Th12: Ito Integration and Calculus, Concept and Didactical Simulations

Itô calculus, named after Kiyosi Itô, extends the methods of calculus to stochastic processes such as Brownian motion (see Wiener process). It has important applications in mathematical finance and stochastic differential equations.
The integrands and the integrators are now stochastic processes:

$$\int_0^T h_t dW_t = \sum_{i=0}^{n-1} h_i \left(W_{t_{i+1}} - W_{t_i} \right).$$

where h_t is a locally square-integrable process adapted to the filtration generated by W, which is a Brownian motion. The result of the integration is then another stochastic process.

the properties of the Itô Integral are the following:

1) Linearity: $I_t(aV + bU) = a\, I_t(V)+b\, I_t(U)$.
2) Measurability: $I_t(V)$ is adapted to W.
3) Continuity: $t \to I_t(V)$ is continuous.
4) Mean zero: $\mathbb{E}[\int_0^T h_t dW_t] = 0;$
5) Ito isometry: $\mathbb{E}[\left(\int_0^T h_t dW_t\right)^2] = \mathbb{E}[\int_0^T h_t^2 dt].$

where F is an admissible filtration and I_t is a notation that represents the integral.

Ito integration is a way to define the integral of a stochastic process with respect to a stochastic time parameter. Traditional calculus deals with deterministic functions and their derivatives, but in stochastic calculus, we deal with random processes and their stochastic differentials.

Ito's Lemma is a fundamental result in stochastic calculus that generalizes the chain rule for functions involving stochastic processes. It is used to find the differential of a function of a stochastic process.

For a function f(t,X(t)), where X(t) is a stochastic process, Ito's Lemma is expressed as:
$df(t,X(t))= \partial f/\partial t\ \ dt + \partial f/\partial X\ dX+ ½\ \partial^2 f/\partial X^2\ \ (dX)^2$
where dX represents the stochastic differential of X(t), and the term $(dX)^2$ is interpreted in the sense of Ito's multiplication rules, where $dW \cdot dW=dt$.

Ito integration is designed for stochastic processes and accounts for the randomness inherent in the processes.

The integration by parts formula for the Itô integral differs from the standard result due to the inclusion of a quadratic covariation term. This term comes from the fact that Itô calculus deals with processes with non-zero quadratic variation, which only occurs for infinite variation processes (such as Brownian motion). If X and Y are semimartingales then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{s-}\, dY_s + \int_0^t Y_{s-}\, dX_s + [X,Y]_t$$

where [X, Y] is the quadratic covariation process.

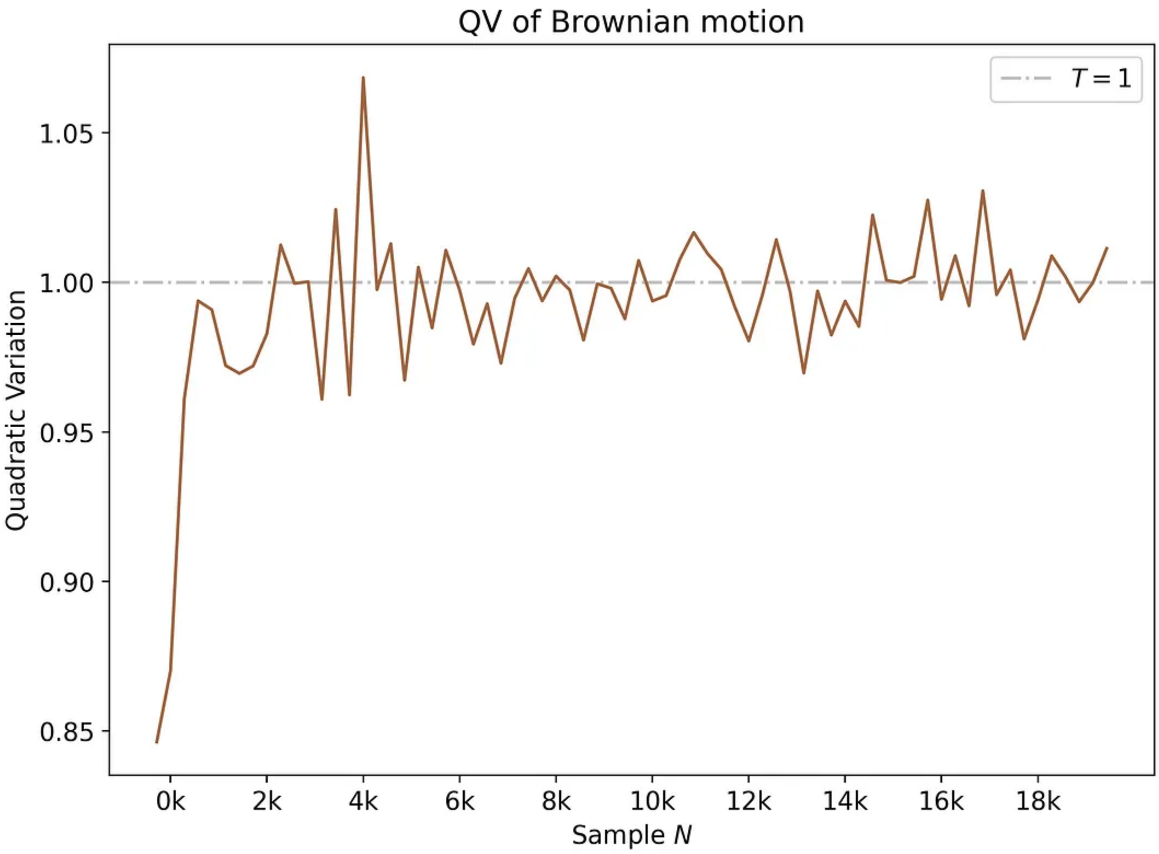
The quadratic variation quantifies the cumulative sum of the squares of the increments in the Wiener process, providing a suitable framework for accommodating the irregular nature of Brownian motion. There are three important quadratic variations in the related Ito calculus, which are

1) $dW_t\, dW_t = d_t$
2) $dW_t\, d_t = 0$
3) $d_t\, d_t = 0$

The concept of quadratic variation entails summing the squared differences between individual points of a function over a given partition, which is described as:

$$Q(T) = \lim_{n \to \inf} \sum_{j=0}^{n-1} (W_{j+1} - W_j)^2 = T$$

The underlying concept implies that as the step size is reduced, the quadratic difference between consecutive step values approaches a close approximation of the total partition. The following graph shows the computation of the quadratic variation directly up to the maturity T (1 year), although the function is applicable to any partition interval. The simulation serves to demonstrate this characteristic, as the values rapidly converge towards T.



We can use the Ito to model stock prices.
We must know the price of underlying S at time t in order to compute the current price of an option C. However, we can not know the exact price of the stock, but it can be modeled as a stochastic process X_t and used to compute the probability density at time t. Unfortunately, the Brownian motion X is smooth nowhere, meaning it's not differentiable by using the ordinary Riemann integral, and thus, Ito's Lemma is required.

We will assume that the stock price follows the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t W_t$$

From Ito's Lemma, it follows that a process followed by $f=f(t, S_t)$ is

$$df(t, S_t) = \left(\frac{\partial f}{\partial t} + \mu(t, S_t) \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2(t, S_t) \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma(t, S_t) \frac{\partial f}{\partial S} dX_t$$

By defining the function f as ln(St), which is an intelligent guess based on the distribution of stock prices, we arrive at the following expression after computing all derivatives for the logarithm.

$$df = d(\ln(S_t)) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dX_t$$

This expression can be solved by applying the normal integral as shown below

$$\int_0^t d(\ln(S_t)) = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dX_t$$

$$\ln(S_t) - \ln(S_0) = \left(\mu - \frac{\sigma^2}{2}\right)(t-0) + \sigma(X_t - X_0)$$

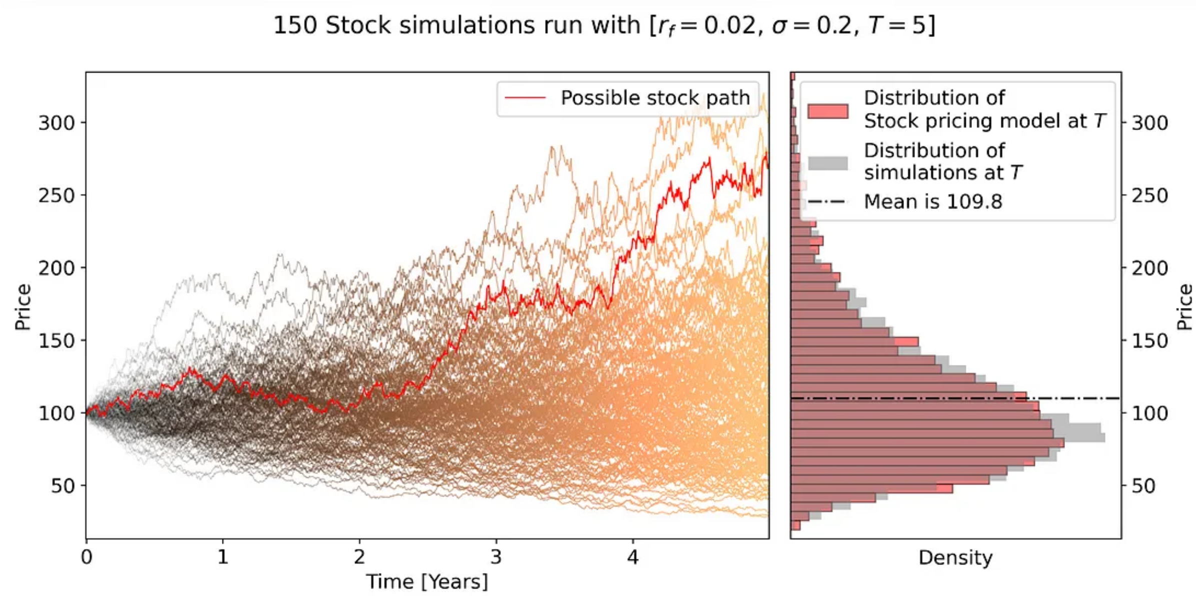
The outcome of this integral yields the stock pricing model as follows:

$$S_t = S_0 e^{\mu - \frac{1}{2}\sigma^2 + \sigma X_t}$$

which can also be expressed as

$$\ln\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$$

This formulation implies that the logarithmic stock returns, following the Geometric Brownian Motion, are normally distributed with a linearly increasing mean and standard deviation. It is evident from the comparison that the red distribution generated by the pricing formula at time T provides similar results as considering all individual price movements together, represented by the gray color in the plot.



The code used to realize this simulation can be found on this [page](#).

Bibliography
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