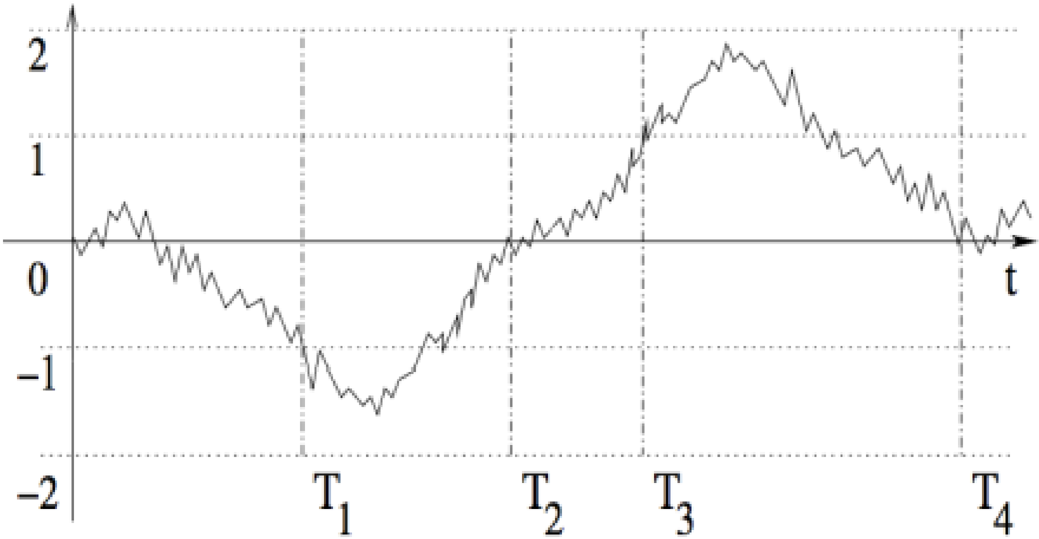
Th11. The functional CLT (Donsker's invariance principle): Proof, Simulations

In probability theory, Donsker's theorem (also known as Donsker's invariance principle, or the functional central limit theorem), named after Monroe D. Donsker, is a functional extension of the central limit theorem.
Let X_1, X_2, X_3, … be a sequence of independent and identically distributed (i.i.d.) random variables with mean 0 and variance 1.
Let $S_n := \sum_{i=1}^{n} X_i$ The stochastic process $S := (S_n)_{n\in\mathbb{N}}$ is known as a random walk. Define the diffusively rescaled random walk (partial-sum process) by $W^{(n)}(t) := \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad t \in [0,1].$
The central limit theorem asserts that $W^{(n)}(1)$ converges in distribution to a standard Gaussian random variable $W(1)$ as n→∞.
Donsker's invariance principle extends this convergence to the whole function $W^{(n)} := (W^{(n)}(t))_{t\in[0,1]}$.
More precisely, in its modern form, Donsker's invariance principle states that: As random variables taking values in the Skorokhod space $\mathcal{D}[0,1]$, the random function $W^{(n)}$ converges in distribution to a standard Brownian motion $W := (W(t))_{t\in[0,1]}$ as n→∞.

In other words, the trajectory of the random walk, after appropriate scaling and shifting, converges to a Brownian motion in the sense of distributional convergence. This result is powerful because it allows one to study the behavior of a wide class of random processes by approximating them with simpler processes, such as Brownian motion.

The invariance principle has applications in various areas of probability theory, statistics, and mathematical finance. It provides a bridge between discrete and continuous stochastic processes, facilitating the analysis of random phenomena

The first step of the proof is to consider a standard Brownian motion B_t and come up with a clever way to think of it as a SRW. One way to do this is to consider the times T_n at which B_t intersects the horizontal integer lines $\{B_t = n : n \in Z\}$.
Formally we set
$T_1 := \inf\{t : |B_t| = 1\}$,
$T_{n+1} := \inf\{t > T_n : |Bt - BT_n| = 1\}$
and we call T_n a stopping time.



From here we proceed by invoking a Skorokhod embedding, which informally, states the existence of a stopping time T such that B_T follows the law of a random variable X for which $E[X] = 0$, $E[X^2] < \infty$, and $E[T] = E[X^2]$.
Now let X be a real random variable with mean 0 and variance 1 and define T_1 to be such that $E[T_1] = 1$ and $BT_1 = X$ in distribution, using the Skorokhod embedding.
Similarly, we then define T'_2 to be such that $E[T'_2] = 1$ and $BT_2 = X$ in distribution, so that $T_2 = T_1 + T'_2$ and $E[T_2] = 2$.
Continuing in this fashion, we inductively define a sequence of stopping times $T_1 < T_2 < \cdots < T_n$ such that $S_n = BT_n$, i.e. the Brownian motion with stopping times T_n has the same distribution of the SRW described by S_n.
Now that we have found a usable embedding that satisfies the correct properties, the conclusion is that, after rescaling our Brownian motion, the difference $\left|\frac{B_{nt}}{\sqrt{nt}} - S^*_n\right|$ becomes negligible as $n \to \infty$.

Bibliography
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