Multivariable Formulas + Derivations

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Chapter 13

Dot Product

The dot product or scalar product of two vectors gives a scalar quantity. It measures the extent to which the two vectors align with each other. The formula is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta).$$

The dot product can also be calculated using

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z.$$

Cross Product

The cross product of A and B is defined as the vector orthogonal to both. Since there are infinitely many along one line, convention is to use the following:

$$A \times B = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

Note that

$$\begin{aligned} |A \times B|^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\ &= |A|^2 |B|^2 - (A \cdot B)^2 \\ &= |A|^2 |B|^2 - |A|^2 |B|^2 \cos^2\theta \\ &= |A|^2 |B|^2 \sin\theta \\ |A \times B| &= |A||B|\sin\theta. \end{aligned}$$

(skipped lots of algebra in line one)

Note that given two vectors, the magnitude of the cross product gives the area of the parallelogram they form (can easily be shown using the sin form).

Finding a plane through 3 points

Note that three points define a plane. Consider the triangle formed by those three points. The normal vector to any point on the triangle will be normal to the whole plane. Thus, the equation is of the form

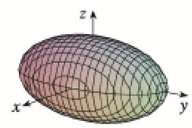
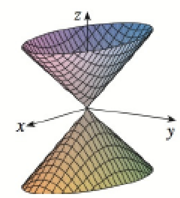
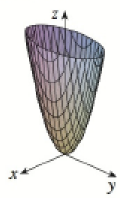
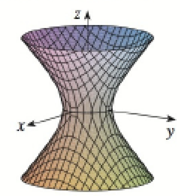
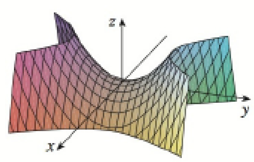
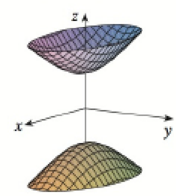
$$\vec{n} \cdot \vec{p} = 0$$

for any point p on the plane (disregarding shifting position vectors). Let the points be a, b, and c. To find n, shift everything so that one point lies on the origin, and then take the cross product of the resulting other two vectors. This is $\vec{n}$. WLOG, $\vec{n} = (b - a) \times (c - a)$. Then, all points p such that

$$\vec{n} \cdot \vec{p} = 0$$

is the plane through the three shifted points. To move it back, just plug in one of the given points to $\vec{p}$ and take the dot product with the normal vector. That will give you the constant on the right side.

Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Chapter 14

Unit Tangent Vector

$$\hat{T}(t)=\frac{v(t)}{|v(t)|}.$$

We can rewrite this as

$$\frac{\frac{dr}{dt}}{\frac{ds}{dt}} = \frac{dr}{ds} = \hat{T}(t).$$

Unit Normal Vector

$$\hat{N}(t) = \frac{\hat{T}'(t)}{\left|\hat{T}'(t)\right|} \text{ or } \frac{\hat{T}'(s)}{\left|\hat{T}'(s)\right|}.$$

Note that N and T are orthogonal. We can derive this by noticing

$$\left|\hat{T}\right| = 1 \implies \hat{T}\cdot\hat{T} = 1 \implies \hat{T}'\cdot\hat{T} + \hat{T}\cdot\hat{T}' = 0 \implies \hat{T}\cdot\hat{T}' = 0 \implies \hat{T}\cdot N = 0.$$

Note that $\left|\hat{T}'(s)\right| = \kappa$, so

$$\kappa\hat{N}(t) = \frac{d\hat{T}}{ds}.$$

Curvature

Curvature is defined as $\kappa = \left|\frac{d\hat{T}}{ds}\right|$. Using the chain rule, we can write this as

$$\left|\frac{d\hat{T}}{dt}\frac{dt}{ds}\right| = \frac{\left|\frac{d\hat{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \frac{\left|\hat{T}'(t)\right|}{|v(t)|}.$$

Alternatively,

$$\kappa = \frac{|v \times a|}{|v|^3}.$$

Tangential and Normal Components of Acceleration

For a twice-differentiable curve r, we have

$$a(t) = r''(t) = a_{\hat{T}}\hat{T} + a_{\hat{N}}\hat{N}$$

where

$$a_{\hat{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|v| \text{ and } a_{\hat{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|v|^2 = \frac{\left|\hat{T}'\right|}{|v|}|v|^2 = \left|\hat{T}'\right||v|.$$

To derive this, note that

$$v = \frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = \hat{T}\frac{ds}{dt}.$$

Then, take the derivative to get

$$\begin{aligned} a &= \hat{T}'\frac{ds}{dt} + \hat{T}\frac{d^2s}{dt^2} \\ &= \frac{d\hat{T}}{ds}\frac{ds}{dt}\frac{ds}{dt} + \hat{T}\frac{d^2s}{dt^2} \\ &= \kappa\hat{N}\left(\frac{ds}{dt}\right)^2 + \hat{T}\frac{d^2s}{dt^2}. \end{aligned}$$

Binormal Vector

$$B(t) = \hat{T} \times \hat{N}.$$

Can rewrite as

$$\frac{v \times a}{|v \times a|}.$$

Torsion

$$\tau = -\frac{dB}{ds} \cdot \hat{N}.$$

Rewrite the derivative as $\frac{\frac{dB}{dt}}{\frac{ds}{dt}}$ to obtain

$$\tau = -\frac{B'}{|v|} \cdot \hat{N}.$$

Can also be written as

$$\frac{(v \times a) \cdot a'}{|v \times a|^2}.$$

Chapter 15

Limits

If the limits differ when approching from different paths, the limit doesn't exist.

Partial Differentiation

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

Notation:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = D_x f.$$

Differentiation

A multivariable function $f(x,y)$ is differentiable at (h,k) ($f(x,y)$ is define at some disk around (h,k)) if, for the tangent plane to $f(x,y)$, which we call $L(x,y)$, we have have

$$\lim_{(x,y)\to(h,k)} \frac{|f(x,y) - L(x,y)|}{|(x,y) - (h,k)|} = 0.$$

Then we have this nice chart:

Suppose f is a multivariable function in $x_1, x_2, \ldots, x_n$, where x_i is a function in t. Then,

$$\frac{df}{dt} = \sum \frac{\partial w}{\partial x_i} \frac{dx_i}{dt}.$$

It's easy to show this using implicit differentiation with respect to t.

Implicit Differentiation

Suppose we have $F(x,y,z(x,y)) = 0$. Then we have

$$2\frac{\partial F}{\partial x} = 0 \implies \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \implies \frac{-F_x}{F_z} = \frac{\partial z}{\partial x}.$$

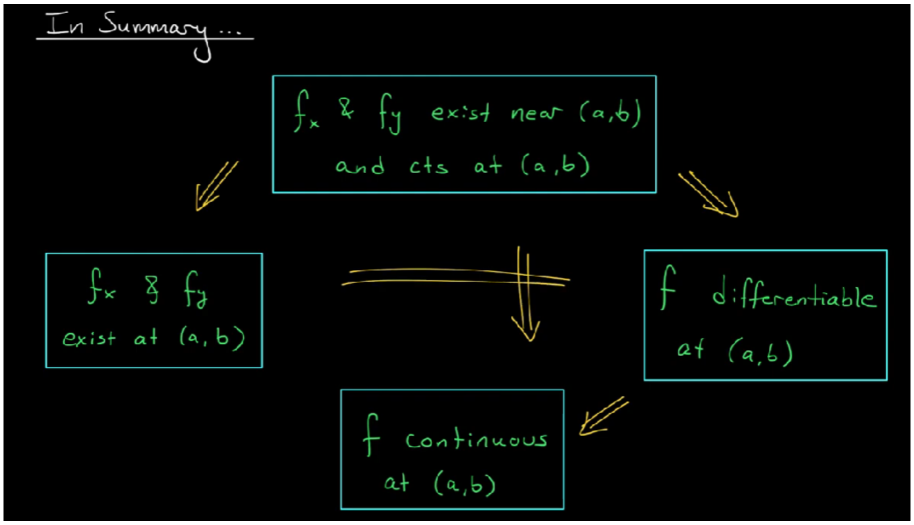


Fig. 1: Middle arrow is going to the left

Directional Derivative

Gradient:

$$\nabla f(x_1, \ldots, x_n) = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$$

Directional Derivative: change in f when moving in the direction of a vector (vector is always a unit vector) from a point $(a_1, \ldots, a_2)$.

$$D_{\vec{u}}(x_1, \ldots, x_n) = \nabla f(x_1, \ldots, x_n) \cdot \vec{u}.$$

Plug in $(a_1, \ldots, a_n)$.

To maximize the gradient, note that we can write the directional derivative as

$$|\nabla f(x_1, \ldots, x_n)|\,|\vec{u}| \cos \theta = |\nabla f(x_1, \ldots, x_n)| \cos \theta.$$

This means that the max is $|\nabla f(x_1, \ldots, x_n)|$, when $\vec{u}$ points in the same direction as ∇f. If we don't use the unit vector then the derivative will be scaled by the magnitude of the vector.

Gradient is perpendicular to level curve

This proof also works for 2D level curves.

Let $w = f(x, y, z)$ and $r(t) = \langle x(t), y(t), z(t)\rangle$. Let (x_0, y_0, z_0) be a point on level curve $c = f(x, y, z)$, and let $g(t) = f(x(t), y(t), z(t))$. Differentiating g yields

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\bigg|_P \frac{dx}{dt}\bigg|_{t_0} + \frac{\partial f}{\partial y}\bigg|_P \frac{dy}{dt}\bigg|_{t_0} + \frac{\partial f}{\partial z}\bigg|_P \frac{dz}{dt}\bigg|_{t_0} = 0.$$

Rewriting yields

$$\nabla f|_P \cdot r'(t)|_{t_0} = 0.$$

Since $f'(t)$ is the tangent vector to a curve, and since the dot product is 0, the gradient
$nabla\ f|_P$ must be perpendicular to the level curve.

Tangent Plane

Tangent plane to curve $x = f(x, y, z)$ at a point $P = (x_0, y_0, z_0)$ is

$$0 = f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0).$$

Optimization

The **second partial derivative test** tells us how to verify whether this stable point is a local maximum, local minimum, or a saddle point. Specifically, you start by computing this quantity:

$$H = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$$

Then the second partial derivative test goes as follows:

- If $H < 0$, then (x_0, y_0) is a saddle point. [See a picture]
- If $H > 0$, then (x_0, y_0) is either a maximum or a minimum point, and you ask one more question:
  - If $f_{xx}(x_0, y_0) < 0$, (x_0, y_0) is a local maximum point. [See a picture]
  - If $f_{xx}(x_0, y_0) > 0$, (x_0, y_0) is a local minimum point. [See a picture]

  (You could also use $f_{yy}(x_0, y_0)$ instead of $f_{xx}(x_0, y_0)$, it actually doesn't matter)
- If $H = 0$, we do not have enough information to tell.

Lagrange Multipliers

Diff alg spam

$$\mathcal{L}(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n} \lambda_i g_i(x_1, x_2, \ldots, x_n)$$

This can be more compactly represented as

$$\nabla f = \lambda \nabla g$$

Chapter 16

Mass stuff

Center of mass for a thin rod with density $\rho(x)$:

$$\overline{x} = \frac{M}{m},$$

where M is the moment and m is the mass.

$$M = \int_a^b x\rho(x)\,dx$$

$$m = \int_a^b \rho(x)\,dx.$$

To show this, note that the center of mass for discrete particles on a line is

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i}.$$

Taking i to infinity and letting the particles become a line yields

$$x_{cm} = \frac{\int x\,dm}{m_{tot}}.$$

Note that $\rho = \frac{dm}{dx}$, which means

$$x_{cm} = \frac{\int x\rho\,dx}{mtot}.$$

On 2D plate with density $\rho(x,y)$ over the region D, the center of mass is given by

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right),$$

where

$$m = \iint_D \rho(x,y)\,dA,$$

$$M_x = \iint_D y\rho(x,y)dA,$$

and

$$M_y = \iint_D x\rho(x,y)dA.$$

For 3D, the coordinate of the center of mass is

$$\overline{x} = \frac{M_{yz}}{m},$$

where m is found by integrating the density function over the region bound by the solid, and M_yz is

$$\iiint_D x\rho(x,y,z)\,dV.$$

The others are found by cyclically shifting variables.

Jacobian

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Chapter 17

A vector field $\vec{F}$ is called conservative if there exists a function f such that $\vec{F} = \nabla f$. f is called the potential function of $\vec{F}$.

The line integral of a function $f(x,y)$ with respect to arc length on a curve C is

$$\int_C f\,ds.$$

Using a parametrization $r(t) = \langle h(t), g(t)\rangle$ for C gives the equivalent integral

$$\int_a^b f(h(t),g(t))\,|r'(t)|\,dt,$$

where the curve is parameterized from $a \leq t \leq b$. Essentially gives the surface area of a scalar function along a curve in the plane.

If we project the surface area onto a plane, we can find the area of the projection using the line integral with respect to one variable.

$$\int_C f\,dx \text{ and } \int_C f\,dy,$$

which can be computed with

$$\int_a^b f(r\vec{(t)})g'(t)\,dt \text{ and } \int_a^b f(\vec{r}(t))h'(t)\,dt,$$

where $\vec{r}(t) = \langle g(t), h(t)\rangle$.

A line integral over a curve in a vector field is given by

$$\int_C \vec{F}\cdot d\vec{r},$$

where $d\vec{r}$ is basically a small change in position. This computes the work of a particle in a force field $\vec{F}$ moving along C. We can rewrite this as

$$\int_C \vec{F}\cdot\hat{T}\,ds,$$

which we can further rewrite as

$$\int_a^b \vec{F}(\vec{r}(t))\cdot r'(t)\,dt.$$

If $\vec{F} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$, then the integral is

$$\begin{aligned}\int_a^b P(x,y)g'(t) + Q(x,y)h'(t)\,dt &= \int_a^b P(x,y)g'(t)\,dt + \int_a^b Q(x,y)h'(t)\,dt\\ &= \int P\,dx + \int Q\,dy.\end{aligned}$$

Flow is the same as line integral over vector field, flux is

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_C P\,dy - \int_C Q\,dx,$$

where $\vec{F} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$.

Curl = circulation density =

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

where $\vec{F} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$.

Divergence = flux density =

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$

where $\vec{F} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$.

Green's Theorem:

Let C be a positively oriented, piecewise smooth, simple, closed curve, let D be the region enclosed by the curve, and let $\vec{F} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$. If P and Q have continuous first order partial derivatives on D, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

and

$$\int_C \vec{F} \cdot \vec{n}\, ds = \iint\limits_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA.$$

Let the ∇ operator be

$$\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{\jmath}.$$

so passing in a function as an argument yields

$$\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{\jmath}.$$

Then

$$\text{curl}\vec{F} = \nabla \times \vec{F}$$

and

$$\mathrm{div}\vec{F} = \nabla \cdot \vec{F}.$$

We can describe a surface parametrically by

$$\vec{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}.$$

If D is the surface, then the surface area is given by

$$\iint_D |\vec{r}_u \times \vec{r}_v| \, dA.$$

If you are given the surface explicitly, $z = g(x,y)$, then you can evaluate a surface integral for a function f as

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA,$$

where D is the region $g(x,y)$ ranges over.

If we have a parametric representation of a surface S, $\vec{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}$, then

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v| \, dA.$$

If we have an implicit description $F(x,y,z) = c$, then

$$\iint_S f(x,y,z)\, dS = \iint_R f(x,y,z)\frac{\nabla F}{\nabla F \cdot \hat{k}}\, dA.$$

The flux over a vector field $\vec{F}$ is

$$\iint_S \vec{F} \cdot \vec{n}\, dS.$$

This is equal to all of the following:

$$\iint\limits_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA,$$

$$\iint\limits_R \vec{F} \cdot \frac{\nabla g}{\left|\nabla g \cdot \hat{k}\right|}\, dA,$$

where $\vec{r}$ is the parametric description of the surface, and g is an implicit description of the surface.

Stokes' Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint\limits_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS,$$

where C is the curve that bounds a surface S.

Divergence Theorem:

$$\iint\limits_S \vec{F} \cdot \vec{n}\, dS = \iiint\limits_D \nabla \cdot \vec{F}\, dV,$$

where S is the surface of a region D.