Th9: The Poisson Process: Meaning, Properties,Simulations

Poisson point process

In probability, statistics and related fields, a Poisson point process is a type of random mathematical object that consists of points randomly located on a mathematical space with the essential feature that the points occur independently of one another. When the process is defined on the real line, it is often called the Poisson process.
Its name derives from the fact that if a collection of random points in some space forms a Poisson process, then the number of points in a region of finite size is a random variable with a Poisson distribution.

The Poisson point process has the property that each point is stochastically independent to all the other points in the process, which is why it is sometimes called a purely or completely random process.
The point process depends on a single mathematical object, which, depending on the context, may be a constant, a locally integrable function or, in more general settings, a Random measure. In the first case, the constant, known as the rate or intensity, is the average density of the points in the Poisson process located in some region of space. The resulting point process is called a homogeneous or stationary Poisson point process.
In the second case, the point process is called an inhomogeneous or nonhomogeneous Poisson point process, and the average density of points depends on the location of the underlying space of the Poisson point process.

We say that X is Poisson random variable with mean μ if it has probability mass function given by $\mathbb{P}(X=n)=\dfrac{e^{-\mu}\mu^n}{n!}$ for $n=0,1,2,\ldots$.
We say that Z is an exponential random variable with mean μ (rate λ=1/μ) if it is a continuous random variable with probability density function given by $z\mapsto\frac{1}{\mu}e^{-\frac{z}{\mu}}$ for $z\geq 0$.
Some of the properties of the Poisson process are:

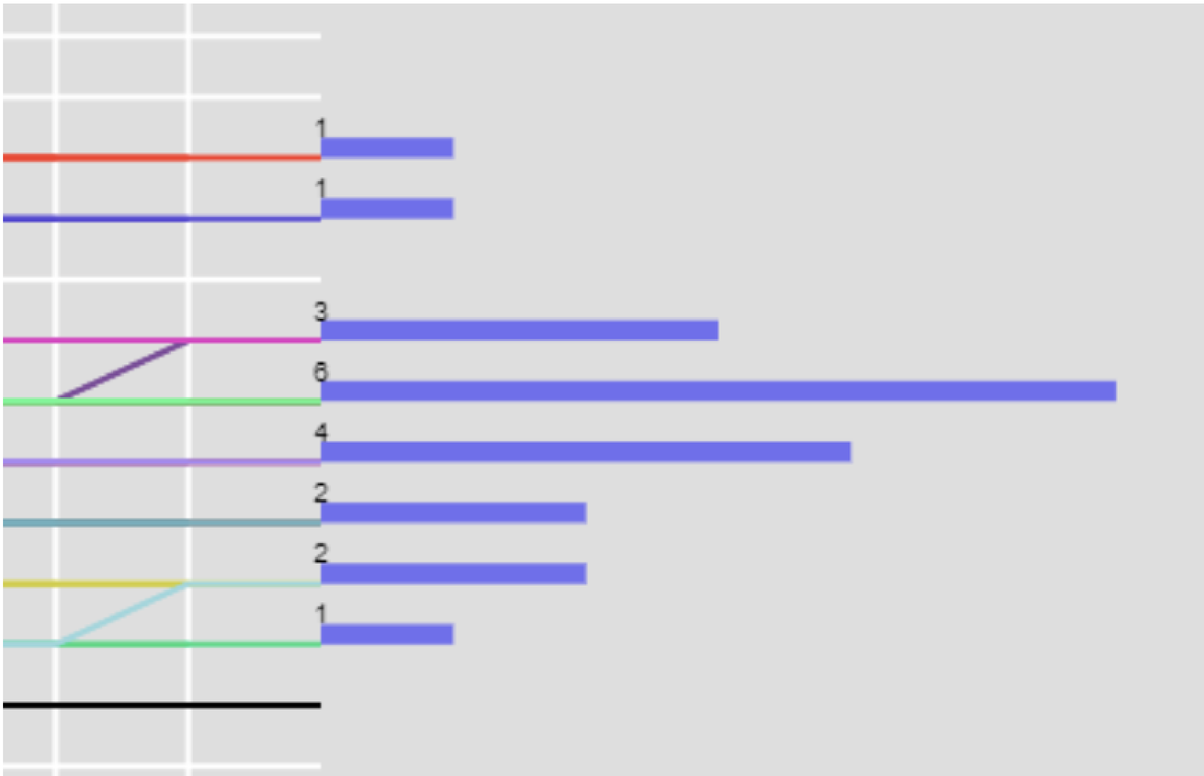
- Superposition: The sum of two independent Poisson point processes is again a Poisson point process.
- Coloring: Given a Poisson point process if we color the points red or blue independently with probability p∈(0,1), the resulting blue and red point processes are Poisson and independent.
- Scaling: Given a Poisson point process Π on Rd of intensity λ, the scaled point process cΠ formed by multiplying each Π-point by c>0, is a Poisson point process of intensity $c^{-1}\lambda$.
- Independence: Events occur independently in disjoint intervals or regions. For example, in a spatial Poisson point process, the occurrence of points in one region is independent of the occurrence of points in another region.
- Homogeneity: The event rate is constant. In a temporal Poisson process, this means that events occur at a constant average rate over time. In a spatial Poisson process, it implies that events occur uniformly across the space.
- Memorylessness: The probability of an event occurring in a given interval depends only on the length of the interval and not on when the process started. In other words, future events are independent of the past given the current state.

A counting process N(t) is a stochastic process defined by the number of occurrences of a random event before time t.
A Poisson process is a counting process satisfying the following conditions:

- $N(0) = 0$.
- For all $t_1 \leq t_2 \leq s_1 \leq s_2$ the random variables $N(t_2) - N(t_1)$ and $N(s_2) - N(s_1)$ are independent.
- There exists a $\lambda > 0$ such that given any $0 \leq t_1 < t_2$, $E[N(t_2) - N(t_1)] = \lambda(t_2 - t_1)$.
- If $P(s) = P\{N(t + s) - N(t) > 2\}$, then $\lim_{s \to 0} \frac{P(s)}{s} = 0.$

Condition 2 can be restated as “the number of occurrences in one time interval is independent of the number of occurrences in any other time interval,” known as independent increments. Condition 3 assures that the expected number of occurrences between intervals of the same size is constant, known as stationary increments. Condition 4 formalizes the idea that occurrences only happen one at a time. The limit represents this notion by saying that the probability of two or more events happening in an interval of length s is much smaller than s.

In homework 5, as for the asymptotic behavior for large N and a sufficient number of systems (M), I observe that the distributions resemble a Poisson distribution.
Using N=50 and M=20 I obtained the following histogram.



In this histogram we can see that the mean is far from the Poisson one. The first one is 6 successful attacks instead the Poisson’s one is 9 attacks.
As N becomes larger and the number of systems M increases, the distribution becomes more and more symmetric and bell-shaped, resembling a Poisson distribution. This is in line with the properties of a Poisson process when the number of subintervals is sufficiently large and the systems are numerous enough.
This can be seen in the following graph where N = 100 and M = 100.

The average of the distribution will converge towards the mean of a Poisson distribution, that is 9 successful attacks, and the one calculated in our histogram is 8.
If we keep incrementing N and M we will see that the average of successful attacks will be near 9.

Bibliography

-https://en.wikipedia.org/wiki/Poisson_point_process
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-https://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Mcquighan.pdf