

CSDS 391 Programming Assignment 2

Writeup

Ningjia Huang, Tianxi Zhao

Due On: December 4, 2020

Problem 1:

Problem 2:

(a)

The mean squared error is calculated using the following equation:

$$MSE = \frac{1}{n} \sum_n (Y_i - \hat{Y}_i)^2$$

, where Y_i is the actual category of the i th item and \hat{Y}_i is the predicted category by using our neural network.

The following codes compute the **mean squared error** for iris data. The parameter "data_vector" are the attributes we would like to take into account, it should be a dataframe of attributes. w_0, w_1, w_2 define the weights of neural network. The parameter "pattern_classes" is a dataframe of the category corresponding to the data_vectors. The **mean_square_error** makes use of the logistic non-linearity function in Problem 1.

```
1 # data vectors in dataframe, pattern classes in list
2 def mean_square_error(data_vectors, w0, w1, w2,
3   pattern_classes):
4     n = data_vectors.shape[0]
5     data_vectors_list = data_vectors.values.tolist()
6     pattern_classes_list = pattern_classes.tolist()
7     temp_mse = 0
```

```

7     for i in range(n):
8         temp_mse = temp_mse + np.square(pattern_classes_list[
          i] - one_layer_network(w0, w1, w2, data_vectors_list[i
            ] [0], data_vectors_list[i] [1]))
9         mse = temp_mse/n
10    return mse

```

Listing 1: Mean Squared Error Calculation

(b)

To plot the linear boundary, we find the x-intercept and y-intercept by using the following code:

```

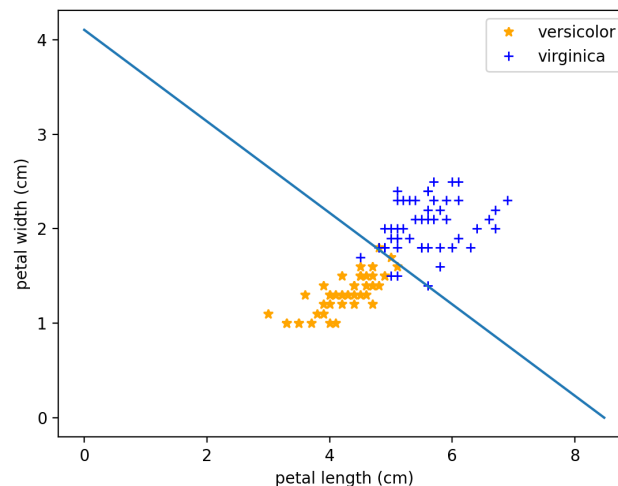
1 plot([0, -w0/w1], [-w0/w2, 0])

```

Listing 2: Plot Mean Squared Error

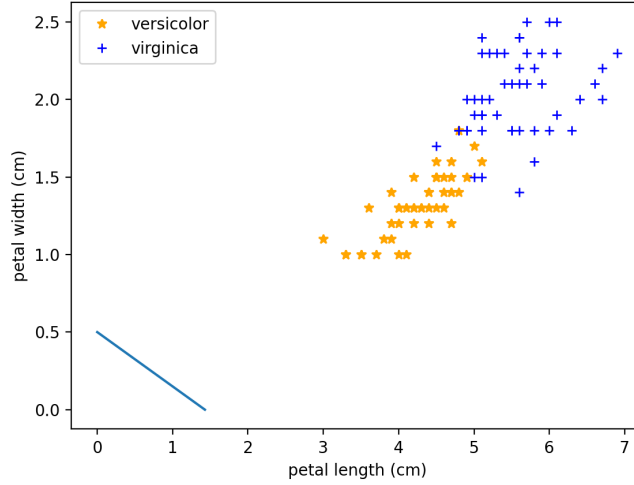
2.b.1

The first set of weights we've chosen is $[-3.9, 0.46, 0.95]$. The mean squared error we concluded is 0.13235245817007218. The following figure shows the linear boundary with our chosen set of weights. As we can tell from the graph, the linear boundary generally separates the two categories of iris.



2.b.2

The second set of weights we've chosen is $[-1, 0.7, 2]$. The mean squared error we concluded is 0.48802189762271864. The following figure shows the linear boundary with our chosen set of weights. As we can tell from the graph, the linear boundary does not separate these two categories at all.



(c)

To compute the gradient of the objective function, we need to take the derivative of the objective function. Assume \mathbf{x}_i is the i th row with attribute petal length and petal width such that $x_i = [1, x_{i1}, x_{i2}]$ (1 is placed here because w_0 requires 1 to be its coefficient). \mathbf{w} is the set of weights for the attributes. The logistic function we get is:

$$\sigma(z) = \frac{1}{1+e^{-z}} = \frac{1}{1+e^{w \cdot x}}$$

Therefore, the mean squared function becomes:

$$MSE = \frac{1}{N} \sum_{i=1}^n (\sigma(w \cdot x_i) - y_i)^2$$

By taking the derivative of this objective function with respect to w_0 , we have:

$$\begin{aligned}\frac{\partial MSE}{\partial w_0} &= \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) \frac{\partial \sigma}{\partial w_0} \\ \frac{\partial MSE}{\partial w_0} &= \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i)) \frac{\partial(w \cdot x)}{\partial w_0} \\ \frac{\partial MSE}{\partial w_0} &= \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i))\end{aligned}$$

Similarly, by only changing the last derivative $\frac{\partial(w \cdot x)}{\partial w_0}$ to $\frac{\partial(w \cdot x)}{\partial w_1}$ and $\frac{\partial(w \cdot x)}{\partial w_2}$, we can compute the derivative of the objective function with respect to w_1 and w_2 :

$$\begin{aligned}\frac{\partial MSE}{\partial w_1} &= \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i)) x_{i1} \\ \frac{\partial MSE}{\partial w_2} &= \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i)) x_{i2}\end{aligned}$$

Therefore, the gradient of MSE is:

$$\nabla MSE = \begin{bmatrix} \frac{\partial MSE}{\partial w_0} \\ \frac{\partial MSE}{\partial w_1} \\ \frac{\partial MSE}{\partial w_2} \end{bmatrix} = \begin{bmatrix} \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i)) \\ \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i)) x_{i1} \\ \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i)) x_{i2} \end{bmatrix}$$

(d)

scalar form:

The scalar form of the gradient is computed in Problem 2 (c):

$$\nabla MSE = \begin{bmatrix} \frac{\partial MSE}{\partial w_0} \\ \frac{\partial MSE}{\partial w_1} \\ \frac{\partial MSE}{\partial w_2} \end{bmatrix} = \begin{bmatrix} \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i)) \\ \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i)) x_{i1} \\ \frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i)) x_{i2} \end{bmatrix}$$

vector form:

Let $X = \begin{bmatrix} x_{11}, x_{12}, 1 \\ \dots \\ x_{n1}, x_{n2}, 1 \end{bmatrix}$, which is a matrix represents the attributes (1 corresponds for bias). Since the term $\frac{2}{N} \sum_{i=1}^N (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i))$ is independent of the dimension j and common for $\frac{\partial MSE}{\partial w_0}$, $\frac{\partial MSE}{\partial w_1}$, and $\frac{\partial MSE}{\partial w_2}$, we can first express this term as $z_i = (\sigma(w \cdot x_i) - y_i) (1 - \sigma(w \cdot x_i)) (\sigma(w \cdot x_i))$.

The gradient of MSE becomes:

$$\nabla MSE = \frac{2}{m} \begin{bmatrix} \sum_{i=1}^N z_i \\ \sum_{i=1}^N z_i x_{i1} \\ \sum_{i=1}^N z_i x_{i2} \end{bmatrix} = \frac{2}{m} \begin{bmatrix} 1, \dots, 1 \\ x_{11}, \dots, x_{n1} \\ x_{12}, \dots, x_{n2} \end{bmatrix} \begin{bmatrix} z_1 \\ \dots \\ z_n \end{bmatrix} = \frac{2}{m} \mathbf{X}^T \begin{bmatrix} z_1 \\ \dots \\ z_n \end{bmatrix}$$

Then, we attempt to expand the vector $z_i = \begin{bmatrix} z_1 \\ \dots \\ z_n \end{bmatrix}$:

$$z_i = \begin{bmatrix} (\sigma(\mathbf{w} \cdot \mathbf{x}_1) - y_1)(1 - \sigma(\mathbf{w} \cdot \mathbf{x}_1))(\sigma(\mathbf{w} \cdot \mathbf{x}_1)) \\ \dots \\ (\sigma(\mathbf{w} \cdot \mathbf{x}_n) - y_n)(1 - \sigma(\mathbf{w} \cdot \mathbf{x}_n))(\sigma(\mathbf{w} \cdot \mathbf{x}_n)) \end{bmatrix} =$$

$$\begin{bmatrix} \sigma(\mathbf{w} \cdot \mathbf{x}_1) - y_1 \\ \dots \\ \sigma(\mathbf{w} \cdot \mathbf{x}_n) - y_n \end{bmatrix} \star \begin{bmatrix} 1 - \sigma(\mathbf{w} \cdot \mathbf{x}_1) \\ \dots \\ 1 - \sigma(\mathbf{w} \cdot \mathbf{x}_n) \end{bmatrix} \star \begin{bmatrix} \sigma(\mathbf{w} \cdot \mathbf{x}_1) \\ \dots \\ \sigma(\mathbf{w} \cdot \mathbf{x}_n) \end{bmatrix}$$

The \star operation is defined as multiplying the i th row's elements. To simplify the expression, let $\sigma(\mathbf{X}\mathbf{w}) = \begin{bmatrix} \sigma(\mathbf{w}^T \mathbf{x}_1) \\ \dots \\ \sigma(\mathbf{w}^T \mathbf{x}_n) \end{bmatrix}$. Thus, z_i can be expressed as $z_i = (\sigma(\mathbf{X}\mathbf{w}) - \mathbf{y}) \star (1 - \sigma(\mathbf{X}\mathbf{w})) \star \sigma(\mathbf{X}\mathbf{w})$. The vector form of gradient can be expressed as:

$$\nabla MSE = \frac{2}{m} \mathbf{X}^T (\sigma(\mathbf{X}\mathbf{w}) - \mathbf{y}) \star (1 - \sigma(\mathbf{X}\mathbf{w})) \star \sigma(\mathbf{X}\mathbf{w})$$

(e)

The code we used to calculate the gradient is as following:

```

1 def summed_gradient(x, w0, w1, w2, y):
2     sg_w0, sg_w1, sg_w2 = [], [], []
3     n = x.shape[0]
4     x = x.values.tolist()
5     y = y.tolist()
6     sigma = one_layer_network(w0, w1, w2, x[0][0], x[0][1])
7     sg_w0.append((2/n) * (sigma - y[0]) * (1 - sigma) * sigma
8 )
9     sg_w1.append((2/n) * ((sigma - y[0]) * (1 - sigma) *
10 sigma) * x[0][0])
11     sg_w2.append((2/n) * ((sigma - y[0]) * (1 - sigma) *
12 sigma) * x[0][1])
13     for i in range(1, n):
14         sigma = one_layer_network(w0, w1, w2, x[i][0], x[i]
15 ] [1])

```

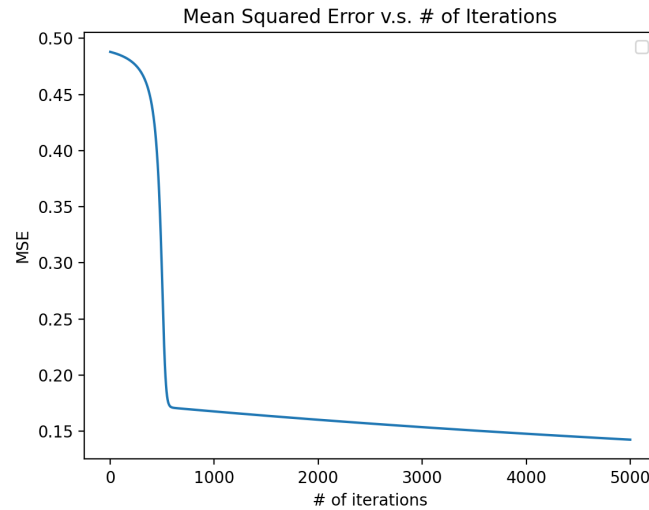
```

12     sg_w0.append(sg_w0[i-1] + (2/n) * (sigma - y[i]) * (1
    - sigma) * sigma)
13     sg_w1.append(sg_w1[i-1] + (2/n) * (sigma - y[i]) * (1
    - sigma) * sigma * x[i][0])
14     sg_w2.append(sg_w2[i-1] + (2/n) * (sigma - y[i]) * (1
    - sigma) * sigma * x[i][1])
15     return sg_w0[len(sg_w0)-1], sg_w1[len(sg_w1)-1], sg_w2[
    len(sg_w2)-1]

```

Listing 3: Calculate Gradient

To illustrate the gradient by showing how the decision boundary changes for a small step, we decreased the gradient by a rate of 0.01 each time for 5000 iterations and plotted MSE v.s. number of iterations after each iteration. The weight we used are the same as the worse case in part (b), which is [-1, 0.7, 2]. The graph we concluded is as following:



The code we used to do this experiment is:

```

1 def illustrate_summed_gradient(x, w0, w1, w2, y):
2     mse_list = []
3     num_of_iter = []
4     print("MSE before iterations: ", mean_square_error(x, w0,
5     w1, w2, y), "w0, w1, w2: ", w0, w1, w2)
6     for i in range(5000):
7         num_of_iter.append(i + 1)
8         temp1, temp2, temp3 = summed_gradient(x, w0, w1, w2,
9         y)

```

```

8         w0 = w0 - 0.01 * temp1
9         w1 = w1 - 0.01 * temp2
10        w2 = w2 - 0.01 * temp3
11        mse = mean_square_error(x, w0, w1, w2, y)
12        mse_list.append(mse)
13        print("MSE after iterations:", mse, "w0, w1, w2: ",
w0, w1, w2)
14        plt.xlabel("# of iterations")
15        plt.ylabel("MSE")
16        plt.title("Mean Squared Error v.s. # of Iterations")
17        plt.plot(num_of_iter, mse_list)
18        plt.legend()
19        plt.show()

```

Listing 4: Illustrate the Change of Gradient's Impact on MSE

Partial outputs are as following:

MSE after iterations: 0.14766977305500825

...

MSE after iterations: 0.14238528888944216

MSE after iterations: 0.14238028834557848

To further illustrate the relationship between gradient and the decision boundary, we also plotted the decision boundary before and after the 5000 iterations. The plot before 5000 iteration can be found in section 2.b.2 and for the sake of simplicity we will not include it again in this section. The plot after 5000 iterations is as following: