Finding the N^{th} Fibonacci Number

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Abstract

The Fibonacci Numbers are a sequence of positive integers defined by a recursive relationship. That is, $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$ and $F_1 = 1$. Likely owning to its simplicity, the Fibonacci Numbers appear frequently in the natural world and a wide range sciences. However, the value of F_n grows very quickly, and the recursive method described becomes impossibly slow as n increases. This paper will discuss some uses of Fibonacci Numbers, then examine two methods of finding the n^{th} Fibonacci Number, given n. The two methods examined will be the previously described recursive approach and the dynamic programming approach. By comparing the theoretical and experimental runtimes of both approaches, dynamic programming will be shown to be the better method.

Problem Description

Given an integer, $n > 1$, the value of the n^{th} Fibonacci Number, F_n, is sought. This is found by applying the recurrence relation $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$ and $F_1 = 1$. There is discrepancy in this definition, as some problem descriptions may choose $F_1 = 0$ and $F_2 = 1$, or $F_1 =$ and $F_2 =1$, or other values and labels for the first two terms. As the values of n increase, the values of F_n grow rapidly and calculation becomes very time and space intensive. Because the Fibonacci Numbers have a wide array of uses and appear frequently in many scientific fields, an efficient method of calculating larger members of this set is necessary.

Practical Applications

Leonardo of Pisa, commonly known as Fibonacci, introduced a series of recursive numbers to Western mathematics in his book Liber Abaci, published in 1202. It wasn't until the 19th century, though, that number theorist Édouard Lucas gave the Fibonacci numbers their name (Knott). The concept of Fibonacci Numbers, and their generating function, had been known to mathematics in other parts of the world as well. Donald Knuth cites Indian scholars and their writings, saying:

> "Before Fibonacci wrote his work, the sequence F_n had already been discussed by Indian scholars, who had long been interested in rhythmic patterns... both Gopala (before 1135 AD) and Hemachandra (c. 1150) mentioned the numbers 1,2,3,5,8,13,21 explicitly." (p. 100)

Appearing frequently in mathematics and science for centuries, it is clear that these numbers are important and have intriguing uses. This section will discuss a few common applications of the Fibonacci Numbers and the general rule that it follows.

The Golden Ratio, φ, is a number that often results when establishing a "beautiful" or "natural" ratio between two measurements. Mathematically, two numbers exhibit this relationship if their ratio is the same as the ratio of their sum to the larger of the two (Wikipedia). That is,

$$\frac{a}{b} = \frac{a+b}{a} = \phi \approx 1.618$$

This ratio was used frequently in classical architecture and it is observed in many instances of biological growth patterns. The value of φ can be calculated by taking any sufficiently large Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$, where $F_{n-1} = a$ and $F_{n-2} = b$, as shown:

$$\frac{a}{b} = \frac{F_{n-1}}{F_{n-2}} = \frac{a+b}{a} = \frac{F_{n-1} + F_{n-2}}{F_{n-1}} = \frac{F_n}{F_{n-1}} = \phi$$

Taking n = 25, for example, the calculation is:

$$\frac{F_{n-1}}{F_{n-2}} = \frac{46368}{28657} \approx 1.618 = \phi$$

$$\frac{F_n}{F_{n-1}} = \frac{75025}{46368} \approx 1.618 = \phi$$

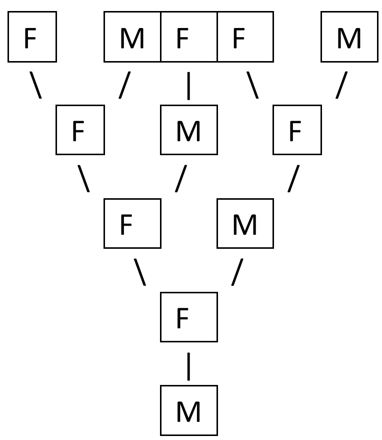
If a highly precise value of φ is needed, large Fibonacci Numbers can be used to calculate it.

In computer science, the heap data structure provides an ordering of its elements such that maximum or minimum values can be quickly accessed. Chapter 19 of the CLRS book is dedicated to a special heap implementation, called the Fibonacci Heap, whose value is described in the opening paragraph:

> "The Fibonacci heap data structure serves a dual purpose. First, it supports a set of operations that constitutes what is known as a "mergeable heap." Second, several Fibonacci-heap operations run in constant amortized time, which makes this data structure well suited for applications that invoke these operations frequently." (p. 505)

The Fibonacci heap is actually a collection of trees that are min-heap ordered, and whose roots are connected in a list. It was named after Fibonacci because the numbers in the series appear in the runtime analysis of the data structure. Specifically, the Fibonacci heap uses a forest of trees, given an order, n, such that the number of nodes in the trees is $\geq F_{n+2}$ (Growing the Web). Operational runtime depends on summing the numbers of nodes in the trees, which is a set of Fibonacci numbers.

In nature, the Fibonacci numbers are found in what is popularly called the "Bee Ancestry Code." A point of bee biology is that female bees produce males when unfertilized and females when fertilized (Bee Ancestry). From this, tracing 1 male bee's ancestry shows he has 1 parent, 2 grandparents, 3 great-grandparents, 5 great-great-grandparents, and so on, which are the Fibonacci numbers. The following table illustrates that n generations back, a male bee has F_n ancestors:



Generation 5: 3 + 2 = 5 bees	F_5
Generation 2: 2 + 1 = 3 bees	F_4
Generation 3: 1 + 1 = 2 bees	F_3
Generation 2: 1 + 0 = 1 bee	F_2
Generation 1: 1 bee	F_1

The Fibonacci Numbers are used frequently in the sciences, so much so that a dedicated research community has developed around them. This is the Fibonacci Association, and it publishes a journal called the Fibonacci Quarterly, which is at www.fq.math.ca. The journal has been in regular publication since 1963, with new volumes released four times per year. The Fibonacci Association also regularly hosts conferences around the world focused on recent research related to the Fibonacci Numbers and sequence. Considering such long-running work dedicated to applications of Fibonacci Numbers, it's clear that they have great importance in mathematics and beyond.

Competing Algorithms

Experiments

Programming Implementation

Conclusions

Sources

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