Complex Analysis Notes

Contents

1. Complex Number Review

$\mathfrak{R}(z)$ gives the real part of z and $\mathfrak{I}(z)$ gives the imaginary part of z.

$z\overline{z} = |z|^2$.

$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$.

Proposition (triangle inequality and reverse triangle inequality): For $z, w \in \mathbb{C}$, we have

$$|z + w| \leq |z| + |w|$$

and

$$|z \pm w| \geq ||z| - |w||.$$

Example: Let $f(z) = a_n z^n + \cdots + a_1 z + a_0$. By the reverse triangle inequality we have

$$|f(z)| \geq \left||a_n z^n| - |a_{n-1}z^{n-1} + \cdots + a_0|\right| \geq |a_n z^n| - |a_{n-1}z^{n-1} + \cdots + a_0| \geq |a_n z^n| - \sum_0^{n-1} |a_i z|^i.$$

Dividing by $|z^n|$ yields

$$\frac{|f(z)|}{|z^n|} \geq |a_n| - \sum_0^{n-1} \frac{|a_i|}{|z|^{n-1}}.$$

Then choose R such that for $|z| > R$, each term in the sum is at most $\frac{|a_n|}{2n}$ (make R sufficiently large). Then we have

$$\frac{|f(z)|}{|z|^n} \geq |a_n| - \frac{|a_n|}{2} = \frac{|a_n|}{2}.$$

Then we have

$$|f(z)| \geq \frac{|a_n|}{2}|z^n| > \frac{|a_n|}{2}R^n.$$

Thus for sufficiently large z (in the sense of its magnitude), the leading term of a polynomial will dominate. A similar method shows that for z sufficiently close to 0, the constant term will dominate.

1.1. Branches

The complex square root and logarithms are multivalued, so often we restrict output to a certain set or "branch" of their range. In particular, we can choose branches so that the functions are continuous (ex. $\sqrt{-9} = 3i$, but we can have $\sqrt{9e^{\pi+\varepsilon}} \approx -3i$ because of multivaluedness).

For example, a popular branch cut for $\sqrt{z}$ is taking all $z \in \mathbb{C} \setminus \{x : x \in \mathbb{R}, x \leq 0\}$ and choosing $\mathfrak{R}(\sqrt{z}) > 0$. We can use the same domain for the logarithm and require that $-\pi < \mathfrak{I}(\log(z)) < \pi$.

1.2. Problems

Problem (1): Show that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Solution: We have

$$\frac{e^{ix} + e^{-ix}}{2} = \frac{\cos x + i \sin x + \cos(-x) + i \sin(-x)}{2} = \frac{2 \cos x + i \sin x - i \sin x}{2} = \cos x.$$

This same method shows the second equality.

Problem (2): Find a complex number $z \neq 1$ such that $z^3 = 1$ and the number of them.

Solution: We have $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$, and since by the fundamental theorem of algebra we can have at most three solutions, these are the only other solutions.

Problem (3): $\sqrt{-4}\sqrt{-9} = (2i)(3i) = -6$, but $\sqrt{(-4)(-9)} = \sqrt{36} = 6$. Reconcile with with the notion that $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Solution: The identity $\sqrt{ab} = \sqrt{a}\sqrt{b}$ should only be defined when a and b are nonnegative reals, and the output should also always be nonnegative (essentially restricting to a certain branch).

Problem (4): Find the locus of $z \in \mathbb{C}$ for both $|z - a| = |z - b|$ and $|z - a| = 2|z - b|$ for complex numbers a, b.

Solution: The locus of $|z - a| = |z - b|$ is the set of points which are equidistant from a and b, but this is the perpendiuclar bisector of AB.

For the second equation, we first find the locus with $a = 0$ and $b = 1$. In particular, we need to find the set of $z \in \mathbb{C}$ such that $|z| = 2|z - 1|$. We claim that this set is given by $\left|z - \frac{4}{3}\right| = \frac{2}{3}$, which is a circle centered at $\frac{4}{3}$ with radius $\frac{2}{3}$.

To see this, note that

$$|z| = 2|z - 1| \Rightarrow z\overline{z} = 4(z - 1)(\overline{z} - 1)$$
$$\Rightarrow 0 = 3z\overline{z} - 4z - 4\overline{z} + 4 \Rightarrow \frac{4}{9} = z\overline{z} - \frac{4}{3}z - \frac{4}{3}\overline{z} + \frac{16}{9}$$
$$\Rightarrow \left(\frac{2}{3}\right)^2 = \left(z - \frac{4}{3}\right)\left(\overline{z} - \frac{4}{3}\right) = \left|z - \frac{4}{3}\right|^2,$$

and square rooting gives the desired conclusion. We can go in reverse to show that every point in $\left|z - \frac{4}{3}\right| = \frac{2}{3}$ does indeed satisfy $|z| = 2|z-1|$.

Now suppose we have arbitrary a and b. We can use $z \to \frac{z}{b-a}$ and multiply by $|b-a|$ to obtain $|z| = 2|z-1|$ by $b-a$ to get $|z| = 2|z-(b-a)|$, and then shift by a to obtain $|z-a| = 2|z-b|$. Doing the same thing to the locus yields

$$\left|z - \frac{4}{3}\right| = \frac{2}{3} \Rightarrow \left|z - \frac{4}{3}(b-a)\right| = \frac{2}{3}|b-a| \Rightarrow \left|z - \left(\frac{4b+a}{3}\right)\right| = \frac{2}{3}|b-a|.$$

This is the desired locus, which is a circle centered at $\frac{4b+a}{3}$ with radius $\frac{2}{3}|b-a|$.

Problem (5): Describe the locus of $z \in \mathbb{C}$ such that $\mathfrak{I}\left(\frac{z}{(z+1)^2}\right) = 0$.

Solution: The condition implies $\frac{z}{(z+1)^2}$ is real. Thus we have

$$\frac{z}{(z+1)^2} = \frac{\overline{z}}{\overline{(z+1)^2}} \Rightarrow z\overline{(z+1)^2} = \overline{z}(z+1)^2 \Rightarrow |z|\overline{z} + 2|z| + z = |z|z + 2|z| + \overline{z}$$
$$\Rightarrow (|z|-1)(\overline{z} - z) = 0.$$

Thus the locus consists of the real line and the unit circle (the unit circle is the first factor, the real line is the second factor since for all real $r = \overline{r}$).

Problem (8): Let $f(x) = e^x \cos x$. Compute $f^{(100)}(x)$.

Solution: Let $g(x) = e^x \sin x$. Then we have

$$\frac{d^{100}}{dx^{100}} f(x) + i \cdot \frac{d^{100}}{dx^{100}} g(x) = \frac{d^{100}}{dx^{100}}(f(x) + ig(x)) = \frac{d^{100}}{dx^{100}}(e^x(\cos x + i \sin x)) = \frac{d^{100}}{dx^{100}} e^{x(i+1)}.$$

Thus $f^{100}(x) = \mathfrak{R}\left(\frac{d^{100}}{dx^{100}} e^{x(i+1)}\right)$. We have that

$$\frac{d^{100}}{dx^{100}} e^{x(i+1)} = (i+1)^{100} e^{x(i+1)}.$$

Letting $\zeta = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ (an 8th root of unity), we have

$$(i+1)^{100} = \left(\sqrt{2}\zeta\right)^{100} = 2^{50}\zeta^4 = -2^{50}.$$

Thus

$$f^{(100)}(x) = -2^{50} e^x \cos(x).$$

Problem (10):
a) Solve the equation $z^4 + z^3 + z^2 + z + 1 = 0$ in two ways.
b) Express $\cos \frac{2\pi}{5}$ in radicals.

Solution:
a) Multiply by $z - 1$ to get $z^5 - 1 = 0$. The solutions of this are the 5th roots of unity (in particular, $\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$ is a solution). We can also write the equation as

$$(z^2 + \varphi z + 1)\left(z^2 - \frac{1}{\varphi} z + 1\right) = 0.$$

This can be found by factoring the equation as $(z^2 + az + 1)(z^2 + bz + 1)$, equating coefficients, and obtaining the system $a + b = 1, ab = -1$, which has solution $a = \varphi$ and $b = -\frac{1}{\varphi}$. Solving each of the quadratic factors yields

$$z = \frac{-\varphi \pm \sqrt{\varphi^2 - 4}}{2}, \frac{\frac{1}{\varphi} \pm \sqrt{\frac{1}{\varphi^2} - 4}}{2}.$$

a) Note that $\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$ corresponds to the positive solution og the second quadratic. This is because both $\cos \frac{2\pi}{5}$ and $\sin \frac{2\pi}{5}$ are positive, and the only solutions with both positive real and imaginary parts is that solution. Thus we have

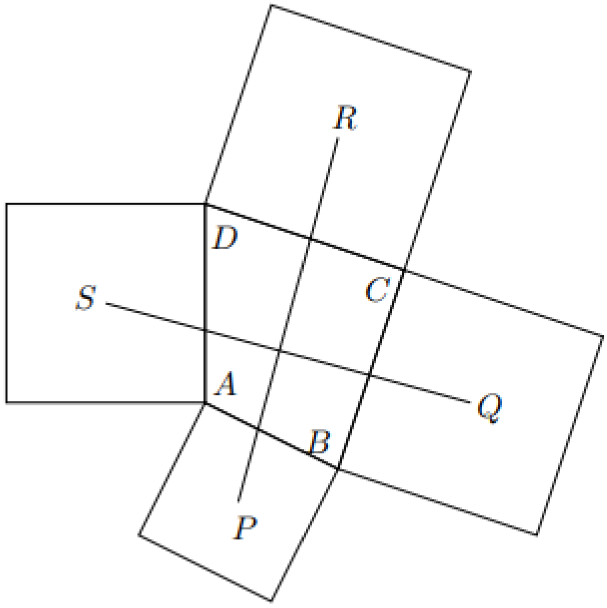
$$\cos \frac{2\pi}{5} = \Re\left(\frac{\frac{1}{\varphi} + \sqrt{\frac{1}{\varphi^2} - 4}}{2}\right) = \frac{1}{2\varphi} = \frac{1}{1 + \sqrt{5}} = \frac{\sqrt{5} - 1}{4}.$$

Problem (13): Show that $\left|1 - e^{i\theta}\right| = 2 \sin \frac{\theta}{2}$.

Solution: The left side is the distance between 1 and $e^{i\theta}$. Using the Pythagorean theorem, this distance is equal to $\sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} = \sqrt{2 - 2\cos \theta} = 2\sqrt{\frac{1 - \cos \theta}{2}} = 2 \sin \frac{\theta}{2}$, as desired.

Problem (17): Let $ABCD$ be any quadrilateral. Draw squres on each edge of $ABCD$ and let P, Q, R, S be the centers of the squares based on AB, BC, CD, and DA respectively. Show that PR and QS have the same length and are perpendicular.

Solution:



Orient the quadrilateral as show in the image and place it in the complex plane. Let a be the complex number at A and so on. First we calculate R

Note that the top left corner of the square with R at the center is given by $(c-d)i+d$ (we shift DC to the origin, rotate 90° by multiplying by i, and shift back to D). Since R is the midpoint of this corner and C, we have

$$r = \frac{(c-d)i+c+d}{2}.$$

We get similar values for the other centers. Then we have

$$r-p = \frac{(c-d-a+b)i+c+d-a-b}{2} \quad \text{and} \quad q-s = \frac{(b-c-d+a)i+b+c-a-d}{2}.$$

Note that $i(q-s) = r-p$, so the two segments are indeed perpendicular. This also implies that $|i(q-s)| = |q-s| = |QS| = |RP| = |r-p|$, as desired.

Problem (20): Show that if $z^{10} = (z-1)^{10}$, then $\mathfrak{R}(z) = \frac{1}{2}$, and find the number of solutions.

Solution: Note that there are nine solutions, since expanding the right side and simplifying yields a degree 9 polynomial. Taking 10th roots yields

$$z \cdot \omega^k = z - 1,$$

where ω is the 10th root of unity $\cos\frac{2\pi}{10} + i\sin\frac{2\pi}{10}$. Note that $k \neq 0$, since then $z = z-1$, which has no solutions. Thus

$$z = \frac{1}{1-\omega^k} = \frac{1}{1-\cos\frac{2\pi k}{10} - i\sin\frac{2\pi k}{10}} = \frac{1-\cos\frac{2\pi k}{10} + i\sin\frac{2\pi k}{10}}{\left(1-\cos\frac{2\pi k}{10}\right)^2 + \sin^2\frac{2\pi k}{10}} = \frac{1-\cos\frac{2\pi k}{10} + i\sin\frac{2\pi k}{10}}{2-2\cos\frac{2\pi k}{10}}.$$

Note that $\mathfrak{R}(z) = \frac{1-\cos\frac{2\pi k}{10}}{2-2\cos\frac{2\pi k}{10}} = \frac{1}{2}$, as desired.

Problem (22): Show that if m and n are integers can be written as the sum of two squares of integers, then mn can also be written as the square of two integers.

Solution: If $m = a^2 + b^2$ and $n = c^2 + d^2$, then

$$mn = (a^2 + b^2)(c^2 + d^2) = (a + bi)(a - bi)(c + di)(c - di)$$
$$= (ac - bd + i(ad + bc))(ac - bd - i(ad + bc)) = (ac - bd)^2 + (ad + bc)^2.$$

Problem (24): Let $z, w \in \mathbb{C}$ with magnitude less than 1. Show that

$$\left|\frac{z - w}{1 - \overline{w}z}\right| < 1.$$

Then fix w and let $f(z)$ be the function inside the absolute value. Show that f maps the unit disk to itself, swaps 0 and w, and is bijective.

Solution: For the inequality, we have

$$(1 - |w|^2)(1 - |z|^2) > 0 \Rightarrow 1 - w\overline{w} - z\overline{z} + w\overline{w}z\overline{z} > 0$$
$$\Rightarrow 1 - \overline{w}z - w\overline{z} + w\overline{w} + z\overline{z} > z\overline{z} - \overline{w}z - w\overline{z} + w\overline{w} \Rightarrow |1 - \overline{w}z|^2 > |z - w|^2$$
$$\Rightarrow \left|\frac{z - w}{1 - \overline{w}z}\right| < 1,$$

as desired.

For the second part, 0 and w swapping is obvious, since $f(0) = w$ and $f(w) = 0$. Clearly f maps the unit disk to itself by the inequality we showed. Then note that f is its own inverse, so $f(f(z)) = z$. Thus f is an involution, which implies it's bijective (in particular it will transform the unit disk into itself, since we already showed that the unit disk is closed under f).

2. Limits and Continuity

Pretty much port everything over from real analysis.

2.1. Limits

Definition (limit): Let $z_1, z_2, \ldots$ be a sequence of complex numbers. Then $\lim_{n \to \infty} z_n = L$ if for any $\varepsilon > 0$, there exists N such that $n \geq N \Rightarrow |z_n - L| < \varepsilon$.

Note that from this definition we can get that the limit of a sequence is unique using a simple contradiction argument.

Definition (Cauchy): A sequence is *Cauchy* is for every $\varepsilon > 0$, there exists N such that $n, m \geq N \Rightarrow |z_n - z_m| < \varepsilon$.

Definition (functional limit): Let $U \subset \mathbb{C}$ and let $f : U \to \mathbb{C}$. Let $z_0 \in \mathbb{C}$ be such that U contains a punctured disk of some radius around z_0. Then $\lim_{z \to z_0} f(z) = L$ if for any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Example: Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be $f(z) = \frac{z}{\overline{z}}$. Approaching on the real line, we have $z = \overline{z}$, so f will be 1. Approaching in on the imaginary line, $z = -\overline{z}$, so f will be -1. Thus approaching along two different paths yields a different limit for f. Thus, the limit of f as z approaches 0 doesn't exist.

2.2. Topology

Definition (open): A set $U \subseteq \mathbb{C}$ is *open* if for every $z_0 \in U$, there exists some disk centered at z_0 contained in U.

Definition (closed): A set is *closed* if its complement is open.

Definition (bounded): A set is *bounded* if the set is contained with a disk of finite size.

Definition (compact): A set $X \subseteq \mathbb{C}$ is *compact* if it's closed and bounded.

Definition (boundary point): Let $S \in\subseteq \mathbb{C}$ be a set. Then $z \in \mathbb{C}$ is a *boundary point* of S is for every $\varepsilon > 0$, $B_\varepsilon(z)$ contains points in S and S^c.

2.3. Continuity

Definition (continuity): Let $U \subseteq \mathbb{C}$ contain an open set around z_0, and let $f : U \to \mathbb{C}$ be a function. Then f is *continuous* at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Proposition: Let $f : U \to V$ and $g : V \to \mathbb{C}$ be continuous. Then $g \circ f : U \to \mathbb{C}$ is also continuous.

This can easily be proved by just throwing the definition of continuity at it (points close in U will be close in V, and so will be close in $\mathbb{C}$).

Proposition: Let U be an open set, and let $f : U \to \mathbb{C}$. Then f is continuous if and only if for every open subset V, $f^{-1}(V)$ is open.

Proof: Suppose f is continuous and let V be open. Pick some $w \in f^{-1}(V)$. Since V is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(w)) \subseteq V$. Since f is continuous, there exists some $\delta > 0$ such that $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$. Thus $B_\delta(w) \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is open.

Now suppose for every open subset V that $f^{-1}(V)$ is open. Pick some $z_0 \in U$ and some $\varepsilon > 0$. Let $V = B_\varepsilon(f(z_0))$. Then V is open, so $f^{-1}(V)$ is open. Since $z_0 \in f^{-1}(V)$, there's some $\delta(z_0) > 0$ such that $B_{\delta(z_0)}(z_0) \subseteq f^{-1}(V)$. Thus whenever $|z - z_0| < \delta(z_0)$, then $|f(z) - f(z_0)| < \varepsilon$, which is continuity, as desired. ■

2.4. Problems

Problem (1): Find a family of infinitely many open sets which have an intersection which is not open.

Solution: Consider the family $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Note that each interval contains 0, and that any other real number will eventually no longer be an in interval (since $\frac{1}{n}$ gets arbitrarily small). Thus

$\bigcap_{n=1}^{\infty} I_n = \{0\}$. This is a closed set, and since each interval is open, we have the desired family of sets.

Problem (2): Find a clopen set and a set which is neither closed nor open.

Solution: Note that $\mathbb{C}$ is clopen. Any neighborhood about a point z in $\mathbb{C}$ is contained in $\mathbb{C}$, so it's open. Note that the complement of $\mathbb{C}$ is $\emptyset$, which is also open. Thus $\mathbb{C}$ is closed as well.

The interval $(0, 1]$ is neither closed nor open. No neighborhood of 1 is entirely contained in the interval. Similarly for the complement $(-\infty, 0] \cup (1, \infty)$, no neighborhood of 0 is entirely contained in the interval. Thus the interval is neither closed nor open.

Problem (3): Suppose $a_1, a_2, \ldots$ is a sequence of real numbers. Let A_k be the subsequence $a_k, a_{k+1}, \ldots$. Show that $\inf A_k \leq \inf A_{k+1}$ and $\sup A_k \geq \sup A_{k+1}$.

Solution: We prove the first inequality, as the second one follows similarly. By definition, we have $\inf A_k \leq a_i$ for $i = k, k+1, \ldots$. Thus $\inf A_k$ is less than or equal to every term in A_{k+1}, so it's a lower bound of A_{k+1}. Since $\inf A_{k+1}$ is by definition the greatest lower bound, we have $\inf A_k \leq \inf A_{k+1}$, as desired.

Problem (5): Suppose we write $f : \mathbb{C} \to \mathbb{C}$ in terms of its real and imaginary parts as $f(x + iy) = u(x, y) + iv(x, y)$. Show that f is continuous at $x + iy$ if and only if both u and v are continuous at (x, y).

Solution: First suppose f is continuous at $x + iy$. We show that u is continuous at (x, y), and v follows similarly. Pick arbitrary $\varepsilon > 0$. We need to show that there exists $\delta > 0$ such that

$$d((a, b), (x, y)) < \delta \Rightarrow |u(a, b) - u(x, y)| < \varepsilon,$$

where d is the Euclidean distance on $\mathbb{R}^2$. Note that since f is continuous at $x + iy$, there exists $\delta' > 0$ for which

$$|(a + ib) - (x + iy)| < \delta' \Rightarrow |f(a + ib) - f(x + iy)| = |u(a, b) - u(x, y) + i(v(a, b) - v(x, y))| < \varepsilon.$$

Note that $|(a + ib) - (x + iy)| < \delta'$ is equivalent to $d((a, b), (x, y)) < \delta'$ and that we have

$$|u(a, b) - u(x, y)| \leq |u(a, b) - u(x, y) + i(v(a, b) - v(x, y))| < \varepsilon.$$

Thus we can let $\delta = \delta'$.

Now suppose u and v are both continuous at (x, y). We show that f is continuous at $x + iy$. Pick arbitrary $\varepsilon > 0$. We need to show there exists δ such that

$$|(a + ib) - (x + iy)| < \delta \Rightarrow |f(a + ib) - f(x + iy)| < \varepsilon.$$

Since u and v are continuous at (x, y), there exists $\delta_1, \delta_2 > 0$ such that

$$d((a,b),(x,y)) < \delta_1 \Rightarrow |u(a,b) - u(x,y)| < \frac{\varepsilon}{2},$$
$$d((a,b),(x,y)) < \delta_2 \Rightarrow |v(a,b) - v(x,y)| < \frac{\varepsilon}{2}.$$

We claim that $\delta = \min\{\delta_1, \delta_2\}$ works. Indeed, if we have $|(a+ib) - (x+iy)| < \delta$, then we also have $d((a,b),(x,y)) < \delta < \delta_1, \delta_2$. Thus we have

$$|f(a+ib) - f(x+iy)| = |u(a,b) - u(x,y) + i(v(a,b) - v(x,y))| \leq |u(a,b) - u(x,y)| + |v(a,b) - v(x,y)| < \varepsilon.$$

Problem (9): A set $S \subseteq \mathbb{C}$ is closed if and only if it contains all its boundary point, if and only if every Cauchy sequence with all terms in S converges to a term in S.

Solution: First we prove that if S is closed, then it contains all its boundary points. Suppose for the sake of contradiction that the boundary point z is not in S. Thus $z \in S^c$ and S^c is open. However, since z is a boundary point, $B_\varepsilon(z)$ contains points in S and S^c for all $\varepsilon > 0$. This contradicts z being contained in the open set S^c.

Now suppose S contains all its boundary points. Suppose for the sake of contradiction that some sequence with all its terms in S converges to $L \notin S$ (Cauchy sequences and convergent sequences are equivalent in $\mathbb{C}$). Note that this implies L must be a boundary point, since otherwise there's some arbitrary disk which is no longer contained in S, but must contain elements from the sequence by definition. However this contradicts S containing all its boundary points. This if S contains all its boundary points, then every sequence in S converges to a term in S, as desired.

Now suppose every sequence in S converges into S. For the sake of contradiction suppose S is not closed. Then S^c is not open. Thus we can pick a point $z \in S^c$ which is a boundary point (since this point doesn't have disk around it which lies entirely in S^c). Thus we can construct a sequence entirely out of points in S that get arbitrarily close to z, which implies the sequence converges to z. This is a contradiction, as desired.

Problem (11): Let $s_1 = \sqrt{2}$ and define

$$s_{n+1} = \sqrt{2 + s_n}.$$

Prove that the sequence converges and find its limit.

Solution: Note that the sequence is increasing and each term is less than 2 (these can both be proved by induction). Thus by the monotone convergence theorem, (s_n) converges. Suppose is converges to L. Then

$$L = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sqrt{2 + s_{n-1}} \Rightarrow L^2 = \lim_{n\to\infty} (2 + s_{n-1}) = 2 + L.$$

Thus $L^2 - L - 2 = 0$, which has solutions $-1, 2$. Since the sequence is increasing and starts at $\sqrt{2}$, we must have $L = 2$.

Problem (13): Show that a convergent sequence is also Cauchy.

Solution: Suppose $(z_n) \to L$. Pick N such that $n \geq N \Rightarrow |z_n - L| < \frac{\varepsilon}{2}$. Then for any $n, m \geq N$, we have

$$|z_n - z_m| \leq |z_n - L| + |L - z_m| < \varepsilon,$$

so (z_n) is indeed Cauchy.

Problem (14): Let $z_1, z_2, \ldots$ be a bounded sequence in $\mathbb{C}$. Show that the sequence has a convergent subsequence $z_{n_1}, z_{n_2}, \ldots$.

Solution: Since the sequence is bounded, we can enclose all the terms in a square centered at the origin. Pick a random term to start at as z_{n_1} and throw away all terms that have index smaller than n_1. Note we will only throw away a finite number of terms. Now divide the square into four equally sized quadrants. One of these quadrants will have to have infinitely many terms. Choose a random term from such a quadrant to be z_{n_2}. Then throw away all terms in the other quadrants and all terms with index less than n_2. We can keep repeating this process and we will get a subsequence $z_{n_1}, z_{n_2}, \ldots$. Note that at every step, the terms get closer and closer to each other (in particular, if the initial square has side length s, then every term in $z_{n_i}, z_{n_{i+1}}, \ldots$ will be within $\frac{s}{2^i}\sqrt{2}$ of each other). Thus the subsequence is Cauchy and converges.

Problem (15): Suppose $z_1, z_2, \ldots$ is a sequence in $\mathbb{C}$, and $L \in \mathbb{C}$ such that $\lim_{n \to \infty} z_{2n} = L = \lim_{n \to \infty} z_{2n+1}$. Prove that $\lim_{n \to \infty} z_n = L$.

Solution: Note for $\varepsilon > 0$, there exists N_1 and N_2 such that $n \geq N_1 \Rightarrow |z_{2n} - L| < \varepsilon$ and $n \geq N_2 \Rightarrow |z_{2n+1} - L| < \varepsilon$. Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, both of the inequalities above hold. However, that implies that all z_n with $n \geq 2N$ are within ε of L. This works for arbitrary ε, so we do indeed have $\lim_{n \to \infty} z_n = L$.

Problem (16): Show that if (z_n) converges, then $(|z_n|)$ converges and find a counterexample for the other direction.

Solution: Suppose $(z_n) \to L$. Then for $\varepsilon > 0$, there exists N such that $n \geq N \Rightarrow |z_n - L| < \varepsilon$. We then have

$$||z_n| - |L|| \leq |z_n - L| < \varepsilon$$

for all $n \geq N$. Thus $(|z_n|) \to |L|$. For the reverse direction, note that $(-1)^n$ doesn't converge regularly but does converge under absolute value to 1.

3. Functional convergence

3.1. Compactness

Theorem (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

Proof: Proved in earlier problems. ■

Proposition: Let X_1, X_2 be closed nonempty nested intervals with X_1 bounded. Then the intersection is nonempty.

Proof: Choose $z_i \in X_i$. Since these all lie in X_1, by BW we have a convergent subsequence $z_{n_1}, \ldots$. Let its limit be z. Suppose $z \notin X_{n_k}$ for some k. Since its closed, we have $B_\varepsilon(z) \subseteq X_{n_k}^c$ for some ε. However, we have $\left|z_{n_m} - z\right| < \varepsilon$ for sufficiently large m, which implies $z_{n_m} \notin X_{n_k}$. For $m > k$, we have $z_{n_m} \in X_{n_m} \subseteq X_{n_k}$, contradiction. ■

Theorem (Heine-Borel): A set is compact if and only if it's closed and bounded.

Proof: Suppose X is closed and bounded by the square $|\mathfrak{R}|, |\mathfrak{I}| \leq B$. Let $\{U_\alpha\}$ be an open cover of X. Suppose there's no finite subcover. Divide the square into four subsquares. Since there's no fininte subcover, some subsquare has no finite subcover, say S_1. Now divide S_1 into four subsquares, and again note one of them, say S_2, doesn't have a finite subcover. Continue doing this, getting squares $S_1, S_2, \ldots$. Since the squares are a bounded decreasing sequence of nonempty closed subsets, their intersection is nonempty. However, since their diameter is tending to 0, their intersection can't be more than just a single point, so their intersection must be a single point, say z. Since the U_α's cover X, and thus z, some U_i, say U_1, covers z. Since U_1 is open, $B_\varepsilon(z) \subseteq U_1$ for some ε. But for sufficiently large n, $S_n \subseteq B_\varepsilon(z) \subseteq U_1$, which contradicts the S_i not having a fininte subcover.

Now suppose X is compact. Suppose it's unbounded. For each $n \in \mathbb{N}$, we can find z_n with $|z_n| > n$. Consider an open cover $U_1, \ldots$ of $\mathbb{C}$, with U_n being the ball of radius n around the origin. We have $z_n \notin U_n$. Thus there cannot be a finite subcover, contradiction.

Now we show X is closed. Let $z_0 \notin X$. Let $U_n = \left\{z \in \mathbb{C} : |z - z_0| > \frac{1}{n}\right\}$. Note that union of all of these sets is $\mathbb{C} \setminus \{z_0\}$, so they form an open cover of X. Since X is compact, it has a finite subcover. Since the U_n's are nested, we have $X \subseteq U_n$ for some n. Thus $B_{\frac{1}{n}}(z_0) \cap X = \emptyset$. Thus X^c is open, as desired. ■

Proposition: A set X if compact if and only if for every sequence in X, there exists some subsequence which converges into X.

Proof: Suppose X is compact and $(z_n) \in X$. We need to show that there's some $z \in X$ such that every neighborhood of z contains infinitely terms of the sequence. Suppose not. Then for every $z \in X$, some neighborhood U_z contains finitely many z_n's. Because $X \subseteq \bigcup_{z \in X} U_z$ is an open cover of X, there is a finite subcover. But each set in the finite cover only contains finitely many terms, so they cannot cover X, contradiction.

Now suppose every sequence has a convergent subsequence that converges into X. Suppose X is unbounded. Then for every n, we can find some $z_n \in X$ with $|z_n| > n$. Thus there cannot be a convergent subsequence.

Now suppose X^c is not open. Then there's some $z \in X^c$ such that for all $\varepsilon > 0$, $B_\varepsilon(z) \cap X \neq \emptyset$. For each n, we can find some $z_n \in B_{\frac{1}{n}}(z) \cap X$. The sequence $z_1, z_2, \ldots$ converges to z, so any subsequence does as well. Thus this sequence has no subsequence converging to a point in X. ■

Proposition: Let X be compact, U be an open set containing X, and let $f : U \to \mathbb{C}$ be a continuous function. Then $f(X)$ is also compact.

Proof: Let $\{V_\alpha\}$ be an open cover of $f(X)$. Let $U_\alpha = f^{-1}(V_\alpha) \subseteq X$. Thus each U_α is open, and they together form a open cover of X. Thus there exists a fininte subcover $U_1, \ldots, U_n$. Thus $V_1, \ldots, V_n$ is an finite subcover of $f(X)$, as desired. ■

Corollary (extreme value theorem): A continuous function with codomain $\mathbb{R}$ will always attain a min/max on a nonempty compact set.

Proof: The image of the compact set will also be compact. Thus the image is bounded, so there's a sup and inf. Since we can construct a sequence with these two numbers as their limits, they are limit points, and so must be contained in the image. By definition they're upper and lower bounds, so we're done. ■

3.2. Uniform Continuity

Definition (uniform continuity): A function $f : U \to \mathbb{C}$ is said to be *uniformly continuous* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $z, w \in U$ with $|z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon$.

Theorem: Let $f : U \to \mathbb{C}$ be continuous and $X \subseteq U$ be compact. Then f is uniformly continuous on X.

Proof: Same as proof on $\mathbb{R}$. ■

3.3. Pointwise and Uniform Convergence

Definition (functional convergence): Let $U \subseteq \mathbb{C}$ be an open set, and let $f_n : U \to \mathbb{C}$.

a) If there exists $f : U \to \mathbb{C}$ such that for every $z \in U, \lim_{n\to\infty} f_n(z) = f(z)$, then $f_n \to f$ *pointwise.*
b) If there exists $f : U \to \mathbb{C}$ such that for all $\varepsilon > 0$ there's some N for which $n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon$, then $f_n \to f$ *uniformly.*

Proposition: Let $U \subseteq \mathbb{C}$ be an open set and (f_n) be a sequence of functions on U converging uniformly to f. Then f is continuous.

Proof: Let $z_0 \in U$ and $\varepsilon > 0$. We need to show that there's some $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Note that

$$|f(z) - f(z_0)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)|.$$

Since (f_n) converges uniformly, there's some N such that $n \geq N$ and $w \in U$ implies that $|f_n(w) - f(w)| < \frac{\varepsilon}{3}$. Pick some $n > N$. Let $w = z_0$ and $w = z$. Thus we have

$$|f_n(z_0) - f(z_0)| < \frac{\varepsilon}{3} \quad \text{and} \quad |f_n(z) - f(z)| < \frac{\varepsilon}{3}.$$

The middle term can be made less than $\frac{\varepsilon}{3}$ by continuity, so we're done. ■

3.4. Equicontinuity

Definition (equicontinuity): Let $\mathcal{F}$ be a familiy of functions with each $f : U \to \mathbb{C}$.

- $\mathcal{F}$ is *equicontinuous* at $z_0 \in U$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in \mathcal{F}$ and all z with $|z - z_0| < \delta$ we have $|f(z) - f(z_0)| < \varepsilon$ (also called pointwise equicontinuity).
- $\mathcal{F}$ is *uniformly equicontinuous* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in \mathcal{F}$ and all z, w with $|z - w| < \delta$ we have $|f(z) - f(w)| < \varepsilon$.

Example: Let $[a, b] \in \mathbb{R}$ (so compact) and let $M > 0$. Let $\mathcal{F}$ be the set of functions $f : [a, b] \to \mathbb{R}$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$. We claim that $\mathcal{F}$ is uniformly equicontinuous. Note that $x, y \in [a, b]$ with $y \leq x$, then

$$|f(x) - f(y)| = \left| \int_y^x f'(t)\, dt \right| \leq \int_y^x |f'(t)|\, dt \leq M(x - y).$$

Thus $\varepsilon > 0$ is arbitrary, we can take $\delta = \frac{\varepsilon}{M}$ such that $|y - x| < \delta \Rightarrow |f(y) - f(x)| \leq M|y - x| < M\delta = \varepsilon$.

Theorem (Arzela-Ascoli theorem): Let $X \subseteq \mathbb{C}$ be a bounded set, and let $\mathcal{F}$ be an infinite family of uniformly bounded and uniformly equicontinuous functions $f : X \to \mathbb{C}$. Then there exists a uniformly convergent subsequence $f_1, f_2, \ldots \in \mathcal{F}$.

Proof: We first find a countable and dense subset $z_1, z_2, \ldots$ of X. We can extract countably many of the functions in $\mathcal{F}$ and make a sequence out of them, say $g_1, g_2, \ldots$. Because $\mathcal{F}$ is uniformly bounded, say by M, some subsequence $g_{1,1}, g_{1,2} \ldots$ exists such that $g_{1,1}(z_1), g_{1,2}(z_1), \ldots$ forms a convergent subsequence.

Now, out of $g_{1,1}, g_{1,2} \ldots$, find a subsequence $g_{2,1}, g_{2,2}, \ldots$ such that $g_{2,1}(z_2), g_{2,2}(z_2)$ for a convergent sequence. Note also that $g_{2,1}(z_1), g_{2,2}(z_1)$ also converges, since its a subsequence of a convergent sequence. We can keep doing this, and so we have that $g_{k,1}(z_k), g_{k,2}(z_k), \ldots$ converges for all k.

Let $f_n = g_{n,n}$. Note that for each k, the sequence $f_1(z_k), f_2(z_k) \ldots$ is a convergent sequence, since $f_k(z_k), f_{k+1}(z_k) \ldots$ is a subsequence of the convergent sequence $g_{k,1}(z_k), g_{k,2}(z_k)$ (remember that each $(g_{k,i})$ is a subsequence of $(g_{k-1,i})$), and the first finitely many terms don't matter. Thus $f_1(z_k), f_2(z_k), \ldots$ converges.

Now we show that $f_1, f_2, \ldots$ is uniformly convergent. Let $\varepsilon > 0$, and choose $\delta > 0$ such that $z, w \in X$ are such that $|z - w| < \delta \Rightarrow |f(z) - f(w)| < \frac{\varepsilon}{3}$ for all $f \in \mathcal{F}$, in particular $|f_n(z) - f_n(w)| < \frac{\varepsilon}{3}$ (which exists by uniform equicontinuity). Find some N such that for every $z \in X$, there exists some n with $1 \leq n \leq N$ such that $|z - z_n| < \delta$ (we can show N must be finite by first taking the closure of X, which makes it compact, and then taking the union of all $B_\delta(z_n)$. Since $z_1, z_2, \ldots$ is dense, this union must cover X. Thus there exists a finite subcover of the closure. Since they cover $\overline{X}$ and $X \subseteq \overline{X}$, they also cover X. Thus there is a maximal n among the cover, and we can set N to be that maximum). Because $f_1, f_2, \ldots$ converges at each z_n, there is some K such that if $\ell, k \geq K$, then

$$|f_\ell(z_n) - f_k(z_n)| < \frac{\varepsilon}{3} \quad \text{for all } n \text{ with } 1 \leq n \leq N.$$

Now pick $z \in X$. There exists some $n \leq N$ such that $|z - z_n| < \delta$, so if $\ell, m > K$, then

$$|f_k(z) - f_\ell(z)| \leq |f_k(z) - f_k(z_n)| + |f_k(z_n) - f_\ell(z_n)| + |f_\ell(z_n) - f_\ell(z)|.$$

The first and third terms are $< \frac{\varepsilon}{3}$ because of the choice of δ, and the second term is $< \frac{\varepsilon}{3}$ by the above. Thus we have $|f_k(z) - f_\ell(z)| < \varepsilon$ for all $z \in X$ and all $k, \ell \geq K$. Thus $f_1, f_2, \ldots$ is uniformly convergent on X.

∎

3.5. Problems

Problem (1): Let X and Y be two nonempty compact subsets of $\mathbb{C}$. Define their distance $d(X, Y)$ to be

$$d(X, Y) = \inf_{\substack{x \in X \\ y \in Y}} |x - y|.$$

Show that the infinum is obtained, and provide a counterexample for noncompact X and Y.

Solution: Define $f : Y \to \mathbb{R}$ by

$$f(y) = \inf_{x \in X} |x - y|.$$

Note that since X is compact and that function $|x - y|$ is continuous in X, $|x - y|$ will achieve a minimum on X by the extreme value theorem. Thus we can write

$$f(y) = \min_{x \in X} |x - y|.$$

Next we show that $f(y)$ is continuous, and then again by the extreme value theorem we get than $f(y)$ has a minimum. By construction, this value will be smaller than $|x - y|$ for all $x \in X, y \in Y$, and thus must be the infinum, implying the infinum is obtainable.

Pick $y_0 \in Y$ and $\varepsilon > 0$. Let $\delta = \varepsilon$. Then, for all $y \in Y$ such that $|y - y_0| < \delta$, we have

$$\min_{x \in X} |x - y| \leq \min_{x \in X} (|x - y_0| + |y_0 - y|) < \delta + \min_{x \in X} |x - y_0|.$$

From this we obtain

$$\left| \min_{x \in X} |x - y| - \min_{x \in X} |x - y_0| \right| < |\delta| = \varepsilon.$$

Thus $f(y)$ is continuous at y_0. This holds for any $y_0 \in Y$, so we have that f is continuous on Y, as desired.

Problem (4): Show that a uniformly continuous function $f : (0, 1) \to \mathbb{C}$ is bounded.

Solution: Pick some arbitrary $\varepsilon > 0$. By uniform continuity, there exists $\delta > 0$ such that for all z with $\left|z - \frac{1}{2}\right| < \delta$, we have $\left|f(z) - f\left(\frac{1}{2}\right)\right| < \varepsilon$. Thus all $z \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right)$ are bounded by either $\left|f\left(\frac{1}{2}\right) - \varepsilon\right|$ or $\left|f\left(\frac{1}{2}\right) + \varepsilon\right|$ (depending on which one is larger). We can now do the same thing at the points $\frac{1}{2} - \delta, \frac{1}{2} + \delta$, so f will be bounded on the interval $\left(\frac{1}{2} - 2\delta, \frac{1}{2} + 2\delta\right)$. We can keep doing this, and eventually we'll cover all of $(0, 1)$. Thus f is bounded on that interval, as desired.

Problem (5): Suppose $f, g : U \to \mathbb{C}$ are uniformly continuous. Show that $f + g$ is uniformly continuous. Give an example that shows that fg need not be uniformly continuous.

Solution: Let $\varepsilon > 0$. By uniform continuity, there exists $\delta_1, \delta_2 > 0$ such that for all $z, w \in U$ with $|z - w| < \delta_1$, we have $|f(z) - f(w)| < \frac{\varepsilon}{2}$ and for all $z, w \in U$ with $|z - w| < \delta_2$, we have $|g(z) - g(w)| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all $z, w \in U$ with $|z - w| < \delta$, we have

$$|f(z) + g(z) - f(w) - g(w)| \leq |f(z) - f(w)| + |g(z) - g(w)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $f + g$ is uniformly continuous on U

Consider $f : \mathbb{C} \to \mathbb{C}$ as $f(x) = x$. Then $f^2 = x^2$, which is not uniformly continuous on $\mathbb{C}$, while x is uniformly continuous on $\mathbb{C}$.

Problem (9): Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n & x < \frac{1}{n}, \\ 0 & x \geq \frac{1}{n}. \end{cases}$$

Show that f_n converges pointwise to an integrable function f on $[0, 1]$, but that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

Proof: Note that for $x_0 \in [0, 1]$, we have $\lim_{n\to\infty} f_n(x_0) = 0$, since by the Archimedean principle there's some N such that $\frac{1}{N} < x_0$, and for all $n \geq N$, we have $\frac{1}{n} < x$. Thus $f_n \to 0$ pointwise, so the integral on the right is 0. However, we have

$$\int_0^1 f_n(x)\,dx = \int_0^{\frac{1}{n}} n\,dx + \int_{\frac{1}{n}}^1 0\,dx = 1.$$

This clearly has limit 1, so the two sides of the equation don't agree. ■

Problem (11): Prove the Weierstrass M-test.

Solution: Suppose $f_1, f_2, \ldots : \mathbb{C} \to \mathbb{C}$ is a sequence of functions, and that $|f_i(z)| \leq M_i$ for all $z \in \mathbb{C}$ and some $M_i \geq 0$. Pick $\varepsilon > 0$. Note that the partial sums of (M_n) (let them be (S_n)) converge, so there exists N such that $n \geq m \geq N \Rightarrow |S_n - S_m| < \varepsilon \Rightarrow |M_n + M_{n-1} + \cdots + M_{m+1}| = M_n + M_{n-1} + \cdots + M_{m+1} < \varepsilon$. However, from this we obtain

$$|f_n(z) + \cdots + f_{m+1}(z)| \leq |f_n(z)| + \cdots + |f_{m+1}(z)| \leq M_n + \cdots + M_{m+1} < \varepsilon$$

for all $z \in \mathbb{C}$. Thus the partial sums of (f_n) are Cauchy, which proves that the sum of all f_n converges uniformly.

Problem (14): Find a sequence of differentiable functions $f_n : [0, 1] \to \mathbb{R}$ such that f_n converges uniformly to 0 on $[0, 1]$, but such that f_n' does not converge to 0.

Solution: Consider $f_n(x) = \frac{x}{1+nx^2}$. Using derivatives we can calculate that the this function has a maximum of $\frac{1}{2\sqrt{n}}$ at $\frac{1}{\sqrt{n}}$ and similarly a minimum of $-\frac{1}{2\sqrt{n}}$ at $-\frac{1}{\sqrt{n}}$. Thus for any $\varepsilon > 0$, we can let $N = \frac{1}{\varepsilon^2}$, and for all $n > N$, we will have $\left|\frac{x}{1+nx^2} - 0\right| < \frac{1}{\sqrt{n}} < \varepsilon$. Thus $f_n \to 0$ uniformly.

We have $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$. Note that $f_n'(0) = 1$ for all n, which implies f_n' doesn't converge to 0.

Problem (15): Let M be a positive real number, and let $\mathcal{F}$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$. Show that $\mathcal{F}$ is a uniformly equicontinuous family.

Solution: Pick $\varepsilon > 0$, and let $\delta = \frac{\varepsilon}{M}$. Then for all $x, y \in \mathbb{R}$ such that $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq M|x - y| < M\delta = \varepsilon.$$

Thus $\mathcal{F}$ is uniformly equicontinuous.

Problem (16): Find a family of functions that is pointwise equicontinuous but not uniformly equicontinuous.

Solution: Let $\mathcal{F}$ be the family of functions $x^2 + c$ with $c \in \mathbb{C}$ with domain $\mathbb{C}$. Clearly these are pointwise equicontinuous (since they're basically all the same function). However, none of them are uniformly continuous on $\mathbb{C}$, so they can't be uniformly equicontinuous.

Problem (17): Show that if $\mathcal{F}$ is a pointwise equicontinuous family on a compact set, then it is uniformly equicontinuous.

Solution: Let X be the compact set and $\varepsilon > 0$. By equicontinuity, for each $z \in X$, there exists $\delta_z > 0$ such that for all $f \in \mathcal{F}$ and all $w \in B_{\delta_z}(z)$, we have $|f(w) - f(z)| < \frac{\varepsilon}{2}$. Note clearly we have

$$X \subseteq \bigcup_{z \in X} B_{\frac{\delta_z}{2}}(z),$$

so by compactness there's some finite subcover using $z_1, z_2, ..., z_n$. Let $\delta = \frac{1}{2}\min\left(\delta_{z_1}, \delta_{z_2}, ..., \delta_{z_n}\right)$. Now suppose $z, w \in X$ such that $|z - w| < \delta$. We can pick i such that $|z - z_i| < \frac{\delta_{z_i}}{2}$ (from the finite subcover), and then from the triangle inequality obtain $|w - z_i| \leq |w - z| + |z - z_i| < \frac{\delta_{z_i}}{2} + \delta < \delta_{z_i}$. Thus we have

$$|f(z) - f(w)| = |f(z) - f(z_i)| + |f(z_i) - f(w)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This holds for all $f \in \mathcal{F}$ and for arbitrary ε, so we do indeed have uniform equicontinuity.

Problem (19): Suppose $\mathcal{F}$ is an equicontinuous family of functions $f : \mathbb{R} \to \mathbb{R}$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous. Is $\{\varphi \circ f : f \in \mathcal{F}\}$ necessarily equicontinuous?

Solution: Let $\mathcal{F} = \{x + c : c \in \mathbb{R}\}$. Clearly this is uniformly equicontinuous. Then let $\varphi(x) = x^2$. Thus the new family of functions is $\{(x+c)^2 : c \in \mathbb{R}\}$. Note that at a point like $x = 0$, as c increases, the absolute value of the derivate of the function increases. Thus as c increases, we need smaller and smaller δ to stay within ε of $f(0)$. Thus this family isn't equicontinuous.

4. Holomorphic Functions and Power Series

4.1. Differentiation

The definition of a derivative on $\mathbb{C}$ is basically the same as on $\mathbb{R}$.

Definition (differentiable): Let $U \subseteq \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$ be a function. Then f is *differentiable* at a point $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, and is denoted $f'(z_0)$. Equivalently,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Remark: Differentiability on $\mathbb{C}$ is much stronger than on $\mathbb{R}$, since the necessary limit must hold along any path approaching z_0, rather than just left or right. In fact, it's stricter than the real and imaginary parts being differentiably on $\mathbb{R}^2$ as well.

Example: Let $f(z) = \overline{z}$. Then

$$f'(z_0) = \lim_{h \to 0} \frac{\overline{z_0 + h} - \overline{z_0}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}.$$

Thus is 1 along $\mathfrak{I}(h) = 0$ and -1 along $\mathfrak{R}(h) = 0$, so f is not differentiable anywhere.

Example: Let $f(z) = |z|^2$. Manipulating the difference quotient yields

$$\overline{z_0} + \overline{h} + z_0 \frac{\overline{h}}{h}.$$

As we saw earlier, the third term causes the limit to never exist for $z_0 \neq 0$. In the case when $z_0 = 0$, then the limit is simply 0. Thus $f'(0) = 0$, and the derivative exists nowhere else.

Theorem (Cauchy-Riemann equations): Let $U \subseteq \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$ be a complex function on U with $f(z) = u(x, y) + iv(x, y)$. Suppose at some point $z_0 \in U$, u and v have continuous partial derivatives with respect to both x and y in a neighborhood of z_0. Then f is differentiable at z_0 if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold at z_0.

Proof: Suppose suppose f is differentiable at $z_0 = x_0 + iy_0$, and write $\Delta z = \Delta z + i\Delta y$. Then

$$\frac{\Delta w}{\Delta z} = \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}.$$

Letting $\Delta y = 0$ and letting Δx arrow 0 (so approaching $\Delta z \to 0$ horizontally), the right side becomes

$$\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Swapping the roles of Δx and Δy (so approaching $\Delta z \to 0$ vertically), the right side becomes

$$-i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0).$$

In order for $\frac{\Delta w}{\Delta z}$ to exist as $\Delta z \to 0$, we must get the same limit by approaching 0 in any way, so we necessarily obtain

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Taking real and imaginary parts yields the Cauchy Riemann equations.

Now suppose the partial derivatives are continuous in a neighborhood of z_0 and the Cauchy-Riemann equations hold. We need to show that f is differentiable.

Pick small h and let $h = \Delta x + i\Delta y$. We write $\Delta f(z_0) = f(z_0 + h) - f(z_0)$, so $\Delta f(z_0) = \Delta u + i\Delta v$, where

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0),$$
$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).$$

By the continuity of partial derivatives, we can write

$$\Delta u = \frac{\partial u}{\partial x}(x_0, y_0)\Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y,$$
$$\Delta v = \frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0$ as $(\Delta x, \Delta y) \to 0$ (basically u and v are well approximated by tangent planes).

Thus, using the Cauchy-Riemann equations to write all the partials with respect to x, we obtain

$$\Delta f(z_0) = \frac{\partial u}{\partial x}(x_0, y_0)h + i\frac{\partial v}{\partial x}(x_0, y_0)h + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y.$$

Since $|\Delta x|, |\Delta y| \leq h$, the quotients, $\frac{\Delta x}{h}, \frac{\Delta y}{h}$ are at most 1, so dividing the above equation by h and using this plus the triangle inequality, we obtain

$$\frac{\Delta f(z_0)}{h} \leq \frac{\partial u}{\partial x}(x_0, y_0)h + i\frac{\partial v}{\partial x}(x_0, y_0)h + |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| + |\varepsilon_4|.$$

Letting $h \to 0$, we obtain

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0),$$

so the derivate exists, as desired. ■

4.2. Holomorphic Functions

Definition (holomorphic): Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be differentiable at every point on U. Then f is *holomorphic*. For $z_0 \in U$, we say that f is holomorphic at z_0 if it's holomorphic in some neighborhood of z_0.

Definition (entire): A holomorphic function with domain $\mathbb{C}$ is called *entire*.

Proposition: If f and g are two holomorphic functions on an open set U, then $f \pm g$ and fg are holomorphic on U. If $g(z) \neq 0$ for all $z \in U$, then $\frac{f}{g}$ is also holomorphic. If $f : U \to V$ and $g : V \to \mathbb{C}$ are holomorphic, then $g \circ f : U \to \mathbb{C}$ is also holomorphic.

Proof: Problem 10 ■

4.3. Taylor Series

Theorem (Taylor's theorem): Let $f : U \to \mathbb{C}$ be differentiable, and let $z_0 \in U$. If $B_r(z_0) \subseteq U$, then the power Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges to $f(z)$ in $B_r(z_0)$.

Theorem (Cauchy-Hadamard theorem): Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. Let

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}.$$

Then the power series converges on $|z - z_0| < R$ and diverges on $|z - z_0| > 0$.

Proof: Problem 11. ■

4.4. Laurent Series

Definition (Laurent series): A series of the form $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ is called a *Laurent series.*

Example: $f(z) = \frac{e^z}{z}$ has a Laurent series

$$\sum_{n=-1}^{\infty} \frac{z^n}{(n+1)!}.$$

Example: Let $f(z) = e^{\frac{1}{z}}$. Then plugging $\frac{1}{z}$ into the MacLaurin series for e^z and rewriting the sum yields

$$\sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}.$$

4.5. Singularities

Definition (singularities): Suppose $z_0 \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ is some open subset such that $z_0 \in U$ by U contains a deleted neighborhood of z_0.

- If there's a continuous function $f : U \cup \{z_0\} \to \mathbb{C}$ such that $g(z) = f(z)$ for any $z \in U$, then z_0 is a *removable singularity.*
- If $\lim_{z \to z_0} f(z) = \infty$, then we say that z_0 is a *pole* of f.
- If $\lim_{z \to z_0} f(z)$ does not exist and is not infinity, then z_0 is an *essential signularity.*

4.6. Problems

Problem (2): Find the power series of $\frac{1}{(1-z)^n}$ about $z = 0$ for positive integers n.

Solution: Using Taylor's theorem yields

$$1 + nz + \frac{n(n+1)}{2!} z^2 + \frac{n(n+1)(n+2)}{3!} z^3 + \cdots = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} z^k.$$

Problem (5): Find the power series expansion of $\frac{z}{e^z - 1}$. Without explicitly calculating, what's the radius of convergence?

Solution: The radius of convergence is 2π, since at $z = 2\pi i$, the denominator of the function is 0 while the numerator is nonzero ($z = 0$ isn't an issue since we just have a removable discontinuity there).

Note that

$$\coth z = \frac{e^z + e^{-z}}{e^z - e^{-z}} = \frac{e^{2z} + 1}{e^{2z} - 1} = 1 + \frac{2}{e^{2z} - 1}.$$

Thus

$$\frac{z}{e^z - 1} = \frac{z}{2}\left(\coth \frac{z}{2} - 1\right).$$

The power series of $\coth z$ is

$$\sum_{n=0}^{\infty} \frac{2^{2n} B_{2n} x^{2n-1}}{(2n)!},$$

where B_k are the Bernoulli numbers. Plugging this into our equation, we get that

$$\frac{z}{e^z - 1} = -\frac{z}{2} + \sum_{n=0}^{\infty} \frac{B_{2n} x^{2n}}{(2n)!}.$$

To find the radius of convergence, we need to find

$$\limsup_{n \to \infty} \left| \frac{B_{2n}}{(2n)!} \right|^{\frac{1}{2n}}.$$

Using the asymptotic for B_n and Stirling's approximation, we have

$$\left| \frac{B_{2n}}{(2n)!} \right|^{\frac{1}{2n}} = \left(\frac{4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}}{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}} \right)^{\frac{1}{2n}} = \frac{2^{\frac{1}{n}}}{2\pi}.$$

Clearly the lim sup of this is 2π, so the radius of convergence is indeed 2π.

Problem (6): Find the Laurent series expansion for $\frac{e^z + e^{\frac{1}{z}}}{z(z+1)}$ centered at 0.

Solution: Let $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!}$.

First we find the Laurent series for $\frac{e^z}{z+1} + \frac{e^{\frac{1}{z}}}{z+1}$. For the first term, note that we are multiplying the Taylor series for e^z and $\frac{1}{1+z}$. Thus

$$\frac{e^z}{z+1} = \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)(1 - z + z^2 - z^3 + \cdots).$$

We can use convlutions to determine the coefficient of z^n, and we see that the coefficient will be a_n.

Now we do the same process with $\frac{e^{\frac{1}{z}}}{z+1}$ to obtain

$$\frac{e^{\frac{1}{z}}}{z+1} = \left(1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots\right)(1 - z + z^2 - z^3 + \cdots).$$

Let b_n be the coefficient of z^n in this expansion. First we deal with nonnegative n.

We can calculate that

$$\begin{aligned} b_0 &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = \frac{1}{e}, \\ b_1 &= -1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots = -\frac{1}{e}, \\ b_2 &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = \frac{1}{e}, \\ &\vdots \end{aligned}$$

Thus, for nonnegative n, we have $b_n = \frac{(-1)^n}{e}$.

For negative n, we calculate

$$\begin{aligned} b_{-1} &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots = -\frac{1}{e} + a_0, \\ b_{-2} &= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \frac{1}{e} + a_1, \\ b_{-3} &= \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \cdots = -\frac{1}{e} + a_2 \\ b_{-4} &= \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} + \cdots = \frac{1}{e} + a_3, \\ &\vdots \end{aligned}$$

Thus for negative n, we have $b_n = \frac{(-1)^n}{e} + a_{n-1}$.

Combining the series and dividing by z yields

$$\frac{e^z + e^{\frac{1}{z}}}{z(z+1)} = \sum_{n=-\infty}^{-2} \left(\frac{(-1)^{n+1}}{e} + a_{-n-2}\right) z^n + \sum_{n=-1}^{\infty} \left(\frac{(-1)^{n+1}}{e} + a_{n+1}\right) z^n.$$

Problem (7): Find the polar form of the Cauchy-Riemann equations.

Solution: We have

$$f(z) = r(x,y)e^{i\theta(x,y)} = r(x,y)\cos(\theta(x,y)) + ir(x,y)\sin(\theta(x,y)).$$

From here on we'll write r for $r(x,y)$ and likewise for θ. Using Cauchy-Riemann on the above, we obtain

$$\begin{aligned} \frac{\partial}{\partial x}(r\cos\theta) &= \frac{\partial}{\partial y}(r\sin\theta), \\ \frac{\partial}{\partial y}(r\cos\theta) &= -\frac{\partial}{\partial x}(r\sin\theta). \end{aligned}$$

Applying the chain rule and product rule yields

$$\frac{\partial r}{\partial x}\cos\theta - \frac{\partial \theta}{\partial x} r\sin\theta = \frac{\partial r}{\partial y}\sin\theta + \frac{\partial \theta}{\partial y} r\cos\theta,$$
$$\frac{\partial r}{\partial y}\cos\theta - \frac{\partial \theta}{\partial y} r\sin\theta = -\frac{\partial r}{\partial x}\sin\theta - \frac{\partial \theta}{\partial x} r\cos\theta.$$

Problem (8): Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. Where does the series converge? Do the same for $g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

Solution: Note that

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup_{n\to\infty} n^{\frac{1}{n}} = 1.$$

Thus $R = 1$ for f. This implies f converges on $|z| < 1$. For $|z| = 1$, f converges everywhere except $z = 1$, where we get the harmonic series (?).

Similarly we have

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup_{n\to\infty} n^{\frac{2}{n}} = 1.$$

Thus $R = 1$ for g, which means g converges on $|z| < 1$. Suppose $|z| = 1$. Then we have

$$\left|\sum_{n=1}^{\infty} \frac{z^n}{n^2}\right| \leq \sum_{n=1}^{\infty} \left|\frac{z^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Thus g converges for all z with $|z| = 0$, so overall g converges on $|z| \leq 1$.

Problem (9): Define the *Wirtinger derivatives* by

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Show that $\frac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the Cauchy-Riemann equations.

Solution: If $\frac{\partial f}{\partial \bar{z}} = 0$, then

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Writing $f(z) = u(x, y) + iv(x, y)$ and substituting yields

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating real and imaginary parts yields the Cauchy-Riemann equations. Similarly, given the Cauchy-Riemann equations, we can do these steps in reverse and obtain that $\frac{\partial f}{\partial \bar{z}} = 0$.

Problem (11): Prove the Cauchy-Hadamard theorem.

Solution: Without loss of generality suppose the power series is centered at 0, and let $\rho = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Pick $|z| < R = \frac{1}{\rho}$.

Note by properties of lim sup, there exists N such that for all $k \geq N$, we have

$$\rho + \frac{\frac{1}{|z|} - \rho}{2} \geq |a_k|^{\frac{1}{k}} \Rightarrow \left(\frac{\rho + \frac{1}{|z|}}{2}\right)^k \geq |a_k|.$$

Now we write the power series as

$$\sum_{n=0}^{N-1} a_n z^n + \sum_{n=N}^{\infty} a_n z^n.$$

The first sum is finite, so we only need to check the convergence of the second sum. We have

$$\left|\sum_{n=N}^{\infty} a_n z^n\right| \leq \sum_{n=N}^{\infty} |z^n| \left|\left(\frac{\rho + \frac{1}{|z|}}{2}\right)^n\right| = \sum_{n=N}^{\infty} \left(\frac{1 + \rho|z|}{2}\right)^n.$$

Note that the common ratio $\frac{1+\rho|z|}{2} < 1$, so the series does indeed converge.

Now suppose $|z| > R$. Again by lim sup properties, there exist infinitely many k such that

$$|a_k|^{\frac{1}{k}} > \rho - \frac{\frac{1}{|z|} - \rho}{2}.$$

This holds because if there were only finitely many k, then the limsup would be smaller. Thus for infinitely many k, we have

$$\left|a_k z^k\right| > \left|\frac{3\rho|z| - 1}{2}\right|^k.$$

Note that the inside is greater than 1, so infinitely many terms of the series are unbounded. Thus the series diverges.

Problem (12): Determine whether the following functions are the real parts of holomorphic functions on some nonempty subset of $\mathbb{C}$.

a) $x^2 - y^2$
b) $x^2 + y^2$
c) $x^3 - y^3$
d) $\sin(2x)\sinh(2y)$
e) $\sin(2x)\cosh(2y)$

Solution:

a) $f(z) = x^2 - y^2 + i(2xy + c)$

b) None. By Cauchy Riemann, the imaginary part will take the form $\int \left(\frac{\partial u}{\partial x}(x^2 + y^2)\right) dy = 2xy + c$. However, these two do not satisfy the second Cauchy Riemann equation.

c) Same reason as the last one.

d) $f(z) = \sin(2x)\sinh(2y) + i(\cos(2x)\cosh(2y) + c)$

e) Same reason as b) and c).

Problem (15): Let $n \in \mathbb{Z}$. Compute

$$\int_0^{2\pi} e^{inx}\, dx.$$

Solution: If n is nonzero, then

$$\int_0^{2\pi} e^{inx}\, dx = \frac{e^{inx}}{in} \,|_0^{2\pi} = \frac{\left(e^{2\pi i}\right)^n}{in} - \frac{1}{in} = 0.$$

If $n = 0$, then the integral is just 2π. Thus

$$\int_0^{2\pi} e^{inx}\, dx = \begin{cases} 0 & \text{if} \quad n \neq 0 \\ 2\pi & \text{if} \quad n = 0 \end{cases}$$

Problem (16): Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a function with period 2π. Suppose f can be represented as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}.$$

Determine a_n.

Solution: We claim

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx.$$

Substitute the sum for f to get

$$\int_0^{2\pi} \left(\sum_{k=-\infty}^{\infty} a_k e^{i(k-n)x} \right) dx.$$

Since the sum converges on all x, we can swap the sums and obtain

$$\sum_{k=-\infty}^{\infty} \left(\int_0^{2\pi} a_k e^{i(k-n)x}\, dx \right).$$

The integral on the inside is 0 when $k \neq n$ and 2π when $k = \pi$. Thus all the terms of the sum with $k \neq n$ vanish, and we're left with $2\pi a_n$. Thus

$$\int_0^{2\pi} f(x)e^{-inx}\,dx = 2\pi a_n,$$

so we're done.

5. Integration

5.1. Integration in $\mathbb{R}$

Standard Riemann sums. Given a function $f : [a, b] \to \mathbb{C}$, where $f(x) = u(x) + iv(x)$, we can write

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx.$$

5.2. Path Integrals in $\mathbb{R}^2$ and $\mathbb{C}$

We parametrize curves in $\mathbb{C}$ with a continuous function $\gamma : [0, 1] \to \mathbb{C}$ and then integrate along them (just like how we do so for line integrals in $\mathbb{R}^2$). $\gamma(t) = u(t) + iv(t)$ is differentiable if u and v are differentiable.

Definition (contour integral): Let $U \subseteq \mathbb{C}$ be an open set and $\Gamma \subseteq U$ a contour, and let $f : U \to \mathbb{C}$. Then we define

$$\int_\Gamma f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt,$$

where $\gamma : [0, 1] \to \mathbb{C}$ is a differentiable parametrization of Γ.

Example: Suppose Γ is the unit circle, which can be parametrized by $\gamma(t) = e^{2\pi it}$. For $n \in \mathbb{Z}$, we have

$$\begin{aligned}\int_\Gamma z^n\,dz &= \int_0^1 e^{2\pi int} \cdot 2\pi ie^{2\pi it}\,dt \\ &= 2\pi i\int_0^1 e^{2\pi i(n+1)t}\,dt.\end{aligned}$$

We showed in an earlier problem that this is 0 unless $n = -1$, in which case it's $2\pi i$.

Example: Now we integrate $f(z) = \overline{z}$.

$$\int_\Gamma \overline{z}\,dz = \int_0^1 e^{-2\pi it} \cdot 2\pi ie^{2\pi it}\,dz = 2\pi i.$$

5.3. Cauchy Integral Theorem

Theorem (Cauchy integral theorem): Let Γ be a closed contour and suppose $f(z)$ is holomorphic on Γ and its interior. Then

$$\int_\Gamma f(z)\,dz = 0.$$

Proof: Write $f(z) = u(x,y) + iv(x,y)$ and $dz = dx + i\,dy$. Then we have

$$\int_\Gamma f(z)\,dz = \int_\Gamma ((u(x,y)\,dx - v(x,y)\,dy) + i(u(x,y)\,dy + v(x,y)\,dx)).$$

Thus, by Green's Theorem, we have

$$\int_\Gamma f(z)\,dz = \iint_D \left[\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right].$$

By the Cauchy-Riemann equations, the terms in parantheses are 0, so we're done. ∎

Remark: This proof actually needs the assumption that f' is continuous, since Green's theorem requires that the partial derivatives are continuous. However in general, this not need be the case.

5.4. Path Invariance

5.5. Definite Integrals

5.6. Problems

Problem (1): Suppose that U is an open set and $f : U \to \mathbb{C}$ is a function so that for every closed contour Γ in U, we have $\int_\Gamma f(z)\,dz = 0$. Show that $z_1, z_2 \in U$ and Γ_1 and Γ_2 are two paths connecting z_1 and z_2. Show that $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$.

Solution: Let Γ_3 be the backwards traversal of Γ_2. Note that the contour formed by Γ_1 and Γ_3 is closed. Thus

$$0 = \int_{\Gamma_1+\Gamma_3} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_3} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz - \int_{\Gamma_1} f(z)\,dz,$$

as desired.

Problem (8): Let Γ be a closed contour. Evaluate $\int_\Gamma \mathfrak{R}(z)\,dz$ and $\int_\Gamma \mathfrak{I}(z)\,dz$.

Solution: Let D be the region enclosed by Γ. For the first integral, we have

$$\int_\Gamma (x)(dx + i\,dy) = \int_\Gamma x\,dx + ix\,dy = \iint_D \left(\frac{\partial}{\partial x}(ix) - \frac{\partial}{\partial y}(x)\right) dx\,dy = \iint_D i\,dx\,dy = i \cdot \mathrm{Area}(D).$$

We can do the same thing for the second integral to obtain

$$\int_\Gamma \mathfrak{I}(z)\,dz = -i \cdot \mathrm{Area}(D).$$

Problem (9): Suppose $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that for all $z \in \mathbb{C}$, $f(z+1) = f(z+i) = f(z)$. Let S be the contour that travels counterclockwise around the square with vertices at $0, 1, 1+i, i$. Show that $\int_S f(z)\,dz = 0$.

Solution: Let the segment from 0 to 1 be S_1, from 1 to $1+i$ be S_2, and so on. Define $\gamma_1, \gamma_2, \gamma_3, \gamma_4 : [0,1] \to \mathbb{C}$ as follows:

$$\begin{aligned}
\gamma_1(t) &= t, \\
\gamma_2(t) &= 1 + it, \\
\gamma_3(t) &= 1 - t + i, \\
\gamma_4(t) &= (1-t)i.
\end{aligned}$$

Then we have

$$\int_S f(z)\,dz = \int_{S_1} f(z)\,dz + \int_{S_2} f(z)\,dz + \int_{S_3} f(z)\,dz + \int_{S_4} f(z)\,dz =$$
$$\int_0^1 f(t)\,dt + i\int_0^1 f(1+it)\,dt - \int_0^1 f(1-t+i)\,dt - i\int_0^1 f(i-it)\,dt =$$
$$\int_0^1 f(t)\,dt - \int_0^1 f(1-t)\,dt + i\int_0^1 f(it)\,dt - i\int_0^1 f(i-it)\,dt.$$

Note that the second integral traverses the same segment as the first integral, just backwards, so the two are equal in value. Similarly, the fourth integral traverses the same segment as the third integral but backwards (this is essentially taking $u = 1-t$ in the first and fourth integrals). Thus the sum is 0, as desired.

Problem (10): Compute

$$\int_{|z-i|=R} \frac{z^4 + z^2 + 1}{z(z^2+1)}\,dz.$$

Solution: Let Γ be a circle with radius R centered at i. Then by partial fractions the desired integral is equal to

$$\int_\Gamma z\,dz + \int_\Gamma \frac{1}{z}\,dz - \frac{1}{2}\int_\Gamma \frac{1}{z+i}\,dz - \frac{1}{2}\int_\Gamma \frac{1}{z-i}\,dz.$$

The first integral vanishes, since z is entire and Γ is closed. Note for $R < 1$, every integral vanishes except the last one (since their poles aren't contained inside the circle). Using $\gamma(t) = i + e^{2\pi i t}$, we can calculate that the last integral is equal to $2\pi i$. Thus for $R < 1$, the desired integral is equal to $-\pi i$.

For $1 < R < 2$, the contour now contains both i and 0, so the second and last integral no longer vanish. We already know the value of the last integral. For the second integral, note that we can deform the path to be centered 0, and it will still have the same value. Again we calculate that this value is $2\pi i$. Thus for $1 < R < 2$, the desired integral is equal to $2\pi i - \pi i = \pi i$.

We do the same thing for $2 < R$, deforming the contour to be centered at $-i$. For $2 < R$, the desired integral is equal to $2\pi - \pi i - \pi i = 0$.

Thus,

$$\int_{|z-i|=R} \frac{z^4+z^2+1}{z(z^2+1)}\,dz = \begin{cases} -\pi i & \text{for} \quad R < 1 \\ \pi i & \text{for} \quad 1 < R < 2 \\ 0 & \text{for} \quad R < 2 \end{cases}$$

Problem (13): Compute

$$\int_0^\infty \sin(x^2)\,dx.$$

Solution: We integrate $f(z) = e^{iz^2}$ over an eigth circle. Let Γ_1 be the part of the eigth on the real line, let Γ_2 be the arc, and let Γ_3 be the other radius of the eigth circle. Thus

$$\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_3} f(z)\,dz = 0.$$

We're looking for $\Im\left(\int_{\Gamma_1} f(z)\,dz\right)$.

First we bound $\int_{\Gamma_2} f(z)\,dz$. Define $\gamma : \left[0, \frac{\pi}{4}\right] \to \mathbb{C}$ by $\gamma(t) = Re^{it}$. Thus this integral is equal to

$$\int_0^{\frac{\pi}{4}} e^{iR^2(\cos t + i\sin t)^2} iRe^{it}\,dt = iR\int_0^{\frac{\pi}{4}} e^{iR^2(\cos^2 t - \sin^2 t)} e^{-2R^2\cos t\sin t} e^{it}\,dt.$$

Taking the absolute value, we have

$$\left| iR\int_0^{\frac{\pi}{4}} e^{iR^2(\cos^2 t-\sin^2 t)} e^{-2R^2\cos t\sin t} e^{it}\,dt \right| \le R\int_0^{\frac{\pi}{4}} e^{-2R^2\cos t\sin t}\,dt = R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2t}\,dt.$$

Note that $\sin 2t \ge \frac{4}{\pi}t$ for $0 \le t \le \frac{\pi}{4}$, so we have

$$R\int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2t}\,dt \leq R\int_0^{\frac{\pi}{4}} e^{-\frac{4R^2 t}{\pi}}\,dt = R\left(-\frac{\pi}{4R^2}\left(e^{-R^2}-1\right)\right) = \frac{\pi}{4R}\left(1-e^{-R^2}\right).$$

Thus as $R \to \infty$, $\int_{\Gamma_2} f(z)\,dz \to 0$.

Now we look at $\int_{\Gamma_3} f(z)\,dz$. Define $\gamma : [0, R] \to \mathbb{C}$ by $\gamma(t) = te^{i\frac{\pi}{4}}$. Then the integral is equal to

$$\int_R^0 e^{it^2 e^{i\frac{\pi}{2}}} e^{i\frac{\pi}{4}}\,dt = -e^{i\frac{\pi}{4}} \int_0^R e^{-t^2}\,dt.$$

As $R \to \infty$, we get the Gaussian integral, so we have

$$\int_{\Gamma_3} f(z)\,dz = \sqrt{\frac{\pi}{8}} + i\sqrt{\frac{\pi}{8}}.$$

Thus $\int_{\Gamma_1} f(z)\,dz = -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}$. Taking real and imaginary parts, we obtain

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \sqrt{\frac{\pi}{8}}.$$

Problem (14): Compute

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x^2)}\,dx.$$

Solution: Let $f(z) = \frac{1}{\sqrt{z}(1+z^2)}$. We integrate over a keyhole contour. Let Γ_1 be the inner circle, Γ_2 be the line above the positive real axis, Γ_3 be the large outer circle, and Γ_4 be the line below the positive real axis. Then we have

$$\int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4} f(z)\,dz = 2\pi i(\operatorname{Res}(f, i) + \operatorname{Res}(f, -i)).$$

In the limit as we let Γ_2 and Γ_4 approach each other, the two integrals will cancel each other out. However, we can change the direction we travel on one of the lines, and then we get double contribution from the real line. Thus, as we let the radius of the outer circle approach infinity and the radius of the inner circle appraoch zero, we also see that the right side is equal to

$$\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_3} f(z)\,dz + 2\int_0^\infty f(z)\,dz = 2\pi i(\operatorname{Res}(f, i) + \operatorname{Res}(f, -i)).$$

Thus we need to compute the first two integrals.

Let R be the radius of the outer circle. By the ML inequality, we have

$$\left|\int_{\Gamma_1} f(z)\,dz\right| \leq 2\pi R\left(\frac{1}{\sqrt{R}(R^2-1)}\right).$$

As R approaches infinity, this approaches 0.

Let ε be the radius of the inner circle, and parametrize it with $\gamma(t) = \varepsilon e^{it}$, where $0 \le t \le 2\pi$. Then we have

$$\left|\int_{\Gamma_3} f(z)\,dz\right| = \left|\int_0^{2\pi} \frac{\varepsilon i e^{it}}{\sqrt{\varepsilon} e^{i\frac{t}{2}}(1+\varepsilon^{2it})}\,dt\right| \le \left|2\pi\left(\frac{\sqrt{\varepsilon}}{\varepsilon+1}\right)\right|.$$

As $\varepsilon \to 0$, the bound approaches 0. Thus we have $\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_3} f(z)\,dz = 0$. Now we just need to calculate the residues.

We have

$$\operatorname{Res}(f, i) = \lim_{z\to i} \frac{z-i}{\sqrt{z}(z^2+1)} = \frac{1}{\sqrt{i}\cdot 2i} = -\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}i.$$

$$\operatorname{Res}(f, -i) = \lim_{z\to -i} \frac{z+i}{\sqrt{z}(z^2+1)} = \frac{1}{\sqrt{-i}\cdot -2i} = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}i.$$

Thus $2\pi i(\operatorname{Res}(f,i) + \operatorname{Res}(f,-i)) = \pi\sqrt{2}$. Thus we have

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x^2)}\,dz = \frac{\pi\sqrt{2}}{2}.$$

Problem (16): Let $a, b > 0$. Compute

$$\int_0^{2\pi} \frac{1}{a^2\cos^2 t + b^2 \sin^2 t}\,dt.$$

Solution: Let Γ be an ellipse with center 0, semi-major axis of length a on the x-axis, and semi-minor axis of length b on the y-axis. Let $\gamma(t) = a\cos t + ib\sin t$ be its parametrization, where $0 \le t \le 2\pi$. Then we have

$$\int_\Gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{-a\sin t + ib\cos t}{a\cos t + ib\sin t}\,dt = \int_0^{2\pi} \frac{(-a\sin t + ib\cos t)(a\cos t - ib\sin t)}{a^2\cos^2 t + b^2\sin^2 t}\,dt$$
$$= \int_0^{2\pi} \frac{(b^2-a^2)\sin t\cos t + abi}{a^2\cos^2 t + b^2\sin^2 t}\,dt.$$

Thus, $\frac{\Im\left(\int_\Gamma \frac{1}{z}\,dz\right)}{ab}$ is our desired integral. We can deform Γ to be the unit circle, since we don't cross the pole at 0. Thus we just need to compute $\int_{|z|=1} \frac{1}{z}\,dz$, but this is just $2\pi i$. Thus

$$\int_0^{2\pi} \frac{1}{a^2\cos^2 t + b^2\sin^2 t}\,dt = \frac{2\pi}{ab}.$$

Problem (17): Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a holomorphic functions. Suppose that for any closed contour Γ in U, $\int_\Gamma f(z)\,dz = 0$. Show that f has antiderivative on U.

Solution: For $z_0 \in U$, define $F : U \to \mathbb{C}$ as

$$F(z_0) = \int_{\Gamma(z_0)} f(z)\, dz,$$

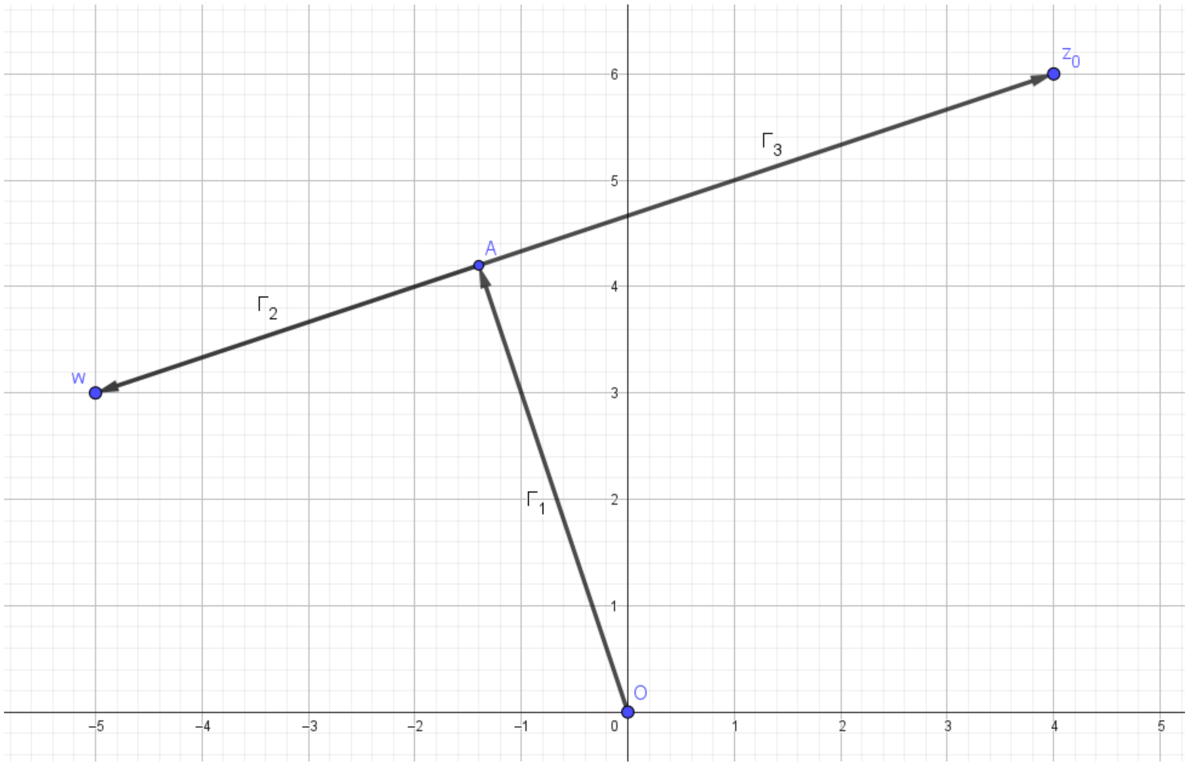
where $\Gamma(z_0)$ is a contour from 0 to z_0. Note that $F(z_0)$ is independent of the choice of contour. Given two contours $\Gamma(z_0)$ and $\Gamma'(z_0)$, both starting at 0 and going to z_0, we can write

$$\int_{\Gamma(z_0)} f(z)\, dz - \int_{\Gamma'(z_0)} f(z)\, dz = \int_{\Gamma(z_0)} f(z)\, dz + \int_{\tilde{\Gamma}'(z_0)} f(z)\, dz.$$

Since $\Gamma(z_0)$ goes from 0 to z_0, and $\tilde{\Gamma}'(z_0)$ goes from z_0 to 0, $\Gamma(z_0) + \tilde{\Gamma}'(z_0)$ is closed, and thus the equation above is equal to 0. Thus the value of $F(z_0)$ is independent of choice of contour.

We now claim that $F'(z) = f(z)$ for all $z \in U$. We need to show that

$$\lim_{w \to z_0} \frac{F(w) - F(z_0)}{w - z_0} = f(z_0).$$



We have $F(w) = \int_{\Gamma_1 + \Gamma_2} f(z)\, dz$ and $F(z_0) = \int_{\Gamma_1 + \Gamma_3} f(z)\, dz$. Let Γ be the straight line path from w to z_0. Then

$$\begin{aligned} F(w) - F(z_0) = \int_{\Gamma_1 + \Gamma_2} f(z)\, dz - \int_{\Gamma_1 + \Gamma_3} f(z)\, dz = \int_{\Gamma_2} f(z)\, dz - \int_{\Gamma_3} f(z)\, dz = \int_{\Gamma_2} f(z)\, dz + \int_{\tilde{\Gamma}_3} f(z)\, dz \\ = -\int_{\Gamma} f(z)\, dz. \end{aligned}$$

Let $\gamma(t) = w + (z_0 - w)t$, where $0 \le t \le 1$. Then

$$F(w) - F(z_0) = -\int_\gamma f(z)\,dz = -\int_0^1 f(w + (z_0 - w)t)(z_0 - w)\,dt.$$

Thus

$$\lim_{w \to z_0} \frac{F(w) - F(z_0)}{w - z_0} = \lim_{w \to z_0} \int_0^1 f(w + (z_0 - w)t)\,dt = \int_0^1 f(z_0) = f(z_0),$$

as desired.

6. More Integration

6.1. Cauchy Integral Formula

Theorem (Cauchy integral formula): Let $U \subseteq \mathbb{C}$ be an open set, let Γ be a positively oriented simple closed contour in U whose interior is contained in U, and let $f : U \to \mathbb{C}$ be a holomorphic function. Then, for any z_0 in the interior of Γ, we have

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz.$$

Proof: Consider the function $F : U \to \mathbb{C}$ given by

$$F(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if} \quad z \neq z_0, \\ f'(z_0) & \text{if} \quad z = z_0. \end{cases}$$

Note that $F(z)$ is continuous at z_0, and note also that F is holomorphic everywhere except at z_0. Thus we can deform Γ to a small circle $|z - z_0| = \varepsilon$ without changing the value of the integral $F(z)$ on Γ. Thus

$$\int_\Gamma F(z)\, dz = \int_{|z-z_0|=\varepsilon} F(z)\, dz.$$

By the ML inequality, we have

$$\left| \int_{|z-z_0|=\varepsilon} F(z)\, dz \right| \leq 2\pi\varepsilon \sup_{|z-z_0|=\varepsilon} F(z).$$

Note that M is bounded on some closed neighborhood of z_0 by the extreme value theorem, so the right hand side is only dependent on ε. Thus as $\varepsilon \to 0$, we get

$$\int_\Gamma F(z)\, dz = 0.$$

Thus we have

$$\int_\Gamma \frac{f(z)}{z - z_0}\, dz = f(z_0) \int_\Gamma \frac{1}{z - z_0}\, dz.$$

We can again deform Γ to be a small circle around z_0, and then we have that the right hand side is equal to $2\pi i f(z_0)$, as desired. ■

Theorem: Under the hypotheses of the Cauchy integral formula, for every integer $n \geq 0$, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

Solution: We just prove the case for $n = 1$. The rest follow by induction. We can deform Γ be to be a small circle $|z - z_0| = \rho$. Suppose $|\Delta z| < \rho$. Then we have

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \int_\Gamma \left(\frac{1}{z - z_0 - \Delta z} - \frac{1}{z - z_0} \right) \frac{f(z)}{\Delta z} \, dz = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} \, dz.$$

We can decompose the integral as

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^2} \, dz + \frac{1}{2\pi i} \int_\Gamma \frac{\Delta z \cdot f(z)}{(z - z_0 - \Delta z)(z - z_0)^2} \, dz.$$

By the ML inequality, the second integral is bounded by

$$\frac{|\Delta z| M}{(\rho - \Delta z)\rho^2} \cdot 2\pi\rho,$$

where M is the maximum of f on Γ. As $\rho \to 0$, the bound goes to 0. Thus the second integral vanishes, so we're done.

6.2. Consequences of the Cauchy Integral Formula

Example: Using the Cauchy integral theorem, we can compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx.$$

Let $f(z) = \frac{e^{iz}}{z+i}$. Using the Cauchy integral formula, we know that

$$2\pi i f(i) = \int_\Gamma \frac{f(z)}{z - i} \, dz,$$

where Γ is a semicircular contour on the upper half plane. Note that the desired integral is the real part of the integral above (the semicircular part of Γ vanishes for R large). Thus the integral is equal to $\frac{\pi}{e}$ (!).

Example: Compute

$$\int_0^{2\pi} \frac{1}{3 + \cos\theta} \, d\theta.$$

We use the substituion $z = e^{i\theta}$. Then $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $d\theta = -\frac{i}{z} \, dz$, and we're integrating over the unit circle (since θ goes from 0 to 2π). Thus we end up needing to compute

$$-2i \int_{|z|=1} \frac{1}{z^2 + 6z + 1} \, dz,$$

which we can just compute using the Cauchy integral formula.

Theorem: If $f : U \to C$ is holomorphic at a point $z_0 \in U$, then its derivative f' is also holomorphic at z_0.

Proof: Suppose that f is holomorphic at z_0. By definition, that means it's holomorphic in some neighborhood of z_0, say $|z - z_0| < \varepsilon$. Let Γ be the circle of radius $\frac{\varepsilon}{2}$ centered at z_0. Then we have

$$f''(z) = \frac{2!}{2\pi i} \int_\Gamma \frac{f(w)}{(w - z)^3} \, dw$$

for every z with $|z - z_0| < \frac{\varepsilon}{2}$. The existence of the second derivative implies that f' is holomorphic in this neighborhood, and thus at z_0. ■

Remark: This implies that if a function is differentiable once, then it's differentiable infinitely many times over $\mathbb{C}$.

Theorem (Morera's theorem): Suppose that $U \subseteq \mathbb{C}$ is an open set and $f : U \to \mathbb{C}$ is a continuous function. Suppose that for every closed contour in U, the integral of f over Γ is 0. Then f is holomorphic.

Theorem: Suppose that f is holomorphic in a closed disk of radius R centerd at z_0. Let

$$M_R = \sup_{|z - z_0| = R} |f(z)|.$$

Then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! M_R}{R^n}.$$

Proof: Letting Γ_R be the circle $|z - z_0| = R$, we have

$$\left| \int_{\Gamma_R} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right| \leq \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{2\pi M_R}{R^n}.$$

Thus

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! M_R}{R^n}.$$

■

6.3. Liouville's Theorem

Theorem (Liouville's theorem): A bounded entire function is constant.

Proof: Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then by the derivative bound, we have

$$|f'(z_0)| \leq \frac{M_R}{R} \leq \frac{M}{R}$$

for all $R > 0$. Since M doesn't depend on R, the right side goes to 0 as $R \to \infty$. Thus $|f'(z_0)| = 0$ for all $z_0 \in \mathbb{C}$, from which it follows that f is constant. ∎

Theorem (fundamental theorem of algreba): Let $p(z)$ be a nonconstant polynomial with complex coefficients. Then there is some $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof: Suppose p is a nonconstant polynomial with no complex roots. Then $\frac{1}{p(z)}$ is entire. Note by our bounds on polynomials, there exists some R such that if $|z| > R$, we have

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|a_n|R^n}.$$

Thus $\frac{1}{p(z)}$ is bounded when $|z| > R$. For the disk $|z| \leq R$, its compact, so any continuous function must take a maximum on it. Since $\frac{1}{p(z)}$ is entire, it's continous. Thus $\frac{1}{p(z)}$ is bounded, and therefore by Liouville's theorem constant, which is a contradiction. ∎

6.4. Gauss Mean Value Theorem and Maximum Modulus Principle

Theorem (Gauss mean value theorem): Suppose that f is holomorphic in $\overline{B_\rho(z_0)}$. Then we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right) d\theta.$$

Proof: Let Γ be the circle $|z - z_0| = \rho$ and parametrize it by $\gamma(\theta) = z_0 + \rho e^{i\theta}$ for $0 \leq \theta \leq 2\pi$. Then we have

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + \rho e^{i\theta}\right)}{\rho e^{i\theta}} \cdot i\rho e^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right) d\theta. \end{aligned}$$

∎

Corollary (maximum modulus principle): Suppose that $U \subseteq \mathbb{C}$ is a connected open set and $f : U \to \mathbb{C}$ is holomorphic. If there exists some z_0 so that $|f(z_0)| \geq |f(z)|$ for all $z \in U$, then f is constant.

Proof: Suppose $z_0 \in U$ is a max for $|f(z)|$. Pick ρ such that $\overline{B_\rho(z_0)} \subseteq U$. Then by the Gauss mean value theorem, we have

$$|f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right) d\theta \right| \leq \frac{1}{2\pi} \cdot |f(z_0)| \cdot 2\pi = |f(z_0)|.$$

Since we have equality, we must have $|f(z)| = |f(z_0)|$ for all $|z - z_0| = \rho$.

Let $g(z) = f(z) + f(z_0)$. Then $|g(z)| \leq |f(z)| + |f(z_0)|$, with equality if and only if $f(z_0)$ is a nonnegative muliple of $f(z)$. Note that $|g(z)|$ also has a max at z_0, so by the previous argument has constant modulus on $|z - z_0| = \rho$. Thus we have

$$2|f(z_0)| = |g(z_0)| = |g(z)| \leq |f(z)| + |f(z_0)| = 2|f(z_0)|,$$

so again we must have $|g(z)| = |f(z)| + |f(z_0)|$. Thus we must have $f(z) = f(z_0)$ by the equality case. Thus f is constant on the circle, and since ρ was arbitrary, f is constant on the disk $\overline{B_\rho(z_0)}$.

Then we can just extend this to the rest of U by picking a point in this disk and showing again by the same method that f is constant. ■

6.5. Problems

Problem (1): For positive integers n and m, compute

$$\int_\Gamma \frac{(1+z)^n}{z^m}\, dz,$$

where Γ is a contour winding once around the origin counterclockwise.

Solution: Using the binomial theorem, we have

$$\int_\Gamma \frac{z^n + \binom{n}{n-1} z^{n-1} + \cdots + \binom{n}{1} z + 1}{z^m}\, dz.$$

Every term vanishes except for when $k - m = -1$, where k represent the exponent of a term in the numerator. For $n + 1 < m$, all exponents are less than -1, so overall the integral is 0. Otherwise, we get a contribution of $2\pi i \binom{n}{m-1}$. Thus we have

$$\int_\Gamma \frac{(1+z)^n}{z^m}\, dz = \begin{cases} 0 & \text{if } \; n < m - 1, \\ 2\pi i \binom{n}{m-1} & \text{otherwise.} \end{cases}$$

Problem (3): Let Γ be a closed contour not passing through $0, 1, -1$. Find all possible values of

$$\int_\Gamma \frac{1}{z^3 - z}\, dz.$$

Solution: By partial fractions, we have

$$\frac{1}{2}\int_\Gamma \frac{1}{z+1} + \frac{1}{2}\int_\Gamma \frac{1}{z-1}\, dz - \int_\Gamma \frac{1}{z}\, dz.$$

If the contour does not contain 0, then the third integral vanished by the Cauchy integral theorem, and similarly for the other two poles. Thus we only get contributions from the poles contained within the contour. In particular, we'll get a contribution of $2\pi i$ from each integral if their pole is within the contour. Trying all possible combinations of integrals, we get the following possible values:

$$0, \pm\pi i, \pm 2\pi i.$$

Problem (4): Let Γ be a closed contour whose interior contains $\overline{B_{|a|}(0)}$. Compute

$$\int_\Gamma \frac{e^z}{z^2 + a^2}\, dz.$$

Solution: Let $f(z) = e^z$. Then by the Cauchy integral theorem, we have

$$f(ai) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - ai} \quad \text{and} \quad f(-ai) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z + ai}\, dz.$$

Thus we have

$$e^{ai} - e^{-ai} = \frac{1}{2\pi i}\int_\Gamma e^z\left(\frac{1}{z - ai} - \frac{1}{z + ai}\right) dz.$$

Multiplying by $2\pi i$ and dividing by $2ai$, we get

$$\frac{\pi}{a}(e^{ai} - e^{-ai}) = \int_\Gamma \frac{e^z}{z^2 + a^2}\, dz.$$

We can rewrite the left side as

$$\frac{2\pi \sin a}{a}.$$

Problem (5): Compute

$$\int_0^\infty \frac{1}{1+x^3}\,dx.$$

Solution: Let I denote the desired integral, and let Γ be a sector of a circle with radius R, where one radii is located on the positive x-axis and the other radius is located on $\theta = \frac{2\pi}{3}$. Thus for $R > 1$, the contour only contains the pole $e^{i\frac{\pi}{3}}$. Thus by the Cauchy integral theorem, we have

$$\int_\Gamma \frac{1}{1+z^3}\,dz = \int_\Gamma \frac{1/(z+1)\left(z-e^{i\frac{5\pi}{3}}\right)}{\left(z-e^{i\frac{\pi}{3}}\right)}\,dz = \frac{2\pi i}{\left(e^{i\frac{\pi}{3}}+1\right)\left(e^{i\frac{\pi}{3}}-e^{i\frac{5\pi}{3}}\right)} = \frac{\pi}{\sqrt{3}} - i\frac{\pi}{3}.$$

Now we compute $\int_\Gamma \frac{1}{1+z^3}\,dz$ on each part of the contour. The part of that lies on the real axis is simply our desired integral when we let $R \to \infty$. For the part on the circumference of the sector, note that $\left|z^3+1\right| \geq R^3 - 1$, and thus by the ML inequality it bounded by $\frac{1}{R^3-1} \cdot \frac{2}{3}\pi R$. As $R \to \infty$, this part goes to 0.

For the radius that lies on $\theta = \frac{2\pi}{3}$, parametrize it with $\gamma(t) = e^{i\frac{2\pi}{3}}t$, where $0 \leq t \leq R$. Letting Γ' be this radius (traversed towards the origin), we have

$$\int_{\Gamma'} \frac{1}{1+z^3}\,dz = \int_R^0 \frac{1}{1+t^3} \cdot e^{i\frac{2\pi}{3}}\,dt = -e^{i\frac{2\pi}{3}} \int_0^R \frac{1}{1+t^3}\,dt.$$

As $R \to \infty$, this becomes $-e^{i\frac{2\pi}{3}} I$. Thus we have

$$\left(1 - e^{i\frac{2\pi}{3}}\right) I = \frac{\pi}{\sqrt{3}} - i\frac{\pi}{3}.$$

Dividing yields

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi}{3\sqrt{3}}.$$

Problem (6): Suppose that a is in the interior of a closed contour Γ. Compute

$$\int_\Gamma \frac{ze^z}{(z-a)^3}\,dz.$$

Solution: Let $f(z) = ze^z$. By the Cauchy integral theorem, we have

$$f''(a) = \frac{2!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-a)^3}\,dz.$$

Note that $f''(z) = 2e^z + ze^z$. Thus the integral is equal to

$$(a+2)e^a \pi i.$$

Problem (8): Let Γ be a contour, and let f be holomorphic on Γ and its exterior. Suppose that $\lim_{z\to\infty} f(z) = a$. Compute

$$\int_\Gamma \frac{f(z)}{z - z_0}\, dz.$$

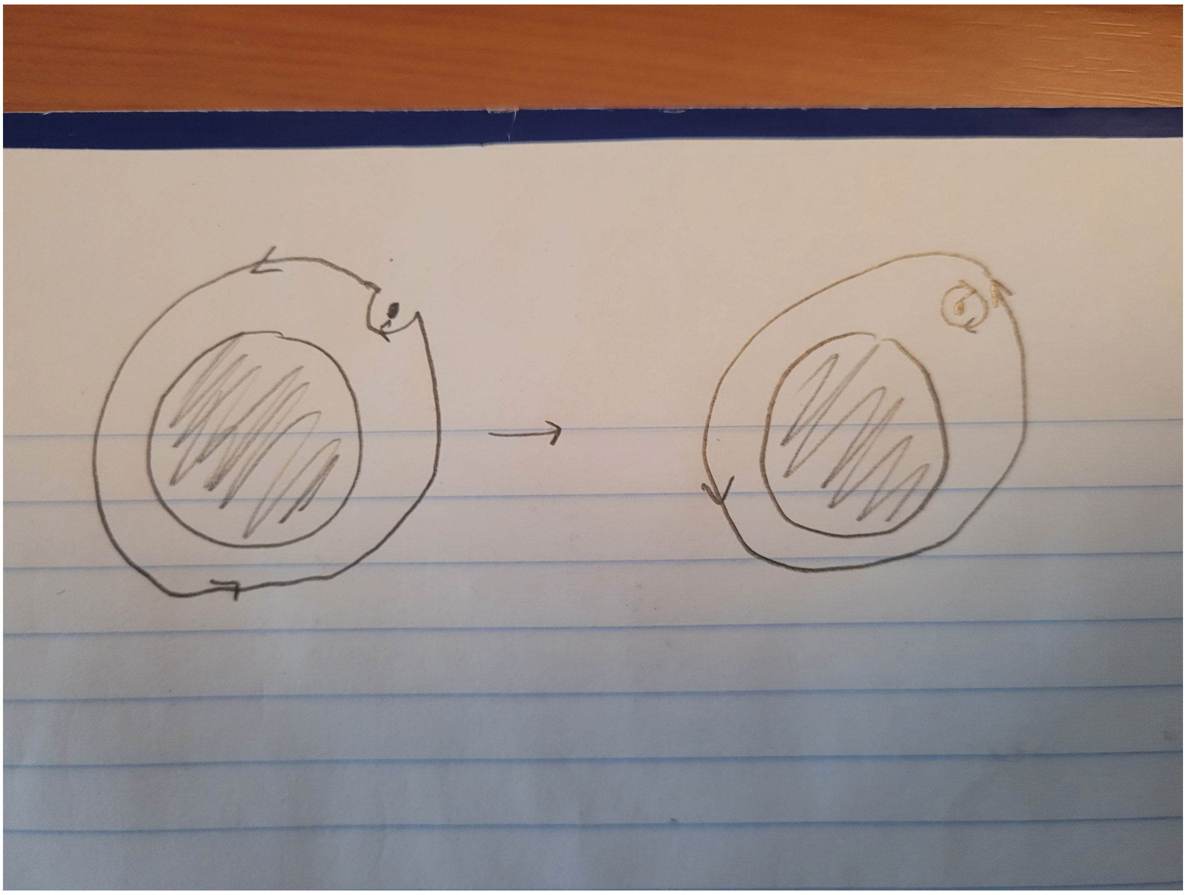
Solution: First suppose z_0 lies in Γ. By path invariance, we can deform Γ to be an circle centered at z_0 (while still keeping the non-holomorphic portion inside). Let R be the arbitrary radius of the circle. Then, letting $\gamma(t) = z_0 + Re^{it}$ where $0 \le t \le 2\pi$, we have

$$\int_\Gamma \frac{f(z)}{z - z_0}\, dz = \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} \cdot iRe^{it}\, dt = i\int_0^{2\pi} f(z_0 + Re^{it})\, dt.$$

Letting $R \to \infty$, from the limit we have

$$i\int_0^{2\pi} f(z_0 + Re^{it})\, dt = i\int_0^{2\pi} a\, dt = 2\pi i a.$$

Thus the initial integral is equal to $2\pi i a$.



Now suppose z_0 lies outside Γ. We can deform the path to contain z_0, but we must have another small contour around z_0 going clockwise. Letting Γ' be this small contour, and deforming the outer contour to be a large circle centered at z_0, we have

$$\int_{\Gamma} \frac{f(z)}{z-z_0}\, dz = \int_{|z-z_0|=R} \frac{f(z)}{z-z_0} - \int_{\tilde{\Gamma}'} \frac{f(z)}{z-z_0}\, dz.$$

$\tilde{\Gamma}'$ is counterclockwise, so we swapped the sign. By the Cauchy integral theorem (since z_0 is outside the non-holomorphic region, and thus f is holomorphic at z_0), the second integral is equal to $2\pi i f(z_0)$. We already know the first integral is equal to $2\pi i a$. Thus the desired integral is equal to $2\pi i(a - f(z_0))$.

Problem (9): Suppose $a^2 < 1$. Compute the integral

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta.$$

Solution: Note that the integrand traverses the same values in $[0, \pi]$ as in $[\pi, 2\pi]$. Thus the desired integral is equal to

$$\frac{1}{2}\int_0^{2\pi} \frac{\cos(2\theta)}{1 + a^2 - 2a\cos\theta}\, d\theta.$$

Letting $z = e^{i\theta}$, we get $-\frac{i}{z}\, dz = d\theta$, $\frac{z+\frac{1}{z}}{2} = \cos\theta$, and $\frac{z^2+\frac{1}{z^2}}{2} = \cos 2\theta$. Thus we obtain

$$\frac{1}{2}\int_{|z|=1} \frac{\frac{z^2+\frac{1}{z^2}}{2}}{1 + a^2 - 2a\left(\frac{z+\frac{1}{z}}{2}\right)} \cdot -\frac{i}{z}\, dz = -\frac{i}{4}\int_{|z|=1} \frac{z^4+1}{-z^2(az^2 - (a^2+1)z + a)}\, dz = \frac{i}{4}\int_{|z|=1} \frac{z^4+1}{az^2(z-a)\left(z-\frac{1}{a}\right)}\, dz.$$

We can now deform the contour to two circles around the poles at 0 and a. Letting f be the integrand, from the residue theorem we obtain that the integral is equal to

$$-\frac{\pi}{2}(\operatorname{Res}(f, 0) + \operatorname{Res}(f, a)) = -\frac{\pi}{2}\left(\lim_{z\to 0} \frac{d}{dz}(z^2 f(z)) + \lim_{z\to a}(z-a)f(z)\right).$$

We have

$$\frac{d}{dz}(z^2 f(z)) = \frac{4z^3(z-a)\left(z-\frac{1}{a}\right) - (z^4+1)\left(2z - a - \frac{1}{a}\right)}{a(z-a)^2\left(z-\frac{1}{a}\right)^2}.$$

Thus the residue at 0 is $\frac{a^2+1}{a^2}$.

Similarly, the second limit is equal to $\frac{a^4+1}{a^2(a^2-1)}$. Adding everything up and simplifying yields

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta = \frac{\pi a^2}{1 - a^2}.$$

Problem (10): For $a > 1$, compute

$$\int_0^{\pi} \frac{1}{(a + \cos\theta)^2}\, d\theta.$$

Solution:

Since $\cos\theta$ traverses the same values in $[\pi, 2\pi]$ and $[0, \pi]$, the desired integral is equal to

$$\frac{1}{2}\int_0^{2\pi} \frac{1}{(a+\cos\theta)^2}\, d\theta.$$

Converting to a contour integral using $z = e^{i\theta}$ yields

$$-2i\int_{|z|=1} \frac{z}{\left(z^2+2az+1\right)^2}\, dz.$$

The roots of the denominator are $\alpha = -a-\sqrt{a^2-1}$ and $\beta = -a+\sqrt{a^2-1}$. Since $\alpha < -1$ and $-1 < \beta < 0$, the only pole in the contour is at β. Thus by the Cauchy integral formula, the integral is equal to

$$-2i(2\pi i f'(\beta)),$$

where $f(z) = \frac{z}{\left(z+a+\sqrt{a^2-1}\right)^2}$.

Computing yields

$$\int_0^{\pi} \frac{1}{(a+\cos\theta)^2}\, d\theta = \frac{\pi a}{(a^2-1)^{\frac{3}{2}}}.$$

Problem (11): For $a > 0$, compute

$$\int_0^{\frac{\pi}{2}} \frac{1}{a+\sin^2\theta}\, d\theta.$$

Solution: We have

$$2\int_0^{\frac{\pi}{2}} \frac{1}{a+\sin^2\theta}\, d\theta = \int_0^{\pi} \frac{1}{a+\sin^2\theta}\, d\theta = \int_0^{\pi} \frac{1}{a+\frac{1-\cos 2\theta}{2}}\, d\theta = \int_0^{\pi} \frac{2}{2a+1-\cos 2\theta}\, d\theta.$$

Thus the integral we want to calculate is equal to

$$\int_0^{\pi} \frac{1}{2a+1-\cos 2\theta}\, d\theta.$$

Let $z = e^{2i\theta}$. Then $\cos 2\theta = \frac{z+\frac{1}{z}}{2}$ and $-\frac{i}{2z}\, dz = d\theta$. Thus this integral is equal to

$$\int_{|z|=1} \frac{1}{2a+1-\frac{z+\frac{1}{z}}{2}} \cdot -\frac{i}{2z}\, dz = i\int_{|z|=1} \frac{1}{z^2-(4a+2)z+1}\, dz.$$

The roots of the denominator are

$$\alpha = 2a+1+2\sqrt{a^2+a},$$
$$\beta = 2a+1-2\sqrt{a^2+a}.$$

Note that $\alpha > 1$ and $0 < \beta < 1$. Thus only one pole lies in the contour. By the Cauchy integral formula, we have

$$i \int_{|z|=1} \frac{1 \,/\, (z-\alpha)}{z-\beta} \, dz = -2\pi \cdot \frac{1}{\beta - \alpha} = \frac{\pi}{2\sqrt{a^2+a}}.$$

Thus

$$\int_0^{\frac{\pi}{2}} \frac{1}{a + \sin^2\theta} \, d\theta = \frac{\pi}{2\sqrt{a^2+a}}.$$

Problem (12): Compute

$$\int_0^{\pi} \log \sin\theta \, d\theta.$$

Solution: Note that sin takes on the same values in both $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$, so we can write

$$2 \int_0^{\frac{\pi}{2}} \log \sin\theta \, d\theta.$$

Let I be the integral above minus the constant factor. Letting $\theta = \frac{\pi}{2} - t$, we get

$$I = -\int_{\frac{\pi}{2}}^{0} \log\cos t \, dt = \int_0^{\frac{\pi}{2}} \log\cos t \, dt.$$

Adding the two integrals yields

$$2I = \int_0^{\frac{\pi}{2}} \log(\sin t \cos t) \, dt = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\sin 2t\right) dt. = -\frac{\pi}{2}\log 2 + \int_0^{\frac{\pi}{2}} \log\sin 2t \, dt.$$

Letting $u = 2t$, we obtain

$$2I = -\frac{\pi}{2}\log 2 + \frac{1}{2}\int_0^{\pi} \log\sin u \, du = -\frac{\pi}{2}\log 2 + I.$$

Thus

$$2I = \int_0^{\pi} \log\sin\theta \, d\theta = -\pi \log 2.$$

Problem (13): Suppose f is entire, and there exist real numbesr a and b and a positive integer n such that for all $z \in \mathbb{C}$, $|f(z)| \le a + b|z|^n$. Show that f is a polynomial of degree at most n.

Solution: Pick $z_0 \in \mathbb{C}$. By the bound on derivatives, we have

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! \max_{|z-z_0|=R}|f(z)|}{R^n},$$

where $R > 0$. Note that the maximum value of $|z|$ on the circle with radius R centered at z_0 is $|z_0| + R$ (by the triangle inequality). We have

$$|f(z)| \leq a + b(|z_0| + R)^n$$

for all $|z| \leq |z_0| + R$. In particular, this implies that $\max_{|z-z_0|=R}|f(z)| \leq a + b(|z_0| + R)^n$. Thus we have

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n}(a + b(|z_0| + R)^n) = n!\left(\frac{a}{R^n} + b\left(\frac{|z_0|}{R} + 1\right)^n\right).$$

Pick R such that $\frac{a}{R^n} < 1$ and $\frac{|z_0|}{R} < 1$. Then the right side is less than

$$n!(1 + 2^n b).$$

This works for all values of z_0. Thus $f^{(n)}$ is bounded, and by Liouville's theorem (since f is entire, $f^{(n)}$ must also be entire) must be constant. Taking n integrals of a constant will yield a polynomial of degree at most n, so we're done.

Problem (14): Suppose that f is an entire function, and $f(z+1) = f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Show that f is constant.

Solution: Note that on the unit square, f is bounded, since the unit square is compact and f is continuous. Note that from the condition, we get that every square with lattice corners is a copy of the unit square (i.e. the square with corners $1+i, 2+i, 2+2i, 1+2i$ takes on the same values as the unit square). Thus f is bounded, and by Liouville's theorem, it must be constant.

Problem (16): Suppose that f is a holomorphic function on some open set U, and suppose that for some $z_0 \in U$, $f(z_0) = 1$. Show that there are infinitely many $z \in U$ so that $|f(z)| = 1$.

Solution: Pick ρ such that $\overline{B_\rho(z_0)} \subseteq U$. By the Gauss mean value theorem, we have

$$1 = f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right) d\theta.$$

Letting Γ be the boundary of $\overline{B_\rho(z_0)}$, note that this boundary is a closed set. Since f is holomorphic, it's continuous, so by the extreme value theorem, $|f(z)|$ has a max and min on Γ. Note that if the max $M < 1$, then by the ML inequality,

$$\left|\frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right) d\theta\right| \leq \frac{1}{2\pi} \cdot M \cdot 2\pi = M < 1,$$

which contradicts the equality above. Similarly, if the min $m > 1$, we have

$$\left|\frac{1}{2}\pi \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta\right| \geq \frac{1}{2\pi} \cdot m \cdot 2\pi = m > 1,$$

which contradicts the equality. Thus we have

$$\min_{|z-z_0|=\rho} |f(z)| \leq 1 \leq \max_{|z-z_0|=\rho} |f(z)|.$$

Since $|z|$ and $f(z)$ are continuous, $|f(z)|$ continuous. Thus, by the intermediate value theorem, some z' with $|z' - z_0| = \rho$ satisfies $|z'| = 1$. Since ρ can be arbitrary (as long as $\overline{B_\rho(z_0)} \subseteq U$), there are infinitely many such values of z'.

Problem (17): Suppose that f is holomorphic on $\overline{B_r(z_0)}$. Show that

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left|f(z_0 + re^{i\theta})\right| d\theta.$$

Solution: From the Gauss mean value theorem, we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

We convert this into a Riemann sum over the boundary of $\overline{B_r(z_0)}$. Let $\gamma(t) = z_0 + re^{it}$, where $0 \leq t \leq 2\pi$. We partition this interval into $\left[0, \frac{2\pi}{n}\right] \cup \left[\frac{2\pi}{n}, \frac{4\pi}{n}\right] \cup \dots \cup \left[\frac{n-1}{n} \cdot 2\pi, 2\pi\right]$. Let

$$I_n = \sum_{j=0}^{n-1} f\left(z_0 + re^{i\frac{2\pi j}{n}}\right)\left(\gamma\left(\frac{2\pi(j+1)}{n}\right) - \gamma\left(\frac{2\pi j}{n}\right)\right).$$

Then by definition, $\lim_{n\to\infty} I_n = 2\pi f(z_0)$. Taking absolute values and applying the triangle inequality, we obtain

$$|I_n| \leq \sum_{j=0}^{n-1} \left|f\left(z_0 + re^{i\frac{2\pi j}{n}}\right)\right| \left|\gamma\left(\frac{2\pi(j+1)}{n}\right) - \gamma\left(\frac{2\pi j}{n}\right)\right|.$$

Note that the second term in the sum is simply the length between two consecutive roots of unity scaled by r. Thus this distance is constant over all values of j. Let the distance be Δ_n. Thus

$$|I_n| \leq \sum_{j=0}^{n-1} \left|f\left(z_0 + re^{i\frac{2\pi j}{n}}\right)\right| \Delta_n.$$

The limit of the right side is the integral $\int_0^{2\pi} \left|f(z_0 + re^{i\theta})\right| d\theta$ (as the differences between consecutive inputs to f is equal to Δ_n). Since $f(z_0 + re^{i\theta})$ is continuous, its absolute value is continuous, and thus integrable. Therefore this integral exists. The limit of the left side is simply $|2\pi f(z_0)|$, so we obtain

$$2\pi |f(z_0)| \leq \int_0^{2\pi} \left|f(z_0 + re^{i\theta})\right| d\theta,$$

as desired.

Problem (18): Suppose that f is entire and $\mathfrak{R}(f(z))$ is bounded. Show that f is constant.

Solution: Suppose $\mathfrak{R}(f(z)) \leq M$. Consider

$$\frac{1}{M+1-f(z)}.$$

Note that $\mathfrak{R}(M+1-f(z)) \geq 1$, so the denominator is never 0. Thus this function is entire. If we can show that it's bounded, then we're done by Liouville's theorem. However, this follows from $\mathfrak{R}(M+1-f(z)) \geq 1$, since then $|M+1-f(z)|^2 \geq \mathfrak{R}(M+1-f(z))^2 + \mathfrak{I}(M+1-f(z))^2 \geq 1$, and thus

$$\left|\frac{1}{M+1-f(z)}\right| \leq 1.$$

Problem (19): Suppose that f is holomorphic on a closed contour Γ and its interior, and suppose $|f(z)|$ is constant on Γ. Show that either f is constant on Γ and its interior, or there is some point z_0 in the interior of Γ so that $f(z_0) = 0$.

Solution: First suppose that $f(z)$ is constant on Γ with value a. Then for any z_0 in its interior, by the Cauchy integral formula we have

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,dz = \frac{a}{2\pi i}\int_\Gamma \frac{1}{z-z_0}\,dz = a.$$

Thus f is constant on Γ and its interior.

Now suppose $f(z)$ is not constant on Γ but has constant modulus. We assume that Γ is simple (which is somewhat required since the Cauchy integral formula requires simple contours). By the Jordan-Schoenflies theorem, the inside of Γ is homeomorphic to the inside of the unit circle. Since homeomorphisms are continuous and since the inside of a circle is open, the interior of Γ must also be open.

Let U be the interior of Γ, and suppose for the sake of contradiction that f is nonzero on U. Thus by the minimum modulus principle, the minimum of $|f(z)|$ on $\Gamma \cup U$ occurs on Γ. Similarly by the maximum modulus principle, the maximum of $|f(z)|$ occurs on Γ. However, since the modulus is constant on Γ, since implies that $|f(z)|$ is constant on $\Gamma \cup U$.

Thus there exists some z_0 in U such that $|f(z_0)| \geq |f(z)|$ for all $z \in U$ (in particular any $z_0 \in U$) will work). However, again by the maximum modulus principle, this implies that f is constant on U, say with value a. Now pick a point b on Γ. Since $\Gamma \cup U$ is closed, every point is a limit point. Since Γ is the boundary of U, we can pick a sequence of points (a_n) entirely within U with limit point b. Then by continuity,

$$a = \lim_{n\to\infty} a = \lim_{n\to\infty} f(a_n) = f\Big(\lim_{n\to\infty} a_n\Big) = f(b).$$

Since this works for all $b \in \Gamma$, f is constant on Γ, contradiction.

7. Residues and Series

7.1. Power and Laurent Series

Theorem (Taylor's theorem): Suppose f is holomorphic on the disk $|z - z_0| < R$. Then f has a power series expansion about z_0 with radius of convergence R.

Proof: WLOG $z_0 = 0$. Suppose $|z_1| = r < R$, and pick some r_0 such that $r < r_0 < R$. Then by the Cauchy integral formula,

$$2\pi i f(z_1) = \int_{|z|=r_0} \frac{f(z)}{z - z_1}\, dz.$$

Writing the integrand as $\frac{f(z)}{z} \frac{1}{1 - z_1 / z}$ and expanding the second term partially using geometric series (since $\left|\frac{z_1}{z}\right| = \frac{r}{r_0} < 1$), we obtain

$$\frac{f(z)}{z - z_1} = \sum_{n=0}^{N-1} \frac{f(z)}{z^{n+1}} z_1^n + \frac{f(z)}{(z - z_1) z^N} z_1^N.$$

Integrating over $|z| = r_0$, we obtain

$$2\pi i f(z_1) = \int_{|z|=r_0} \frac{f(z)}{z - z_1}\, dz = \sum_{n=0}^{N-1} \int_{|z|=r_0} \frac{f(z)}{z^{n+1}} z_1^n + \int_{|z|=r_0} \frac{f(z)}{(z - z_1) z^N} z_1^N.$$

Using the Cauchy integral formula to rewrite the sum, we obtain

$$f(z_1) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z_1^n + \frac{z_1^N}{2\pi i} \int_{|z|=r_0} \frac{f(z)}{(z - z_1) z^N}.$$

Let $N \to \infty$, and using the ML inequality on the second term yields

$$\left| \frac{z_1^N}{2\pi i} \int_{|z|=r_0} \frac{f(z)}{(z - z_1) z^N} \right| < \frac{r^N}{2\pi} \cdot \frac{M}{(r_0 - r) r_0^N} \cdot 2\pi r_0 = \left(\frac{r}{r_0}\right)^N \frac{M r_0}{r_0 - r}.$$

Since $r < r_0$, clearly the right side tends to 0. Thus we obtain

$$f(z_1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z_1^n,$$

as desired. ∎

Remark: The above result shows that holomorphicity implies analyticity.

Remark: Using the Weierstrass M-test, we can show that the power series expansion converges uniformly for $|z - z_0| \leq r$, where $r < R$. Thus we can integrate and differentiate power series termwise and maintain the same radius of convergence (convergence on the boundary might change).

Definition (Laurent series): A *Laurent series* centered at z_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.$$

Theorem: Suppose f is holomorphic in the annulus $r < |z - z_0| < R$ centered at z_0 and let Γ be a contour in the annulus which travels around z_0 once in the counterclockwise direction. Then f has a Laurent series expansion in the annulus, given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_{\Gamma} f(z)(z - z_0)^{n-1}\, dz.$$

Proof: Similar to Taylor's theorem. ■

7.2. The Residue Theorem

Definition (residue): Suppose the Laurent series of $f(z)$ converges in $0 < |z - z_0| \leq r$. Is a_{-1} is the coefficient of the $\frac{1}{z - z_0}$ term, then we say a_{-1} is the *residue* of f at z_0. We write

$$a_{-1} = \operatorname{Res}(f, z_0).$$

Note that if a contour Γ is contained in the region of convergence surrounding z_0 once counterclockwise, then we have

$$\int_{\Gamma} f(z)\, dz = 2\pi i \operatorname{Res}(f, z_0),$$

which comes from just integrating the Laurent series termwise.

Theorem (residue theorem): Suppose that Γ is a closed contour oriented positively, and that f is holomorphic on and inside Γ, except at a finite number of singular points $z_1, ..., z_n$. Then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_n).$$

Proof: Deform contour to be a bunch of small circles around the singular points. ■

7.3. Singularities

Definition (singularities): Suppose $z_0 \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ is some open subset such that $z_0 \in U$ by U contains a deleted neighborhood of z_0.

- If there's a continuous function $f : U \cup \{z_0\} \to \mathbb{C}$ such that $g(z) = f(z)$ for any $z \in U$, then z_0 is a *removable singularity.*
- If $\lim_{z \to z_0} f(z) = \infty$, then we say that z_0 is a *pole* of f.
- If $\lim_{z \to z_0} f(z)$ does not exist and is not infinity, then z_0 is an *essential signularity.*

In particular, given the Laurent series, the singularity at z_0 is

- removable if $a_n = 0$ for $n < 0$.
- a pole if $a_n \neq 0$ for finitely many (and nonzero amount) $n < 0$. The order of the pole is index of the largest nonzero coefficient of a negative power.
- an essential singularity of $a_n \neq 0$ for infinitely many $n < 0$.

With poles, the function blows up near z_0. For essential singularities, $\mathbb{C}$ is dense near z_0, so it also blows up (follows from Casorati-Weierstrass, which we show next). Thus we have the following:

Theorem (Riemann's removable singularity theorem): Suppose $U \subseteq \mathbb{C}$ is an open set, $z_0 \in U$, and $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. Then the following are equivalent:

a) $f(z_0)$ can be defined so as to make the extension holomorphic at z_0.
b) $f(z_0)$ can be defined so as to make the extension continuous at z_0.
c) There is a neighborhood of z_0 on which f is bounded.
d) $\lim_{z \to z_0} (z - z_0) f(z) = 0$.
e) z_0 is a removable singularity of f.

Theorem (Casorati-Weierstrass theorem): Suppose f has an essential singularity at z_0. Let w be any complex number. Then for any $\delta > 0$ and $\varepsilon > 0$, there exists a z with $0 < |z - z_0| < \delta$ and $|f(z) - w| < \varepsilon$.

Proof: Pick some $\delta > 0$ such that f is analytic in $0 < |z - z_0| < \delta$. Suppose that for some $\varepsilon > 0$, we cannot find a point z with $|z - z_0| < \delta$ and $|f(z) - w| < \varepsilon$. Let

$$g(z) = \frac{1}{f(z) - w}.$$

By construction, f is analytic in $0 < |z - z_0| < \delta$, and it bounded by $\frac{1}{\varepsilon}$. From the removable singularity theorem, z_0 must be a removable singularity of g, so we extend g to z_0 so that g is analytic there.

Now suppose $g(z_0) \neq 0$. Since $f(z) = \frac{1}{g(z)} + w$, if we define $f(z_0) = \frac{1}{g(z_0)} + w$, then f becomes analytic at z_0, so f has a removable singularity, contradiction.

If $g(z_0) = 0$, then suppose g has zero of order m at z_0, i.e. $g(z) = (z - z_0)^m h(z)$ for some holomorphic $h(z)$ thats nonzero in a neighborhood of z_0. Then $f(z) = (z - z_0)^{-m} \frac{1}{h(z)} + w$. Thus f has a pole of order m at z_0. ■

7.4. Problems

Problem (1): Compute $\operatorname{Res}(f, z_0)$ for the following choices of f and z_0:

a) $f(z) = \frac{z^2+2}{z^2+1}$ at $z_0 = i$.
b) $f(z) = \frac{e^z}{(z^2-1)^3}$ at $z_0 = 1$.
c) $f(z) = \frac{z}{\sin z}$ at $z = \pi n, n \in \mathbb{Z}$
d) $f(z) = \frac{z}{\sin^2 z}$ at $z = \pi n, n \in \mathbb{Z}$
e) $f(z) = \sin \frac{1}{z}$ at $z = 0$.

Solution:

a) We have a simple pole at $z = i$, so

$$\operatorname{Res}(f, i) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{z^2 + 2}{z + i} = -\frac{i}{2}.$$

b)

c) At $z = 0$, we have a removable singularity, so $\operatorname{Res}(f, 0) = 0$. Otherwise we have a simple pole at πn, so

$$\operatorname{Res}(f, \pi n) = \lim_{z \to \pi n} \frac{(z - \pi n) \cdot z}{\sin z} = \lim_{z \to \pi n} \frac{2z - \pi n}{\cos z} = (-1)^n \pi n.$$

d)

e) There's an essential singularity at 0, so we explicitly calculate the Laurent series:

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} - \cdots.$$

Thus $a_{-1} = \operatorname{Res}(f, 0) = 1$.

Problem (2): Show that the only complex zeros of $\sin z$ are integer multiples of π.

Solution: We already know that on the reals, the only zeros are integer multiples of π, so we just need to show there are no solutions with nonzero imaginary part.

Let $z = x + yi$, where $y \neq 0$. Then we have

$$\sin(x+yi) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2} = \frac{e^{-y+ix} - e^{y-ix}}{2} = \frac{1}{2}(\cos(x)(e^{-y} - e^{y}) + i\sin(x)(e^{-y} + e^{y})).$$

Note that the exponential terms can never be 0 (for the first one we would need $y = 0$, and the second is always positive). Thus this can only be 0 if both $\sin(x)$ and $\cos(x)$ are 0. However, the solutions for $\sin(x)$ on the are the integer multiples of π, while the solutions for $\cos(x)$ are $\frac{\pi}{2} + \pi k$ for $k \in \mathbb{Z}$. Thus there are no complex solutions.

Problem (3): Suppose that f and g are holomorphic at z_0, and g has a simple zero at z_0. Show that $\operatorname{Res}(f/g, z_0) = \frac{f(z_0)}{g'(z_0)}$.

Solution: Since the pole is simple, we have

$$\operatorname{Res}(f/g, z_0) = \lim_{z \to z_0} \frac{(z - z_0)f(z)}{g(z)}.$$

We can apply l'Hopital's rule, and obtain the limit is equal to

$$\lim_{z \to z_0} \frac{f(z) + (z - z_0)f'(z)}{g'(z)} = \frac{f(z_0)}{g'(z_0)}.$$

Problem (6): Compute

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\, dx.$$

Solution: We integrate $f(z) = \frac{1}{(1+z^2)^{n+1}}$ over a semicircular contour. The part of the semicircle that lies on the real axis is equal to our desired integral as R goes to infinity. For the part on the semicircle, we use the ML inequality to obtain

$$\left| \int_{|z|=R, \Im(z) \geq 0} \frac{1}{(1+z^2)^{n+1}} \right| dz \leq \frac{1}{(R^2 - 1)^{n+1}} \cdot \pi R.$$

The degree of the denominator is clearly greater than that of the numerator, so this integral goes to 0 as $R \to \infty$. Thus if we let Γ be the semicircular contour as $R \to \infty$, we have

$$2\pi i \sum_{\text{residues}} \operatorname{Res}(f) = \int_\Gamma f(z)\,dz = \int_{-\infty}^{\infty} \frac{1}{\left(1+x^2\right)^{n+1}}\,dx.$$

The only pole contained within the contour is at $z = i$. Using the formula for residues at higher order poles, we obtain

$$\operatorname{Res}(f,i) = \lim_{z\to i} \frac{d^n}{dz^n}\left(\frac{(z-i)^{n+1}}{\left(1+x^2\right)^{n+1}}\right) = \lim_{z\to i} \frac{d^n}{dz^n}\left(\frac{1}{(i+x)^{n+1}}\right) = \lim_{z\to i} \frac{(-1)^n(2n)!}{(i+z)^{2n+1}n!}.$$

Plugging in i and multiplying by $2\pi i$ yields

$$\int_{-\infty}^{\infty} \frac{1}{\left(1+x^2\right)^{n+1}}\,dz = \frac{\pi(2n)!}{4^n n!}.$$

Problem (7): Compute

$$\int_{|z|=5} z^n e^{1/z}\,dz.$$

Solution: Note that the only pole we have is at $z = 0$. Writing the Laurent series about 0 yields

$$z^n e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!}\cdot z^{n-k}.$$

For negative $n < -1$, the highest order power is -2, so the integral is just 0. Otherwise, the residue of the function at 0 is $\frac{1}{(n+1)!}$. Thus by the residue theorem, we have

$$\int_{|z|=5} z^n e^{1/z}\,dz = \begin{cases} 0 \ \text{ if } & n < -1 \\ \frac{2\pi i}{(n+1)!} & \text{otherwise} \end{cases}$$

Problem (8): Suppose that $\Gamma \subseteq \mathbb{C}$ is a contour, f is holomorphic on Γ and its interior except for some isolated singularities, all of which are removable or poles. Suppose further that f is holomorphic and nonzero on Γ. Show that $\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz$ is equal to the number of zeros minus the number of poles in the interior of Γ, counted with multiplicity.

Solution: First redefine f so there are no removable singularities. By Riemann's removable singularity theorem, the extension will be holomorphic at those points. From path invariance, we can deform Γ to be small circles around each zero and pole such that each circle only contains one pole or zero. Since f is holomorphic everywhere except at its poles, we can write f and f' as convergent power/Laurent series on these small circles.

Suppose z_0 is a zero of f with order n. Then we can write the power series of f as

$$f(z) = (z-z_0)^n h(z),$$

where $h(z_0) \neq 0$. Then we have

$$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1}h(z) + (z-z_0)^n h'(z)}{(z-z_0)^n h(z)} = \frac{n}{z-z_0} + \frac{h'(z)}{h(z)}.$$

Clearly the second term is holomorphic at z_0. Thus integrating on the small circle around z_0, the second term vanishes, and we get a contribution of $2\pi i n$ from the first term. Thus we get a positive contribution from each zero counted with multiplicity.

Now suppose z_0 is a pole of f with oreder n. Then we can write its Laurent series as

$$f(z) = (z-z_0)^{-n}h(z),$$

where $h(z_0) \neq 0$. We have

$$\frac{f'(z)}{f(z)} = \frac{-n(z-z_0)^{-n-1}h(z) + (z-z_0)^n h'(z)}{(z-z_0)^{-n} h(z)} = -\frac{n}{z-z_0} + \frac{h'(z)}{h(z)}.$$

Again the second term is holomorphic at z_0, so it vanishes when we integrate. We a get a contribution from the first term of $-2\pi i n$. Thus we get negative contributions from each pole counted with multiplicity, so we have the desired result.

Problem (9): Suppose that Γ is a closed contour, and f and g are holomorphic on Γ and its interior. Suppose further that $|f(z)| > |g(z)|$ for all $z \in \Gamma$. Show that f and $f + g$ have the same number of zeros in the interior of Γ, counted with multiplicity.

Solution: Note that $f + \lambda g$ for $\lambda \in [0,1]$ has no zeros on Γ, since otherwise $f(z) + \lambda g(z) = 0 \Rightarrow |f(z)| = \lambda|g(z)| \Rightarrow 1 \geq \frac{1}{\lambda} = \frac{|g(z)|}{|f(z)|} < 1$, contradiction.

Consider

$$N(\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{f'(z) + \lambda g'(z)}{f(z) + \lambda g(z)}\, dz,$$

where $\lambda \in [0,1]$. This counts the numbers and zeros and poles, but since f and g are holomorphic on the interior of Γ, $f + \lambda g$ has no poles. Thus it just counts the zeros. We show that $N(\lambda)$ is continuous on $[0,1]$, which then implies that $N(\lambda)$ is constant (since the integral from the argument principle only takes on integer values). The result then follows by taking $\lambda = 0$ and $\lambda = 1$.

Fix arbitrary $c \in [0,1]$ and pick arbitrary $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$|\lambda - c| < \delta \Rightarrow |N(\lambda) - N(c)| < \varepsilon.$$

We have

$$N(\lambda) - N(c) = \frac{1}{2\pi i}\int_\Gamma \left(\frac{f'(z) + \lambda g'(z)}{f(z) + \lambda g(z)} - \frac{f'(z) + c g'(z)}{f(z) + c g(z)}\right) dz = \frac{1}{2\pi i}\int_\Gamma \frac{(\lambda - c)(f(z)g'(z) - f'(z)g(z))}{(f(z) + \lambda g(z))(f(z) + c g(z))}\, dz.$$

Since $f(z) + \lambda g(z)$ is holomorphic on and nonzero on Γ for $\lambda \in [0,1]$, the integrand (modulus the independent $(\lambda - c)$ factor) is continuous on Γ. Since Γ is closed and bounded, its also compact, so by the extreme value theorem the integrand has a max M. Then applying the ML inequality yields

$$|N(\lambda) - N(c)| \leq \frac{\lambda - c}{2\pi} \cdot ML,$$

where L is the length of Γ. Since neither M and L depend on λ, we can make $\lambda - c$ arbitrarily small, and in fact small enough so that the right side is less than ε. In particular, we choose $\delta = \frac{2\pi\varepsilon}{ML}$. Thus $N(\lambda)$ is continuous at c, as desired.

Problem (10): Use Rouche's theorem to prove the fundamental theorem of algebra.

Solution: We prove this by induction on the degree of the polynomial. Clearly we have a solution for degree one polynomials, so suppose the theorem holds for all k up to $n-1$. Let $p(z) = a_n z^n + \dots + a_1 z + a_0$ with $n > 1$. We apply Rouche's theorem to $f(z) = a_n z^n$ and $g(z) = p(z) - f(z)$. Note that if we can find some contour where $|f(z)| > |g(z)|$ for all z on the contour, along with containing 0, we're done, since f has n roots within this contour (namely 0 with multiplicity n).

Let $m = \max(|a_{n-1}|, \dots, |a_0|)$. Then we have

$$|g(z)| \leq |a_{n-1}||z|^{n-1} + \cdots |a_0| \leq m \cdot \frac{|z|^n - 1}{|z| - 1}.$$

Let Γ be a circular contour centered at the origin with radius R and note that $|z| = R$ on this contour. If we can find an R large enough such that

$$m \cdot \frac{R^n - 1}{R - 1} < |a_n| R^n$$

then we're done. Dividing by $|z|^n$ and m yields

$$\frac{|a_n|}{m} > \frac{1 - \frac{1}{R^n}}{R - 1}.$$

Clearly the numerator goes to 1 as $R \to \infty$, and the denominator goes off to infinity. Thus the write side can get arbitrarily small, as desired.

Problem (11): Compute the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}.$$

Solution: We compute the integral of $f(z) = \frac{\pi \sec(\pi z)}{z^3}$ over a square $x, y = \pm N$ for a large integer N. Note that for any $z = x + yi$, we have

$$|\cos \pi z| = \frac{1}{2}\left|e^{i\pi x} e^{-\pi y} + e^{-i\pi x} e^{\pi y}\right| = \frac{1}{2}\left|e^{-i\pi x} e^{\pi y}\right|\left|1 + e^{2i\pi x} e^{-2\pi y}\right| = \frac{1}{2} e^{\pi y}\left|1 + e^{2i\pi x} e^{-2\pi y}\right|.$$

If we're on one of the vertical edges, $x = N$ is fixed. Then $e^{2i\pi N} = 1$, so the above is equal to $\frac{1}{2} e^{\pi y}(1 + e^{-2\pi y}) \geq \frac{1}{2} e^{\pi y} = \cosh(\pi y)$, which has a minimum of 1. Thus using ML on the right edge yields

$$\left| \int_{\Re(z) = N, -N \leq \Im(z) \leq N} f(z)\, dz \right| \leq \frac{\pi}{N^3} \cdot 2N,$$

which goes to 0 as N goes to infinity. The same works for the left vertical edge.

Going back to $\frac{1}{2}e^{\pi y}\left|1+e^{2i\pi x}e^{-2\pi y}\right|$, use the reverse triangle inequality to obtain

$$\frac{1}{2}e^{\pi y}\left|1+e^{2i\pi x}e^{-2\pi y}\right| \geq \frac{1}{2}e^{\pi y}\left||1|-\left|e^{2i\pi x}e^{-2\pi y}\right|\right| = \frac{1}{2}e^{\pi y}\left|1-e^{-2\pi y}\right| = |\sinh(\pi y)|.$$

On the horizontal edges, y is fixed, and sinh is nonzero for all nonzero numbers. Thus we have

$$\left|\int_{\Im(z)=N,-N\leq\Re(z)\leq N} f(z)\,dz\right| \leq \frac{\pi}{N^3\sinh \pi N}\cdot 2N,$$

which also goes to 0 as N goes to infinity. The same works for the bottom horizontal edge. Thus over a large square, $\int_\Gamma f(z)\,dz \to 0$.

Now we compute the residues. f has an order 3 pole at 0 and an order 1 pole at $\frac{2k+1}{2}$ (zeros of $\cos \pi z$) where $k \in \mathbb{Z}$. We have

$$\operatorname{Res}(f,0) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^3\cdot\left(\frac{\pi}{z^3\cos \pi z}\right)\right) = \frac{1}{2}\lim_{z\to 0}(\pi^3\sec \pi z\tan^2 \pi z+\pi^3\sec^3 \pi z) = \frac{\pi^3}{2}.$$

For any $k \in \mathbb{Z}$, we have

$$\operatorname{Res}\left(f,\frac{2k+1}{2}\right) = \lim_{z\to\frac{2k+1}{2}}\frac{\pi\left(z-\frac{2k+1}{2}\right)}{z^3\cos \pi z} = \lim_{z\to\frac{2k+1}{2}}\frac{\pi}{3z^2\cos \pi z-\pi z^3\sin \pi z} = \frac{\pi}{-\pi\left(\frac{2k+1}{2}\right)^3(-1)^k} = (-1)^{k+1}\frac{8}{(2k+1)^3}.$$

Thus from the residue theorem, we have

$$8\left(\cdots-\frac{1}{(-3)^3}+\frac{1}{(-1)^3}-\frac{1}{1^3}+\frac{1}{3^3}+\cdots\right)+\frac{\pi^3}{2} = -16\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}+\frac{\pi^3}{2} = \frac{1}{2\pi i}\int_{x,y=\pm N,N\to\infty} f(z)\,dz = 0.$$

Thus our desired sum is equal to $\frac{\pi^3}{32}$.

Problem (12): Compute

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}.$$

Solution: Let S be the desired sum. We compute $f(z)=\frac{\pi\cot \pi z}{z^2+1}$ over a large square with $x,y=\pm\left(N+\frac{1}{2}\right)$ for large integers N. Note from the discussion in the chapter about bounding $\frac{\pi\cot \pi z}{z^2}$ over the same square, the integral went to 0 as $N\to\infty$. Since in f the only difference is the $+1$ in the denominator, the overall magnitude of the function is essentially the same for large N, so this integral indeed converges to 0.

Note the poles of this function are at $\pm i$ and all $n\in\mathbb{Z}$. Thus by the residue theorem, we have

$$\operatorname{Res}(f,i)+\operatorname{Res}(f,-i)+\cdots\operatorname{Res}(f,-1)+\operatorname{Res}(f,0)+\operatorname{Res}(f,1)+\cdots=0.$$

First we compute $\operatorname{Res}(f,n)$ for some $n\in\mathbb{Z}$. Note that the pole is simple, so we have

$$\operatorname{Res}(f,n) = \lim_{z\to n} \frac{\pi(z-n)\cos \pi z}{(z^2+1)\sin \pi z} = \lim_{z\to n} \frac{\pi \cos \pi z - \pi(z-n)\sin \pi z}{2z \sin \pi z + \pi(z^2+1)\cos \pi z} = \frac{\pi \cos \pi n}{\pi(n^2+1)\cos \pi n} = \frac{1}{n^2+1}.$$

Thus the sum of this over all $n \in \mathbb{Z}$ is $2S+1$ (the 1 comes from $n=0$).

Note that the poles at $\pm i$ are also simple, So

$$\operatorname{Res}(f,i) = \lim_{z\to i} \frac{\pi(z-i)\cot \pi z}{z^2+1} = \frac{\pi \cot \pi i}{2i},$$

$$\operatorname{Res}(f,-i) = \lim_{z\to -i} \frac{\pi(z+i)\cot \pi z}{z^2+1} = \frac{\pi \cot(-\pi i)}{-2i} = \frac{\pi \cot \pi i}{2i}.$$

Thus

$$2S+1 = -\frac{\pi}{i}\cot(\pi i).$$

Since $\cot z = i\frac{e^z+e^{-z}}{e^z-e^{-z}}$, we have

$$\cot(\pi i) = i\frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}} = -i\coth \pi.$$

Thus

$$2S+1 = \pi \coth \pi \Rightarrow S = \frac{\pi}{2}\coth \pi - \frac{1}{2}.$$

Problem (13): For $w \in \mathbb{Z}$, compute

$$\sum_{n=-\infty}^{\infty} \frac{1}{(w+n)^2}.$$

Solution: We compute $f(z) = \frac{\pi \cot \pi z}{(w+z)^2}$ over a large square with $x, y = \pm\left(N + \frac{1}{2}\right)$ for a large integer N. As with the previous problem, the contour integral over this square approaches 0 as $N \to \infty$. We have simple poles at every integer and a double pole at $-w$. Computing, we obtain

$$\operatorname{Res}(f,-w) = \lim_{z\to -w} \frac{d}{dz}\pi \cot \pi z = \lim_{z\to -w} -\pi^2 \csc^2 \pi z = -\frac{\pi^2}{\sin^2(\pi w)},$$

$$\operatorname{Res}(f,n) = \lim_{z\to n} \frac{\pi(z-n)\cos \pi z}{(w+z)^2 \sin \pi z} = \lim_{z\to n} \frac{\pi \cos \pi z - \pi^2(z-n)\sin \pi z}{2(w+z)\sin \pi z + \pi(w+z)^2 \cos \pi z} = \frac{\pi \cos \pi n}{\pi(w+n)^2 \cos \pi n} = \frac{1}{(w+n)^2}.$$

Thus from the residue theorem, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(w+n)^2} - \frac{\pi^2}{\sin^2(\pi w)} = 0,$$

as desired.

Problem (14): Suppose that f has a convergent power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ in a neighborhood $\{z : |z - z_0| < r\}$ of z_0. Show that f is differentiable at any point z_1 with $|z_1 - z_0| < r$.

Solution: We write

$$\sum_{n=0}^{\infty} a_n ((z - z_1) + (z_1 - z_0))^n.$$

Let $d = z_1 - z_0$. Note that after expanding and collecting terms, the degree k term has coefficient

$$\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0)^{n-k}.$$

Denote the coefficients as b_k. Thus

$$\sum_{n=0}^{\infty} b_n (z - z_1)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Note that the b_n's must be finite, since otherwise the right hand side wouldn't be finite for $|z - z_0| < r$. Thus the power series of f centered at converges in some neighborhood of z_1, and in particular $f'(z_1) = b_1$. Thus f is indeed differentiable at z_1.

Problem (16): Suppose that f is holomorphic on $\overline{B_1(0)}$ and that $|z_0| < 1$. Show that

$$f(z_0) = \frac{1}{\pi} \iint_{|z| \le 1} \frac{f(z)}{(1 - \overline{z} z_0)^2} \, dx \, dy.$$

Solution: We convert the integral to polar coordinates. Thus $z = re^{i\theta}$ and $dx \, dy = r \, d\theta \, dr$, yielding

$$\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \frac{f(re^{i\theta}) r}{(1 - re^{-i\theta} z_0)^2} \, d\theta = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \frac{f(re^{i\theta}) r^3 e^{2i\theta}}{(re^{i\theta} - r^2 z_0)^2} \, d\theta \, dr.$$

Now we convert the inner integral to a contour integral. Let $z = re^{i\theta}$. Then $-\frac{i}{z} \, dz = d\theta$, yielding

$$\frac{1}{\pi} \int_0^1 \int_{|z|=r} \frac{f(z) r z^2}{(z - r^2 z_0)^2} \cdot \frac{-i}{z} \, dz \, dr = -\frac{i}{\pi} \int_0^1 \int_{|z|=r} \frac{r z f(z)}{(z - r^2 z_0)^2} \, dz \, dr.$$

Note that $r^2 |z_0| < r \Rightarrow r < \frac{1}{|z_0|}$, which is true, since the right side is greater than 1 from $|z_0| < 1$. Thus the integrand is holomorphic on $|z| = r$ and its interior except at $r^2 z_0$. Letting $g(z) = r z f(z)$ and using the Cauchy integral theorem, we obtain that the double integral is equal to

$$-\frac{i}{\pi} \int_0^1 2\pi i g'(r^2 z_0) \, dr = 2 \int_0^1 r(f(r^2 z_0) + r^2 z_0 f'(r^2 z_0)) \, dr.$$

Letting $u = r^2 \Rightarrow du = 2r \, dr$, we obtain

$$\int_0^1 f(uz_0) + uz_0 f(uz_0)\, dr = \int_0^1 \frac{d}{du}(uf(uz_0))\, du = f(z_0),$$

as desired.

8. Branch Points and Riemann Surfaces

8.1. Analytic Continuation

Consider a nonzero holomorphic function in a neighborhood of z_0, and let n be the smallest integer such that the power series centered at z_0 has $a_n \neq 0$. Then we can write $f(z) = (z - z_0)^n g(z)$, where $g_{z_0} \neq 0$ and is holomorphic in a neighborhood of z_0. Since g is continuous, there's some neighborhood around z_0 such that $g(z) \neq 0$. Thus there exists some neighborhood around z_0 where f is nonzero except at z_0. Thus the zeros of holomorphic functions are isolated (i.e. they don't have a limit point).

Suppose f and g are two holomorphic functions on a connected open set $U \subseteq \mathbb{C}$, and suppose that f and g agree on some set S containing a limit point. Then $f(z) = g(z)$ for all $z \in S$, so $f(z) - g(z) = 0$. From above, we know this is impossible unless $f - g$ is the zero function on U. From this we get the notion of *ananlytic continuation*.

8.2. Branch Points

Definition (branch point): Let $z_0 \in \mathbb{C}$, and suppose f is a holomorphic function defiend on a subset of $\mathbb{C}$ containing points arbitrarily close to z_0. Pick a sufficiently small $\varepsilon > 0$, and let z_1 be a point in the domain of definition of f such that $|z_0 - z_1| < \varepsilon$. We'll take $z_1 = z_0 + \frac{\varepsilon}{2}$. Suppose that there analytic continuations f_1, f_2 of f to regions containing the top and bottom semicircles centered at z_0 with radius $\frac{\varepsilon}{2}$ such that $f_1(z_1) = f_2(z_1) = f(z_1)$ but $f_1(z_0 - \frac{\varepsilon}{2}) \neq f_2(z_0 - \frac{\varepsilon}{2})$. Then z_0 is a *branch point* of f.

8.3. Branch Cuts

8.4. Global Analytic Functions

Definition (function element): A *function element* is a pair (f, Ω), where Ω is an open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is holomorphic.

Definition (analytic continuation): Suppose there exists a chain $(f_1, \Omega_1), ..., (f_n, \Omega_n)$ of function elements such that (f_{j+1}, Ω_{j+1}) is a direct analytic continuation of (f_j, Ω_j) for $1 \leq j \leq n - 1$. Then (f_n, Ω_n) is an *ananlytic continuation* of (f_1, Ω_1).

Definition (global analytic function): A *global analytic function* is an equivalence class of function elements. We write $\mathbf{f}$ for the global analytic function represented by (f, Ω).

8.5. Riemann Surfaces

A Riemann surface consists of several pieces. First a collection of open sets $\{U_\alpha\}_{\alpha \in I}$ in $\mathbb{C}$ called *charts*. Next, we need identification functions between various charts: for each pair α, β with $\alpha \neq \beta$, there is an open subset $U_{\alpha\beta} \subseteq U_\alpha$ (possibly empty) and a holomorphic function $\varphi_{\alpha\beta} : U_{\alpha\beta} \to U_{\alpha\beta}$ such that $\varphi_{\beta\alpha} \circ \varphi_{\alpha\beta}$ and $\varphi_{\alpha\beta} \circ \varphi_{\beta\alpha}$ are the identity function on $U_{\alpha\beta}$ and $U(\beta\alpha)$ respectively. We now obtain a Riemann surface by gluing $z \in U_{\alpha\beta}$ to $\varphi_{\alpha\beta}(z) \in U_{\beta\alpha}$ for all z, α, β.

If X is a Riemann surface and $z \in X$, then there is some chart U_α such that $z \in U_\alpha$. Thus an open set around z is an open set around z in U_α.

Example *(Riemann sphere, $\mathbb{CP}^1$)*: Take two charts U_1, U_2, each of which is the entire complex plane. We now take U_{12} and U_{21} to be the complex plane with the origin deleted. The gluing maps φ_{12} and φ_{21} are both given by $\varphi_{12}(z) = \varphi_{21}(z) = \frac{1}{z}$ for $z \in U_{12}$ or $z \in U_{21}$.

8.6. Problems

Problem (2): The series $\sum_{n \geq 0} z^n$ converges to $\frac{1}{1-z}$ on the unit disk. Find a series that converges $\frac{1}{1-z}$ but on the set $\{z \in \mathbb{C} : |z| > 1\}$.

Solution: The series $-\sum_{n=1}^{\infty} \frac{1}{z^n}$ works. Note that for $|z| > 1$, the common ratio of this series has absolute value $\left|\frac{1}{z}\right| < 1$, and so converges everywhere on this set. In particular, it converges to

$$-\frac{\frac{1}{z}}{1 - \frac{1}{z}} = \frac{1}{1-z}.$$

Problem (3): The integral

$$f(z) = \int_0^\infty e^{-zt} \, dt$$

converges for $\Re(z) > 0$. Find an analytic continuation of f to a larger connected open set. What is the largest open set to which f can be continued?

Solution: For $\Re(z) > 0$, we have

$$\int_0^\infty e^{-zt} \, dt = -\frac{1}{z} e^{-zt} \Big|_0^\infty = \frac{1}{z}.$$

Since f and $\frac{1}{z}$ agree on the right half plane, and since $\frac{1}{z}$ is defined on $\mathbb{C} \setminus \{0\}$, we can extend f to $\mathbb{C} \setminus \{0\}$ by letting it equal $\frac{1}{z}$ everywhere that isn't $\mathfrak{R}(z) > 0$.

Problem (4): The integral

$$f(z) = \int_0^\infty e^{-zt} \sin t \, dt$$

converges for $\mathfrak{R}(z) > 0$. Find an analytic continuation of f to a larger connected open set. What is the largest open set to which f can be continued?

Solution: For $\mathfrak{R}(z) > 0$, we recognize this is the Laplace transform of $\sin t$, so

$$f(z) = \frac{1}{z^2 + 1}.$$

Since f and $\frac{1}{z^2+1}$ agree on the right half plane, and since $\frac{1}{z^2+1}$ is defined on $\mathbb{C} \setminus \{-i, i\}$, we can extend f to $\mathbb{C} \setminus \{-i, i\}$ by letting it equal $\frac{1}{z^2+1}$ on everywhere that isn't $\mathfrak{R}(z) > 0$.

Problem (5): Define the function $\Gamma(z)$ by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt.$$

a) For which $z \in \mathbb{C}$ does the integral converge?
b) Show that, when both sides converge, we have the functional equation

$$\Gamma(z + 1) = z\Gamma(z).$$

c) Find the largest subset of $\mathbb{C}$ on which you can define an analytic continuation of $\Gamma(z)$, and explain how to define it via an explicit formula.

Solution:

a) We claim $\Gamma(z)$ converges for $\Re(z) > 0$ and diverges otherwise. Let $z - 1 = x + yi$. First we show convergence for $\Re(z) \geq 1$. Thus $x \geq 0$. We have

$$\left| \int_0^\infty e^{-t} t^{x+yi} \, dt \right| \leq \int_0^\infty e^{-t} t^x \, dt.$$

Since exponentials eventually get larger than any polynomial, we can write the second integral as

$$\int_0^N e^{-t} t^x \, dt + \int_N^\infty e^{-t} t^x \, dt \leq \int_0^N e^{-t} t^x \, dt + \int_N^\infty e^{-t} e^{\frac{t}{2}} \, dt$$

for some suitably large N. The second integral clearly converges (antidervative is $-2e^{-\frac{t}{2}}$), and the first integral is finite, so $\Gamma(z)$ does converge.

Now suppose $0 < \Re(z) < 1$, which means $-1 < x < 0$. We start with the same absolute value bound. Then we split the integral as

$$\int_0^1 e^{-t} t^x \, dt + \int_1^\infty e^{-t} t^x \, dt.$$

The second integral converges for similar reasons as the first case. In the first integral, we bound

$$\int_0^1 e^{-t} t^x \, dx \leq \int_0^1 t^x \, dt.$$

This has antiderivative $\frac{t^{x+1}}{x+1}$, and since $x > -1$, we don't have divide by zero issues with the lower bound. Thus it converges.

b) We apply integration by parts on $u = t^z$ and $dv = e^{-t}$ to obtain

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z \, dt = -t^z e^{-t} \Big|_0^\infty + \int_0^\infty z t^{z-1} e^{-t} \, dt = z\Gamma(z).$$

c) We can analytically continue $\Gamma(z)$ to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Note we can't continute it to 0 via the functional equation, because plugging in $z = 0$ yields $\Gamma(1) = 0$, so we can never get $\Gamma(0)$ on a side of an equation that's nonzero on the other side. Since we can't hit 0, we can't hit $-1, -2, \ldots$.

Now suppose $z = x + yi \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with $x < 0$. Let $-n = \lfloor x \rfloor$. Applying the functional equation n times on $\Gamma(x + yi)$ yields

$$\Gamma(x+yi)(x+yi)(x+1+yi) \cdots (x+n-1+yi) = \Gamma(x+n+yi).$$

The right side converges, so we have

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1) \cdots (z+n-1)}.$$

Problem (7): Find an open set U of $\mathbb{C}$ on which the function

$$f(z) = \sqrt{z(z-1)(z-2)(z-3)}$$

has a holomorphic branch, such that U is as large as possible. Do the same for $g(z) = \sqrt{z(z-1)(z-2)(z-3)(z-4)}$.

Solution: Note that clearly there are branch points at $z = 0, 1, 2, 3$ for f. Consider

$$\frac{1}{f\left(\frac{1}{z}\right)} = \frac{1}{\sqrt{\frac{1}{z}\left(\frac{1}{z}-1\right)\left(\frac{1}{z}-2\right)\left(\frac{1}{z}-3\right)}} = \frac{\sqrt{z^4}}{\sqrt{(1-z)(1-2z)(1-3z)}}.$$

Neither the top nor bottom have a branch point at $z = 0$, so f has no branch point at ∞. Thus the set $\mathbb{C} \setminus [0,1] \cup [2,3]$ with branch cuts along $[0,1]$ and $[2,3]$ works, since any path encircling 0 and 1 or 2 and 3 will get two sign changes from their corresponding roots in f, and thus will contribute no sign change at the end of the loop.

For $g(z)$, we have obvious branch points at $z = 0, 1, 2, 3, 4$. However, we have

$$\frac{1}{g\left(\frac{1}{z}\right)} = \frac{\sqrt{z^5}}{\sqrt{(1-z)(1-2z)(1-3z)(1-4z)}}.$$

Since the numerator has a branch point at $z = 0$, g has a branch point at infinity. Thus we define g on $\mathbb{C} \setminus (-\infty, 0] \cup [1,2] \cup [3,4]$.

Problem (9): Suppose that f has a branch point of order n at z_0. Show that f can be expressed as a Laurent series in $(z - z_0)^{\frac{1}{n}}$ in a neighborhood of z_0.

Solution: Without loss of generality, suppose $z_0 = 0$. Consider $f(z^n)$. Note that in a neighborhood of 0, this can only have one branch, since otherwise if it has more than one, the original function $f(z)$ would have more than n branches around 0. Then we can write $f(z^n)$ as a Laurent series about 0, so subbing in $z = w^{\frac{1}{n}}$ yields the desired result.

Problem (11): Say that an analytic function element (f, Ω) has a *natural boundary* at the boundary of Ω if it has no direct analytic continuation to a larger set. Show that the function $f : \mathbb{D} \to \mathbb{C}$ given by $f(z) = \sum_{n=0}^{\infty} z^{n!}$ has the unit circle as its natural boundary.

Solution: We show that there is a dense set S on the unit circle such that if we approach a point in S on a specific path from inside $\mathbb{D}$, then f blows up to infinity. Then, for the sake of contradiction, if there were to exist some direct analytic continuation (f_1, Ω), the intersection $\mathbb{D} \cap \Omega$ would contain some point in S (and some small neighborhood around those points). Since f_1 is analytic on $\mathbb{D} \cap \Omega$, it must be analytic at a point in S within this intersection as well. However, since there's some path for which the limit of f at this point blows up, this point is either a pole or essential singularity. Thus f_1 is not analytic on Ω, contradiction.

We claim that S is the set of roots of unity. In particular, this is the set of complex numbers $e^{2\pi i \frac{p}{q}}$, where p and q are relatively prime. Note that the rationals are dense in the reals, so $2\pi i \frac{p}{q}$ is dense in $[0, 2\pi]$.

Fix some number in S, say $e^{2\pi i \frac{p}{q}}$. Consider the limit $\lim_{r \to 1} f\left(re^{2\pi i \frac{p}{q}}\right)$. Note that the terms in f that have index at least q will just contribute 1 multiplied by some large power of r. Thus the tail end of the series is

$$r^{q!} + r^{(q+1)!} + \cdots$$

As $r \to 1$, these terms are get closer to 1, and thus approach infinity, as desired.

9. Products and the Gamma Function

9.1. Infinite Products

Given a sequence $z_1, \ldots$, we define $\prod(n = 1)^{\infty} z_n$ to be the limit of the partial products if that limit exists and is nonzero (to avoid issues with terms being zero). Clearly in order for the product to converge, $\lim z_n = 1$, so write $z_n = 1 + a_n$. Taking the logarithm of the product yields (choosing a branch cut that doesn't go through any z_n's)

$$\log \prod_{n=1}^{\infty} (1 + a_n) = \sum_{n=1}^{\infty} \log(1 + a_n).$$

Using $\log(1 + z) = z - \frac{z^2}{2} + \cdots$, note that if $|z|$ is small, then $\log(1 + z) \approx z$, so $\lim_{a \to 0} \frac{\log(1+a)}{a} = 1$. Thus from the comparison, either both $\sum \log(1 + |a_n|)$ and $\sum |a_n|$ converge or diverge.

Definition (absolute convergence): An infinite product $\prod(1 + a_n)$ *converges absolutely* if $\prod(1 + |a_n|)$ converges.

Proposition: The product $\prod(1 + a_n)$ converges absolutely if and only if the series $\sum |a_n|$ converges.

Proof: Follows from previous discussion. ■

9.2. Weierstrass Product

Suppose $a_1, a_2, \ldots$ is a sequence tending to ∞ sorted in order of magnitude, and suppose the first m terms are 0. The product

$$z^m \prod_{j=m+1}^{\infty} \left(1 - \frac{z}{a_j}\right)$$

has zeros with correct multuplicities at exactly the terms in the sequence. However, this isn't guranteed to converge. We only have converges is $\sum \frac{1}{|a_n|}$ converges. In the case where it does, we're fine, but otherwise we need some term to damp the product while keeping the zeros the same.

Theorem (Weierstrass product formula): Define

$$p_n(z) = z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}$$

to be the truncation of the Taylor series of $-\log(1-z)$ for small z. Let $0, \ldots, 0, a_1, a_2, \ldots$ with m zeros be a sequence sorted by magnitude with $|a_n| \to 0$ as $n \to \infty$. Then

$$f(z) = z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z/a_n)}$$

is an entire function that absolutely converegs and that vanishes at only numbers in the sequence with the right multiplicity.

Remark: The p_n's are chosen to make $e^{p_n(z/a_n)}$ mimic $\left(1 + \frac{z}{a_n}\right)^{-1}$ so that all terms for large enough a_n are close to 1 (add details later).

Proposition: Suppose h is nonvanishing and entire. Then there exists an entire function g such that $h(z) = e^{g(z)}$.

Solution: Let $\varphi(z) = \frac{h'(z)}{h(z)}$. Note that $\varphi(z) = \frac{d}{dz} \log h(z)$. Then

$$\int_0^{z_0} \varphi(z)\, dz = [\log h(z)]_0^{z_0}$$

along any path connecting 0 to z_0 (since φ is entire). Thus

$$h(z_0) = e^{\int_0^{z_0} \varphi(z)\, dz + \log h(0)},$$

so we can take $g(z_0)$ to be the exponent.

9.3. Hadamard Factorization Theorem and Growth of Entire Functions

Definition (order): Let f be an entire function. The order of f is defined to be

$$\inf\left\{a : |f(z)| \leq e^{|z|^a}\right\}$$

for sufficiently large $|z|$.

Equivalently equal to

$$\limsup_{|z| \to \infty} \frac{\log \log |f(z)|}{\log |z|}.$$

Proof: Let $\rho = \limsup_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|}$, and let $\alpha = \inf\{a : |f(z)| \leq e^{|z|^a}\}$. By definition, for $\varepsilon > 0$, there exists some r such that

$$|z| > r \Rightarrow \rho + \varepsilon > \frac{\log\log|f(z)|}{\log|z|}.$$

Exponentiating twice and rearranging yields

$$e^{|z|^{\rho+\varepsilon}} > |f(z)|.$$

This holds for all $\varepsilon > 0$ and large enough $|z|$, so we have $\rho \geq \alpha$.

Note by definition, we also have that

$$\frac{\log\log|f(z)|}{\log|z|} > \rho - \varepsilon$$

infinitely often (since otherwise the limsup would be lower than it actually is). Thus $|f(z)| > e^{|z|^{\rho-\varepsilon}}$ infinitely often, so clearly $\rho - \varepsilon$ is not in the set of α's construction for any ε. Thus we obtain $\alpha \geq \rho$, and thus we have equality. ∎

Theorem (Hadamard factorization theorem): Let f an entire function of finite order ρ and suppose it has infinitely many zeros $0, \ldots, 0, a_1, a_2, \ldots$, where there are m zeros and the a_n's are sorted by increasing magnitude. Let $\varkappa \leq \lfloor\rho\rfloor$ be the largest nonnegative integer k such that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^k}$$

diverges. Then f can be written as

$$e^{g(z)} z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_\varkappa(z/a_n)},$$

where g is a polynomial of degree at most $\lfloor\rho\rfloor$.

9.4. Meromorphic Functions

Definition (meromorphic): Let $U \subseteq \mathbb{C}$ be open. A *meromorphic function* on U is a function $f(z)$ that is holomorphic except at a set of points without a limit point, where it has poles or removable singularities.

Proposition: A function f is meromorphic on U if and only if for each $z_0 \in U$, there is a neighborhood of z_0 on which f is a quotient of two holomorphic functions.

Proof:

∎

Proposition: A meromorphic function f on all of $\mathbb{C}$ is a rational function if and only if ∞ is a removable singularity or pole.

9.5. Gamma Function

9.6. Problems

Problem (1): Find product representations for the following entire functions:

a) $e^z - 1$
b) $(z^3 - 1)\cos(z^2)$
c) $1 - \cos(z)$

Solution:

a) This has zeros $0, \pm 2\pi i, \pm 4\pi i, \ldots$ Clearly the sum of reciprocal of the magnitudes of the nonzero terms in this sequence diverge for $k = 1$ (since its just the even terms of the harmonic series times 2). Thus in Hadamard's factorization theorem, $\varkappa = 1$. Clearly the order of $e^z - 1$ is 1, so the product form of it is equal to

$$e^z - 1 = e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z}{2n\pi i}\right) e^{z/\ 2n\pi i} \left(1 + \frac{z}{2n\pi i}\right) e^{-z\ /2n\pi i} = e^{g(z)} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4n^2\pi^2}\right),$$

where g has at most degree 1. Let $g(z) = a + bz$. For $z = 0$ (we divide by z so that the left side is $\frac{e^z - 1}{z}$, which has a removable singularity with value 1 at 0), we have $1 = e^a$, so $a = 0$. Letting $z = 2$, we obtain

$$e^2 - 1 = 2e^{2b} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2\pi^2}\right) = 2e^{2b} \cdot \frac{\sin(i)}{i} = e^{2b}(e^1 - e^{-1}) \Rightarrow e = e^{2b},$$

where we used the product form of $\frac{\sin(z)}{z}$ in the second equality. Then $b = \frac{1}{2}$, so we have

$$e^z - 1 = e^{z/2} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4n^2\pi^2}\right).$$

b)

c) We can write $1 - \cos z = 2\sin^2 \frac{z}{2}$, and then using the product formula for sin yields

$$1 - \cos z = \frac{1}{2} z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{4n^2\pi^2}\right)^2.$$

Problem (4): Compute $\Gamma\left(\frac{1}{2}\right)$.

Solution: From the reflection formula, we have

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi}{2}\right)}=\pi.$$

Thus $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

Problem (7): Compute $\sum \frac{1}{n^4}$ by mimicing the solution to the Basel problem, but using a different entire function.

Solution: Consider $f(z)=\frac{\sin(z)}{z}\cdot\frac{\sin(iz)}{iz}$. This has the product expansion

$$\frac{\sin(z)}{z}\cdot\frac{\sin(iz)}{iz}=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)\left(1+\frac{z^2}{n^2\pi^2}\right)=\prod_{n=1}^{\infty}\left(1-\frac{z^4}{n^4\pi^4}\right).$$

The z^4 coefficient on the right side is $-\frac{1}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$. Looking at the power series expansion of the left side, we have

$$\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)\left(1+\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)=1-\frac{z^4}{90}+\cdots$$

Thus, $-\frac{1}{90}=-\frac{1}{\pi^4}\zeta(4)\Rightarrow\zeta(4)=\frac{\pi^4}{90}$.

Problem (9): Prove that

$$|\Gamma(1/2+it)|=\sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}}$$

for all $t\in\mathbb{R}$.

Solution: Using the product formulation of the Gamma function, note that

$$\overline{\Gamma(z)}=\overline{e^{-\gamma z}z^{-1}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}}=e^{-\gamma\overline{z}}\overline{z}^{-1}\prod_{n=1}^{\infty}\left(1+\frac{\overline{z}}{n}\right)^{-1}e^{\overline{z}/n}=\Gamma(\overline{z}).$$

Thus

$$\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2=\Gamma\left(\frac{1}{2}+it\right)\overline{\Gamma\left(\frac{1}{2}+it\right)}=\Gamma\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}-it\right)=\frac{\pi}{\sin\left(\frac{\pi}{2}+i\pi t\right)}=\frac{\pi}{\cos i\pi t}=\frac{2\pi}{e^{\pi t}+e^{-\pi t}}.$$

Taking the square root of both sides gives us the desired conclusion.

Problem (11): Find an entire function order $\frac{1}{3}$.

Solution: Let $\omega = e^{\frac{2\pi i}{3}}$, and define the entire function

$$f(z) = \frac{e^z + e^{\omega z} + e^{\omega^2 z}}{3}.$$

Note that $f(z) = f(\omega z) = f(\omega^2 z)$. Thus the function $g(z) = f(\sqrt[3]{z})$ is entire (since $\sqrt[3]{z}$ has the multiple values $\sqrt[3]{z}, \omega\sqrt[3]{z}, \omega^2\sqrt[3]{z}$, and so the function is well defined).

We claim that $g(z)$ has order $\frac{1}{3}$. We use the formula in problem 12 for order. Consider $z^n + (\omega z^n) + (\omega^2 z)^n$. If n is not divisible by 3, then the coefficients will be a permutation of $\{1, \omega, \omega^2\}$, which when summed add to 0. Otherwise we get $3z^n$. Thus, when we add the power series of $e^z, e^{\omega z}, e^{\omega^2 z}$, only the exponents divisible by 3 will survive, while picking up a factor of 3. Thus we have

$$f(z) = 1 + \frac{z^3}{3!} + \frac{z^6}{6!} + \cdots \Rightarrow f(\sqrt[3]{z}) = 1 + \frac{z}{3!} + \frac{z^2}{6!} + \cdots.$$

Thus $a_n = \frac{1}{(3n)!}$. Therfore, the order is equal To

$$\limsup_{n\to\infty} \frac{n \log n}{\log(3n)!}.$$

Applying Stirling's approximation, we have

$$\limsup_{n\to\infty} \frac{n \log n}{\log\left(\sqrt{6\pi n}\left(\frac{3n}{e}\right)^{3n}\right)} = \limsup_{n\to\infty} \frac{n \log n}{\frac{1}{2}\log(6\pi) + \frac{1}{2}\log(n) + 3n\log(3n) - 3n}$$

$$= \limsup_{n\to\infty} \frac{1}{\frac{\log 6\pi}{2n\log n} + \frac{1}{2n} + \frac{3\log 3}{\log n} + 3 - \frac{3}{\log n}}.$$

As $n \to \infty$, every term in the bottom goes to 0, except for 3, so we have

$$\limsup_{n\to\infty} \frac{n \log n}{\log(3n)!} = \frac{1}{3}.$$

Problem (12): Suppose that f is an entire function of finite order ρ. Suppose that the Taylor series of f is $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let

$$\beta = \limsup_{n\to\infty} \frac{n \log n}{-\log|a_n|}.$$

a) Show that $\beta \leq \rho$.
b) Show that $\rho \leq \beta$.
c) Show that for any positive real number α, there is an entire function or order α.

Solution:

Lemma:

$$\lim_{n\to\infty} |a_n| \to 0$$

Proof: Follows from f has radius of convergence infinity, and thus Cauchy-Hadamard gurantees $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$. ■

Thus $-\log|a_n|$ is positive for sufficiently large n.

a) Note from the bound on derivatives, we have

$$\left|f^{(n)}(0)\right| \leq \frac{n! M_R}{R^n} \Rightarrow |a_n| \leq \frac{M_R}{R^n},$$

where $M_R = \max_{|z|=R}|f(z)|$. This holds for all n and R. Now pick $\varepsilon > 0$. For sufficiently large R, we have by definition

$$|a_n| \leq \frac{M_R}{R^n} \leq \frac{e^{R^{\rho+\varepsilon}}}{R^n} \Rightarrow \log|a_n| \leq R^{\rho+\varepsilon} - n\log R.$$

Let $R = n^{1/(\rho+\varepsilon)}$ for sufficiently large n. Then

$$\log|a_n| \leq n - \frac{n}{\rho+\varepsilon}\log n.$$

Multiplying by -1, reciprocating, and multiplying by $n\log n$ yields

$$\frac{n\log n}{-\log|a_n|} \leq \frac{\rho+\varepsilon}{1 - \frac{\rho+\varepsilon}{\log n}}.$$

For large enough n, we have $1 - \frac{\rho+\varepsilon}{\log n} \approx 1$, so we obtain

$$\frac{n\log n}{-\log|a_n|} \leq \rho+\varepsilon.$$

This holds for arbitrary $\varepsilon > 0$, so $\beta \leq \rho$.

b) Pick $\varepsilon > 0$. For sufficiently large n (large enough such that the lemma applies as well), we have

$$\beta + \varepsilon > \frac{n\log n}{-\log|a_n|} \Rightarrow \log|a_n|^{-(\beta+\varepsilon)} > \log n^n \Rightarrow |a_n| < n^{-n/(\beta+\varepsilon)}.$$

Now, from the definition of order, there exist infinitely many R such that

$$e^{R^{\rho-\varepsilon}} \leq M_R \leq \sum_{n=0}^{\infty} |a_n| R^n.$$

Now, for sufficiently large R, and letting $M = \max\left\{|a_i|, 0 \leq i \leq (2R)^{\beta+\varepsilon}\right\}$ we have

$$
\begin{aligned}
e^{R^{\rho-\varepsilon}} \leq \sum_{n=0}^{\infty} |a_n| R^n \leq \sum_{n=0}^{(2R)^{\beta+\varepsilon}} |a_n| R^n + \sum_{n=(2R)^{\beta+\varepsilon}}^{\infty} n^{-\frac{n}{\beta+\varepsilon}} R^n \\
\leq M \cdot \frac{R^{(2R)^{\beta+\varepsilon}+1} - 1}{R-1} + \sum_{n=(2R)^{\beta+\varepsilon}}^{\infty} (2R)^{-n} R^n \\
\leq R^{(2R)^{\beta+\varepsilon}+1+\sigma} + 2\left(\frac{1}{2}\right)^{(2R)^{\beta+\varepsilon}},
\end{aligned}
$$

where $\sigma = \log_R\left(\frac{M}{R-1}\right)$. For large R, this is just $\log_R M - \log_R(R-1) \approx -1$, and thus $1 + \sigma \approx 0$. For large enough R, we also have that the second term is small, so we can ignore it. Taking logs yields

$$
\begin{aligned}
R^{\rho-\varepsilon} \leq (2R)^{\beta+\varepsilon} \log R \Rightarrow (\rho - \varepsilon) \log R \leq \log \log R + (\beta + \varepsilon) \log 2R \\
\Rightarrow \rho - \varepsilon \leq \frac{\log \log R}{\log R} + \frac{(\beta + \varepsilon) \log 2}{\log R} + \beta + \varepsilon.
\end{aligned}
$$

Again for large R, the first two terms are small, so we obtain $\rho - \varepsilon \leq \beta + \varepsilon$. This holds arbitrary ε, so we have $\rho \leq \beta$.

c) Let $a_n = n^{-\frac{n}{\alpha}}$. From Cauchy-Hadamard, we have $\limsup_{n\to\infty} n^{-\frac{1}{\alpha}} = 0$, so the series with coefficients a_i has infinite radius of convergence, and thus is entire. Then we have

$$
\limsup_{n\to\infty} \frac{n \log n}{-\log n^{-\frac{n}{\alpha}}} = \limsup_{n\to\infty} \alpha = \alpha,
$$

and thus this power series has order α.

Problem (13): Γ has poles as the nonpositive integers. Show these poles are simple and find the residues at them.

Solution: Consider the pole at $-n$ for $n \geq 0$. We calculate

$$
\lim_{z\to -n} (z+n)\Gamma(z),
$$

which will be nonzero and finite. Thus the pole at $-n$ is simple, and this limit is the value of the residue. Using the recurrence for Γ, we obtain

$$
\lim_{z\to -n} \frac{(z+n)\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} = \lim_{z\to -n} \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)}.
$$

Plugging in $-n$ directly yields

$$
\frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)} = \frac{(-1)^n}{n!}.
$$

Problem (15): Let $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Prove that

a) $\psi(z+1) - \psi(z) = \frac{1}{z}$.
b) $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$.

Solution:
a) Take the logarithmic derivative of both sides of $\Gamma(z+1) = z\Gamma(z)$ to obtain

$$\psi(z+1) = \frac{1}{z} + \psi(z).$$

Rearrange and done.
b) Take the logarithmic derivative of both sides of $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$ to obtain

$$\psi(z) - \psi(1-z) = -\frac{\pi\csc(\pi z)\cot(\pi z)}{\csc(\pi z)} = -\pi\cot(\pi z).$$

Problem (18): Show that

$$\Gamma(z) = \lim_{n\to\infty} \frac{N! e^{\sum_{n=1}^{N} z/n - \gamma z}}{z(z+1)\cdots(z+N)}.$$

Solution: Fix $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with $|z| = R$, and let $P(N)$ be the function in the limit. We show that

$$\lim_{N\to\infty} \log(\Gamma(z)) - \log(P(N)) = 0,$$

from which taking the exponential of both sides gives the desired result (both $\Gamma(z)$ and $P(N)$ are nonzero and finite given the conditions, so we can take logs. Further $\Gamma(z)$ converges absolutely, so we can manipulate terms in the sum of its log freely). We can write $\log(P(N))$ as

$$-\gamma z - \log(z) + \sum_{n=1}^{N} \frac{z}{n} - \log\left(1 + \frac{z}{n}\right).$$

Note that $\log(\Gamma(z))$ is basically just $\log(P(\infty))$. Thus the difference of the two is equal to

$$\sum_{n=N+1}^{\infty} \frac{z}{n} - \log\left(1 + \frac{z}{n}\right).$$

For sufficiently large N (in particular $N > |z|$), we can write $\log\left(1 + \frac{z}{n}\right)$ as

$$\frac{z}{n} - \frac{z^2}{n^2} \cdot \frac{1}{2} + \frac{z^3}{n^3} \cdot \frac{1}{3} - \cdots.$$

Thus the our original sum is equal to

$$\sum_{n=N+1}^{\infty} \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \cdot \frac{z^k}{n^k}.$$

We can bound the inner sum as

$$\left|\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\cdot\frac{z^k}{n^k}\right| \leq \sum_{k=2}^{\infty}\frac{R^k}{n^k}.$$

Since N is sufficiently large, this is a convergent geometric series, so the double sum becomes

$$\sum_{n=N+1}^{\infty}\frac{\frac{R^2}{n^2}}{1-\frac{R}{n}} = R^2\sum_{n=N+1}^{\infty}\frac{1}{n^2-Rn}.$$

The summand is on the order of $O\left(\frac{1}{n^2}\right)$, so it converges to some constant c. Thus we can write

$$R^2\left(c-\sum_{n=M}^{N}\frac{1}{n^2-Rn}\right),$$

where M is some large integer such that $M > R$, so we don't hit any divide by zero issues in the sum. Since the inner term converges, there exists some N such that $k > N$ implies the inner sum is less than $\frac{\varepsilon}{R^2}$. Thus we've bounded the initial difference $\log(\Gamma(z)) - \log(P(N))$ for sufficiently large N to less than some arbitrary ε, so it indeed converges to 0.

Problem (20): Show that if f is meromorphic on all of $\mathbb{C}$, then it can be written as a quotient of two entire functions.

Solution: Suppose first f has finitely many poles $a_1, \dots, a_n$, where a pole is included m times for its multiplicity. Then we can write

$$f(z) = \frac{f(z)(z-a_1)\cdots(z-a_n)}{(z-a_1)\cdots(z-a_n)}.$$

Clearly the denominator is entire, and the numerator is entire as well, since the linear factors get rid of the poles of f. Thus f is globally the quotient of two entire functions.

Now suppose f has infinitely many poles $0, \dots, 0, a_1, a_2, \dots$ arranged in increasing magnitudes and written as many times as their multiplicity (there are m zeros). From the Weierstrass product formula, the function

$$g(z) = z^m\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{p_n(z\,/a_n)}$$

is entire and vanishes at $0, \dots, 0, a_1, a_2$, where there are m zeros. Thus $f(z)g(z)$ is an entire function. Therefore

$$f(z) = \frac{f(z)g(z)}{g(z)}$$

is indeed globally the quotient of two entire functions.

Problem (21): Prove Picard's Little Theorem for functions of finite order.

Solution: If f misses no values, then we're done. Now uppose f is an entire function with finite order ρ that misses at least one value $a \in \mathbb{C}$. Note that $f(z) - a$ has no zeros, and thus the order ρ is an integer (since entire functions with finitely many zeros have integer order (contrapositive of corollary 4.2 in chapter 10)). Then from problem 3, we can write

$$f(z) - a = e^{g(z)}$$

for some degree ρ polynomial g. Since $g(z)$ is a polynomial, it has image $\mathbb{C}$ (by the fundamental theorem of algebra). Thus $e^{g(z)}$ has image $\mathbb{C} \setminus \{0\}$, which implies $f(z) = a + e^{g(z)}$ has image $\mathbb{C} \setminus \{a\}$. Thus f misses only one value, as desired.

10. The Hadamard Product

10.1. Blaschke products

Definition (Blaschke factor): Let $a \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The *Blaschke factor* of a is

$$B(a, z) = \frac{a - z}{1 - \overline{a}z}.$$

Proposition: Let $a \in \mathbb{D}$.

a) We have $B(a, a) = 0$ and $B(a, 0) = a$.
b) $B(a, z)$ is a bijective function from $\mathbb{D}$ to itself.
c) If $|z| = 1$, then $|B(a, z)| = 1$, and $B(a, \cdot)$ is a bijective function from the unit circle to itself.

Sometimes we need a function that send $B_\rho(0)$ bijectively to itself and swaps 0 and a with $|a| < \rho$, in which case

$$B\left(\frac{a}{\rho}, \frac{z}{\rho}\right) = \frac{\rho(a - z)}{\rho^2 - \overline{a}z}$$

works.

Definition (Blaschke product): A product of Blaschke factors times a complex numeber ζ with $|\zeta| = 1$ is called a *Blaschke product*.

Proposition: An analytic function $f : \mathbb{D} \to \mathbb{D}$ can be extended to a continuous function $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that f has modulus 1 on the unit circle if and only if f is a finite Blaschke product.

Proof: The backwards direction is obvious from properties of Blaschke factors. Now suppose f can be extended to a continuous function on the closed disk sending the unit circle to itself. Then f has finitely many zeros in the open unit disk, say $a_1, ..., a_j$. Let $B(z) = \prod_{n=1}^{j} B(a_n, z)$. Then f and B have the same zeros in the unit disk, so $g(z) = \frac{f(z)}{B(z)}$ is holomorphic on nonzero on th unit disk. We also have $|g(z)| = 1$ whenever $|z| = 1$.

From the maxmimum modulus principle, we have that $|g(z)| \leq 1$ on the unit disk. But by the same token, we have that $|1 \ /g(z)| \leq 1$ on the unit disk (or by the minimum modulus principle). Thus $|g(z)| = 1$ on the unit disk. However, again by the maximum modulus principle, this implies that g must be constant on the unit disk, where the constant has modulus 1, as desired. ∎

Proposition: Define $\tilde{B}(a, z)$ as $\frac{|a|}{a} B(a, z)$. Then the infinite product

$$\prod_{n=1}^{\infty} \tilde{B}(a, z)$$

converges absolutely in the unit disk if and only if the a_n's satisfy the *Blaschke condition* $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$.

Proof:

■

10.2. Jensen's Formula

Proposition: Suppose f is a holomorphic function on some open set U containing $\overline{B_\rho(0)}$, and that $f(z) \neq 0$ for all $z \in U$. Then we have

$$\log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(\rho e^{i\theta}\right)\right| \, d\theta.$$

Proof: Since f is nonzero on U, $\log f(z)$ is holomorphic on U. Note that $\log|f(z)| = \Re(\log f(z))$, so we can write $\log f(z) = \log|f(z)| + ig(z)$, where $g(z) = \Im(\log f(z))$. Then by the Gauss MVT, we have

$$\log f(0) = \frac{1}{2\pi} \int_0^{2\pi} \log f\left(\rho e^{i\theta}\right) d\theta.$$

Taking real parts gives us the desired conclusion. ■

We can improve this to functions with zeros on $\overline{B_\rho(0)}$. Note there can only be finitely many, since otherwise the infinitely many zeros would have a limit point, forcing f to be zero. So suppose f has zeros at $\rho e^{i\theta_1}, \ldots, \rho e^{i\theta_n}$. Then

$$g(z) = \frac{f(z)}{\prod_{j=1}^{n} \left(z - \rho e^{i\theta_j}\right)}$$

is holomorphic and nonzero on $\overline{B_\rho(0)}$. Thus using the proposition, we have

$$\log|g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|g\left(\rho e^{i\theta}\right)\right| \, d\theta.$$

The left side is

$$\log\left|\frac{f(0)}{\left(-\rho e^{i\theta_i}\right) \cdots \left(-\rho e^{i\theta_n}\right)}\right| = \log|f(0)| - n \log \rho.$$

The integral on the right is

$$\int_0^{2\pi} \log\left|\frac{f(\rho e^{i\theta})}{(\rho e^{i\theta} - \rho e^{i\theta_1})\cdots(\rho e^{i\theta} - \rho e^{i\theta_n})}\right| d\theta =$$

$$\int_0^{2\pi} \log\left|f(\rho e^{i\theta})\right| d\theta - 2\pi n \log \rho - \sum_{j=1}^{n} \int_0^{2\pi} \log\left|e^{i\theta} - e^{i\theta_j}\right| d\theta.$$

The integral is clearly independent on θ_j, so we can write

$$\int_0^{2\pi} \log\left|e^{i\theta} - 1\right| d\theta = \int_0^{2\pi} \log\left(2 \sin \frac{\theta}{2}\right) d\theta = 2\pi \log 2 + 2 \int_0^{\pi} \log \sin \theta = 2\pi \log 2 - 2\pi \log 2 = 0.$$

Adding back in the factor of $\frac{1}{2\pi}$ and canceling, we see the formula stays the same even with zeros on the boundary.

Theorem (Jensen's formula): Let f be holomorphic on $\overline{B_\rho(0)}$ with $f(0) \neq 0$, with zeros at $a_1, \ldots, a_n$. Then we have

$$\log|f(0)| = -\sum_{j=1}^{n} \log \frac{\rho}{|a_j|} + \frac{1}{2\pi} \int_0^{2\pi} \log\left|f(\rho e^{i\theta})\right| d\theta.$$

Proof: Divide f by Blaschke factor $B\left(\frac{a_j}{\rho}, \frac{z}{\rho}\right)$, which swaps zero at a_j with zero at $\frac{\rho^2}{\overline{a_j}}$, which lies outside $\overline{B_\rho(0)}$. Do this for all a_j, apply the earlier proposition, expand. In particular, in the integral, every Blaschke factor vanished since on the boundary of $\overline{B_\rho(0)}$, they have modulus 1.■

10.3. Weierstrass Factors bounds

Write

$$E_h(z) = (1 - z)e^{p_h(z)} = (1 - z)e^{z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}}.$$

Lemma: For all $z \in \mathbb{C}$, we have

$$\log|E_h(z)| \leq (2h + 1)|z|^{h+1}.$$

Proof:

■

10.4. Proof of Hadamard Product

[INSERT STATEMENT HERE]

Proof: WLOG suppose f is nonzero 0 (we can divide by z^m if necessary) Write $\nu(r)$ for the number of a_n;s with modulus at most r. ■

10.5. Problems

Problem (1): Show that $f(z) = \sin^2(\sqrt{z})$ is entire, and write down its Hadamard product expansion.

Solution: Note that since $\sqrt{\cdot}$ has both positive and negative output, $\sin(\sqrt{\cdot})$ is either positive or negative. Thus $\sin^2(\sqrt{z})$ is uniquely defined at each point, and so we have no branch point issues. Thus f is entire. Then, using the product expansion for sin, we have

$$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2 \pi^2}\right)^2.$$

Problem (5): Let $a \in \mathbb{D}$, and prove the following:

a) We have $B(a, a) = 0$ and $B(a, 0) = a$.
b) $B(a, z)$ is a bijective function from $\mathbb{D}$ to itself.
c) If $|z| = 1$, then $|B(a, z)| = 1$, and $B(a, \cdot)$ is a bijective function from the unit circle to itself.

Solution:

a) We have

$$B(a, a) = \frac{a - a}{1 - \overline{a}a} = 0 \quad \text{and} \qquad B(a, 0) = \frac{a - 0}{1} = a.$$

b) Pick some $z \in \mathbb{D}$. We want to show that $\frac{a-z}{1-\overline{a}z} \in \mathbb{D}$. Thus all we need to show is that

$$|a - z| < |1 - \overline{a}z| \Rightarrow (a - z)(\overline{a} - \overline{z}) < (1 - \overline{a}z)(1 - a\overline{z}) \Rightarrow$$
$$|a|^2 - a\overline{z} - \overline{a}z + |z|^2 < 1 - a\overline{z} - \overline{a}z + |a|^2|z|^2 \Rightarrow \left(|a|^2 - 1\right)\left(|z|^2 - 1\right) > 0,$$

which is true since both a and z has modulus less than 1.

Now pick some $w \in \mathbb{D}$. We need to show their exists $z \in \mathbb{D}$ such that $B(a, z) = w$. Solving for z, we obtain that

$$z = \frac{a - w}{1 - \overline{a}w}.$$

Since $\frac{1}{a}$ is outside $\mathbb{D}$, the denominator is nonzero, and thus z exists.

c) Looking at the last inequality in the last part, note that if $|z| = 1$, then we have $\left(|a|^2 - 1\right)\left(|z|^2 - 1\right) = 0$. Thus, we can reverse all the manipulations to obtain that $|a - z| = |1 - \overline{a}z|$, which implies that $|B(a, z)| = 1$.

Problem (6): Let $a_1, ..., a_n$ be a finite sequence of points in $\mathbb{D}$, and let

$$B(z) = \prod_{j=1}^{n} B(a_j, z)$$

be their Blaschke product. Prove the following:

a) If $z \in \mathbb{D}$, then $B(z) \in \mathbb{D}$.
b) If $|z| = 1$, then $|B(z)| = 1$.
c) If $|z| > 1$ and z is in the domain of B, then $|B(z)| > 1$.
d) If z and $1/\overline{z}$ are in the domain of B, then

$$B(z) = \frac{1}{\overline{B(1/\ \overline{z})}}.$$

Solution:

a) This holds true for each Blaschke factor, and thus the product of each one will clearly have modulus less than 1.
b) Again, this holds for each Blaschke factor, so the product will have modulus 1.
c) Look at the inequality in b) of the last problem. With $|z| > 1$, we have instead $(|a|^2 - 1)(|z|^2 - 1) < 0$. Reversing the manipulations with the inequality in this direction now, we obtain that $|a - z| > |1 - \overline{a}z|$, and thus $|B(z)| > 1$. Since this holds for each Blaschke factor, the product of each of them will clearly have modulus greater than 1.
d) We prove this for each Blaschke factor, after which its clear this holds for the product. We have

$$B\left(\frac{1}{\overline{z}}\right) = \frac{a - \frac{1}{\overline{z}}}{1 - \frac{\overline{a}}{\overline{z}}} = \frac{a\overline{z} - 1}{\overline{z} - \overline{a}}.$$

Taking the conjugate of this yields

$$\overline{B\left(\frac{1}{\overline{z}}\right)} = \frac{1 - \overline{a}z}{a - z}.$$

Taking the reciprocal yields the desired result.

Problem (7): Let $a_1, ..., a_n \in \mathbb{D}$. Let

$$B(z) = \prod_{j=1}^{n} \frac{a_j - z}{1 - \overline{a_j} z}$$

and

$$B_k(z) = \prod_{j=1}^{k-1} \frac{a_j - z}{1 - \overline{a_j} z}.$$

Prove that if $|z| \neq 1$ and z is in the domain of $B(z)$, then

$$\frac{1 - |B(z)|^2}{1 - |z|^2} = \sum_{k=1}^{n} |B_k(z)|^2 \frac{1 - |a_k|^2}{|1 - \overline{a_k} z|^2}.$$

Solution: We prove this by induction. For the base case, we have $B_1(z) = 1$. Then we need to show that

$$\frac{1 - \frac{|a_1 - z|^2}{|1 - \overline{a_1} z|^2}}{1 - |z|^2} = \frac{1 - |a_1|^2}{|1 - \overline{a_1} z|^2} \Rightarrow |1 - \overline{a_1} z|^2 - |a_1 - z|^2 = \left(1 - |a_1|^2\right)(1 - |z|^2).$$

We can write the left side as

$$(1 - \overline{a_1} z)(1 - a_1 \overline{z}) - (a_1 - z)(\overline{a_1} - \overline{z}) = 1 + |a_1|^2 |z|^2 - |a_1|^2 - |z|^2,$$

which is equal to the right side, so the base case is true.

Now suppose this holds up until $n - 1$. We want to show that

$$\frac{1 - |B(z)|^2}{1 - |z|^2} = \sum_{k=1}^{n} |B_k(z)|^2 \frac{1 - |a_k|^2}{|1 - \overline{a_k} z|^2}.$$

Note that we can split the sum as

$$\left(\sum_{k=1}^{n-1} |B_k(z)|^2 \frac{1 - |a_k|^2}{|1 - \overline{a_k} z|^2} \right) + |B_n(z)|^2 \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|^2}.$$

Then by the induction hypothesis, this is equal to

$$\frac{1 - |B_n(z)|^2}{1 - |z|^2} + |B_n(z)|^2 \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|^2}.$$

Now we need to prove that

$$\frac{1 - |B(z)|^2}{1 - |z|^2} = \frac{1 - |B_n(z)|^2}{1 - |z|^2} + |B_n(z)|^2 \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|^2} \Rightarrow$$
$$\frac{|B_n(z)|^2 - |B(z)|^2}{1 - |z|^2} = |B_n(z)|^2 \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|^2}.$$

Dividing through by the $B_n(z)$ yields

$$\frac{1 - |B(a_n, z)|^2}{1 - |z|^2} = \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|^2},$$

but this is just the base case, which we already proved.

Problem (9): Let $P(z)$ be a polynomial of degree n, all of whose roots are in $\mathbb{D}$, and let

$$f(z) = \frac{P(z)}{z^n \overline{P(1/\overline{z})}}.$$

Prove that $f(z)$ is a finite Blaschke product.

Solution: Let $P(z) = c(z - a_1) \cdots (z - a_n)$, where $|a_i| < 1$ for all $1 \leq i \leq n$. Then we have

$$f(z) = \frac{c(z - a_1) \cdots (z - a_n)}{c z^n \left(\frac{1}{z} - \overline{a_1}\right) \cdots \left(\frac{1}{z} - \overline{a_n}\right)} = \frac{(z - a_1) \cdots (z - a_n)}{(1 - \overline{a_1} z) \cdots (1 - \overline{a_n} z)} = (-1)^n B(a_1, z) \cdots B(a_n, z),$$

as desired.

Problem (12): Prove the Poisson-Jensen formula:

$$\log|f(z)| = -\sum_{j=1}^{n} \log\left|\frac{\rho^2 - \overline{a_j} z}{\rho(z - a_j)}\right| + \frac{1}{2\pi} \int_0^{2\pi} \Re\left(\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z}\right) \log\left|f\left(\rho e^{i\theta}\right)\right| d\theta.$$

Proof: First we prove a modified version of the Gauss mean value theorem. Let f be holomorphic on $\overline{B_\rho(0)}$, and fix z with $|z| < \rho$. Then by Cauchy's integral theorem, we have

$$f(z) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(w)}{w - z} - \frac{f(z)}{w - \rho^2/\overline{z}} \, dw.$$

We get a contribution of $f(z)$ from the first term, and no contribution from the second, since $\left|\frac{1}{\overline{z}}\right| > \rho$. Letting Γ be $|z| = \rho$, we can rewrite this integral as

$$\begin{aligned}
\int_\Gamma f(w) \left(\frac{z - \rho^2/\overline{z}}{w^2 - zw - \rho^2 w \,/\overline{z} + \rho^2 z/\overline{z}}\right) dw &= \int_\Gamma \left(\frac{|z|^2 - \rho^2}{w\overline{z} - |z|^2 - \rho^2 + \rho^2 z \,/w}\right) \frac{f(w)}{w} \, dw \\
&= \int_\Gamma \left(\frac{|z|^2 - \rho^2}{w\overline{z} - |z|^2 - \rho^2 + z\overline{w}}\right) \frac{f(w)}{w} \, dw \\
&= \int_\Gamma \left(\frac{\rho^2 - |z|^2}{\rho^2 - 2\Re(z\overline{w}) + |z|^2}\right) \frac{f(w)}{w} \, dw,
\end{aligned}$$

where the second equality came from $w\overline{w} = \rho^2$. Parametrizing with $\gamma(t) = \rho e^{it} \Rightarrow \gamma'(t) = i\rho e^{i\theta}$, and letting $z = re^{i\theta}$, we obtain

$$\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(t - \theta) + r^2} \cdot \frac{f(\rho e^{it})}{\rho e^{it}} \cdot i\rho e^{it} \, dt.$$

Thus we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(t - \theta) + r^2} \cdot f(\rho e^{it})\, dt.$$

Note that we have

$$\begin{aligned}\Re\left(\frac{\rho e^{it} + re^{i\theta}}{\rho e^{it} - re^{i\theta}}\right) &= \Re\left(\frac{(\rho\cos t + r\cos\theta) + i(\rho\sin t + r\sin\theta)}{(\rho\cos t - r\cos\theta) + i(\rho\sin t - r\sin\theta)}\right) \\ &= \frac{\rho^2\cos^2 t - r^2\cos^2\theta + \rho^2\sin^2 t - r^2\sin^2 t}{\rho^2 + r^2 - 2\rho r(\cos t\cos\theta + \sin t\sin\theta)} \\ &= \frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(t-\theta) + r^2}.\end{aligned}$$

Thus

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{\rho e^{it} + z}{\rho e^{it} - z}\right) f(\rho e^{it})\, dt.$$

Now suppose f is nonzero on $\overline{B_\rho(0)}$. Then $\log f(z)$ is holomorphic there as well, so applying the above equation, we obtain

$$\log f(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{\rho e^{it} + z}{\rho e^{it} - z}\right) \log f(\rho e^{it})\, dt.$$

Taking real parts yields

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{\rho e^{it} + z}{\rho e^{it} - z}\right) \log\left|f(\rho e^{it})\right|\, dt.$$

Now suppose f has zeros at $a_1, ..., a_n$ and $f(z_0) \neq 0$. Let $g(z) = f(z)\prod_{j=1}^n \frac{1}{B(a_j/\rho, z/\rho)} = f(z)\prod_{j=1}^n \frac{\rho^2 - \overline{a_j}z}{\rho(z-a_j)}$. This is nonzero on $\overline{B_\rho(0)}$, so we can apply the above formula to obtain

$$\begin{aligned}\log|f(z_0)| + \sum_{j=1}^n \left|\frac{\rho^2 - \overline{a_j}z_0}{\rho(z-a_j)}\right| = \log|g(z_0)| &= \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{\rho e^{it} + z}{\rho e^{it} - z}\right) \log\left|g(\rho e^{it})\right|\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{\rho e^{it} + z}{\rho e^{it} - z}\right) \log\left|\frac{f(\rho e^{it})}{\prod_{j=1}^n B\left(\frac{a_j}{\rho}, e^{it}\right)}\right| dt.\end{aligned}$$

Since Blaschke factors have modulus 1 on the unit circle, the factors in the integral above can be ignored. Rearranging, we obtain

$$\log|f(z_0)| = -\sum_{j=1}^n \left|\frac{\rho^2 - \overline{a_j}z_0}{\rho(z-a_j)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{\rho e^{it} + z}{\rho e^{it} - z}\right) \log\left|f(\rho e^{it})\right| dt,$$

as desired.

■

Problem (13): Prove the following version of Jensen's Formula: Let f be an entire function that is not identically zero, and let $\nu(t)$ be the number of zeros of f with modulus $< t$, counted with multiplicity. Then

$$\log|f(0)| = -\int_0^r \frac{\nu(t)}{t}\,dt + \frac{1}{2\pi}\log\left|f\left(re^{i\theta}\right)\right|\,d\theta.$$

Solution: Let $a_1, ..., a_n$ be the zeros with modulus less than r and in increasing modulus. Let $b_1, ..., b_k$ be the values of the modulus of the zeros in order of increasing magnitude, with b_i being repeated m_i times, with $m_1 + \cdots + m_k = n$.

All we need to show is that

$$\sum_{j=1}^{n} \log\frac{r}{|a_j|} = \int_0^r \frac{\nu(t)}{t}\,dt.$$

We can write the sum on the left as

$$m_1 \log\frac{r}{b_1} + \cdots + m_k \log\frac{r}{b_k}.$$

We can write the integral on the right as

$$\int_0^{b_1} \frac{0}{t}\,dt + \int_{b_1}^{b_2} \frac{m_1}{t}\,dt + \cdots + \int_{b_{k-1}}^{b_k} \frac{m_1 + \cdots + m_{k-1}}{t}\,dt + \int_{b_k}^{r} \frac{m_1 + \cdots + m_k}{t}\,dt.$$

The first integral vanished, and the antiderivative of $\frac{1}{t}$ is $\log t$. Thus we obtain

$$\begin{gathered} m_1[\log b_2 - \log b_1] + (m_1 + m_2)[\log b_3 - \log b_2] + \cdots \\ +(m_1 + \cdots + m_{k-1})[\log b_k - \log b_{k-1}] + (m_1 + \cdots + m_k)[\log r - \log b_k]. \end{gathered}$$

We get lots of telescoping terms, leaving

$$\begin{gathered} -m_1 \log b_1 - m_2 \log b_2 - \cdots - m_{k-1}\log b_{k-1} - m_k \log b_k + n \log r = \\ m_1 \log\frac{r}{b_1} + \cdots + m_k \log\frac{r}{b_k}, \end{gathered}$$

as desired.

Problem (14): Let f be an entire function with $f(0) = 1$, and let $M_f(r) = \sup_{|z|=r}|f(z)|$. Show that $\nu(r)\log 2 \leq \log M(2r)$ for all $r > 0$.

Solution: Applying Jensen's formula with $2r$, we obtain

$$\sum_{j=1}^{\nu(2r)} \log\frac{2r}{|a_j|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2re^{i\theta}\right)\right|\,d\theta \leq \log M(2r),$$

where we used the ML inequality. We can write the left as

$$\sum_{j=1}^{\nu(r)} \log \frac{2r}{|a_j|} + \sum_{j=\nu(r)+1}^{\nu(2r)} \log \frac{2r}{|a_j|} \geq \sum_{j=1}^{\nu(r)} \log \frac{2r}{|a_j|}.$$

Since the upper bound is $\nu(r)$, we have $|a_j| \leq r \Rightarrow \log \frac{2r}{|a_j|} \geq \log 2$. Thus

$$\sum_{j=1}^{\nu(r)} \log \frac{2r}{|a_j|} \geq \nu(r) \log 2,$$

as desired.

11. Conformal Mappings

11.1. Conformal Equivalence

Definition (conformal equivalence): Let U and V be open subsets of $\mathbb{C}$. A holomorphic function $f : U \to V$ is called a *conformal map*. Is such an f is bijective, we say that U and V are *conformally equivalent*.

Proposition: Let $f : U \to V$ be a conformal map. Then its inverse $f^{-1} : V \to U$ is also a conformal map.

Proof: Use the Lagrange inversion theorem to write down a power series of f^{-1} at $f(z)$. ■

Theorem (Lagrance inversion theorem): Suppose $w = f(z)$ and f is holomorphic at a with $f'(a) \neq 0$. Then there's a holomorphic function g such that $g(w) = z$ is the inverse of f in a neighborhood of $f(a)$. As a power series in a neighborhood of $f(a)$, we have

$$g(w) = a + \sum_{n=1}^{\infty} g_n \frac{(w - f(a))^n}{n!},$$

where

$$g_n = \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[\left(\frac{z - a}{f(z) - f(a)} \right)^n \right].$$

Remark: This tells us that conformal equivalence is indeed an equivalence relation.

Conformal means "angle preserving", in the sense that that if we have two curves intersect, and the tangents at the intersection form some angle, then the transformation of the curves by f preserve that angle. This is essentially a local rotation matrix, and since we're looking at tangents, essentially multiplying by the derivative of f at that point. Thus, as long as the derivative is nonzero, we obtain a conformal map.

Proposition: Let $f : U \to V$ be continuous. Then f is conformal if and only if it's holomorphic and $f'(z_0) \neq 0$ for all $z_0 \in U$.

11.2. Conformal Equivalences

Example: Let $\mathbb{D} = \{Z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{H} = \{z \in \mathbb{C} : \mathfrak{I}(z) > 0\}$. Then the function $f : \mathbb{D} \to \mathbb{H}$ given by $f(z) = i\frac{1-z}{1+z}$ is a conformal map. Thus the two regions are conformally equivalent.

Example: $f(z) = e^z$ maps $\{z \in \mathbb{C} : 0 < \mathfrak{I}(z) < 2\pi\}$ to $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

11.3. Schwarz Reflection Principle

Theorem (Schwarz reflection principle): Let f be an analytic function defiend on the upper half disk such that for every $a \in (-1, 1)$, $\lim_{z \to a} f(z)$ exists, is real, is a continuous function of $a \in \mathbb{R}$. Then there exists an analytic function F on all of $\mathbb{D}$ such that $F(z) = f(z)$ for all z in the upper half disk, and $F(\overline{z}) = \overline{F(z)}$ for all $z \in \mathbb{D}$.

11.4. Schwarz-Christoffel Formula

11.5. Problems

Problem (3): Find a conformal map from $\{z \in \mathbb{H} : |z| > 1\}$ to $\mathbb{D}$.

Solution: Let the first set be A. We claim that $f(z) = \frac{i-z-\frac{1}{z}}{i+z+\frac{1}{z}}$ works. In particular, this is the composition of the two maps $z \to z + \frac{1}{z}$ and $z \to \frac{i-z}{i+z}$. The second one maps $\mathbb{H}$ to $\mathbb{D}$, so we just need to show that the first one maps A to $\mathbb{H}$ and has an inverse.

Let $z = x + yi$ with $x^2 + y^2 > 1$ and $y > 0$. Then

$$\mathfrak{I}\left(z + \frac{1}{z}\right) = y - \frac{y}{x^2 + y^2} = \frac{y(x^2 + y^2 - 1)}{x^2 + y^2}.$$

The right side is positive, so $z + \frac{1}{z}$ maps into $\mathbb{H}$. We also have the inverse map $\frac{z+\sqrt{z^2-4}}{2}$, which is holomorphic on $\mathbb{H}$, since the set doesn't loop around 0.

Problem (4): Find a conformal map from $\{z \in \mathbb{C} : \mathfrak{R}(z) > 0, 0 < \mathfrak{I}(z) < 1\}$ to $\mathbb{H}$.

Solution: Let B be the first set. We claim that $f(z) = e^{\pi z} + e^{-\pi z}$ works. This is the composition of the two maps $z \to e^{\pi z}$ and $z \to z + \frac{1}{z}$. The second one maps A from the previous problem to $\mathbb{H}$, so we just need to show that the first one maps B to A. Clearly the map is invertible with inverse $\frac{1}{\pi} \log z$ (which is well defined, since B doesn't loop around 0). Now suppose $z = x + yi$, where $x > 0$, $0 < y < 1$. Then

$$e^{\pi z} = e^{\pi x} e^{i\pi y} = e^{\pi x}(\cos(\pi y) + i\sin(\pi y)).$$

Since $0 < y < 1$, $0 < \sin(\pi y) < 1$, which means the imaginary part of this is positive. We also have that $|e^{\pi z}| = e^{\pi x}$, and since $x > 0$, this is greater than 1. Thus $z \to e^{\pi z}$ does ideed map B to A.

Problem (10): Let $\lambda \in \mathbb{R} \setminus \{0, 1\}$. Show that the function

$$f(z) = \int_0^z \frac{1}{\sqrt{w(w-1)(w-\lambda)}}\, dw$$

is a conformal map from $\mathbb{H}$ to a rectangle. Which rectangle is it?

Solution: Without loss of generality, suppose $\lambda > 1$ (if $\lambda < 0$ or $0 < \lambda < 1$, then the process is similar). We know this function maps to a rectangle since $\beta_k = \frac{1}{2}$ for each term in the denominator. If we let $z_1 = 0$, $z_2 = 1$, $z_3 = \lambda$, and $z_4 = \infty$, then the corresponding vertices of the rectange are are $f(0), f(1), f(\lambda), f(\infty)$. Note that $f(0) = 0$ and $f(1)$ is a real number, since on the interval $(0, 1)$ the inside of the radical is positive. Then note that

$$\begin{aligned} f(\lambda) &= \int_0^1 \frac{1}{\sqrt{w(w-1)(w-\lambda)}}\, dw + \int_1^\lambda \frac{1}{\sqrt{w(w-1)(w-\lambda)}}\, dw \\ &= f(1) + \int_1^\lambda \frac{1}{i\sqrt{w(w-1)(\lambda-w)}}\, dw. \end{aligned}$$

Ignoring the i, the integrand in the second equation is real, and thus the integral is real. Therefore, $f(\lambda) = f(1) - ci$, where c is the above integral excluding the i. Then $f(\infty)$ is forced to be $-ci$, since the Schwarz-Christoffel has $\beta_k = \frac{1}{2}$ for each angle, and thus we must have right angles at $f(\lambda)$ and $f(0)$.

Problem (11): Find all values of λ in problem 10 such that the image of $\mathbb{H}$ is a square. What sidelengths do these squares have?

Solution: Again suppose $\lambda > 1$. From the last problem, in order for the image to be a square, we need

$$\int_1^\lambda \frac{1}{\sqrt{w(w-1)(\lambda-w)}}\, dw = \int_0^1 \frac{1}{\sqrt{w(w-1)(w-\lambda)}}\, dw.$$

In the first integral, let $u = \frac{\lambda - w}{\lambda - 1}$. Transforming yields

$$\int_0^1 \frac{1}{\sqrt{u(u-1)((\lambda-1)u-\lambda)}}\, du.$$

The bounds for this integral and the one on the right side of the equation are the same, so we just need the integrands to agree (since both integrands are positive). Thus, we need $(\lambda - 1)w - \lambda = w - \lambda \Rightarrow \lambda = 2$.

Problem (16): Let $a, b \in \mathbb{R}_{>0}$ with $a \geq b > 0$. Let $a_1 = \frac{a+b}{2}$ and $b_1 = \sqrt{ab}$. Let $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$.

a) Prove that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are equal. Denote this limit by $M(a, b)$.

b) Show that

$$\frac{1}{M(a,b)} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}\, d\theta.$$

Solution:

a) We claim that $a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ for all n. The middle inequality follows from AM-GM. Then we have that

$$a_{n+1} = \frac{a_n + b_n}{2} \leq \frac{a_n + a_n}{2} = a_n, \quad b_{n+1} = \sqrt{a_n b_n} \geq \sqrt{b_n^2} = b_n.$$

Thus (a_n) is bounded from below and decreasing, and (b_n) is bounded from above and increasing. Then both sequences converge by monotone convergence. Letting the limits of the sequences be A and B respectively, the inequality also implies that $A \geq B$.

Now suppose for the sake of contradiction that $A > B$, and let $\varepsilon = \frac{A-B}{2}$. Then by convergence, there exists N such that $|a_n - A| < \varepsilon$ and $|b_n - B| < \varepsilon$, which implies

$$A - \varepsilon < a_n < A + \varepsilon,$$
$$B - \varepsilon < b_n < B + \varepsilon.$$

Averaging the two inequalities yields

$$B = \frac{A+B}{2} - \varepsilon < a_{n+1} < \frac{A+B}{2} + \varepsilon = A,$$

but this is a contradiction, since A is the limit of a monotone decreasing sequence, so $a_n \geq A$ for all n.

12. Riemann Mapping Theorem

Theorem (Riemann mapping theorem): Let $\Omega \subseteq \mathbb{C}$ be a nonempty simply connected open set that is not all of $\mathbb{C}$. The for any $z_0 \in \Omega$, there exists a unique conformal equivalence $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.

12.1. Schwarz's Lemma

Lemma (Schwarz's lemma): Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for any $z \neq 0$ or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

12.2. Montel's Theorem

Definition (uniformly bounded): Let Ω be an open subset of $\mathbb{C}$, adn let $\mathcal{F}$ be a set of holomorphic functions from $f : \Omega \to C$. We say that $\mathcal{F}$ is *uniformly bounded on compact subsets* if for every compact subset $K \subseteq \Omega$, there is a constant B such that $|f(z)| \leq B$ for all $z \in K$ and $f \in \mathcal{F}$.

Definition (normal): Let $\Omega \subseteq \mathbb{C}$ be an open subset, and let $\mathcal{F}$ be a family of holomorphic functions from Ω to $\mathbb{C}$. We say that $\mathcal{F}$ is a *normal family* if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on all compact subsets of Ω.

Theorem (Montel's first theorem): Let $\Omega \subseteq \mathbb{C}$ be an open subset, and let $\mathcal{F}$ be a family of holomorphic functions on Ω which is uniformly bounded on compact subsets. Then $\mathcal{F}$ is a normal family.

Proof: By Arzela-Ascoli, its enough to show that $\mathcal{F}$ is equicontinuous on compact sets. Let K be a compact subset of Ω, and choose $r > 0$ such that $B_{3r}(z) \subseteq \Omega$ for all $z \in K$. Now let $z_0, z_1 \in K$ be such that $|z_0 - z_1| < r$, and let Γ be the boundary circle of $B_{2r}(z_1)$. Then we have

$$f(z_0) - f(z_1) = \frac{1}{2\pi i} \int_\Gamma f(z) \left[\frac{1}{z - z_0} - \frac{1}{z - z -_1} \right] dz.$$

The bracketed term is bounded by $\frac{|z_0 - z_1|}{r^2}$, so

$$|f(z_0) - f(z_1)| \leq \frac{1}{2\pi} \frac{2\pi r}{r^2} B|z_0 - z_1| < C|z_0 - z_1|,$$

where B is the uniform bound for all $f \in \mathcal{F}$ on the compact set of points withins distance $2r$ of K and C is a constant that only depends on K. This estimate holds for all $z_0, z_1 \in K$ of distance at most r and all $f \in \mathcal{F}$, so $\mathcal{F}$ is equicontinuous. ∎

12.3. Sequences of Holomorphic Functions

Proposition: The uniform liimit of holomorphic function is holomorphic.

Proposition: Let $f_1, \ldots$ be a sequence of injective holomorphic functions on a path-connected open set $\Omega \subseteq \mathbb{C}$ which converges uniformly on compacts subsets to f. Then f is either injective or constant.

Proof: Suppose f is not injective, so there exists some $z_1, z_2 \in \Omega$ such that $f(z_1) = f(z_2)$. We need to show that f is constant. Let $g_n(z) = f_n(z) - f_n(z_1)$. Since f_n is injective, the only zero of g_n is z_1, and the sequence $g_1, \ldots$ converges uniformly to $g(z) = f(z) - f(z_1)$. Note that g has a zero at z_2, and if g isn't the zero function, then this zero is isolated. Thus if we let Γ be a small enough circle around z_2, we have

$$\frac{1}{2\pi i} \int_\Gamma \frac{g'(z)}{g(z)}\, dz = d,$$

where d is the order of the zero of g at z_2 by the argument principle. From uniform convergence, we have

$$\lim_{n \to \infty} \frac{1}{2\pi i} \int_\Gamma \frac{{g_n}'(z)}{g_n(z)}\, dz = \frac{1}{2\pi i} \int_\Gamma \frac{g'(z)}{g(z)}\, dz = d \neq 0.$$

However, the integral on the left is always 0, contradiction. ∎

12.4. Interesting Problems

Problem (1): Show that if $\Omega \subseteq \mathbb{C}$ is path connected and $a \notin \Omega$, then $\log(z - a)$ is holomorphic on Ω.

Solution: Without loss of generality, suppose $a = 0$. For any loop in Ω, it can be continuously shrunk to a point in Ω while staying in Ω. Note that if the initial loop contains 0, then while shrinking, the loop must pass through 0, but this is impossible, since $0 \notin \Omega$. Thus any loop in Ω does not contain 0, and thus $\log(z - a)$ is holomorphic on Ω. To make it more precise, we can make some branch cut on $\mathbb{C}$ that doesn't hit any points in Ω where we define a holomorphic logarithm, which is possible since no loops in Ω contain 0.

Problem (3): Let $f_n(z) = z^n$, where n is a positive integer. Show that $\{f_n\}$ is a normal family on $\mathbb{D}$, but not on any subset of $\mathbb{C}$ containing any point in the complement of $\mathbb{D}$.

Solution: From the converse of Montel's theorem, we just need to show that $\{f_n\}$ is uniformly bounded on compact subsets of $\mathbb{D}$. However, this is clear, since we can take the maximal $|z|$ in K, and then we have $|z| \geq |z|^n$ for all $n \geq 1$.

Now suppose some subset Ω contains a point not in $\mathbb{D}$. Letting K contain only this point (note K is compact), we see that $\mathcal{F}$ is unbounded, since the modulus of this point is greater than 1.

Problem (4): If $\{f_n'\}$ from the previous problem a normal family on $\mathbb{D}$?

Solution: Yes. Take some compact $K \subset . = \mathbb{D}$. We need to show that $\{f_n'\}$ is uniformly bounded on K. Cleary we only need to look at the maximal $a = |z|$ in K. Thus we need to show that na^n is bounded over all n, where $a < 1$. Taking the derivative of this with respect to n, we obtain $a^n(1 + n \log a)$. Since $a < 1$, the term inside the paratheses is eventually negative. Thus na^n has a maximum, as desired.

Problem (11): Prove the converse of Montel's theorem: If $\mathcal{F}$ is a normal family of holomorphic functions from an open subset $\Omega \subseteq \mathbb{C}$ to $\mathbb{C}$, then $\mathcal{F}$ is uniformly bounded on compact subsets of Ω.

Solution: Pick some compact $K \subseteq \Omega$. For $f \in \mathcal{F}$, let

$$b(f) = \min\{M : |f(z)| \leq M \text{ for all } z \in K\}.$$

Note we can take a minimum here since f is continuous and K is compact. Now let $B = \sup_{f \in \mathcal{F}} b(f)$. We need to show that B is finite. Suppose otherwise for the sake of contradiction. Then there exists some sequence of functions $f_1, f_2, \ldots$ such that $b(f_i) \leq b(f_{i+1})$ for all i and such that $b(f_i)$ is unbounded.

Since $\mathcal{F}$ is normal, there's some subsequence $f_{k_1}, f_{k_2}, \ldots$ that converges uniformly to some holomorphic function f. Note that the sequence $b\left(f_{k_i}\right)$ must necessarily be unbounded, since its a subsequence of an unbounded sequence.

Since f is holomorphic on K, it's bounded by some A. Now pick $\varepsilon > 0$. By uniform convergence, there exists N_1 such that $n \geq N_1$ implies

$$\left|f_{k_n}(z) - f(z)\right| < \varepsilon \text{ for all } z \in K.$$

We also know that there exists some N_2 such that $n \geq N_2$ implies that $b\left(f_{k_n}\right) \geq A + \varepsilon$. In particular, $b\left(f_{k_n}\right)$ is attained for some $z_n \in K$, which depends on f_{k_n}. Then, for all $n \geq \max\{N_1, N_2\}$, we have

$$\varepsilon > \left|f_{k_n}(z_n) - f(z_n)\right| \geq \left|\left|f_{k_n}(z_n)\right| - |f(z_n)|\right| \geq \left|b\left(f_{k_n}\right) - A\right| \geq \varepsilon,$$

contradiction.

Problem (12): Let $\mathcal{F}$ be the family of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that $f(0) = 0$, $f'(0) = 1$, and for all $n \geq 2$, we have $\frac{|f^{(n)}(0)|}{n!} \leq n$. Show that $\mathcal{F}$ is normal.

Solution: By Montel's theorem, we just need to show that for any compact $K \subseteq \mathbb{D}$, $\mathcal{F}$ is uniformly bounded. Now from holomorphicity, we can write each $f \in \mathcal{F}$ as

$$f(z) = z + \sum_{n=2}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

Bounding using the triangle inequality, we obtain

$$|f(z)| \leq \sum_{n=1}^{\infty} n|z|^n = \frac{|z|}{(1 - |z|)^2}.$$

Since the function on the right is continuous on K and thus bounded, $f(z)$ is also bounded by the max of the function on the right. This max applies to all $f \in \mathcal{F}$, so $\mathcal{F}$ is indeed uniformly bounded on K.

Problem (13): Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic, and that $f\left(\frac{1}{2}\right) = 0$. Show that $\left|f\left(\frac{3}{4}\right)\right| \leq \frac{2}{5}$.

Solution: Let $g(z) = f\left(B\left(\frac{1}{2}, z\right)\right)$. Then $g(0) = f\left(B\left(\frac{1}{2}, 0\right)\right) = f\left(\frac{1}{2}\right) = 0$. Since $B\left(\frac{1}{2}, z\right)$ is bijective from the disk to itself, we also have that $g : \mathbb{D} \to \mathbb{D}$, and thus $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Then by Schwarz's lemma, we have $|g(z)| \leq z$ for all $z \in \mathbb{D}$. Letting $z = -\frac{2}{5}$, we obtain

$$|g(z)| = \left|f\left(B\left(\frac{1}{2}, -\frac{2}{5}\right)\right)\right| = \left|f\left(\frac{3}{4}\right)\right| \leq \frac{2}{5}.$$

13. Elliptic Functions

Complex functions can either be periodic in two directions by basically giving basis vectors $w_1, w_2 \in \mathbb{C}$, where $f(z) = f(z + w_1) = f(z + w_2)$ (as long as w_1, w_2 are linearly independent).

Definition (elliptic function): A meromorphic function f on $\mathbb{C}$ such that there exist two nonzero $w_1, w_2 \in \mathbb{C}$ with $w_1 \ / \ w_2 \notin \mathbb{R}$ such that $f(z + w_1) = f(z + w_2) = f(w)$ for all $z \in \mathbb{C}$ is said to be an *elliptic function.*

Proposition: Entire elliptic functions are constant.

Proof: Focus on paralleogram with sides w_1, w_2. Since compact, f is bounded on that paralleogram, but this paralleogram repeats, thus f is and bounded, and so by Liouville's theorem is constant. ■

Proposition: There are no elliptic functions with a single simple pole in each fundamental parallelogram.

Proof: Suppose one did exist, say f. Suppose the pole is in the interior. Let Γ be the boundary of the parallelogram. Then

$$\int_\Gamma f(z)\, dz = 0,$$

since opposite sides cancel (they have the same values, and Γ travels in the opposite direction). However, the residue theorem implies that this integral is nonzero. ■

Proposition: If f is a nonzero elliptic function, then it has the same number of zeros and poles in each fundamental parallelogram.

Proof: By the argument principle on the perimeter of a parallelogram, we have

$$\int_\Gamma \frac{f'(z)}{f(z)}\, dz = Z - P.$$

However, on opposite sides, the integrand is equal, and we're traveling the opposite direction. Thus the integral is 0. ■

13.1. The Weierstrass $\wp$ function

Definition (lattic): Let ω_1, ω_2 be nonzero linearly independent complex numbers. Then the *lattice* they generate is $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$.

Let $\Lambda^* = \Lambda \setminus \{0\}$. Then

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right).$$

Note that $\wp$ depends on the chosen lattice.

Proposition: $\wp$ is doubly periodic with period ω_1 and ω_2. It has a double pole at each lattice point $\omega \in \Lambda$ and no poles anywhere else.

Proof: Poles are obvious. We have

$$\wp'(z) = -2 \sum_{\omega \in \Lambda^*} \frac{1}{(z+\omega)^3},$$

which is clearly convergent and periodic with periods ω_1, ω_2. Thus the derivative is periodic, which implies $\wp(z + \omega_1) = \wp(z) + \alpha$ and $\wp(z + \omega_2) = \wp(z) + \beta$ for some α, β. Note that $\wp$ is even, so $\wp(\frac{\omega_1}{2}) = \wp(-\frac{\omega_1}{2})$, which implies that $\alpha = 0$, and similarly for β. ■

Note that $\wp$ has a triple pole at the lattice points, and thus there must be three zeros in each parallelogram. Since $\wp$ is even, $\wp'$ is odd, and thus $\wp'(\frac{\omega_1}{2}) = -\wp'(-\frac{\omega_1}{2})$, but from periodicity, we have $\wp'(\frac{\omega_1}{2}) = \omega'(-\frac{\omega_1}{2})$. Thus there's a zero at $\frac{\omega_1}{2}$. Similar reasoning gives zeros at $\frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$.

Let $e_1 = \wp(\frac{\omega_1}{2}), e_2 = \wp(\frac{\omega_2}{2}), e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$. Then $\wp(z) - e_i$ with $i \in [1, 3]$ has double roots at the inputs of e_i, since the derivative there is 0.

Proposition:

Proof:

■

Proposition:

$$(\wp)'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

Proof:

■

13.2. The Space of Elliptic Functions

Lemma: Every even elliptic function with respect to Λ is a rational function in $\wp(z)$.

Proof:

■

Theorem: Let Λ be a lattice and let $\wp(z)$ be the Weierstrass $\wp$ function with respect to Λ. Then every elliptic function with respect to Λ is a ration function in $\wp(z)$ and $\wp'(z)$.

Proof: Decompose into odd and even ■

13.3. Einsenstein Series

Theorem: The Laurent series of $\wp$ centered at $z = 0$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) E_{2k+2} z^{2k},$$

where

$$E_k = \sum_{\omega \in \Lambda^*} \frac{1}{\omega^k}.$$

Proof:

■

Proposition: We have

$$(\wp')^2 = 4\wp^3 - 60E_4\wp - 140E_6.$$

13.4. Problems

Problem (2): Let f be an elliptic function. Show that the sum of the residues in any fundamental parallelogram is 0.

Solution: We can shift f if necessary so that no poles or zeros lie on the boundary of any fundamental parallelogram. Then the contour integral on opposite sides of the parallelogram cancel each other out, since they take on the same values by the contour travels in the opposite direction. Thus the integral over the parallelogram is zero. But from the residue theorem, we have

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{a \text{ is a pole of } f} \operatorname{Res}(f, a) = 0,$$

so the sum of residues is indeed 0.

Problem (3): Show that if Λ is a lattice in $\mathbb{C}$, then for all $z \in \mathbb{C}$, the sum

$$\sum_{\omega \in \Lambda} \frac{1}{(z+\omega)^2}$$

diverges.

Solution: We can just show that $\sum_{\omega\in\Lambda} \frac{1}{\omega^2}$ diverges, since then by the limit comparison test, the initial series diverges. We can write this sum as

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m\omega_1 + n\omega_2)^2},$$

where ω_1, ω_2 make the basis of the lattice. Again by limit comparison, we only need to show that

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m+n)^2}$$

diverges. Since all the terms are positive, we can finish by showing

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n)^2}$$

diverges. Note that the integral $\int_1^\infty \frac{1}{(m+n)^2}\,dm$ well approximates the sum, so we obtain

$$\sum_{n=1}^{\infty} \int_1^\infty \frac{1}{(m+n)^2}\,dm = \sum_{n=1}^{\infty} \frac{1}{n+1},$$

which diverges, as desired.

Problem (4): Let $\wp_\Lambda(z)$ denote the Weierstrass $\wp$ function with respect to a lattice Λ. For $a \in \mathbb{C} \setminus \{0\}$, write $a\Lambda = \{a\omega : \omega \in \Lambda\}$. Express $\wp_{a\Lambda}(z)$ in terms of some value of $\wp_\Lambda(z)$.

Solution:

$$\wp_{a\Lambda}(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*} \left(\frac{1}{(z+a\omega)^2} - \frac{1}{(a\omega)^2} \right) = \frac{1}{a^2} \left(\frac{1}{\left(\frac{z}{a}\right)^2} + \sum_{\omega\in\Lambda^*} \left(\frac{1}{\left(\frac{z}{a}+\omega\right)^2} - \frac{1}{\omega^2} \right) \right) = \frac{1}{a^2} \wp_\Lambda\left(\frac{z}{a}\right).$$

Problem (5): Express $\wp_{\Lambda/2}(z)$ as a rational function in $\wp_\Lambda(z)$.

Solution:

$$\frac{\left(\wp(z)-\wp\left(\frac{\omega_1+\omega_2}{4}\right)\right)^2\left(\wp(z)-\wp\left(\frac{3\omega_1+\omega_2}{4}\right)\right)^2}{\wp(z)\left(\wp(z)-\wp\left(\frac{\omega_1}{2}\right)\right)\left(\wp(z)-\wp\left(\frac{\omega_2}{2}\right)\right)}.$$

Problem (7): Show that

$$\pi\cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{z+n}\right).$$

Solution: From problem 13 of chapter 7, we have

$$\frac{1}{z^2}+\sum_{n=1}^{\infty}\left(\frac{1}{(z+n)^2}+\frac{1}{(z-n)^2}\right)=\sum_{n=-\infty}^{\infty}\frac{1}{(z+n)^2}=\frac{\pi^2}{\sin^2(\pi z)}.$$

Integrating both sides and canceling negatives yields

$$\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{z+n}\right)=\pi\cot(\pi z)+C$$

for some constant C. We can plug in $\frac{1}{2}$ on both sides to obtain

$$0=2+\left(-2+\frac{2}{3}\right)+\left(-\frac{2}{3}+\frac{2}{5}\right)+\left(-\frac{2}{5}+\frac{2}{7}\right)+\cdots=0+C,$$

which implies $C=0$, as desired.

Problem (13): Show that

a) The series for $E_k(\tau)$ converges absolutely for all integers $k\geq 3$.
b) $E_k(\tau)=0$ for all odd $k\geq 3$.
c) $E_k(\tau+1)=E_k(\tau)$.
d) $E_k\left(-\frac{1}{\tau}\right)=\tau^k E_k(\tau)$
e)
$$E_k\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k E_k(\tau),$$
whenever $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$.

Solution:

a) We already know that the series converges absolutely absolutely for $k = 3$. Note that there are only finitely many numbers in the lattice with modulus less than 1, and since $\left|\frac{1}{(m+n\tau)^3}\right| \geq \left|\frac{1}{(m+n\tau)^k}\right|$ for all $k \geq 3$. Thus these series clearly converges absolutely.

b) Note that the lattice is rotationally symmetric about the origin, so when raised to an odd power, a lattice point will cancel with it's reflection about the origin.

c) We can write the series as

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m+n(\tau+1))^k} = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m+n+n\tau)^k}.$$

The additional n does nothing to the range of integer parts that are being summed over, so the series stays the same.

d)
$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{\left(m-\frac{n}{\tau}\right)^k} = \tau^k\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m\tau-n)^k}.$$

Again here the integers being summed over are the same, so the series is still equal to $E_k(\tau)$.

e)
$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{\left(m+n\cdot\frac{a\tau+b}{c\tau+d}\right)^k} = (c\tau+d)^k\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{((md+nb)+(mc+na)\tau)^k}.$$

We need to show that the denominator still generates $\{m+n\tau : (m,n)\in\mathbb{Z}^2\}$. Note if we look at the denomiator as a vector in $\mathbb{Z}^2$ with the second term being x and the first term being y, we have

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} mc+na \\ md+nb \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, it and it's transpose also have inverses in $\mathrm{SL}_2(\mathbb{Z})$. Thus the matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ bijectively maps the lattice in $\mathbb{Z}^2$ to the lattice generated by τ (and $(0,0)$ in $\mathbb{Z}^2$ maps to $(0,0)$ in Λ, so that's not an issue). Thus our sum is still the same.

Problem (15):

a) Express E_8, E_{10}, E_{12}, and E_{14} in terms of E_4 and E_6.

b) Show that

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m).$$

c) Find other related identities.

Solution:

a) Using the Laurent series for $\wp$, we have

$$\wp(z) = \frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + \cdots$$

$$\wp(z)^3 = \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 + (21E_8 + 27E_4^2)z^2 + (27E_{10} + 90E_4E_6)z^4 + \cdots$$

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 + (36E_4^2 - 168E_8)z^2 + (240E_4E_6 - 288E_{10})z^4 + \cdots$$

Then from the differential equation, we compare terms to obtain

$$36E_4^2 = 4(21E_8 + 27E_4^2) - 60E_4(3E_4) \Rightarrow \frac{3}{7}E_4^2 = E_8$$

$$240E_4E_6 - 288E_{10} = 4(27E_{10} + 90E_4E_6) - 60E_4(5E_6) \Rightarrow \frac{5}{11}E_4E_6 = E_{10}.$$

b) Using theorem 5.6 on the first identity above, we have

$$\frac{3}{7} \cdot \frac{\pi^8}{45^2}\left(1 + 240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^2 = \frac{\pi^8}{4725}\left(1 + 480\sum_{r=1}^{\infty}\sigma_7(r)q^r\right) \Rightarrow$$

$$1 + 480\sum_{r=1}^{\infty}\sigma_3(r)q^r + 240^2\left(\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^2 = 1 + 480\sum_{r=1}^{\infty}\sigma_7(r)q^r.$$

Note that each coefficient terms in the square sum will be a convolution, so the coefficient of q^r would be $\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m)$. Equating coefficients and dividing by 480 yields

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m).$$

c) Using the same process as above for the second identity, we obtain

$$10\sigma_3(n) - 21\sigma_5(n) + 11\sigma_9(n) = 7!\sum_{m=1}^{n-1}\sigma_3(m)\sigma_5(n-m).$$

14. Picard's Little Theorem

14.1. The Elliptic Modular Function

Recall $e_1 = \wp(\frac{\omega_1}{2}), e_2 = \wp(\frac{\omega_2}{2}), e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$. If we let $e_1(\Lambda)$ denote $\wp(\frac{\omega_1}{2})$ with respect to Λ and similarly for e_2, e_3, and if we scale the lattice to have periods $a\omega_1, a\omega_2$, then $e_i(a\Lambda) = \frac{1}{a^2}e_i(\Lambda)$.

To remove the dependence on a, look at

$$\lambda(\omega_1, \omega_2) = \frac{e_3 - e_2}{e_1 - e_2}.$$

This only depends one $\frac{\omega_2}{\omega_1} = \tau$, so we can write

$$\tau = \frac{e_3 - e_2}{e_1 - e_2}.$$

Doing som analysis, we can show that τ is a holomorphic function from $\mathbb{H}$ to $\mathbb{C} \setminus \{0, 1\}$.

Definition: Matrices from the modular group $\mathrm{SL}_2(\mathbb{Z})$ interact with $\tau \in \mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau = \frac{a\tau + b}{c\tau + d}.$$

The modular group acts nicely with λ.

Proposition: Suppsoe $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is such that $a, b, c, d \equiv 1, 0, 0, 1 \pmod 2$ respectively. Then $\lambda(\tau) = \lambda(\gamma\tau)$.

14.2. Mapping Properties of λ

Let Ω be the set in between $\mathfrak{R}(z) = 0, \mathfrak{R}(z) = 1$, and the upper semicircle $\left|z - \frac{1}{2}\right| = \frac{1}{2}$. Then λ is a bijective holomorphic function from Ω to $\mathbb{H}$. It extends continuously to the boundary such that $\lambda(0) = 1, \lambda(1) = \infty$, and $\lambda(\infty) = 0$.

14.3. Covering Spaces

14.4. Interesting Problems

Problem (1): Show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\tau\right] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]\tau$$

for matrices in $\mathrm{SL}_2(\mathbb{Z})$.

Solution: We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\tau\right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{e\tau + f}{g\tau + h} = \frac{(ac + bg)\tau + (af + bh)}{(ce + dg)\tau + (cf + dh)} = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]\tau.$$

Problem (2): Show that if $\tau \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then $\gamma\tau \in \mathbb{H}$.

Solution: Letting $\tau = x + yi$ with $y > 0$, we have

$$\Im\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau\right) = \Im\left(\frac{ax + ayi + b}{cx + cyi + d}\right) = \frac{ay(cx + d) - cy(ax + b)}{(cx + d)^2 + (cy)^2} = \frac{1}{(cx + d)^2 + (cy)^2} > 0.$$

Problem (8): Suppose that f, g, and h are nonvanishing entire functions with $f(z) + g(z) = h(z)$. Show that f and g are constant multiples of h.

Solution: Divide through by g and set $F(z) = \frac{f}{g}$ and $H(z) = \frac{h}{g}$ to obtain $F + 1 = H$. From the conditions in the problem, we see that F and H are nonzero and entire. Now suppose that neither are constant. From Picard's little theorem, this implies that they achieve every value in $\mathbb{C} \setminus \{0\}$. In particular, $H(z_0) = 1$ for some z_0. However, from the equation, this would imply that $F(z_0) = 0$, contradiction. Thus F and H must both be constants, which implies that f and h are constant multiples of g, which is equivalent to the problem statement.

Problem (11): Prove that if f is a nonvanishing entire function, then it has an enture n^{th} root for every positive integer n.

Solution: Since f is entire and nonzero, there exists an entire g such that $f(z) = e^{g(z)}$. Then simply take $e^{g(z)/n}$ to be the n^{th} root.

Problem (12): Deduce Picard's little theorem from Landau's theroem.

Solution: Let f be a nonconstant entire function that misses 0 and 1. Since f is nonzero, we can write $f(z) = e^{2\pi i h(z)}$ for some entire h. Note that h can never be an integer since otherwise f would be 1.

Since h is never an integer, it also is never 0 or 1, and thus h and $h - 1$ are nonvanishing. Thus from the previous problems, they have entire square roots, u and v respectively. Clearly u and v can never be equal, so $u - v$ is nonvanishing, and again we can write $u - v = e^{g(z)}$ for some entire g.

Since h is never an integer, $u \neq \pm\sqrt{m}$ and $v \neq \pm\sqrt{m-1}$ for all $m \geq 1$. Thus g misses all points $\pm\log\left(\sqrt{m} + \sqrt{m-1}\right) + \frac{i\pi n}{2}$, where $m, n \in \mathbb{Z}$ and $m \geq 1$ (the $\pm$ in front the log determines whether $\sqrt{m}$ and $\sqrt{m-1}$ have different signs (since if we bring the negative inside the log and rationalize the denominator, the sum turns into a difference), and the $\frac{i\pi n}{2}$ determines what that sign is).

Next we show every disk of radius 1 contains a point that g misses. We only look at disks that are mostly contained in the first quadrant, since the points are symmetric about the x and y axes. Note that the consecutive points are separated vertically by $\frac{\pi}{2} \approx 1.57$, so if some circle contains a point near its center, no vertical shift will make it so that there is not point in the disk, since there would have to be two consecutive points at least 2 unit aparts. Note also that the difference between consecutive horizontal points rapidly decreases, since $\log\left(\sqrt{m+1} - \sqrt{m}\right) - \log\left(\sqrt{m} - \sqrt{m-1}\right)$ is small for m not too large. In particular, for $m = 1$, the distance is about 0.88, so no shifting of the unit circle horizontally will cause no points to be within for similar reasons of the vertical shift.

15. Analytic Number Theory

15.1. Arithmetic Functions

15.2. Asymptotic Analysis

15.3. Dirichlet Series

Definition (Dirichlet series): Let f be an arithmetic function. Its *Dirichlet series* is the function

$$F(s) = \sum_{n=1}^{\infty} \frac{f(s)}{n^s}.$$

15.4. Wiener-Ikehara Theorem

Theorem (Wiener-Ikehara Theorem): Suppose that $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is a Dirichlet series, where each a_n is a nonnegative real number and $F(s)$ is analytic on $\mathfrak{R}(s) > 1$ and extends analytically to $\mathfrak{R}(s) \geq 1$ except as $s = 1$, where we have

$$F(s) = \frac{H(s)}{(s-1)^{1-\alpha}}$$

for some real number α and some $H(s)$ holomorphic on $\mathfrak{R}(s) \geq 1$ and nonzero there. Then we have

$$\sum_{n \leq x} a_n \sim \frac{H(1)x}{\Gamma(1-\alpha)(\log x)^{\alpha}}.$$

15.5. Problems

Problem (1): Prove the following identities involving Dirichlet convolutions:

a) $\mathbb{1} * \mathbb{1} = d$.
b) $N * \mathbb{1} = \sigma$.
c) $N^k * \mathbb{1} = \sigma_k$.
d) $\varphi * \mathbb{1} = N$.

Solution:

a)
$$\sum_{d|n} \mathbb{1}(d)\mathbb{1}\left(\frac{n}{d}\right) = \sum_{d|n} 1 = d(n).$$

b)
$$\sum_{d|n} N(d)\mathbb{1}\left(\frac{n}{d}\right) = \sum_{d|n} d = \sigma(n).$$

c)
$$\sum_{d|n} N^k(d)\mathbb{1}\left(\frac{n}{d}\right) = \sum_{d|n} d^k = \sigma_k(n).$$

d)
$$\sum_{d|n} \varphi(d)\mathbb{1}\left(\frac{n}{d}\right) = \sum_{d \min n} \varphi(d) = N(n),$$

where the last inequality is a well known identity.

Problem (2): Prove that

$$\sum_{d|n} \Lambda(d) = \log n.$$

Solution: Let $n = p_1^{e_1} \cdots p_k^{e_k}$. Note that $\Lambda(d)$ is nonzero only when $d = p_i^k$ for some i and some $1 \leq k \leq e_i$, and is equal to $\log p_i$ at these divisors. Thus we get a $\log p_i$ contribution e_i times, and so summing all together yields

$$\log p_1^{e_1} \cdots p_k^{e_k} = \log n.$$

Problem (5): Prove that if f and g are multiplicative functions, then $f * g$ is also multiplicative.

Solution: Pick m, n such that $\gcd(m, n) = 1$. We have

$$\begin{aligned}(f * g)(mn) &= \sum_{d|mn} f(d)g\left(\frac{mn}{d}\right) = \sum_{d_1|m} \sum_{d_2|n} f(d_1 d_2)g\left(\frac{mn}{d_1 d_2}\right) = \sum_{d_1|m} f(d_1)g\left(\frac{m}{d_1}\right) \sum_{d_2|n} f(d_2)g\left(\frac{n}{d_2}\right) \\ &= \left(\sum_{d_1|m} f(d_1)g\left(\frac{m}{d_1}\right)\right)\left(\sum_{d_2|n} f(d_2)g\left(\frac{n}{d_2}\right)\right) = (f * g)(m)(f * g)(n).\end{aligned}$$

Problem (6): Let f be an arithmetic function, and let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ be its Dirichlet series. What are the coefficients of $F'(s)$ whe expressed as a Dirichlet series?

Solution: $f(n)$ is constant with respect to s, and $\frac{d}{ds}\frac{1}{n^s} = -\frac{\log n}{n^s}$, so the coefficients are $-f(n)\log n$.

Problem (12): Show that

$$\sum_{n \leq x} \Lambda(n) \left\lfloor \frac{x}{n} \right\rfloor = \log(\lfloor x \rfloor !).$$

Solution: First we show that $\left\lfloor \frac{x}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor$. Let $x = kn + a + s$, where $k \geq 1$ is an integer, $0 \leq a < n$ is an integer, and $0 \leq s < 1$. Then $\frac{x}{n} = k + \frac{a+s}{n}$ and $\frac{\lfloor x \rfloor}{n} = k + \frac{a}{n}$. We have $a, a + s < n$, so the floor of both these numbers is k, as desired. Thus in the sum, we can let x be an integer.

Note the summand is nonzero only when $n = p^k$ for some p and nonzero k. Fix p. Then we can write the sum as

$$\sum_{k=1}^{\infty} \sum_{p} \log(p) \left\lfloor \frac{x}{p^k} \right\rfloor = \sum_{p} \log(p) \sum_{k=1}^{\infty} \left\lfloor \frac{x}{p^k} \right\rfloor.$$

The inner sum is equal to $\nu_p(x)$, so we can write the sum as

$$\sum_{p} \log(p) \nu_p(x) = \sum_{p} \log\left(p^{\nu_p(x)}\right) = \log(x!),$$

as desired.

Problem (15): Let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ and $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ be two Dirichlet series, which are both convergent whenever $\Re(s) > \sigma_0$. Show that if $a, b > \sigma_0$, then

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} F(a + it) G(b - it)\, dt = \sum_{n=1}^{\infty} \frac{f(n) g(n)}{n^{a+b}}.$$

Solution: We assume that the series converge absolutely for $a, b > \sigma_0$. Multiplying out the series in the integrand, we obtain

$$\sum_{n=1}^{\infty} \frac{f(n) g(n)}{n^{a+b}} + \sum_{\substack{m \neq n \\ m,n \geq 1}} \frac{f(m) g(n)}{m^{a+it} n^{b-it}}.$$

Separating out the first and second sums, we obtain

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sum_{n=1}^{\infty} \frac{f(n) g(n)}{n^{a+b}}\, dt + \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sum_{\substack{m \neq n \\ m,n \geq 1}} \frac{f(m) g(n)}{m^{a+it} n^{b-it}}\, dt.$$

In the first term, the sum is independent, and the integral is equal to $2T$, so the limit just converges to the sum. Thus, all we need to show is that the second limit converges to 0.

Note we have

$$\sum_{\substack{m\neq n\\ m,n\geq 1}}\left|\frac{f(m)g(n)}{m^{a+it}n^{b-it}}\right|\leq\left(\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^{a+it}}\right|\right)\left(\sum_{n=1}^{\infty}\left|\frac{g(n)}{n^{b-it}}\right|\right).$$

Since the right side contains both convergent series, the left also converges. Thus the integrand in the second limit converges absolutely, so we can apply Fubini's theorem and swap the sum and integral. Thus we have

$$\lim_{T\to\infty}\frac{1}{2T}\sum_{\substack{m\neq n\\ m,n\geq 1}}\int_{-T}^{T}\frac{f(m)g(n)}{m^{a+it}n^{b-it}}\,dt=\lim_{T\to\infty}\sum_{\substack{m\neq n\\ m,n\geq 1}}\frac{1}{2T}\cdot\frac{f(m)g(n)}{m^{a+it}n^{b-it}}\cdot\frac{1}{\log\left(\frac{n}{m}\right)^{i}}\left(\left(\frac{n}{m}\right)^{iT}-\left(\frac{n}{m}\right)^{-iT}\right).$$

Note that $\left(\frac{n}{m}\right)^{i}$ is some number on the unit circle not equal to 1. Thus we can write it as $e^{i\theta}$ for some $\theta\in(0,2\pi)$. Then the terms barring the f and g terms in the sum combine to make $\frac{e^{iT\theta}-e^{-iT\theta}}{2iT\theta}=\frac{\sin T\theta}{T\theta}$, which has magnitude at most 1. Then we can apply dominated convergence to the sum and limit (using the same bounding for the $\frac{f(m)g(n)}{m^{a+it}n^{b-it}}$ term as we did for Fubini's) to swap them to obtain

$$\sum_{\substack{m\neq n\\ m,n\geq 1}}\frac{f(m)g(n)}{m^{a+it}n^{b-it}}\cdot\frac{1}{\log\left(\frac{n}{m}\right)^{i}}\cdot\lim_{T\to\infty}\frac{1}{2T}\left(\left(\frac{n}{m}\right)^{iT}-\left(\frac{n}{m}\right)^{-iT}\right).$$

The term in the parantheses has magnitude at most 2, so the limit converges to 0, which means the sum converges to 0, as desired.

16. The Riemann Zeta Function

16.1. Introduction

We can use the Dirichlet eta function $\eta(s)$ to get an analytic continuation to $\mathfrak{R}(s) > 0$:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

The functions are related by the equation

$$(1 - 2^{1-s})\zeta(s) = \eta(s).$$

There the points where $(1-s)\log 2 = 2\pi i n$, we have to be a bit careful, but thankfully $\eta(s)$ has zeros at these points, so the only pole we have is still at $s = 1$.

16.2. Analtytic Continuation

For $\mathfrak{R}(s) > 1$, we can write

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt,$$

where this follows from expanding the denominator as a geometric series and swapping the integral and sum.

Now we write use a contour integral to get another functional equation. Let C be a keyhole contour about the positive real axis. Then we have

$$\int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz = \int_{\infty}^{0} \frac{e^{(s-1)(\log t - i\pi)}}{e^t - 1}\, dz + \int_0^{\infty} \frac{e^{(s-1)(\log t + i\pi)}}{e^t - 1}\, dz + \int_{|z|=\varepsilon} \frac{(-z)^{s-1}}{e^z - 1}\, dz.$$

The last integral goes to zero as $\varepsilon \to 0$ by the ML inequality. Thus we have

$$\begin{aligned}
\int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz &= \left(e^{(s-1)\pi i} - e^{-(s-1)\pi i}\right) \int_0^{\infty} \frac{e^{(s-1)\log t}}{e^t - 1}\, dt \\
&= 2i \sin((s-1)\pi) \int_0^{\infty} \frac{e^{(s-1)\log t}}{e^t - 1}\, dt \\
&= -2i \sin(\pi s) \int_0^{\infty} \frac{e^{(s-1)\log t}}{e^t - 1}\, dt \\
&= -2i \sin(\pi s)\Gamma(s)\zeta(s).
\end{aligned}$$

This gives an analytic continuation to $\mathbb{C} \setminus \{1\}$, since the integral is analytic on this domain. Rearranging and use the Γ reflection formula, we obtain

$$\zeta(s) = \frac{i\Gamma(1-s)}{2\pi} \int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz.$$

16.3. Functional Equation

We deform the contour C to pick up some residues. Let C_n be the contour that start above the real axis at ∞, travels to $(2n+1)\pi$, then along the square with vertices $(\pm 1 \pm i)(2n+1)\pi$ until it reaches below $(2n+1)\pi$, after which it goes back to ∞.

We change contours from C to C_n, where we pick up the residues at $2\pi i k$, with $0 < |k| \leq n$. All these poles are simple, so we have

$$\operatorname{Res}\left(\frac{(-z)^{s-1}}{e^z - 1}, 2\pi i k\right) = (-2\pi i k)^{s-1}.$$

When k is positive, this is $(2k\pi)^{s-1} i e^{-\frac{1}{2} i\pi s}$, and when k is negative, this is $-(2k\pi)^{s-1} i e^{\frac{1}{2} i\pi s}$. Adding, we obtain

$$2(2k\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right).$$

Thus we have

$$\frac{1}{2\pi i} \int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1}\, dz = \sum_{k=1}^{n} 2(2k\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) = 2\sin\left(\frac{\pi s}{2}\right)(2\pi)^{s-1} \sum_{k=1}^{n} k^{s-1}.$$

Taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1}\, dz = 2\sin\left(\frac{\pi s}{2}\right)(2\pi)^{s-1} \zeta(1-s).$$

Now assume $\sigma = \Re(s) < 0$. We show that the integral of the funciton on C_n goes to zero as $n \to \infty$. Thus

$$\frac{1}{2\pi i} \int_{-C} \frac{(-z)^{s-1}}{e^z - 1}\, dz = 2\sin\left(\frac{\pi s}{2}\right)(2\pi)^{s-1} \zeta(1-s).$$

This holds for $\sigma = \Re(z) < 0$, but since these are meromorphic functions that agree on an open set, they must agree everywhere. Thus this holds for all $s \in \mathbb{C}$. Subbing in the expression we had for the integral and simplifying, we obtain

$$\sin(\pi s)\Gamma(s)\zeta(s) = \sin\left(\frac{\pi s}{2}\right)(2\pi)^s \zeta(1-s).$$

Letting $\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$ and using the Γ duplication formula, we obtain $\xi(s) = \xi(1-s)$, or

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

16.4. Zeros of $\zeta(s)$

From the functional equation, it's easy to see that ζ has zeros at the negative even integers, and these are the only zeros with $\Re(s) < 0$. For $\Re(s) > 1$ there are also no zeros. The prime number theorem is equivalent to the nonvanshing on ζ on $\Re(s) = 1$ (or $\Re(s) = 0$ by the functional equation), and the Riemann hypothesis is about the zeros in $0 < \Re(z) < 1$.

16.5. $\zeta(s)$ Near $s = 1$

16.6. The Mobius Function

16.7. Problems

Problem (1): Show that $\varphi = \mu * N$.

Solution: Note that the inverse of μ is $\mathbb{1}$. Thus we just need to show that $\varphi \star \mathbb{1} = \sum_{d|n} \varphi(d) = n$, but this is a well known identity.

Problem (2): Estimate $\sum_{n \le x} \varphi(n)$,

Solution: We can write

$$\sum_{n \le x} \varphi(n) = \sum_{n \le x} \sum_{d|n} \mu(d) \frac{n}{d}$$

using the identity in the previous problem. Swapping sums yields

$$\sum_{d \le x} \sum_{k=1}^{\frac{x}{d}} \mu(d) \frac{kd}{d} = \sum_{d \le x} \mu(d) \sum_{k=1}^{\frac{x}{d}} k.$$

The inside sum is equal to $\frac{\lfloor \frac{x}{d} \rfloor (\lfloor \frac{x}{d} \rfloor + 1)}{2} = \frac{x^2}{2d^2} + O(\frac{x}{d})$, so we have

$$\sum_{d \le x} \mu(d) \frac{x^2}{2d^2} + \sum_{d \le x} O\left(\frac{x}{d}\right) = \frac{x^2}{2} \sum_{d \le x} \frac{\mu(d)}{d^2} + O(x \log x).$$

Note that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)}$ from the Dirichlet series for $\mu(n)$, and the error term will be the same as for $\sum_{n \le x} \frac{1}{n^2}$, since $\left|\frac{\mu(n)}{n^2}\right| \le \frac{1}{n^2}$. Thus we have

$$\frac{3}{\pi^2} x^2 + \frac{x^2}{2} \cdot O\left(\frac{1}{x}\right) + O(x \log x) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

Problem (3): Let $c, y \in \mathbb{R} > 0$. Show that

$$\lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s} \, ds = \begin{cases} 1 & y > 1 \\ \frac{1}{2} & y = 1 \\ 0 & y < 1 \end{cases}$$

Solution: First we do the case when $y = 1$. Then we simply have

$$\frac{1}{2\pi i}\log s\Big|_{c-iT}^{c+iT} = \frac{1}{2\pi i}\log\left(\frac{c+iT}{c-iT}\right) = \frac{1}{2\pi i}\log\left(\frac{re^{i\arctan\left(\frac{T}{c}\right)}}{re^{-i\arctan\left(\frac{T}{c}\right)}}\right) = \frac{1}{\pi}\arctan\left(\frac{T}{c}\right),$$

and as $T \to \infty$, this approaches $\frac{1}{2}$.

Now suppose $y > 1$. Parametrizing the integral with $\gamma(t) = c + it$, with $t \in [-T, T]$ yields

$$\frac{1}{2\pi i} \cdot i \int_{-T}^{T} \frac{y^{c+it}}{c+it}\,dt = \frac{1}{2\pi}\left(\int_0^T \frac{y^{c+it}}{c+it}\,dt + \int_{-T}^{0} \frac{y^{c+it}}{c+it}\,dt\right) = \frac{1}{2\pi}\left(\int_0^T \frac{y^{c+it}}{c+it} + \overline{\frac{y^{c+it}}{c+it}}\right)dt.$$

Since $\Re(z) = \frac{z+\overline{z}}{2}$, we have

$$\frac{1}{\pi}\int_0^T \Re\left(\frac{y^{c+it}}{c+it}\right)dt.$$

Computing the real part and subbing in yields

$$\frac{1}{\pi}\int_0^T \frac{y^c}{t^2+c^2}(c\cos(t\log y) + t\sin(t\log y))\,dt.$$

We show that the improper integrals

$$\frac{cy^c}{\pi}\int_0^\infty \frac{\cos(t\log y)}{t^2+c^2}\,dt \quad\text{and}\quad \frac{y^c}{\pi}\int_0^\infty \frac{t\sin(t\log y)}{t^2+c^2}\,dt$$

exist, thus allowing us to split the limit of the previous integral.

First we look at the first integral, which we can rewrite as

$$\frac{cy^c}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\cos(t\log y)}{t^2+c^2}\right)dt.$$

Ignoring the constant, if we integrate over a semicirlcle, this is integral is the integral of $\frac{e^{iz\log y}}{z^2+c^2}$ on the real line. From computing $\int_C \frac{\cos z}{z^2+1}\,dz$, we know the part of the integral on the semicircular part vanishes as the semicircle grows large, so the only the part on the real line is left. The only residue in the semicircle is as ci, so from the residues theorem, we have

$$\frac{cy^c}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\cos(t\log y)}{t^2+c^2}\right)dt = \frac{cy^c}{2\pi}\cdot 2\pi i \cdot \frac{e^{-c\log y}}{2ci} = \frac{1}{2}.$$

We do a simlar thing with the second integral, except we use imaginary part of $\frac{ze^{iz\log y}}{z^2+c^2}$, and we again get $\frac{1}{2}$. Thus we have that the limit is equal to 1, as desired.

Now suppose $y < 1$. We do the same thing up until before the splitting. We have

$$y^{2c}\cdot\frac{1}{\pi}\int_0^T \frac{\left(\frac{1}{y}\right)^c}{t^2+c^2}\left(c\cos\left(t\log\frac{1}{y}\right) - t\sin\log\frac{1}{y}\right)dt.$$

From the above work, we know that the limits of the separate integrals exists, so we can split them up. Since $\frac{1}{y} > 1$, both of these integrals equal $\frac{\pi}{2}$, and thus the limit is 0, as desired.

Problem (5): Let $c, x \in \mathbb{R}_{>0}$, let a_n be an arithmetic function, and let $A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$. Show that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} A(s) \frac{x^s}{s} \, ds = \sum_{n<x} a_n + \frac{1}{2} a_x,$$

where the latter term is 0 unless x is a positive integer.

Solution: Suppose $A(s)$ converges absolutely for $\mathfrak{R}(s) = c$. We have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} a_n \frac{\left(\frac{x}{n}\right)^s}{s} \, ds.$$

From the absolute convergence of the summand, we can swap the sum and integral to obtain

$$\sum_{n=1}^{\infty} a_n \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{x}{n}\right)^s}{s} \, ds.$$

From problem 3, this integral is zero for every $n > x$, 1 for every $n < x$, and $\frac{1}{2}$ when $x = n$, which only occurs when x is a positive integer. Thus we have the desired sum.

Problem (9): Given a Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, let

$$\sigma_a = \inf\left\{ \sigma : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges absolutely for some } s \text{ with } \mathfrak{R}(s) = \sigma \right\}.$$

Prove that if $\mathfrak{R}(s) > \sigma_a$, then $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges absolutely.

Solution: Pick s such that $\mathfrak{R}(s) = \sigma > \sigma_a$, and let s' be the number with $\mathfrak{R}(s') = \sigma$ for which the Dirichlet series converges absolutely. We have that

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n^{s'}} \right| = \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}}$$

converges absolutely, but the last sum is also equal to

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n^s} \right|.$$

Thus the Dirichlet series converges for all s with $\mathfrak{R}(s) = \sigma$. This holds for all $\sigma > \sigma_a$, as desired.

Problem (11): Prove that $\sigma_c \le \sigma_a \le \sigma_c + 1$.

Let A be the set we're taking the infinum of in problem 9, and let C be the set in problem 10. Note we clearly have $A \subseteq C$, so this immediately implies $\sigma_c = \inf C \le \inf A = \sigma_a$. Now consider $C + (1 + \varepsilon)$ for some $\varepsilon > 0$. Pick some $\sigma + (1 + \varepsilon) \in C + (1 + \varepsilon)$, and let s be the number with $\mathfrak{R}(s) = \sigma$ for which the Dirichlet series converges. From convergence, we know the limit of the terms is 0, so there exists some N such that $n \geq N \Rightarrow \left|\frac{a_n}{n^s}\right| \leq 1 \Rightarrow \left|\frac{a_n}{n^{s+1+\varepsilon}}\right| \leq \frac{1}{n^{1+\varepsilon}}$. Thus we have

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{n^{s+1+\varepsilon}}\right| = \sum_{n=1}^{N-1}\left|\frac{a_n}{n^{s+1+\varepsilon}}\right| + \sum_{n=N}^{\infty}\left|\frac{a_n}{n^{s+1+\varepsilon}}\right| \leq C + \sum_{n=N}^{\infty}\frac{1}{n^{1+\varepsilon}},$$

where C is some finite constant. The second series converges for all $\varepsilon > 0$. Thus, for any $\sigma \in C$, and for any $\varepsilon > 0$, the Dirichlet series for s with $\mathfrak{R}(s) = \sigma + 1 + \varepsilon$ converges absolutely. Thus we have

$$C + (1 + \varepsilon) \subseteq A,$$

which implies $\sigma_a = \inf A \leq \inf C + (1 + \varepsilon) = \sigma_c + 1 + \varepsilon$. This holds for all $\varepsilon > 0$, so we have $\sigma_a \leq \sigma_c + 1$, as desired.

17. Prime Number Theorem

Theorem (prime number theorem): Let $\pi(x)$ denote the number of primes $\leq x$. Then

$$\pi(x) \sim \frac{x}{\log x}.$$

The proof of the prime number theorem has two main steps: showing that $\zeta(s)$ isn't zero on $\Re(s) \geq 1$, and showing that the nonvanishing is equivalent to $\pi(x) \sim \frac{x}{\log x}$.

17.1. Nonvanishing on $\Re(s) \geq 1$

Lemma: If $\Re(s) > 1$, then $\zeta(s) \neq 0$.

Proof: Consider

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

This converges absolutely for $\Re(s) > 1$, since

$$\sum_{p \text{ prime}} \frac{1}{|p^s|} = \sum_{p \text{ prime}} \frac{1}{p^{\Re(s)}} \leq \zeta(\Re(s)),$$

which is finite. Since the definition for absolute convergence of a product means it doesn't converge to 0, we have the desired result. ■

Lemma: If $\Re(s) > 1$, then

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}.$$

Proof: Take the log of the Euler product and expand using the Taylor series for $\log(1 - x)$. ■

Thus $\log \zeta(s)$ is a Dirichlet series with coefficients c_n, where $c_n = \frac{1}{k}$ if $n = p^k$ for a prime p and zero otherwise.

Lemma: If $\theta \in \mathbb{R}$, then $3 + 4\cos\theta + \cos 2\theta \geq 0$.

Lemma: If $\sigma > 1$ and $t \in \mathbb{R}$, then

$$\log\left|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)\right| \geq 0.$$

Proof: Let $s = \sigma + it$. We have $\Re\left(\frac{1}{n^s}\right) = n^{-\sigma}\cos(t\log n)$. Thus

$$\begin{aligned}\log\left|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)\right| &= 3\Re(\log\zeta(\sigma)) + 4\Re(\log\zeta(\sigma + it)) + \Re(\log\zeta(\sigma + 2it)) \\ &= \sum_{n=1}^{\infty}\frac{c_n(3 + 4\cos(t\log n) + \cos(2t\log n))}{n^\sigma},\end{aligned}$$

which from the previous lemma implies that the sum is at least 0. ∎

Theorem: If $\Re(s) \geq 1$, then $\zeta(s) \neq 0$.

Proof: We already did $\Re(s) > 1$, so suppose there exists some $t_0 \in \mathbb{R}$ such that $\zeta(1 + it_0) = 0$. Since ζ is analytic in a neighborhood of $1 + it_0$, we have that $\zeta(s) = (s - (1 + it_0))f(s)$ is a neighborhood of $1 + it_0$ where f is also analytic and nonzero in that neighborhood. Thus $f(s)$ is bounded in that neighborhood, so we have

$$|\zeta(\sigma + it_0)|^4 \leq C(\sigma - 1)^4$$

for some $C > 0$. Similarly, at the pole at $s = 1$, we have

$$|\zeta(\sigma)|^3 \leq C'(\sigma - 1)^{-3}$$

for some $C' > 0$. Since ζ is analytic at $s + 2it_0$, $|\zeta(\sigma + 2it_0)|$ is bounded as $\sigma \to 1$. Thus

$$\lim_{\sigma \to 1}\left|\zeta^3(\sigma)\zeta^4(\sigma + it_0)\zeta(\sigma + 2it_0)\right| = 0.$$

However, this contradicts the previous lemma, since that implies $\left|\zeta^3(\sigma)\zeta^4(\sigma + it_0)\zeta(\sigma + 2it_0)\right| \geq 1$ for all $\sigma > 1$. ∎

17.2. Smoothing out $\pi(x)$

When trying to prove the PNT, it's easier to work with the smoother function of $\pi(x)$. We define the Chebyshev ψ function as

$$\psi(x) = \sum_{n \leq x}\Lambda(n).$$

We want to show that $\pi(x) \sim \frac{x}{\log x}$ is equivalent to $\psi(x) \sim x$. We will also need the function

$$\theta(x) = \sum_{p \leq x}\log p$$

to show this equivalence.

Proposition: We have

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = \lim_{x\to\infty} \frac{\theta(x)}{x},$$

in the sense that if one of the limits exists, the other must as well and they are equal.

Proof: We have

$$\begin{aligned}\psi(x) - \theta(x) &= \sum_{n\leq x} - \sum_{p\leq x} \log p \\ &= \sum_{\substack{p^n \leq x \\ n\geq 2}} \log p \\ &= \sum_{p\leq x} \left(\lfloor \log_p x \rfloor - 1\right) \\ &\leq \sum_{p\leq \sqrt{x}} \log_2 x \\ &\leq \sqrt{x} \log_2 x.\end{aligned}$$

Since $\lim_{x\to\infty} \frac{\log_2 x}{\sqrt{x}} = 0$, we have desired equality of limits (we are able to split the limits at the end since we assume that one of them exists). ■

Proposition: The prime number theorem is equivalent to $\lim_{x\to\infty} \frac{\psi(x)}{x} = \lim_{x\to\infty} \frac{\theta(x)}{x} = 1$.

Proof: Let a_n be the prime indicator function, and let $\varphi(u) = \log u$. Using Abel summation, we have

$$\theta(x) = \sum_{p\leq x} \log p = \sum_{n\leq x} a_n \varphi(n) = \pi(x) \log x - \int_2^x \frac{\pi(u)}{u}\, du.$$

Thus if $\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$, then we have

$$\theta(x) = x + o(x) - \int_2^x \frac{1}{\log u} + o\left(\frac{1}{\log u}\right) du = x + o(x),$$

where you can use l'Hopitals rule to show that $\int_2^x \frac{1}{\log u}\, du = o(x)$.

For the reverse direction, use $a_n = \log n$ if n is prime and 0 otherwise, and let $\varphi(u) = \frac{1}{\log u}$. ■

17.3. Zeta and Psi

The final step of proving the PNT is showing that the nonvanishing of ζ on $\Re(s) \geq 1$ implies $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$.

17.3.1. Tauberian Theorem

17.3.2. Final Steps

17.4. Problems

Problem (2): Prove that

$$\sum_{n \leq x} \log n = x \log x - x + O(\log x).$$

Solution: Using Abel summation with $a_n = 1$, we have

$$\begin{aligned}\sum_{n \leq x} \log n = \log 1 + \sum_{1 < n \leq x} \log n &= \lfloor x \rfloor \log x - \int_1^x \frac{\lfloor u \rfloor}{u} \, du \\ &= x \log x - \{x\} \log x - \int_1^x 1 - \frac{\{u\}}{u} \, du \\ &= x \log x - x + O(\log x) - \int_1^x O\left(\frac{1}{u}\right) du \\ &= x \log x - x + O(\log x).\end{aligned}$$

Problem (4): Let $\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} \, dt$. Show that

$$\mathrm{Li}(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Conclude that PNT is equivalent to $\pi(x) \sim \mathrm{Li}(x)$.

Solution: From integration by parts, we have

$$\int_2^x \frac{1}{\log t} \, dt = \frac{t}{\log t}\Bigg|_2^x + \int_2^x \frac{1}{(\log t)^2} \, dt = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{1}{(\log t)^2} \, dt.$$

We can show that the integral is $O\left(\frac{x}{(\log x)^2}\right)$ (which the constant gets absorbed into) using l'Hopitals rule:

$$\lim_{x \to \infty} \frac{\int_2^x \frac{1}{(\log t)^2} \, dt}{\frac{x}{(\log x)^2}} = \lim_{x \to \infty} \frac{\frac{1}{(\log x)^2}}{\frac{(\log x)^2 + 2 \log x}{(\log x)^4}} = \lim_{x \to \infty} \frac{(\log x)^2}{(\log x)^2 + 2 \log x} = 1.$$

Thus we have

$$\mathrm{Li}(x) \sim \frac{x}{\log x} \sim \pi(x),$$

as desired.

Problem (6): Let $M(x) = \sum_{n \leq x} \mu(x)$. Let σ be a real number such that $M(x) = O(x^\sigma)$. Show that if $\zeta(s) = 0$, then $\mathfrak{R}(s) < \sigma$.

Solution: From Abel summation using $\varphi(n) = n^{-s}$ and $a_n = \mu(n)$, we obtian

$$\sum_{n \leq x} \frac{\mu(n)}{n^s} = \frac{M(x)}{x^s} + s \int_1^x \frac{M(u)}{u^{s+1}} \, du.$$

Since we clearly have $|M(x)| \leq x$, and supposing $\mathfrak{R}(s) > 1$, when we let $x \to \infty$, the first term vanishes, and we have

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^\infty \frac{M(u)}{u^{s+1}} \, du.$$

Suppose $M(x) = O(x^\sigma) \Rightarrow |M(x)| \leq Cx^\sigma$ for some C. We show that if $\mathfrak{R}(s) \geq \sigma$, then $\zeta(s) \neq 0$. We can bound the integral by

$$|s| \int_1^\infty \left| \frac{M(u)}{u^{s+1}} \right| du \leq |Cs| \int_1^\infty \frac{1}{u^{\mathfrak{R}(s) - \sigma + 1}} \, du,$$

which converges when $\mathfrak{R}(s) - \sigma + 1 > 1 \Rightarrow \mathfrak{R}(s) > \sigma$. Thus the integral $s \int_1^\infty \frac{M(u)}{u^{s+1}} \, du$ is finite for $\mathfrak{R}(s) > \sigma$, i.e. has modulus in $[0, \infty)$ which implies that $|\zeta(s)| \in (0, \infty]$ for $\mathfrak{R}(s) > \sigma$, which is the desired conclusion.

Problem (8): Show that

$$\mathrm{Li}(x) = \sum_{k=1}^{n} (k-1)! \frac{x}{(\log x)^k} + O\left(\frac{x}{(\log x)^{k+1}} \right).$$

Solution: We claim that

$$\mathrm{Li}(x) = \sum_{k=1}^{n} \left((k-1)! \frac{x}{(\log x)^k} + (k-1)! \frac{2}{(\log 2)^k} \right) + \int_2^x \frac{k!}{(\log t)^{k+1}} \, dt.$$

The integral is $O\left(\frac{x}{(\log x)^{k+1}} \right)$ using l'Hopital's rule like we did in problem 4, and the constant part of the sum gets absorbed into the asymptotic. Now we prove this by induction, and we already have the base case from problem 4.

Suppose the equation holds for n. We do integration by parts using $u = \frac{k!}{(\log t)^{k+1}} \Rightarrow du = -\frac{(k+1)!}{t(\log t)^{k+2}} \, dt$ and $v = t \Rightarrow dv = dt$. Thus we have

$$\int_2^x \frac{k!}{(\log t)^{k+1}} \, dt = k! \frac{t}{(\log t)^{k+1}} \Bigg|_2^x + \int_2^x \frac{(k+1)!}{(\log t)^{k+2}} \, dt = k! \frac{x}{(\log x)^{k+1}} - \frac{2k!}{(\log 2)^{k+1}} + \int_2^x \frac{(k+1)!}{(\log t)^{k+2}} \, dt,$$

which adds to the sum as desired.

18. Theta Functions

18.1. Problems

Problem (1): Prove that

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

Solution: We can expand each term in the product as a geometric series to obtain

$$\prod_{n=1}^{\infty} (1 + x^n + x^{2n} + x^{3n} + \cdots).$$

Consider the x^k term in the expansion. We get one contribution from x^k in the $n = 1$ term, one contribution from x^{k-2} in $n = 1$ and x^2 in $n = 2$, and so on. Each contribution comes from some number of 1's, some number of 2's, and so on, that sum up to k. The number of contributions is then the partition function, as desired

Problem (3): Let $P(x) = \sum_{n=0}^{\infty} p(n)x^n$. Show that

$$\log P(x) \sim \frac{\pi^2}{6(1-x)}$$

as $x \to 1^-$.

Solution: From problem 1, we have

$$\begin{aligned}
\log P(x) &= -\sum_{n=1}^{\infty} \log(1 - x^n) \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^{kn}}{k} \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(x^k\right)^n \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{x^k}{1-x^k}.
\end{aligned}$$

Dividing this by $\frac{1}{1-x}$ yields

$$\sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{x^k}{1 + x + \cdots + x^{k-1}}.$$

Thus as $x \to 1^-$, we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

so we have the desired asymptotic equivalence.

Problem (7): Use the Pentagonal Number Theorem to compute $p_{ed}(n) - p_{od}(n)$.

Solution: Consider $\prod_{n=1}^{\infty}(1 - x^n)$. For the exponent k, we get a contribution of $+1$ if an even number of non 1 terms are used, and a contribution of -1 is an odd number of non 1 terms are used. Since each of these non 1 terms have distinct exponents, the coefficient of x^k is $p_{ed}(n) - p_{od}(n)$. Then from the Pentagonal Number Theorem, $p_{ed}(n) - p_{od}(n)$ is 0 unless n is a generalized pentagonal number, in which case the coefficient is $(-1)^j$, where $\frac{3j^2+j}{2} = n$.

Problem (12): Let $f(x) = e^{-\pi x^2}$. Show that $\hat{f}(x) = f(x)$.

Solution: Splitting up the Fourier transform, we have

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-\pi t^2} \cos(2\pi x t) + i e^{-\pi t^2} \sin(2\pi x t)\, dt.$$

The second term is an odd function, so it vanishes. Thus we're left with

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-\pi t^2} \cos(2\pi x t)\, dt.$$

We now compute

$$\int_{-\infty}^{\infty} e^{-a t^2} \cos(bt)\, dt$$

for any $a, b > 0$. Let the integral above be $I(b)$ and differentiate with respect to b to obtain

$$I'(b) = -\int_{-\infty}^{\infty} t e^{-a t^2} \sin(bt)\, dt = \frac{1}{2a} e^{-a t^2} \sin bt \Big|_{-\infty}^{\infty} - \frac{b}{2a} \int_{-\infty}^{\infty} e^{-a t^2} \cos(bt)\, dt.$$

The first term is just zero, and the last term is $-\frac{b}{2a} I(b)$, so we're left with

$$I'(b) = -\frac{b}{2a} I(b).$$

Solving using separation yields

$$I(b) = C e^{-\frac{b^2}{4a}}$$

for some constant C. If we choose $b = 0$, then the integral is just the Gaussian integral with a factor of a, which is then equal to $\sqrt{\frac{\pi}{a}}$. Thus

$$I(b) = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}.$$

Plugging in the values of the integral we want to compute, we obtain

$$\hat{f}(x) = e^{-\pi x^2},$$

as desired.

Problem (13): If $f_u(x) = e^{-\pi u x^2}$, compute $\hat{f}_u(x)$.

Solution: Using the same method as in the previous problem, we just need to compute

$$\int_{-\infty}^{\infty} e^{-\pi u t^2} \cos(2\pi x t)\, dt.$$

Plugging into our formula yields

$$\hat{f}_u(x) = \frac{1}{\sqrt{u}} e^{-\frac{\pi x^2}{u}}.$$

Problem (14): What is the result of applying the Poisson Summation Formula to $\hat{f}_u(x)$?

Solution: We obtain

$$\sum_{k=-\infty}^{\infty} e^{-\pi u k^2} = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{u}} e^{-\frac{\pi k^2}{u}}.$$

Problem (17): Show that if $\mathfrak{R}(s) > 1$, then

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} \int_0^{\infty} u^{\frac{s}{2}-1} \left[\sum_{n=-\infty}^{\infty} f_u(n) - 1\right] du.$$

Solution: In the bracketed term, the -1 cancels with the $n = 0$ term, and then $f_u(x)$ is even, so the negative n terms are equal to the positive n terms. Thus the $\frac{1}{2}$ cancels, so this is equal to

$$\int_0^{\infty} u^{\frac{s}{2}-1} \left[\sum_{n=1}^{\infty} e^{-\pi u n^2}\right] du = \sum_{n=1}^{\infty} \int_0^{\infty} u^{\frac{s}{2}-1} e^{-\pi n^2 u}\, du.$$

Letting $t = \pi n^2 u \Rightarrow dt = \pi n^2\, du$, so we have

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left(\frac{t}{\pi n^2}\right)^{\frac{s}{2}-1} e^{-t} \cdot \frac{1}{\pi n^2}\, dt = \sum_{n=1}^{\infty} \left(\frac{1}{\pi n^2}\right)^{\frac{s}{2}} \int_0^{\infty} t^{\frac{s}{2}-1} e^{-t}\, dt = \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \pi^{-\frac{s}{2}} n^{-s} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Problem (18): Let

$$\psi(u) = \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} f_u(n) - 1\right).$$

Show that

$$\psi(u) = \frac{1}{\sqrt{u}}\psi\left(\frac{1}{u}\right) + \frac{1}{2\sqrt{u}} - \frac{1}{2}.$$

Solution: From problem 14, we obtain

$$\frac{1}{\sqrt{u}}\psi\left(\frac{1}{u}\right) = \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{u}}e^{-\frac{\pi n^2}{u}} - \frac{1}{\sqrt{u}}\right) = -\frac{1}{2\sqrt{u}} + \frac{1}{2} + \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} e^{-\pi u n^2} - 1\right) = -\frac{1}{2\sqrt{u}} + \frac{1}{2} + \psi(u).$$

Rearranging yields the desired conclusion.