(e)$ can be any element in $X(*)$.

Through above examples, it is obivious that for any natural transformation $\alpha : Hom_{\mathcal{C}}(c, -) \Rightarrow F$, it corresponds to an unique element in the image of F on the representing object c. Conversely, for any element in Fc , we can construct a nat from $Hom_{\mathcal{C}}(c, -)$ to F.

Proof

Proof of Yoneda's Lemma.

We directly prove that :

$$\begin{aligned} \Phi_{F,c}: \ Nat(Hom_{\mathcal{C}}(c,-),F) &\longrightarrow Fc \\ \alpha &\mapsto \alpha_c(1_c) \end{aligned}$$

is a bijection.

1.Injection:Let $\alpha_c(1_c) = \eta_c(1_c)$, by the naturality of α, we have:

$$\begin{tikzcd} Hom(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f_*"] & Fc \arrow[d, "Ff"'] \\ Hom(c,d) \arrow[r, "\alpha_d"] & Fd \end{tikzcd}$$

The same for η. Chasing 1_c we obtain:

Proof of Yoneda's Lemma.

$$\alpha_d(f) = \eta_d(f)$$

for any $f \in Hom(c, d)$, $d \in \mathcal{C}$. So $\alpha = \eta$.
2.Surjectivity: Let $x \in Fc$, we construct a natural formation:

$$\beta : Hom(c, -) \to F$$

and $\beta_d : f \mapsto Ff(x)$. So $\beta_c(1_c) = x$. Let verify that β is natrual. For any morphism $g : b \to b' \in \mathcal{C}$, we gonna prove that the following diagram commutes:

$$\begin{tikzcd}
Hom(c, b) \arrow[r, "\beta_b"] \arrow[d, "g_*"] & Fb \arrow[d, "Fg"'] \\
Hom(c, b') \arrow[r, "\beta_{b'}"] & Fb'
\end{tikzcd}$$

Proof of Yoneda's Lemma.

For any $h \in Hom(c, b)$, by the definition of β:

$$Fg(\beta_b(h)) = Fg(Fh(x)) = F(gh)(x) = \beta_{b'}(g_*(h))$$

So β is a natural transformation.

3.Naturality: We prove that $\Phi_{F,c}$ is natural in c, namely for any morphism $f: c \to d \in \mathcal{C}$:

$$\begin{tikzcd}
Nat(Hom(c,-),F) \arrow[r, "\Phi_{F,c}"] \arrow[d, "(f^*)^*"] & Fc \arrow[d, "Ff"'] \\
Nat(Hom(d,-),F) \arrow[r, "\Phi_{F,d}"] & Fd
\end{tikzcd}$$

Proof of Yoneda's Lemma.

We should evaluate on $\alpha : Hom(c, -) \Rightarrow F$. Going clockwise we get $Ff(\alpha_c(1_c))$. Going counter clockwise: $(\alpha f^*)_d(1_d) = \alpha_d(f)$. So they are equal by the naturality of α.

Next we prove that $\Phi_{F,c}$ is natural in F. Given any natural transformation $\beta : F \Rightarrow G$, consider the following diagram:

$$\begin{tikzcd} Nat(Hom(c,-),F) \arrow[r, "\Phi_{F,c}"] \arrow[d, "\beta_*"] & Fc \arrow[d, "\beta_c"'] \\ Nat(Hom(c,-),G) \arrow[r, "\Phi_{G,c}"] & Gc \end{tikzcd}$$

Also we evaluate on $\alpha : Hom(c, -) \Rightarrow F$:

$$\Phi_{G,c}(\beta \cdot \alpha) = (\beta \cdot \alpha)_c(1_c) = \beta_c(\alpha_c(1_c)) = \beta_c(\Phi_{F,c}(\alpha))$$

So we complete the proof. □

Application

Now we are going to use Yoneda's Lemma to prove a theorem which is of great importance in homological algebra.

Definition

Let $\mathcal{C}$ be a category. For $c, d \in \mathcal{C}$, and if there exists morphisms $f : c \to d$ and $g : d \to c$ such that $fg = 1_d$, $gf = 1_c$, then we say c is isomorphic to d, denoted by $c \cong d$, and f, g is isomorphisms.

Definition

Let $\alpha : F \Rightarrow G$ be a nat, and $F, G : \mathcal{C} \to \mathcal{D}$. If for any $c \in \mathcal{C}$, α_c is a isomorphism, then we say that F is natural isomorphic to G, and α is a naturally isomorphism.

In the following parts, we assume $\mathcal{C}$ be a category, and $c, c' \in obj(\mathcal{C})$

Lemma

Let $\alpha : Hom_{\mathcal{C}}(c, -) \to Hom_{\mathcal{C}}(c', -)$ be a natural transformation, then for all $d \in obj(\mathcal{C})$, we have $\alpha_d = f^*$. $f = \alpha_c(1_c) : c' \to c$. Moreover, f is unique.

Proof: For any $d \in \mathcal{C}$, and $g \in Hom(c, d)$. We use the following commutable diagram:

$$\begin{tikzcd} Hom(c,c) \arrow[r, "\alpha_c"] \arrow[d, "g_*"] & Hom(c',c) \arrow[d, "g_*"'] \\ Hom(c,d) \arrow[r, "\alpha_d"] & Hom(c',d) \end{tikzcd}$$

Lemma

Chasing 1_c we obtain:

$$\alpha_d(g) = g_*(f) = gf = f^*(g)$$

So $\alpha = f^*$. The uniqueness assertion follows from the injevtivity of the $\Phi_{Hom(c',-),c}$. $\square$

Theorem

If $Hom_{\mathcal{C}}(c,-)$ and $Hom_{\mathcal{C}}(c',-)$ are naturally isomorphic functors, then $c \cong c'$.

Proof.

Let

$$\alpha : Hom(c,-) \to Hom(c',-) \quad , \beta : Hom(c',-) \to Hom(c,-)$$

be two natural isomorphisms.

Proof.

And $(\alpha\beta)_d = 1_{Hom(c',d)}$, $(\beta\alpha)_d = 1_{Hom(c,d)}$. Let $\alpha_d = f^*$ and $\beta_d = g^*$ by Lemma. Observe that

$$(\beta\alpha)_c = \beta_c\alpha_c = g^* f^* = (fg)^* = 1_c^*$$

By the uniqueness in Lemma, we get $fg = 1_c$. The same reason we have $gf = 1_{c'}$, so $c \cong c'$. □

Reference:

1. Emily Riehl, Category theory in context.
2. Rotman, An Introduction to homological Algebra.