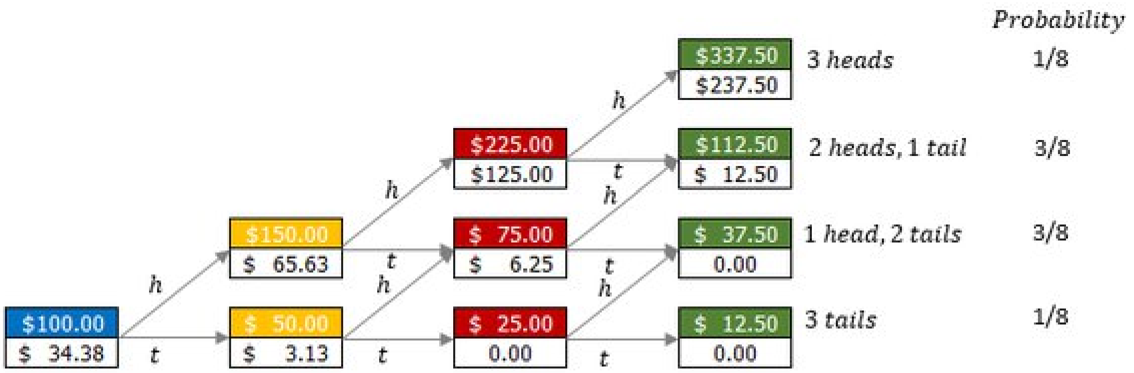
Gambler's Ruin Problem

The Gambler’s Ruin Problem in its most basic form consists of two gamblers A and B who are playing a probabilistic game multiple times against each other. Every time the game is played, there is a probability p ($0 < p < 1$) that gambler A will win against gambler B. Likewise, using basic probability axioms, the probability that gambler B will win is 1 - p. Each gambler also has an initial wealth that limits how much they can bet. The total combined wealth is denoted by k and gambler A has an initial wealth denoted by i, which implies that gambler B has an initial wealth of k - i. Wealth is required to be positive. The last condition we apply to this problem is that both gamblers will play indefinitely until one of them has lost all their initial wealth and thus cannot play anymore.

Imagine that gambler A’s initial wealth is an integer dollar amount and that each game is played for one dollar. That means that gambler A will have to play at least i games for their wealth to drop to zero. The probability that they win one dollar in each game is p, which will be equal to 1/2 if the game is fair for both gamblers.
The main focus of the analysis is to determine the probability that gambler A will end up with a wealth of k dollars instead of 0 dollars. Regardless of the outcome, one of the gamblers will end up in financial ruin, hence the name Gambler’s Ruin.



Now we want to determine the probability a_i that gambler A will end up with k dollars given that they started out with i dollars. We consider that all games are identical and independent.

Mathematically, we can think of each sequence of games that ends in gambler A having j dollars where j = 0, … , k.

Using event notation, we can denote A_1 as the event that gambler A wins game 1. Similarly, B_1 is the event that gambler B wins game 1.
The event W occurs when gambler A ends up with k dollars before they end up with 0 dollars. The probability that this event occurs can be derived using properties of conditional probability.

$$P(W) = P(A_1)P(W \mid A_1) + P(B_1)P(W \mid B_1) = pP(W \mid A_1) + (1-p)P(W \mid B_1)$$

Given that gambler A starts with i dollars, the probability that they win is $P(W)=a_i$.

If gambler A wins one dollar in the first game, then their wealth becomes i+1. If they lost one dollar in the first game, their fortune becomes i - 1. The probability that they win the entire sequence will then depend on whether they won the first game. When we apply this logic to the previous equation, we now have an expression of the probability of winning the entire sequence of games that depends on the probability of winning one dollar each game and the conditional probabilities of winning the sequence given the gambler's wealth.

$$a_i = pa_{i+1} + (1-p)a_{i-1}$$

Given i=1, … , k - 1, we can plug in all possible values of i into the equation above to obtain k - 1 equations that determine the probability of winning based on adjacent values of i. We can use elementary algebra to aggregate these equations into a standardized format that can be simplified into a single formula. This formula specifies a fundamental relation between the probability of winning each game p, the total initial wealth of both gamblers k, and the probability of winning given a wealth of one dollar a_1. Once we have determined a_1, we can iteratively loop through all k - 1 to derive the probability a_i for all possible values of i.

$$1 - a_1 = a_i \sum_{i=1}^{k-1} \left(\frac{1-p}{p} \right)^i$$

We will now consider two possibilities: a fair game and an unfair game, which depends on the value of p that we plug into the equation above. In a fair game, where p=1/2, the base of the exponent in the right side of the equation can be simplified to (1 - p)/p=1. We can then simplify the entire equation as follows: $1 - a_1=(k - 1)a_1$, which can be rearranged to $a_1=1/k$. If we iterate over all previous equations that determine the probabilities of winning for different values of i, we arrive at a general solution for fair games.

$$a_i = \frac{i}{k}, \forall i = 1, \cdots, k-1$$

The equation above is a remarkable result: given that the game is fair, the probability that gambler A will end up with k dollars before they end up with zero dollars is equal to their initial wealth i divided by the total wealth of both gamblers k.

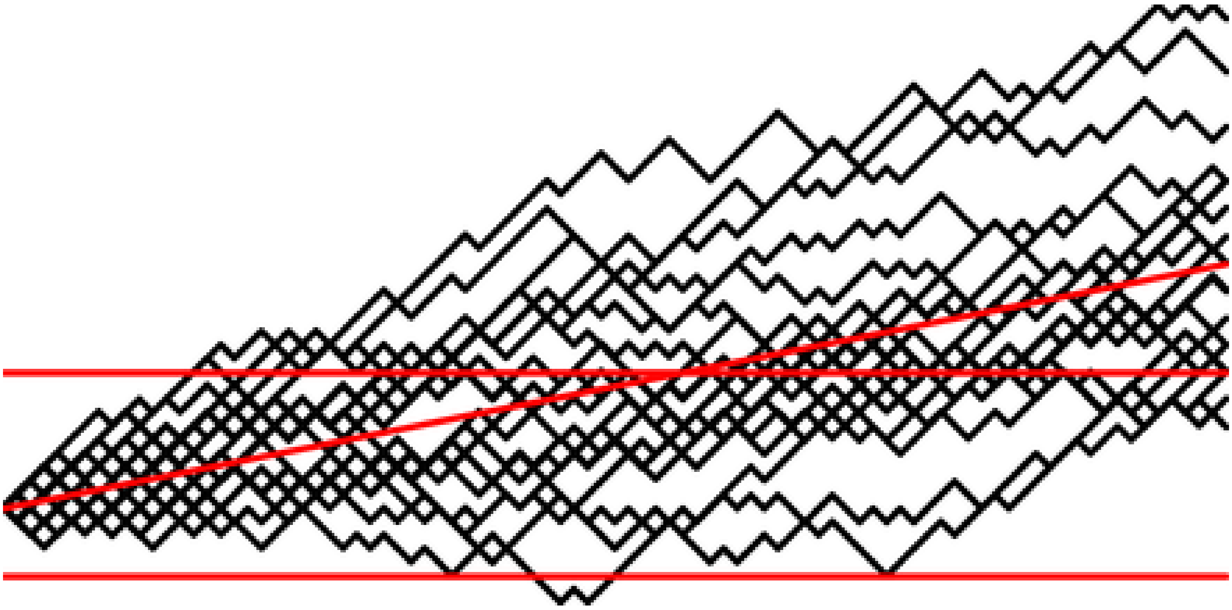
If p is not equal to 1/2 the game is unfair, as one of the two gamblers has a systematic advantage. We can similarly derive a general solution which depends on the value of p and the wealth parameters.

$$a_i = \frac{\left(\frac{1-p}{p}\right)^i - 1}{\left(\frac{1-p}{p}\right)^k - 1}, \forall i = 1, \cdots, k-1$$

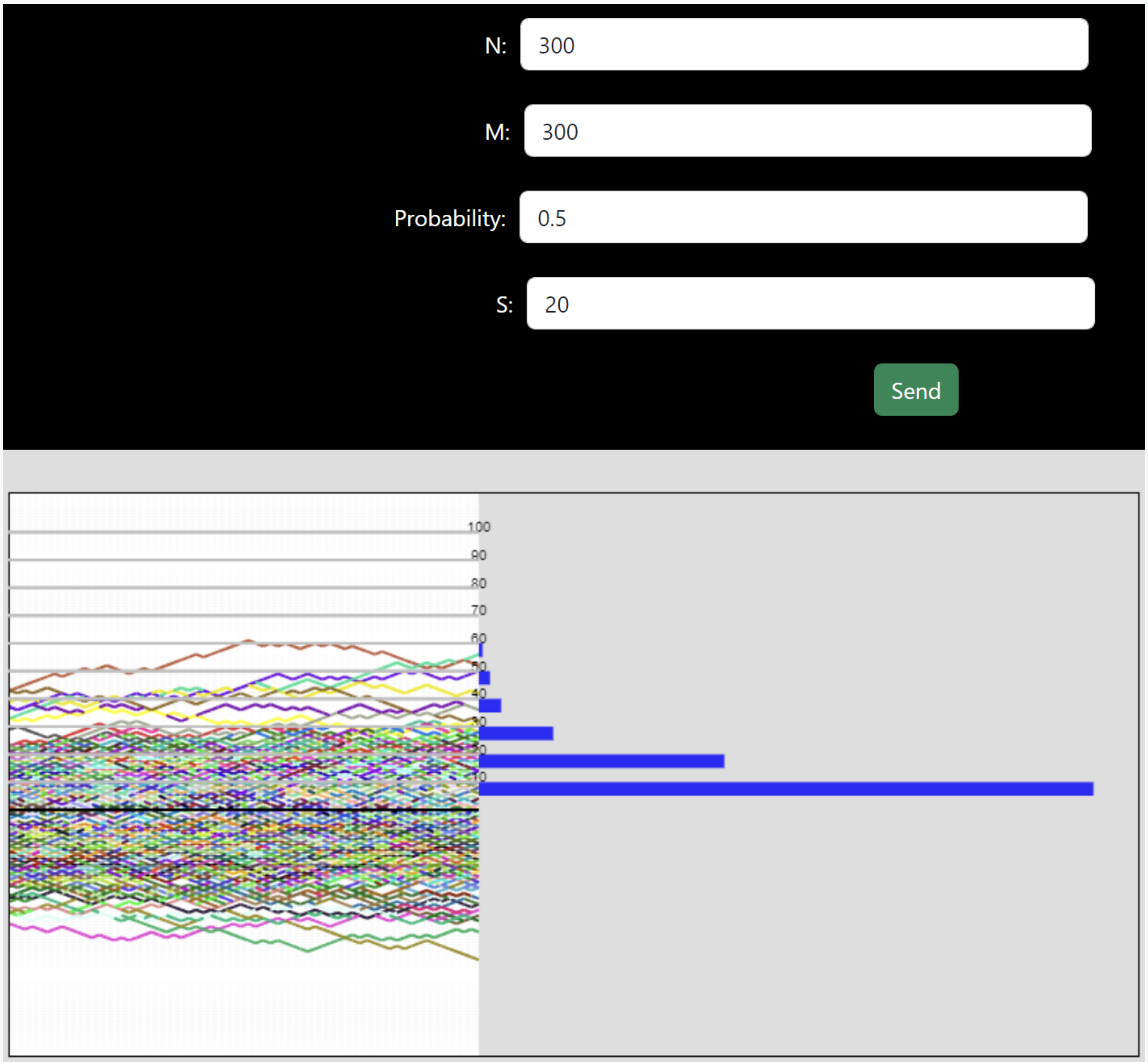
Another example of the Gambler’s Ruin can be seen In the event of an unfair coin, where player one wins each toss with probability p, and player two wins with probability q = 1 − p,

The following graph represents the previous problem for player 1 with P=0.6 starting with 5 pennies and player 2 with10.

The probability of this stochastic process hitting level 15 prior to 0 is $\frac{59049}{67849} \approx 0.8703$ and the sloped line depicts the expected value around which most of the probability mass is clustered. The variance of a Bernoulli process i.e. a binomial distribution is $np(1-p) = npq$ and proportion $\frac{pq}{n}$.

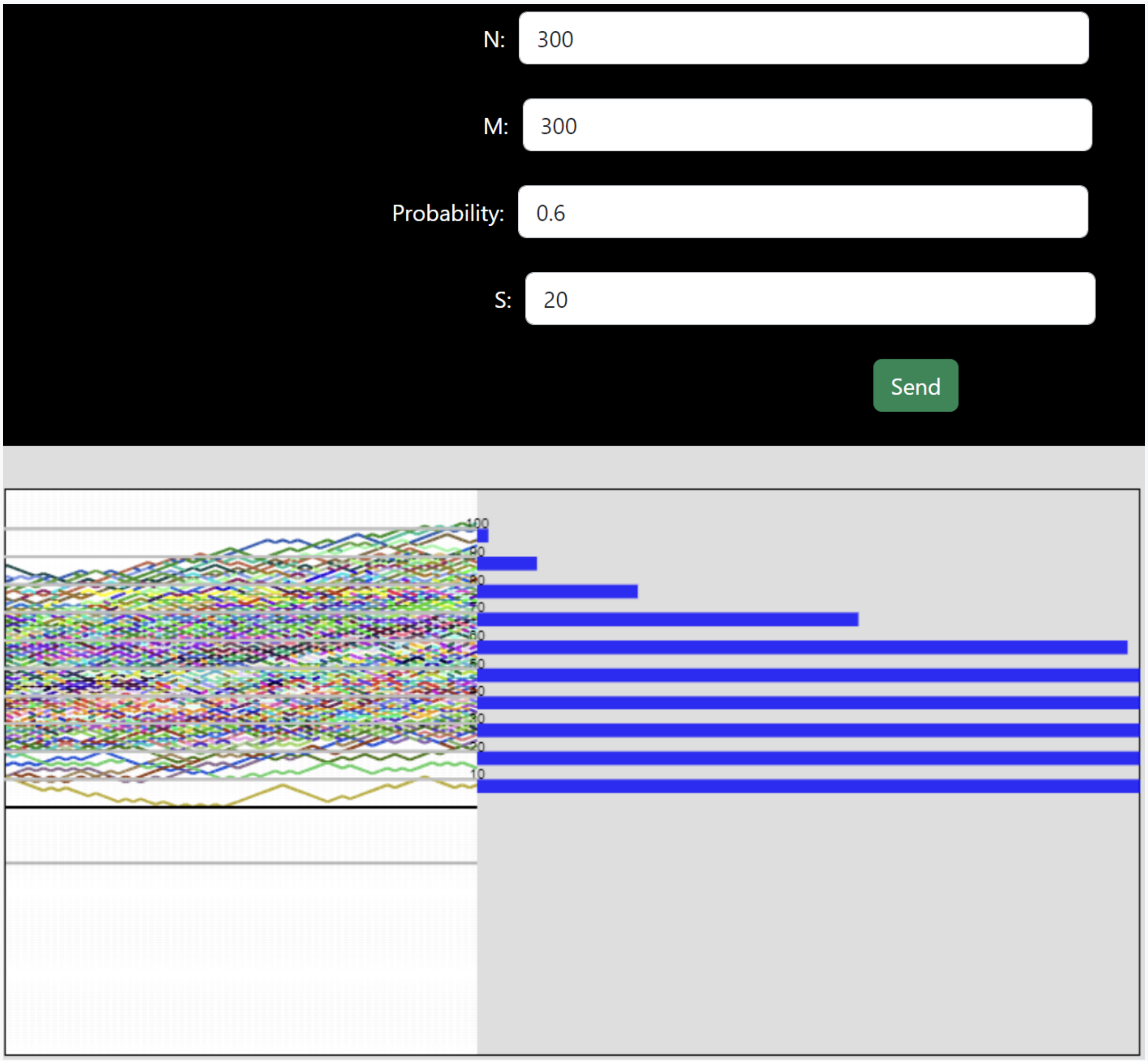


In homework 6 there are two thresholds, one called p and the other s. The first one is located above the x axes and the second one is located below.
If the system reaches p before reaching s, means that the system is unsecure, so it corresponds to the gambler that loses all money.



In the graph we can see that, if we set s=20 and the probability of success of the attack equal to 0.5 (so it is fair), we can see that there are a lot of processes that do not cross the thresholds but instead goes towards S, so there is almost an equal number of unsecure systems and secure ones.

If instead we increase the probability of success of the attack we can see that the number of secure processes decrease.



Instead if we decrease the probability of success of the attack we can see that the number of secure systems increases.

So thanks to the examples shown we can see that if the probability is near 0.5 the game will go on more than if the probability is unfair.

Bibliography

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-https://en.wikipedia.org/wiki/Gambler%27s_ruin