Real Analysis Solutions

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Chapter 3: Sequences

3.1

For $\varepsilon = .0005$, there is some N such that for all $n > N$, $|a_n - .001| < .0005$, which implies that a_n is positive for $n > N$. Thus, finitely many terms are negative.

3.2

(a) Even terms are 0, odd terms are 1

(b) No such sequence exists. Suppose otherwise, and assume it converges to a. Pick $0 < \varepsilon < |a|$. Then, there exists N such that for all $n > N$, $|a_n - a| < \varepsilon$. However, since there are infinitely many 0s, there's some $k > N$ such that $a_k = 0$, in which case we get $|0 - a| < \varepsilon$, a contradiction.

(c) No such sequence exists. Suppose otherwise, and assume it converges to $a < 0$. Pick $0 < \varepsilon < |a|$. Then, there exists N such that for all $n > N$, $|a_n - a|\,\varepsilon$. Unwinding, this yields

$$a - \varepsilon < a_n < a + \varepsilon < 0,$$

a contradiction.

(d) $(a_n) = \frac{\sqrt{2}}{2^n}$

3.4

(a) Fix any $\varepsilon > 0$. Set $N = \frac{1}{\varepsilon^2}$. Then for any $n > N$

$$\left|\left(7 - \frac{1}{\sqrt{n}}\right) - 7\right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{(1/\varepsilon^2)}} = \varepsilon.$$

(b) Fix any $\varepsilon > 0$. Set $N = \frac{12}{25\varepsilon} - \frac{1}{5}$. Then for any $n > N$

$$\left|\left(\frac{2n-2}{5n+1}\right) - \frac{2}{5}\right| = \frac{12}{25n+5} < \frac{12}{25N+5} = \frac{12}{25\left(\frac{12}{25\varepsilon} - \frac{1}{5}\right) + 5} = \varepsilon.$$

(c) Fix any $\varepsilon > 0$. Set $N = \left(\sqrt{\frac{1}{\varepsilon^2} - \frac{51}{4}} - \frac{1}{2}\right)^2$. Then for any $n > N$

$$\left|\left(7 - \frac{1}{\sqrt{n + \sqrt{n} + 13}}\right) - 7\right| = \frac{1}{\sqrt{n + \sqrt{n} + 13}} < \frac{1}{\sqrt{N + \sqrt{N} + 13}} = \varepsilon.$$

3.5

$(a_n) = -\frac{1}{n}$

3.7

(a) No, $\varepsilon = 0.5$ has no working N.

(b) Will never come within .5 of a limit.

(c) Eventually constant.

3.8

(a) Fix $\varepsilon/2 > 0$. There exists some N_1 such that for all $n > N_1$, $|a_n - a| < \varepsilon/2$. Similarly, there exists some N_2 such that for all $n > N_2$, $|b_n - b| < \varepsilon/2$. Pick $N = \max\{N_1, N_2\}$. Then, for any $n > N$,

$$|(a_n + b_n) - (a + b)| \le |a_n - a| + |b_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(b) If $c = 0$, then it holds true. For nonzero c, fix $\varepsilon/|c| > 0$. There exists from N such that for all $n > N$, $|a_n - a| < \varepsilon/|c|$. Then, for any $n > N$,

$$|c \cdot a_n - c \cdot a| = |c| \cdot |a_n - a| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

3.9

Fix $M > 0$. Then there exists N_1 and N_2 such that for all $n > N_1$, $a_n > \frac{M}{2}$, and for all $n > N_2$, $b_n > \frac{M}{2}$. Let $N = \max\{N_1, N_2\}$. Then, for all $n > N$,

$$a_n + b_n > \frac{M}{2} + \frac{M}{2} = M.$$

3.10

Fix $\varepsilon > 0$. Then there exists N_1 such that $|a_{2n} - L| < \epsilon$ for all $n > N_1$ and there exists N_2 such that $|a_{2n-1} - L| < \epsilon$ for all $n > N_2$. Now let $N = \max\{N_1, N_2\}$. Then, for any $k > N$, a_k will be within ε of L, since it will satisfy the other two inequalities, regardless of the parity of k.

3.11

False. Take $(a_n) - \frac{1}{n}$.

3.13

(a) Basically the same as 3.8 (a)

(b) We show that $\left(\frac{1}{b_n}\right) \to \frac{1}{b}$, and then general rule follows from multiplication rule. Since b_n, converges, b_n is bounded. Let C be its bound. Then, there's some N such that for all $n > N$, $|b_n - b| < C|b|\varepsilon$, where $\varepsilon > 0$ is fixed. Then, for any $n > N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| < \frac{1}{C|b|} \cdot C|b|\varepsilon = \varepsilon.$$

3.15

$(a_n) = (-1)^n$, $(b_n) = (-1)^{n+1}$

3.16

$(a_n) = (-1)^n$, $(b_n) = (-1)^{n+1}$

3.17

It's not possible. For sufficiently large n, b_n will be come much larger than a_n minus any limiting value, eclipsing any ε.

3.18

(a) $|a_n| \leq C$. Fix $\varepsilon > 0$. Then there exists N such that for any $n > N$, $|b_n - 0| < \varepsilon/C$. Then, for any $n > N$,

$$|a_n b_n - 0| = |a_n| \cdot |b_n - 0| < C \cdot \frac{\varepsilon}{C} = \varepsilon.$$

(b) $(a_n) = n^2$, $(b_n) = \frac{1}{n}$

3.21

Yes. Use Bolzano-Weierstrass on the subsequence.

3.22

(a) Suppose $L > M$. Then pick $0 < \varepsilon < L - M$. Then, $|a_n - L| \geq L - M < \varepsilon$, so the initial hypothesis can't be true.

(b) Suppose $L > M$. Pick $0 < \varepsilon < (L - M)/2$. Then its easy to show that $a_n > b_n$, a contradiction.

3.23

(a) Reverse triangle inequality.

(b) $(a_n) = (-1)^n$

3.25

Taking the negative of the sequence. This makes it monotone increasing and bounded above. Note that the infinum becomes the supremum, so the monotone convergence theorem for increasing sequences finishes it off.

3.26

Do the same as above.

3.27

Every even term is 1, and every odd term is n.

3.32

(a_n) is a subsequence (a_n).

3.33

Let $M > 0$. Then there exists N such that for all $n > N$, $a_n > M$. Now suppose (a_{n_k}) is a subsequence of (a_n). Note that for any k, $n_k > k$, so clearly $a_{n_k} > M$ holds for all $k > N$.

3.34

$(a_n) = (-1)^n n$, $(b_n) = (-1)^n n$

3.35

$(a_n) = (-1)^n n$

3.37

(a) Induction on n

(b) Show increasing with induction on n, monotone boundedness shows it converges.

(c) We just need so show that $\sup(\{a_n : n \in \mathbb{N}\}) = 2$, after which the monotone convergence theorem will give us the desired result. We already showed 2 is an upper bound. Pick some $\varepsilon > 0$. We need to show there exists some N such that for all $n > N$, $a_n > 2 - \varepsilon$. Sub in the expression for a_n, square, subtract two, there the right side will be less than $2 - \varepsilon$. Rewrite the right as $2 - \varepsilon_1$, and repeat. Eventually it will be less than 0, and the number of iterations gives N.

3.45

Note that $0 \leq b_n \leq a_n$ and that (b_n) is increasing. Thus it converges.