Two dimensional seismic wave simulation

The Acoustic Model for Seismic Waves

Seismic waves are used to infer properties of subsurface geological structures. The physical model is a heterogeneous elastic medium where sound is propagated by small elastic vibrations.

The general mathematical model for deformations in an elastic medium is based on Newton's second law:

$$\rho \boldsymbol{u}_{tt} = \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f} \quad (1)$$

We also use the generalized Hooke's law, which relates $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ to each other *(Hooke's generalized law):*

$$\boldsymbol{\sigma} = K \nabla \cdot \boldsymbol{u} \boldsymbol{I} + G \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T - \frac{2}{3} \nabla \cdot \boldsymbol{u} \boldsymbol{I} \right) \quad (2)$$

$\boldsymbol{u}$ displacement field,

$\boldsymbol{\sigma}$ stress tensor,

I identity tensor,

ρ medium's density,

$\boldsymbol{f}$ body forces (such as gravity),

K medium's bulk modulus,

G shear modulus.

All these quantities may vary in space, while $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ will also show significant variation in time during wave motion

The acoustic approximation to elastic waves arises from a basic assumption that the second term in Hooke's law, representing the deformations that give rise to shear stresses, can be neglected. This assumption can be interpreted as approximating the geological medium by a fluid.

Neglecting also the Body forces : $\boldsymbol{f}$

$$(2) \Rightarrow \quad \boldsymbol{\sigma} = K\nabla \cdot \boldsymbol{u}$$

$$(1) \Rightarrow \quad \rho \boldsymbol{u}_{tt} = \nabla \boldsymbol{\sigma}$$

$$\rho \boldsymbol{u}_{tt} = \nabla(K\nabla \cdot \boldsymbol{u}) \qquad (3)$$

Introducing p as a pressure via

$$p = -K\nabla \cdot \boldsymbol{u} \qquad (4)$$

And dividing (3) by ρ, we get:

$$\frac{(3)}{\rho} \Rightarrow \qquad \boldsymbol{u}_{tt} = \frac{1}{\rho}\, \nabla(K\nabla \cdot \boldsymbol{u}) = -\frac{1}{\rho}\, \nabla\, p \qquad (5)$$

Taking the divergence of this equation:

$$\nabla \cdot\, \boldsymbol{u}_{tt} = \nabla \cdot \left(-\frac{1}{\rho}\, \nabla\, p\right)$$

$$(4) \Rightarrow \quad \nabla \cdot \boldsymbol{u} = -\frac{p}{K}$$

$$-\frac{p_{tt}}{K} = \nabla \cdot \left(-\frac{1}{\rho}\, \nabla\, p\right)$$

We get acoustic approximation to elastic waves:

$$p_{tt} = K\nabla \cdot \left(\frac{1}{\rho}\, \nabla\, p\right) \qquad (6)$$

This is a standard, linear wave equation with variable coefficients. It is common to add a source term s(x,y,z,t) to model the generation of sound waves:

$$p_{tt} = K\nabla \cdot \left(\frac{1}{\rho}\, \nabla\, p\right) + \; S(x,y,z,t) \;\; (7)$$

A common additional approximation of (7) is based on using the chain rule on the right-hand side:

$$K\nabla \cdot \left(\frac{1}{\rho}\, \nabla\, p\right) = \frac{K}{\rho}\nabla^2 p + K\nabla\left(\frac{1}{\rho}\right) \cdot \nabla\, p \approx \frac{K}{\rho}\nabla^2 p \quad (8)$$

Under the assumption that the relative spatial gradient is small. This approximation results in the simplified equation:

$$\nabla \rho{-1} = -\rho{-2} \nabla \rho$$ is small.

$$p_{tt} = \frac{K}{\rho} \nabla^2 p + S \qquad (9)$$

The acoustic approximations to seismic waves are used for sound waves in the ground, and the Earth's surface is then a boundary where p equals the atmospheric pressure p0 such that the boundary condition becomes $p = p_0$.

Anisotropy:

Quite often in geological materials, the effective wave velocity $c = \sqrt{\frac{K}{\rho}}$ is different in different spatial directions because geological layers are compacted, and often twisted, in such a way that the properties in the horizontal and vertical direction differ. With z as the vertical coordinate, we can introduce a vertical wave velocity c_z and a horizontal wave velocity c_h ,and generalize (9) to

$$p_{tt} = c_z^2 p_{zz} + c_h^2 (p_{xx} + p_{yy}) + S(x, y, z. t)$$

Simulate seismic waves in 2D

The goal of this exercise is to simulate seismic waves using the PDE model in a 2D xz domain with geological layers. Introduce m horizontal layers of thickness h_i, $i = 0 , 1, \dots , m - 1$. Inside layer number i we have a vertical wave velocity $c_{z,i}$ and a horizontal wave velocity $c_{h,i}$.

We are going to make a program for simulating such 2D waves and test it on a case with 3 layers where:

$$c_{z,0} = c_{z,1} = c_{z,2}$$

$$c_{h,0} = c_{h,2}$$

$$c_{h,1} \ll c_{h,0}$$

Let s be a localized point source at the middle of the Earth's surface (the upper boundary) and investigate how the resulting wave travels through the medium. The source can be a localized Gaussian peak that oscillates in time for some time interval. Place the boundaries far enough from the expanding wave so that the boundary conditions do not disturb the wave. Then the type of boundary condition does not matter, except that we physically need to have $p = p_0$, where p_0 is the atmospheric pressure, at the upper boundary.

In this problem, the type of boundary conditions is not crucial:

$$p_{tt} = c_z^2 p_{zz} + c_h^2 p_{xx} + S$$

Finite Difference method:

$$P_{tt} = v_x^2 P_{xx} + v_z^2 P_{zz} + f(x, z, t)$$

$$\frac{\partial f}{\partial a} \approx \frac{f_i - f_{i-1}}{\Delta a} \qquad \frac{d^2 f}{da^2} \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta a)^2} \qquad f = f(a)$$

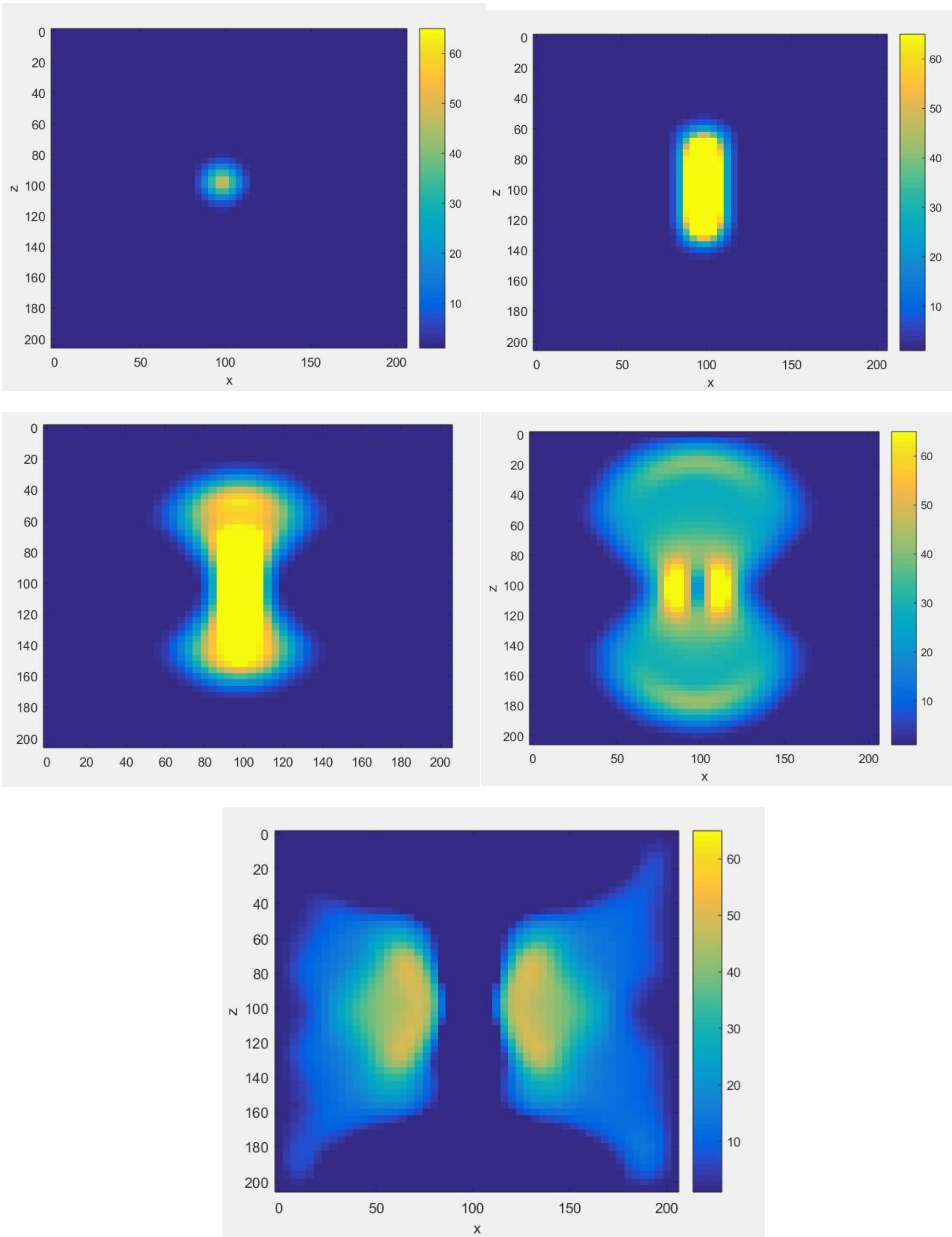
$$\frac{P_{i,k}^{n+1} - 2P_{i,k}^{n} + P_{i,k}^{n-1}}{(\Delta t)^2} = v_x^2 \frac{P_{i+1,k}^{n} - 2P_{i,k}^{n} + P_{i-1,k}^{n}}{(\Delta x)^2} + v_z^2 \frac{P_{i,k+1}^{n} - 2P_{i,k}^{n} + P_{i,k-1}^{n}}{(\Delta y)^2} + f_{i,k}^{n}$$

$$P_{i,k}^{n+1} = 2P_{i,k}^{n} - P_{i,k}^{n-1} + v_x^2 \left(\frac{\Delta t}{\Delta x}\right)^2 \left[P_{i+1,k}^{n} - 2P_{i,k}^{n} + P_{i-1,k}^{n}\right] + $$
$$+ v_z^2 \left(\frac{\Delta t}{\Delta y}\right)^2 \left[P_{i,k+1}^{n} - 2P_{i,k}^{n} + P_{i,k-1}^{n}\right] + (\Delta t)^2 f_{i,k}^{n}$$

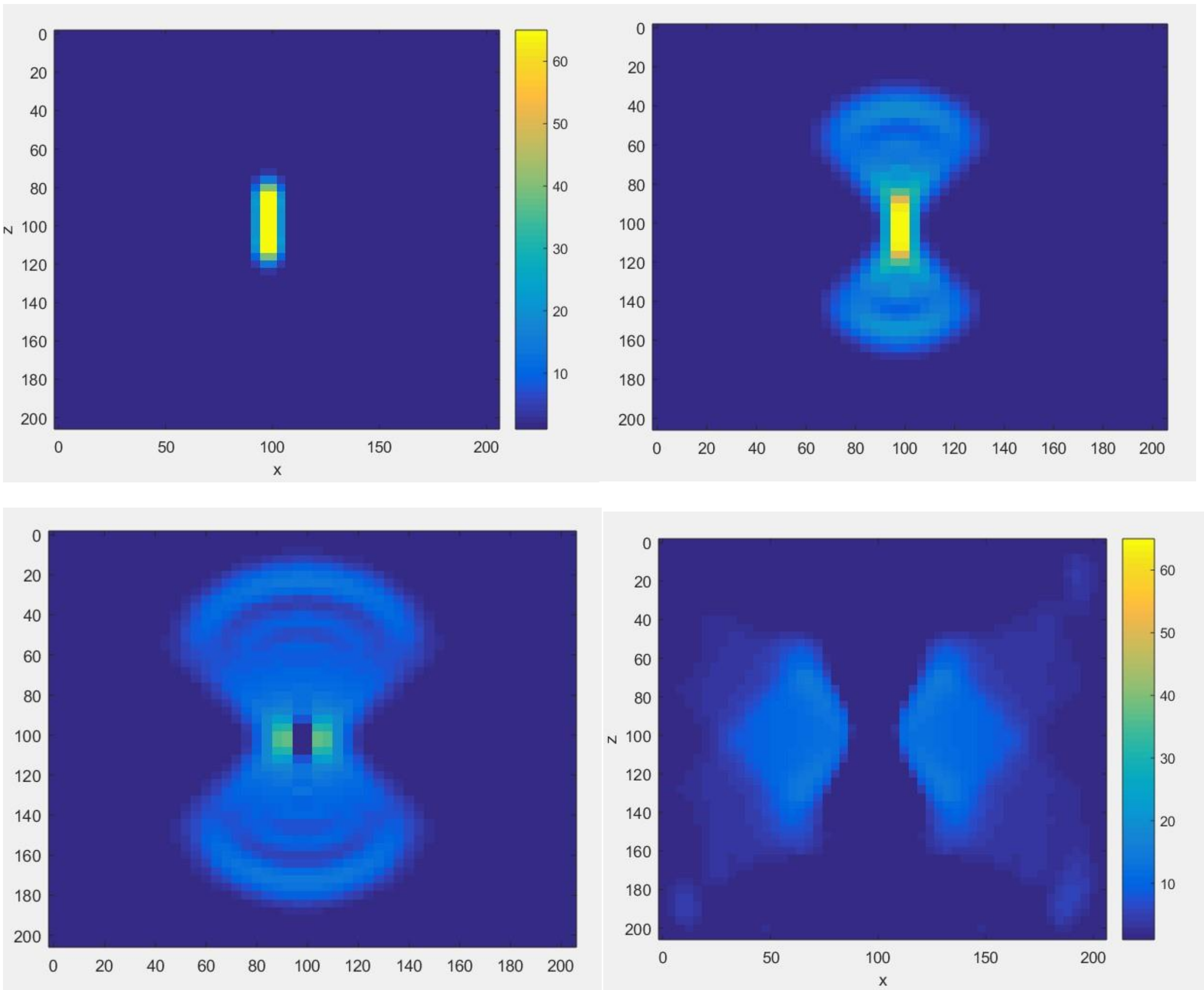
$$\left(v_z \frac{\Delta t}{\Delta y}\right)^2 \equiv C_z^2 \qquad \left(v_x \frac{\Delta t}{\Delta x}\right)^2 \equiv C_x^2$$

$$P_{i,k}^{n+1} = 2P_{i,k}^{n} - P_{i,k}^{n-1} + C_x^2 \left(P_{i+1,k}^{n} - 2P_{i,k}^{n} + P_{i-1,k}^{n}\right) +$$
$$+ C_z^2 \left(P_{i,k+1}^{n} - 2P_{i,k}^{n} + P_{i,k-1}^{n}\right) + (\Delta t)^2 f_{i,k}^{n}$$

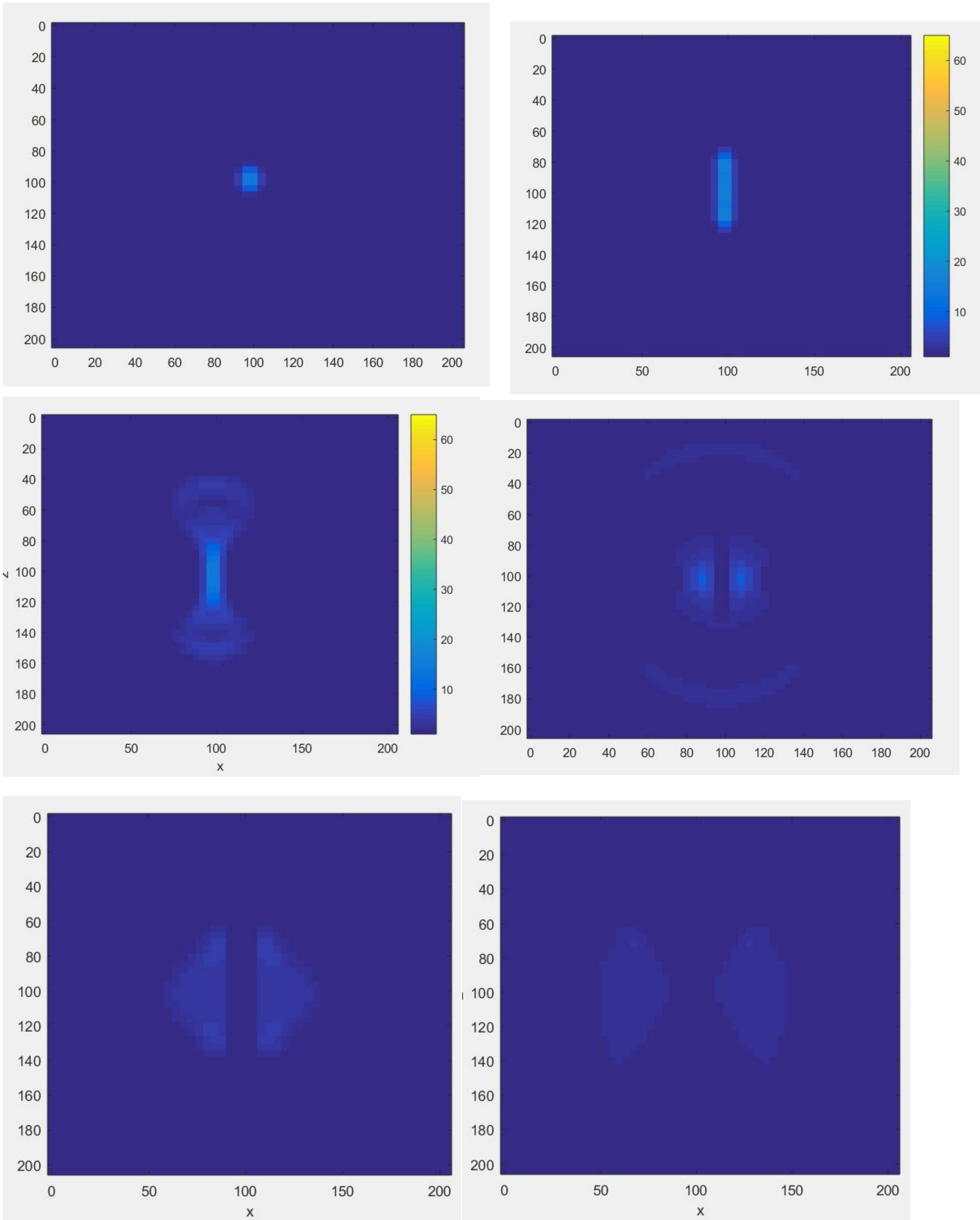
In the case of a less localized Gaussian source



In the case of a more localized Gaussian wave



In the case of a wave with a smaller amplitude



If the speeds were not different: