Th10: The Wiener process and the GBM, Derivations and simulations

The Wiener process is a stochastic process that was first used to describe the random, or "Brownian," motion of particles in a fluid. The Wiener process $W(t)$ is defined for $t \geq 0$ and has the following properties:

1) $W(0) = 0$ with probability 1.
2) For $0 \leq s < t$ the random variable $W(t) - W(s)$, also called the increment of W between s and t, is normally distributed with mean zero and variance $t - s$.
3) For $0 \leq s < t < u < v$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

By setting $s = 0$ in property 2 and using property 1, we immediately see that $W(t)$ is a Gaussian random variable with zero mean and variance t.
The Wiener process has a spectral representation as a sine series whose coefficients are independent $N(0, 1)$ random variables. This representation can be obtained using the Karhunen–Loève theorem.

The Wiener process can be constructed as the scaling limit of a random walk, or other discrete-time stochastic processes with stationary independent increments. This is known as Donsker's theorem.
Like the random walk, the Wiener process is recurrent in one or two dimensions (meaning that it returns almost surely to any fixed neighborhood of the origin infinitely often) whereas it is not recurrent in dimensions three and higher.
Unlike the random walk, it is scale invariant.
The Wiener process exhibits closure properties, meaning if you apply certain operations, you get another Wiener process.
Properties such as independence and stationarity of the increments are so-called distributional properties.
A sample path of a Wiener process is continuous almost everywhere. (the only region where the property does not hold is mathematically negligible.)
The sample paths are nowhere differentiable.
The standard Wiener process process has the Markov property, making it an example of a Markov process.

The Wiener process plays a main role in the theory of probability, the Wiener process is considered the most important and studied stochastic process. It has connections to other stochastic processes and is central in stochastic calculus and martingales. Its discovery led to the development of a family of Markov processes known as diffusion processes.

The Wiener process also arises as the mathematical limit of other stochastic processes such as random walks, which is the subject of Donsker's theorem or invariance principle, also known as the functional central limit theorem.

As already said, the Wiener process is a member of some important families of stochastic processes, including Markov processes, Lévy processes, and Gaussian processes.

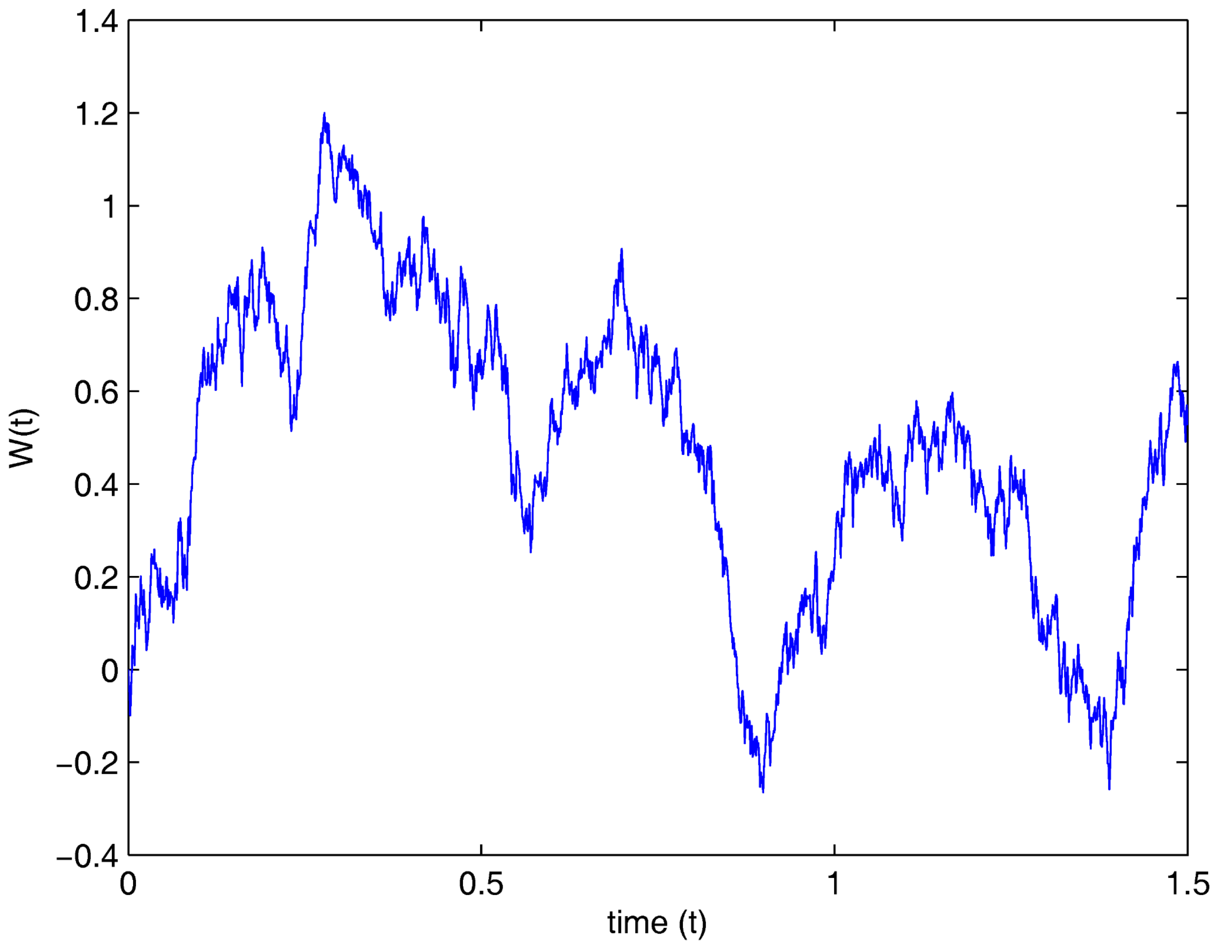
The Wiener process can be derived in the following ways:

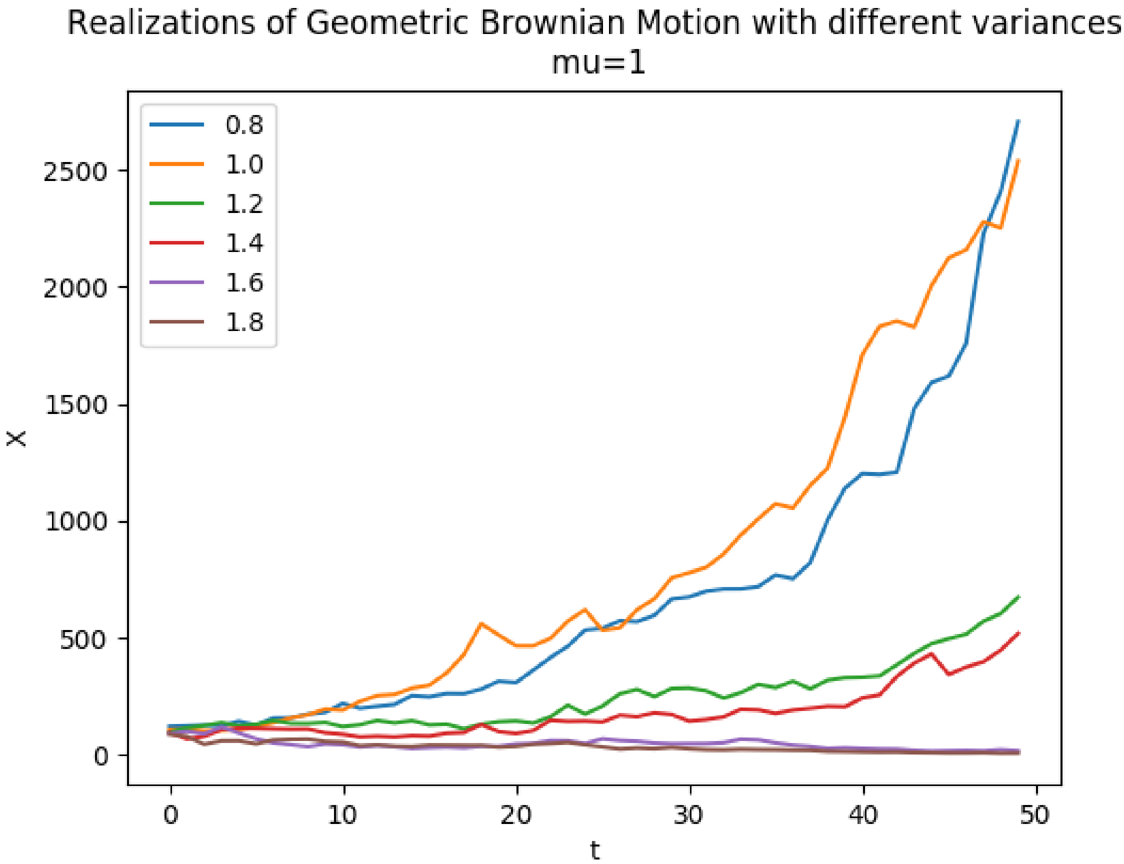
1) Random Walk Derivation: One of the intuitive ways to understand the Wiener process is through a random walk. Consider a particle that moves randomly in one dimension, taking steps of random length in random directions. As the number of steps becomes

very large and the step size becomes very small, the limit of this process is the Wiener process. Mathematically, this can be formulated as the limit of a sum of independent and identically distributed (i.i.d.) random variables with mean zero and finite variance.

2) Stochastic Differential Equation (SDE) Derivation: The Wiener process can also be defined using stochastic calculus and stochastic differential equations. The most common SDE that describes the Wiener process is:
   $dW(t)=(dt)^{1/2} \cdot Z(t)$, where
   $W(t)$ is the Wiener process at time t, dt is an infinitesimally small time step, and $Z(t)$ is a standard normal random variable.
   This formulation captures the continuous and stochastic nature of the Wiener process. The term $(dt)^{1/2} \cdot Z(t)$ represents the random increment at each infinitesimal time step.

One of the possible simulations of the Wiener process are Random Walks.
The Wiener process is essentially a continuous version of a random walk. Simulating a random walk using the Wiener process is a fundamental exercise in probability and statistics. It can be applied in various contexts, such as modeling the movement of particles in physics or the behavior of individuals in a population.





The images above show a representation of a Wiener process.

GBM stands for Geometric Brownian Motion. It is a mathematical model used to describe the stochastic behavior of certain processes over time, and it is particularly commonly applied in finance. GBM is an extension of the standard Brownian motion or Wiener process and is used to model the continuous-time evolution of various variables, such as stock prices.

The key features of a Geometric Brownian Motion are as follows:

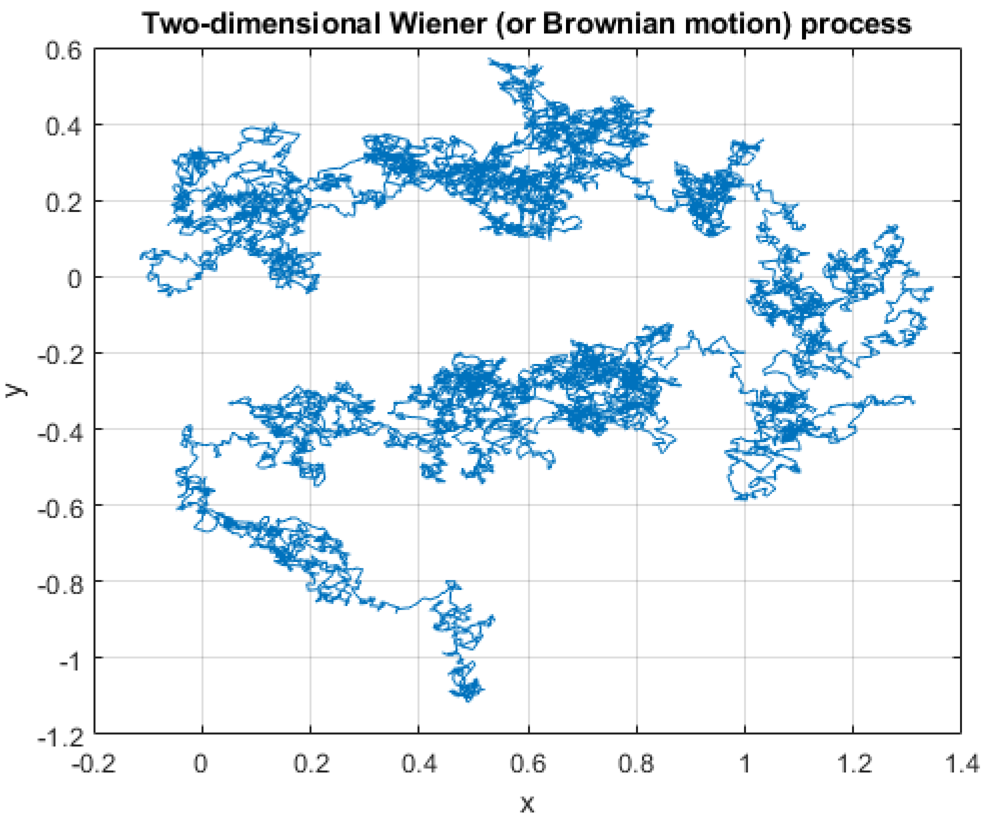
1) Continuous-Time Process: GBM is a continuous-time process, meaning that it is defined for all points in time within a given interval.

2) Stochasticity: Like the standard Brownian motion, GBM incorporates a stochastic (random) component. This randomness reflects the inherent uncertainty and unpredictability in the modeled process.

3) Geometric Drift and Diffusion: GBM is characterized by both a drift term and a diffusion term. The drift term represents the average rate of growth (or decline) of the process, while the diffusion term accounts for the random fluctuations around this average.

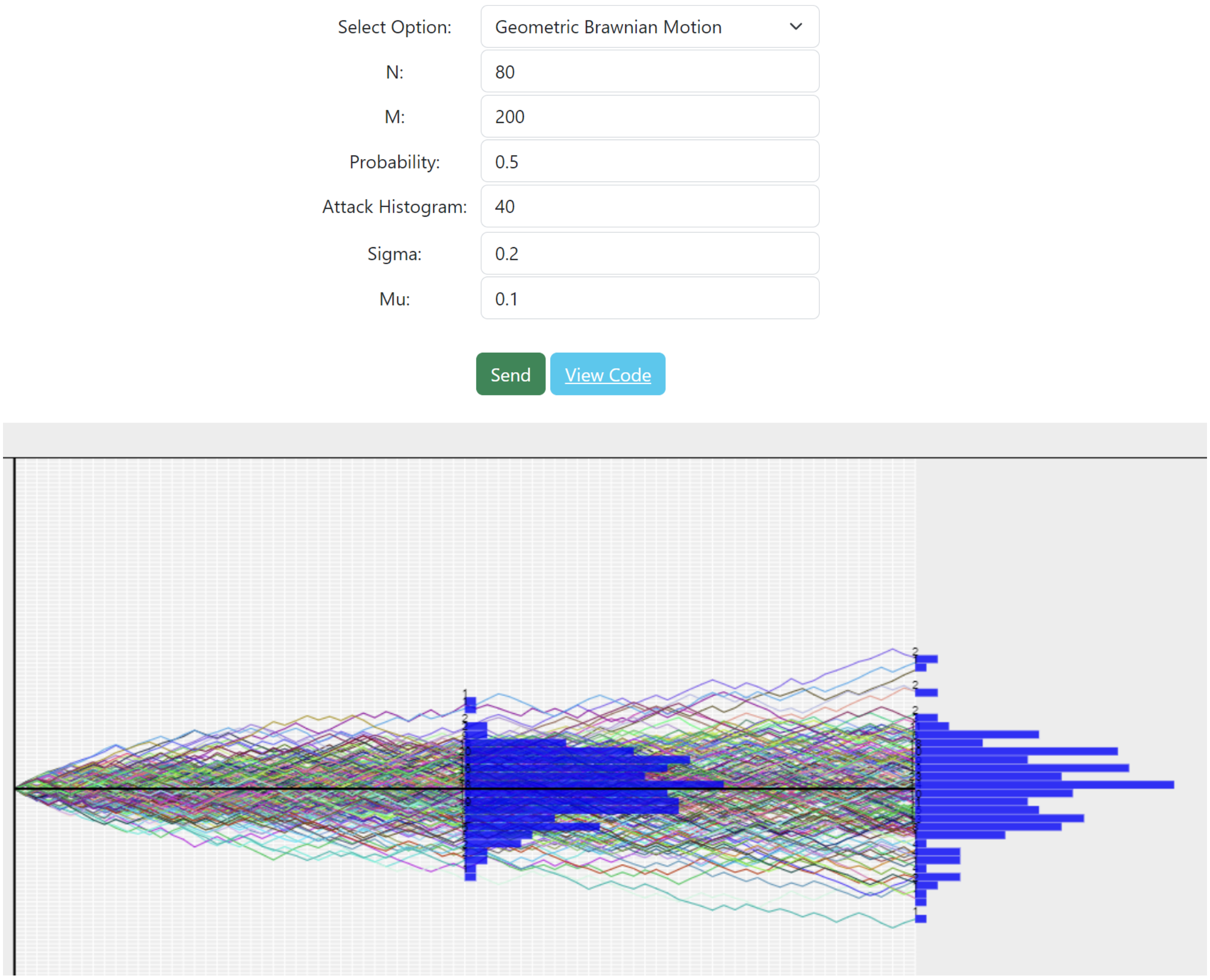
4) Log-Normal Distribution: The logarithm of a variable following a GBM is normally distributed. This log-normal property is particularly useful in finance, where asset prices are often modeled using GBM.

The stochastic differential equation (SDE) that describes a geometric Brownian motion is given by: $dS(t)=\mu S(t)dt+\sigma S(t)dW(t)$ where $S(t)$ is the value of the process at time t, μ is the average rate of return (drift), σ is the volatility (diffusion) and $dW(t)$ is a Wiener process increment.
The solution to this SDE is the GBM process, and it can be used to simulate the evolution of a variable over time.



We can say that the Brownian motion is the generation of random walk with jump according to the normal distribution [-∞, ∞].
We can also say that the Brownian motion is the scaling limit of the random walk
$dx = (t)^{1/2} Z$ where dx is the limit of the random walk as $n \to \infty$ and Z is a normal distribution.

The following graph shows the geometric brownian motion in our simulator.



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