Th10: The Wiener process and the GBM, Derivations and simulations

The Wiener process is a real-valued continuous-time stochastic process that was first used to describe the random, or "Brownian," motion of particles in a fluid. It is a function W(t) that takes a time t as an input and gives a real number as an output. The Wiener process has these characteristics:

1) W(0) is always zero.
2) If we pick any two times s and t, where s is less than t, then the difference W(t) - W(s) has a normal distribution with mean zero and standard deviation equal to the square $\sqrt{t-s}$. We can write this as W(t) − W(s) ~ N(0, t − s), where N(0, t − s) means a normal distribution with mean 0 and variance t − s.
3) If we pick any four times s < t and u < v, then the differences W(t) - W(s) and W(v) - W(u) do not affect each other. This is true for any pair of time intervals that do not overlap.
4) W(t) does not have any gaps or jumps as t changes. It is continuous with probability 1, which means that it is almost certain.
5) W(t) is a martingale, which means that its expected value at any future time is equal to its current value .

By setting s = 0 in property 2 and using property 1, we immediately see that W(t) is a Gaussian random variable with zero mean and variance t.

Another way to describe the Wiener process is by using the Lévy characterisation, which says that the Wiener process is a continuous martingale with W(0)=0 and [W(t), W(t)] = t a.s., where a.s. means almost surely. The Wiener process also has a spectral representation as a sum of sine functions with random coefficients that are independent N(0, 1) random variables. This representation can be derived using the Karhunen–Loève theorem.

Let X_1,X_2,… be i.i.d. random variables with mean 0 and variance 1. For each n, define a continuous time stochastic process $W_n(t)$ by

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} X_k + \frac{nt-\lfloor nt \rfloor}{\sqrt{n}} X_{\lfloor nt \rfloor + 1}$$

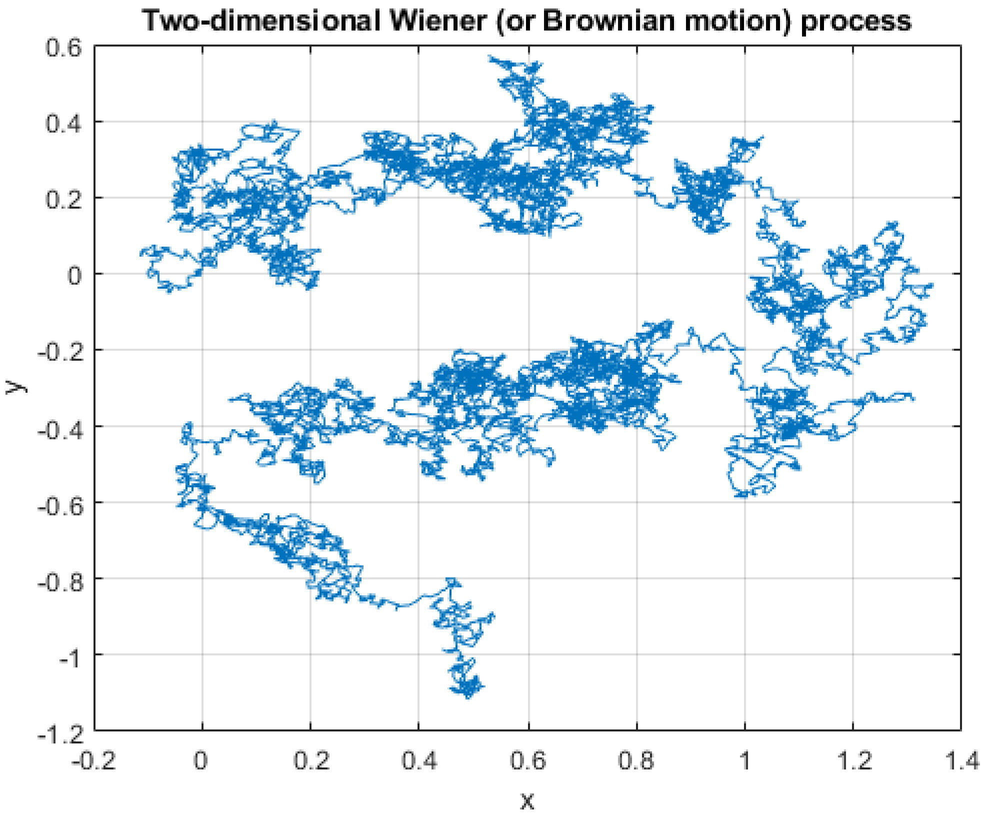
where ⌊x⌋ denotes the largest integer less than or equal to x. This is a random step function that interpolates the partial sums of the X_k's. Increments of Wn are independent because the X_k's are independent. For large n, $W_n(t)-W_n(s)$ is close to N(0,t−s) by the central limit theorem.

Donsker's theorem asserts that as n→∞, W_n converges in distribution to a Wiener process, which is a continuous stochastic process W(t) that satisfies the following properties:

- W(0)=0 with probability 1.
- For 0 ≤ s < t, the increment W(t) − W(s) is normally distributed with mean 0 and variance t−s.
- For 0 ≤ s < t < u < v, the increments W(t)−W(s) and W(v)−W(u) are independent.
- W(t) is almost surely continuous in t.

Donsker's theorem explains the ubiquity of Brownian motion, which is another name for the Wiener process, as a model for various phenomena in physics, biology, finance, and other fields. It shows that Brownian motion is the scaling limit of any random walk with finite variance .

Like the random walk, the Wiener process depends on the dimension of the space, which is the number of independent directions that the particle can move in. For example, a line has one dimension, a plane has two dimensions, and a space has three dimensions.
One property that both models share is recurrence in one or two dimensions. This means that the particle will come back to where it started, or close to it, infinitely many times, with probability one. However, this is not true in three or higher dimensions. In higher dimensions, the particle can wander away forever and never return to its starting point, with probability one. This is called transience.
Another property that only the Wiener process has is scale invariance. This means that the Wiener process looks the same when we zoom in or out, or when we speed up or slow down the time. The random walk does not have this property, because it depends on the size and frequency of the steps, which change when we change the scale. Scale invariance makes the Wiener process more flexible and universal than the random walk.



The Wiener process has some important properties that make it different from other stochastic processes. Here are some of them:

- The Wiener process is closed under certain operations, meaning that if we apply these operations to a Wiener process, we get another Wiener process. For example, if we add a constant to a Wiener process, or multiply it by a constant, or take its absolute value, or reverse its time, we get another Wiener process
- The Wiener process has independent and stationary increments, meaning that the changes in the position of the particle over different time intervals are independent of

each other and have the same distribution. For example, if we look at the difference between the position of the particle at time t and at time s, where s is smaller than t, this difference is independent of the position of the particle at any time before s, and has the same distribution as the difference between the position of the particle at time t-s and at time 0

- The Wiener process has Gaussian increments, meaning that the changes in the position of the particle over different time intervals follow a normal distribution. For example, if we look at the difference between the position of the particle at time t and at time s, where s is smaller than t, this difference has a normal distribution with mean 0 and variance t-s
- The Wiener process is almost surely continuous, meaning that the position of the particle changes smoothly over time, with probability one. This means that there are no sudden jumps or breaks in the motion of the particle, except for a set of times that has zero probability
- The Wiener process is nowhere differentiable, meaning that the position of the particle does not have a well-defined slope or direction at any time, with probability one. This means that the motion of the particle is very erratic and unpredictable, and cannot be approximated by a straight line, except for a set of times that has zero probability
- The Wiener process has the Markov property, meaning that the future position of the particle depends only on the current position, and not on the past positions. For example, if we want to predict the position of the particle at time t, given that we know the position of the particle at time s, where s is smaller than t, we do not need to know the position of the particle at any time before s.

The Wiener process plays a main role in the theory of probability, the Wiener process is considered the most important and studied stochastic process. It has connections to other stochastic processes and is central in stochastic calculus and martingales. Its discovery led to the development of a family of Markov processes known as diffusion processes.

As already said, the Wiener process is a member of some important families of stochastic processes, including Markov processes, Lévy processes, and Gaussian processes.

The Wiener process can be derived in the following ways:

1) Random Walk Derivation: One of the ways to understand the Wiener process is through a random walk. A random walk is a discrete-time stochastic process that models the motion of a particle that takes steps of random size and direction at regular intervals. For example, if we toss a coin and move one unit to the right if it is heads, and one unit to the left if it is tails, we get a random walk on a line. If we repeat this process many times, we get a sequence of positions that represent the random walk. As the number of steps becomes very large and the step size becomes very small, the random walk becomes more and more continuous and smooth, and it converges to the Wiener process in the limit. Mathematically, this can be formulated as the limit of a sum of independent and identically distributed (i.i.d.) random variables with mean zero and finite variance. This is known as Donsker's theorem or the invariance principle .

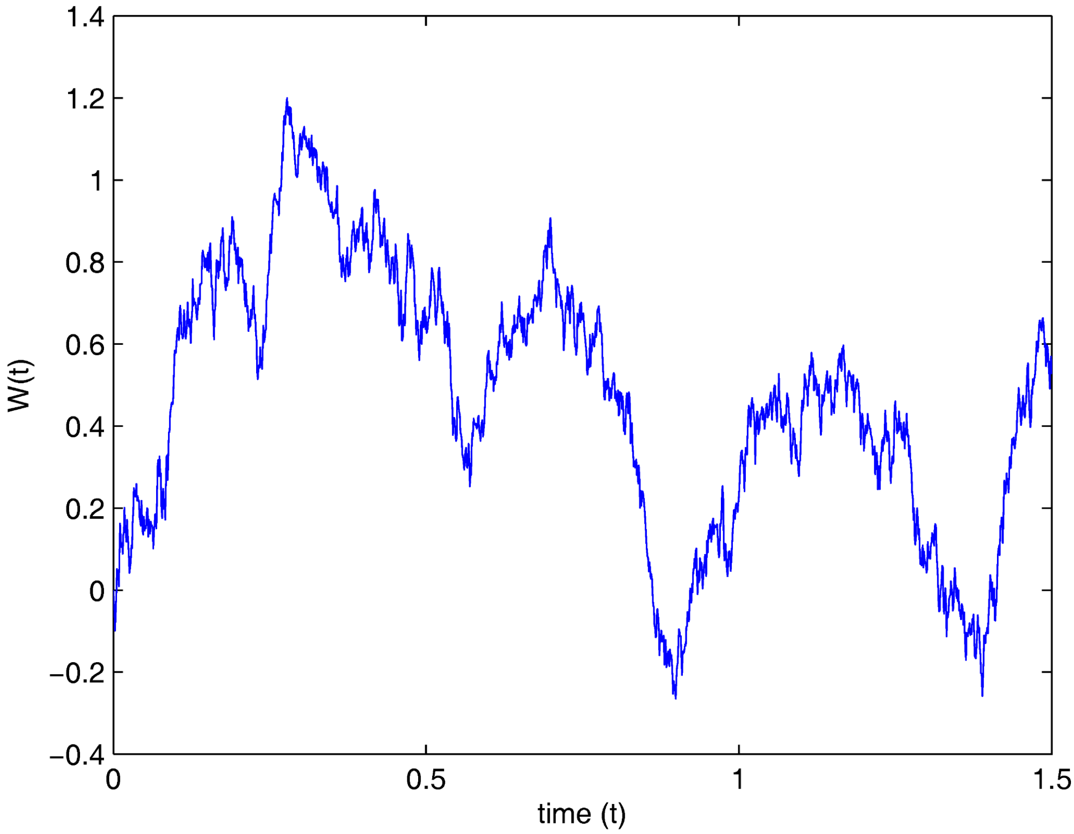
2) Stochastic Differential Equation (SDE) Derivation: The Wiener process can also be defined using stochastic calculus and stochastic differential equations. A stochastic differential equation is an equation that relates the change of a stochastic process to its current value and some random noise. For example, the equation

$$dX(t) = aX(t)dt + bX(t)dW(t)$$

describes the change of a stochastic process X(t) that depends on its current value, a constant factor a, another constant factor b, and the Wiener process W(t). The Wiener process itself can be defined by the simplest SDE, which is:

$$dW(t) = \sqrt{dt} \cdot Z(t),$$

where W(t) is the Wiener process at time t, dt is an infinitesimally small time step, and Z(t) is a standard normal random variable.
This formulation captures the continuous and stochastic nature of the Wiener process.
The term $\sqrt{dt} \cdot Z(t)$ represents the random increment at each infinitesimal time step, which has a normal distribution with mean zero and variance dt.



The image above shows a Wiener process.

GBM stands for Geometric Brownian Motion. It is a mathematical model used to describe the stochastic behaviour of certain processes over time, and it is particularly commonly applied in finance. GBM is an extension of the standard Brownian motion or Wiener process, which is a continuous-time random process that has independent and normally distributed increments. GBM is used to model the continuous-time evolution of various variables, such as stock prices, exchange rates, or commodity prices. The main difference between GBM and standard Brownian motion is that GBM incorporates a drift term and a volatility term, which account for the deterministic and random components of the process, respectively. The mathematical expression for GBM is given by the following stochastic differential equation:

$$ds(t) = \mu S(t)dt + \sigma S(t)dB(t)$$

where S(t) is the value of the variable at time t, μ is the drift rate, σ is the volatility, and B(t) is a standard Brownian motion.

The solution of this equation is:

$$S(t) = S(0)e^{(\mu-\frac{\sigma^2}{2})t+\sigma B(t)}$$

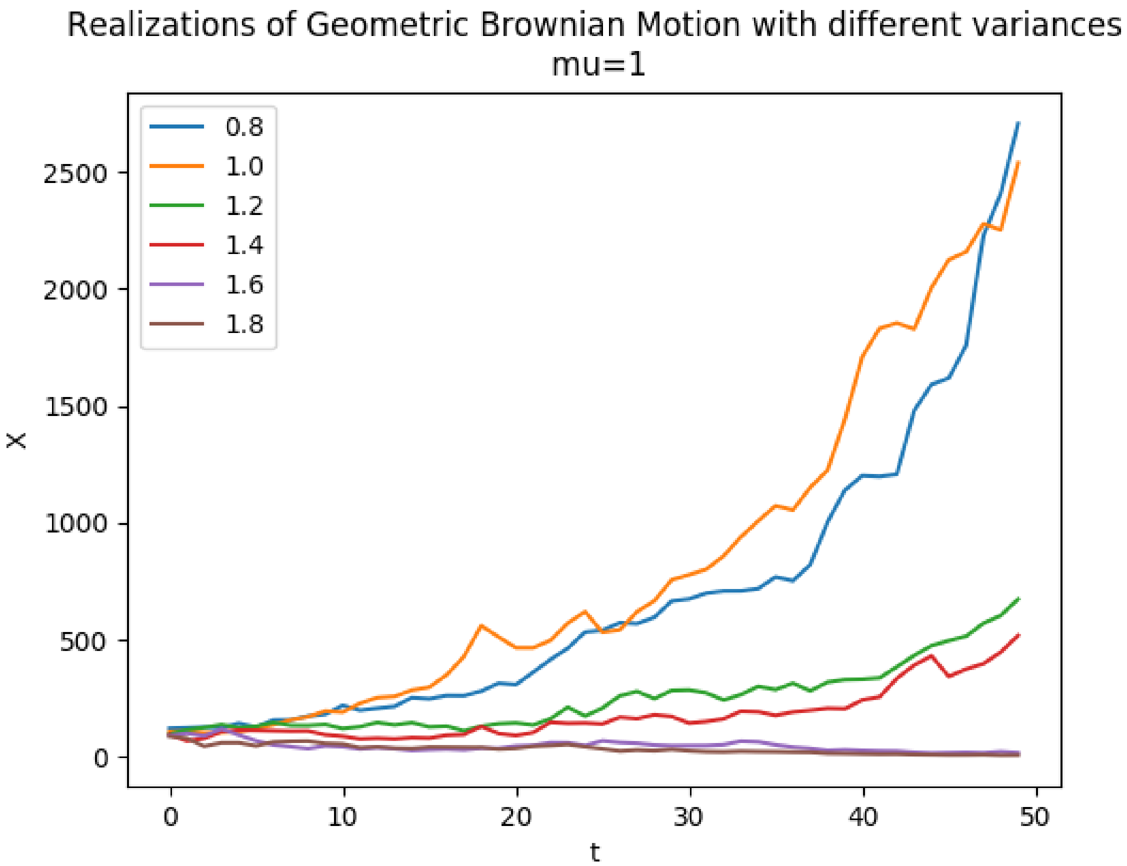
where S(0) is the initial value of the variable, it can be used to simulate the evolution of a variable over time.

This solution shows that S(t) follows a log-normal distribution, with mean and variance given by:

$$E[S(t)] = S(0)e^{\mu t}$$

$$Var[S(t)] = S(0)^2e^{2\mu t}(e^{\sigma^2 t} - 1)$$

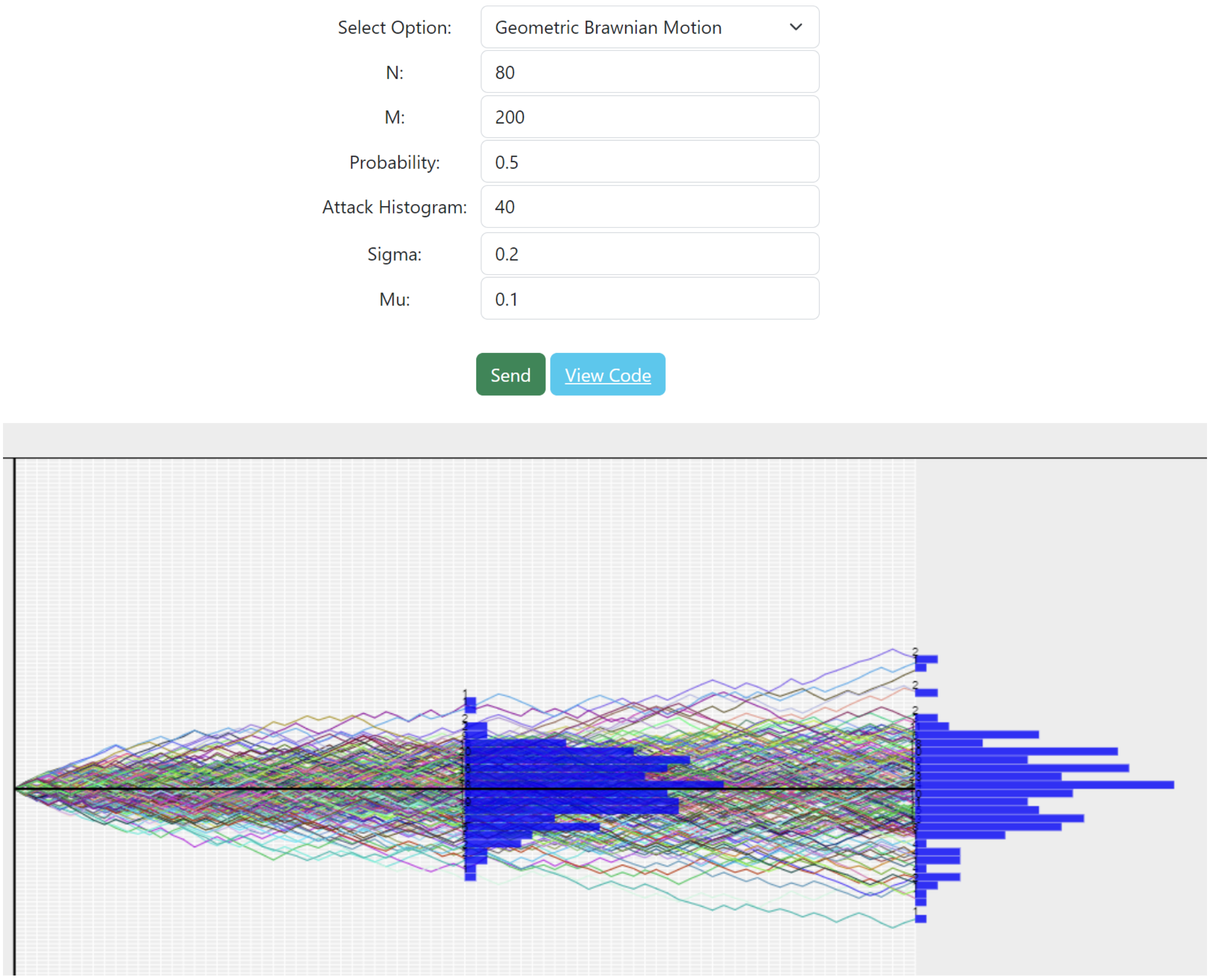
GBM is widely used in financial modelling because it can capture some of the essential features of real-world phenomena, such as the unpredictability of future values, the asymmetry of returns, and the possibility of extreme events. However, GBM also has some limitations and assumptions that may not hold in reality, such as the constant parameters, the geometric scaling property, and the independence of increments.



The key features of a Geometric Brownian Motion are as follows:

1) Continuous-Time Process: GBM is a continuous-time process, meaning that it is defined for all points in time within a given interval.
2) Stochasticity: Like the standard Brownian motion, GBM incorporates a stochastic (random) component. This randomness reflects the inherent uncertainty and unpredictability in the modelled process.
3) Geometric Drift and Diffusion: GBM is characterised by both a drift term and a diffusion term. The drift term represents the average rate of growth (or decline) of the process, while the diffusion term accounts for the random fluctuations around this average.
4) Log-Normal Distribution: The logarithm of a variable following a GBM is normally distributed. This log-normal property is particularly useful in finance, where asset prices are often modelled using GBM.
5) GBM is a Markov process, meaning that its future behaviour depends only on its current state and not on its past history. This property simplifies the analysis and prediction of the process.
6) GBM is a martingale, meaning that its expected future value is equal to its current value. This property implies that there is no arbitrage opportunity in the market, and that the process is fair and unbiased.
7) GBM is a self-similar process, meaning that its statistical properties are invariant under scaling transformations. This property allows us to use the same model for different time scales and units of measurement.
8) GBM is a solution of the Black-Scholes equation, which is a partial differential equation that describes the dynamics of an option price. This equation is the basis of the famous Black-Scholes formula, which gives the value of a European call or put option in terms of the parameters of the underlying GBM.

We can say that the Brownian motion is the generalisation of random walk with jump according to the normal distribution [-∞, ∞].
We can also say that the Brownian motion is the scaling limit of the random walk
$dx = \sqrt(t)Z$
where dx is the limit of the random walk as n -> ∞ and Z is a normal distribution.
The following graph shows the geometric brownian motion in our simulator.



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