Th8: Stochastic processes and SDE's

In probability theory and related fields, a stochastic or random process is a mathematical object usually defined as a sequence of random variables, where the index of the sequence has the interpretation of time. Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner.

A stochastic or random process can be defined as a collection of random variables that is indexed by some mathematical set, meaning that each random variable of the stochastic process is uniquely associated with an element in the set. The set used to index the random variables is called the index set. Each random variable in the collection takes values from the same mathematical space known as the state space. An increment is the amount that a stochastic process changes between two index values, often interpreted as two points in time. A stochastic process can have many outcomes, due to its randomness, and a single outcome of a stochastic process is called a sample function or realization.

A stochastic process can be classified in different ways, for example, by its state space, its index set, or the dependence among the random variables.

When interpreted as time, if the index set of a stochastic process has a finite or countable number of elements, the stochastic process is said to be in discrete time. If the index set is some interval of the real line, then time is said to be continuous.
If the index set is the integers, or some subset of them, then the stochastic process can also be called a random sequence.

the mean value of a stochastic process and its “covariance” are defined by

$$m_V(t) \equiv E[V(t)] \quad \text{and} \quad C_V(t_1, t_2) \equiv E[(V(t_1) - m_V(t_1))(V(t_2) - m_V(t_2))]$$

The stochastic processes have the following properties:

-Stationarity: A large number of stochastic processes have the property that their average statistical properties are independent of where they are formed along the time axis. Such stochastic processes are said to have various types of stationary properties. For example, the mean $m_V(t) = E[V(t)]$ can be independent of t. When in addition the autocovariance function $C_V(t_1, t_2) = E[(V(t_1) - m_V)(V(t_2) - m_V)]$ depends only on the time difference, $\tau = t_2 - t_1$, a stochastic process is called weakly stationary. In this case, the autocovariance function is usually written as $C_V(\tau) = E[(V(t) - m_V)(V(t+\tau) - m_V)]$.

-Ergodicity: Another important property of certain stochastic processes is that averages over the ensemble of values taken at a fixed time can be replaced by an average over time on any sample function $V(\mu_0, t)$ from the stochastic ensemble (with μ_0 fixed).

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process.

In general an SDE is given as $dX(t,\omega) = f(t, X(t,\omega))dt + g(t, X(t,\omega))dW(t,\omega)$ where ω denotes that X = X(t, ω) is a random variable and possesses the initial condition X(0, ω) = X0 with probability one.

Bibliography
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