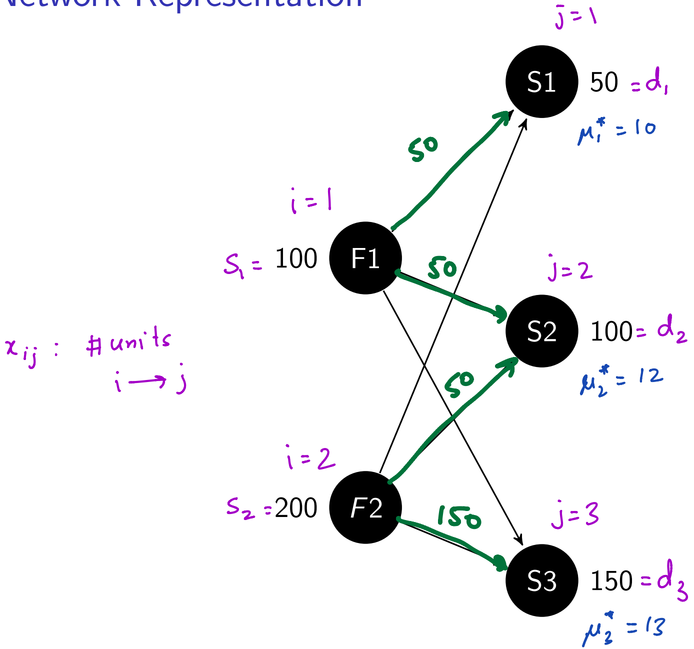
Example: Transportation Problem (minimum cost)

- Two factories F1, F2 with supplies 100, 200 units resp.
- Three shops S1, S2, S3 with demands 50, 100, 150 units resp.
- Shipping costs (in \$ per unit)

	S1	S2	S3
F1	10	12	14
F2	11	12	13

- Problem: Minimize shipping cost

Network Representation



Transportation Problem: LP

$$\min \sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}x_{ij}$$

$$\sum_{j=1}^{3} x_{ij} \le s_i, \quad i = 1, 2$$

$$\sum_{i=1}^{2} x_{ij} \ge d_j, \quad j = 1, 2, 3$$

$$x_{ij} \ge 0, \quad \forall i, j$$

$$\max \sum_{i=1}^{2}\sum_{j=1}^{3} -c_{ij}x_{ij}$$

$$\lambda_i : \sum_{j=1}^{3} x_{ij} \le s_i \quad i=1,2$$

$$\mu_j : -\sum_{i=1}^{2} x_{ij} \le -d_j \quad j=1,2,3$$

$$x_{ij} \ge 0$$

Dual

$$\min \sum_{i=1}^{2} s_i \lambda_i - \sum_{j=1}^{3} d_j \mu_j$$

$$\lambda_i - \mu_j \ge -c_{ij}$$

$$\lambda_i, \mu_j \ge 0$$

Transportation Problem

Primal-Dual Pair: General LP

(Primal)

$$\left[\begin{array}{l} \max \mathbf{c}^T\mathbf{x} \\ \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \quad \mathbf{x} \geq 0 \end{array}\right] \equiv \left[\begin{array}{l} \min \mathbf{b}^T\mathbf{y} \\ \quad \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ \quad \mathbf{y} \geq 0 \end{array}\right]$$

(Dual)

$\max \; c^T x$

$a_i^T x \leq b_i \quad i \in I_1$ $\qquad y_i \geq 0$

$a_i^T x = b_i \quad i \in I_2$ $\qquad y_i$ unrestricted

$x_j \geq 0 \quad j \in J_1$ $\qquad \geq$ constraints

x_j unrestricted $j \in J_2$ $\qquad$ equality constraints

Primal-Dual Pair: General LP

(Primal)

$$\max \mathbf{c}^T\mathbf{x}$$
$$\begin{bmatrix} \mathbf{A}\mathbf{x} \le \mathbf{b} \\ \mathbf{x} \ge 0 \end{bmatrix}$$

(Dual)

$$\min \mathbf{b}^T\mathbf{y}$$
$$\begin{bmatrix} \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \\ \mathbf{y} \ge 0 \end{bmatrix}$$

Weak Duality: Suppose $\mathbf{x}$ is any primal feasible solution and $\mathbf{y}$ is any dual feasible solution, then $\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}$.

Primal-Dual Pair: General LP

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(Dual)

$$\begin{bmatrix} \min \mathbf{b}^T\mathbf{y} \\ \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ \mathbf{y} \geq 0 \end{bmatrix}$$

Weak Duality: Suppose $\mathbf{x}$ is any primal feasible solution and $\mathbf{y}$ is any dual feasible solution, then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Strong Duality: If $\mathbf{x}^*$ is an optimal primal solution, and $\mathbf{y}^*$ is an optimal dual solution, then $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$

Duality: Consequences

- Both *Primal* and *Dual* feasible $\rightarrow$ optimal values are equal

$c^T x^* = b^T y^*$

Duality: Consequences

- Both *Primal* and *Dual* feasible $\rightarrow$ optimal values are equal

must be

- If the *Primal* is infeasible $\rightarrow$ *Dual* unbounded

$$\nexists\, x : \begin{array}{l} Ax \le b \\ x \geq 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} \min\ b^T y \rightarrow -\infty \\ A^T y \geq c \\ y \geq 0 \end{array}$$

must be

- If the *Dual* is infeasible $\rightarrow$ *Primal* unbounded

$$\nexists\, y : \begin{array}{l} A^T y \geq c \\ y \geq 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} \max\ c^T x \rightarrow \infty \\ Ax \le b \\ x \geq 0 \end{array}$$

Complementary Slackness (CS)

- Suppose $\mathbf{x}$ and $\mathbf{y}$ are primal and dual optimal solutions respectively. Then

$$\begin{pmatrix} \max\ c^T x \\ Ax \leq b \\ x \geq 0 \end{pmatrix} \quad \text{(Primal)}$$

$$\left(\mathbf{y}^T \underbrace{(\mathbf{b} - \mathbf{A}\mathbf{x})}_{\text{slack in primal}} = 0, \right) \quad \text{(Primal CS)}$$

$$\left(\mathbf{x}^T \underbrace{(\mathbf{A}^T\mathbf{y} - \mathbf{c})}_{\text{slack in dual}} = 0, \right) \quad \text{(Dual CS)}$$

$$\begin{pmatrix} \min\ b^T y \\ A^T y \geq c \\ y \geq 0 \end{pmatrix} \quad \text{(dual)}$$

$$0 = y^T(b - Ax) = \left[\sum_{i=1}^{m} \underbrace{y_i}_{\geq 0} \underbrace{\left(b_i - \sum_{j=1}^{m} A_{ij} x_j \right)}_{s_i \geq 0} \right]$$

$$s_i > 0 \Rightarrow y_i = 0$$

$$0 = x^T(A^T y - c) = \sum_{j=1}^{n} \underbrace{x_j}_{\geq 0} \underbrace{\left(\sum_{i=1}^{m} A_{ij} y_i - c_j \right)}_{\geq 0 \text{ slack in the } j^{th} \text{ constraint}}$$

Complementary Slackness (CS)

- Suppose $\mathbf{x}$ and $\mathbf{y}$ are primal and dual optimal solutions respectively. Then

$$\begin{pmatrix} \max\ c^T x \\ Ax \le b \\ x \ge 0 \end{pmatrix}$$

(Primal)

$$\left(\mathbf{y}^T \underbrace{(\mathbf{b} - \mathbf{A}\mathbf{x})}_{\text{slack in primal}} = 0,\right) \quad \text{(Primal CS)}$$

$$\left(\mathbf{x}^T \underbrace{(\mathbf{A}^T\mathbf{y} - \mathbf{c})}_{\text{slack in dual}} = 0,\right) \quad \text{(Dual CS)}$$

$$\begin{pmatrix} \min\ b^T y \\ A^T y \ge c \\ y \ge 0 \end{pmatrix}$$

(dual)

$$\boxed{c^T x = b^T y}$$

$$0 \le y^T(b - Ax) = y^T b - y^T A x \underset{\begin{pmatrix} y^T A \ge c^T \\ x \ge 0 \end{pmatrix}}{\le} y^T b - c^T x = 0$$

$$0 \le x^T(A^T y - c) = (Ax)^T y - c^T x \underset{\begin{pmatrix} Ax \le b \\ y \ge 0 \end{pmatrix}}{\le} b^T y - c^T x = 0$$

Complementary Slackness (CS)

- Suppose $\mathbf{x}$ and $\mathbf{y}$ are primal and dual optimal solutions respectively. Then

$$\mathbf{y}^T \underbrace{(\mathbf{b} - \mathbf{A}\mathbf{x})}_{\text{slack in primal}} = 0, \quad \text{(Primal CS)}$$

$$\mathbf{x}^T \underbrace{(\mathbf{A}^T\mathbf{y} - \mathbf{c})}_{\text{slack in dual}} = 0, \quad \text{(Dual CS)}$$

- In other words,
  - If $y_i > 0$, then $\mathbf{x}$ satisfies constraint i with equality ("tight")
  - If $x_j > 0$, then $\mathbf{y}$ satisfies constraint j of the dual with equality

Complementary Slackness

(Primal)

$$\max \mathbf{c}^T\mathbf{x}$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq 0$$

(Dual)

$$\min \mathbf{b}^T\mathbf{y}$$
$$\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$$
$$\mathbf{y} \geq 0$$

Suppose $\mathbf{x}$ and $\mathbf{y}$ are primal and dual feasible respectively and satisfy the CS conditions

$Ax \leq b$ $\qquad A^Ty \geq c$

$x \geq 0$ $\qquad y \geq 0$

$y^T(b - Ax) = 0$

$x^T(A^Ty - c) = 0$

$c^Tx \leq b^Ty$

To prove : $c^Tx \geq b^Ty$

$0 = x^T(A^Ty - c) = (Ax)^Ty - c^Tx = b^Ty - c^Tx$

$(Ax - b)^Ty = 0$

Online Resource Allocation: Using Duality

- m resources with fixed capacities $b_1, \dots, b_m$
- Demand requests arrive sequentially with requirements $a_i, \; i = 1, \dots, m$ and profit p
- Need to make accept or reject decisions without knowing future demands

Online Resource Allocation: Using Duality

- m resources with fixed capacities $b_1, \ldots, b_m$

- Demand requests arrive sequentially with requirements $a_i, \; i = 1, \ldots, m$ and profit p

- Need to make accept or reject decisions without knowing future demands

- Historical data of N demands:

$$(\hat{p}_j, (\hat{a}_{ij})_{i=1,\ldots,m}) \text{ for } j = 1, \ldots, N$$

Online Resource Allocation

Compute shadow prices of resources

$$\max \sum_{j=1}^{N} \hat{p}_j x_j$$

$$\lambda_i : \sum_{j=1}^{N} \hat{a}_{ij} x_j \le h_i, \ \forall i$$

$$\mu_j : \ x_j \le 1, \ \forall j$$

$$\mathbf{x} \ge 0$$

x_j : accept/reject

$0 \le x_j \le 1$

$$\left[\min \sum_{i=1}^{m} \lambda_i h_i + \sum_{j=1}^{N} \mu_j \quad \left(\sum_{i=1}^{m} \lambda_i \hat{a}_{ij} + \mu_j \ge \hat{p}_j \right) \quad \lambda_i, \mu_j \ge 0 \right]$$

$$\rightarrow \left[\mu_j \ge \underbrace{\hat{p}_j - \sum_{i=1}^{m} \lambda_i \hat{a}_{ij}}_{\ge 0} \right]$$

$\mu_j \ge 0$

not tight

$\Downarrow$

$x_j = 0$

$\underset{\smile}{p}, \ \underset{\smile}{a_i} \quad i = 1 \ldots m$

$$\underbrace{p - \sum_{i=1}^{m} \lambda_i a_i}_{} \quad \ge 0 \quad \text{ACCEPT}$$

$$< 0 \quad \text{REJECT}$$