Differential Eq Formulas + Derivations

Nikhil Reddy

Separable Equations

Are in the form

$$N(y)y' = M(x).$$

Integrate both sides to get

$$\int N(y)\frac{dy}{dx}\,dx = \int M(x)\,dx.$$

Then $u = y(x) \implies du = y'(x)\,dx \implies$

$$\int N(u)\,du = \int M(x)\,dx.$$

Solve then back substitute.

Linear Equations

$$y' + P(x)y = f(x).$$

(can reduce to this by dividing by y' coefficient) Solutions:

$$y = \frac{\int e^{\int P(x)\,dx} f(x)\,dx + C}{e^{\int P(x)\,dx}}.$$

To get this get the integrating factor: $\mu = e^{\int P(x)\,dx}$. Note that $\mu' = P(x)\mu$. Multiplying the equation by μ yields

$$\mu y' + \mu P(x)y = \mu f(x).$$

Then reversing product rule on the left yields

$$\frac{d}{dx}(\mu y) = \mu f(x).$$

Integrating and dividing yields

$$y = \frac{\int \mu f(x)\, dx + C}{\mu},$$

ans substituting the expression for μ gives the desired result.

Exact Equations

$$M(x,y)\, dx + N(x,y)\, dy = 0.$$

If $M_y = N_x$, then M and N are the partial derivatives of a stream function. Let $\Psi(x,y)$ be a function such that $\Psi_x = M$ and $\Psi_y = N$. We can write the equation as

$$M(x,y) + N(x,y) \cdot y' = 0.$$

By chain rule, this is the same as

$$\frac{d}{dx}(\Psi(x,y)) = 0.$$

Thus,

$$\Psi(x,y) = C.$$

For some equations, M and N might not be partials of a stream function, but we can multiply by an integrating factor to make them partials of a stream function. There are two cases: when μ is only in x and when μ is only in y.

Suppose μ is only in y. Multiply by μ. Then we have

$$\frac{\partial}{\partial y}(\mu M) = \mu' M + \mu M_y = \mu N_x = \frac{\partial}{\partial x}(\mu N).$$

This is a linear differential equation in μ. Rewriting gives

$$\mu' = \left(\frac{N_x - M_y}{M}\right)\mu.$$

Using separation of variables yields

$$\mu = e^{\int \frac{N_x - M_y}{M}\, dy}.$$

If μ is only in y, then it's an integrating factor that can be used. In a similar spirit,

$$\mu = e^{\int \frac{M_y - N_x}{N}\,dx}$$

is an integrating factor if it's only in x.

Bernoulli Equations

$$y' + P(x)y = f(x)y^n.$$

Divide through by y^n to get

$$\frac{y'}{y^n} + P(x)\frac{1}{y^{n-1}} = f(x).$$

Let $u = \frac{1}{y^{n-1}}$. Then

$$u' = (1-n)\frac{1}{y^n} \cdot y'.$$

Substituting yields

$$\frac{1}{1-n} \cdot u' + P(x)u = f(x) \implies u' + (1-n)P(x)u = (1-n)f(x).$$

This is now a linear equation in u. Solve for u, then go back to y, done.

Homogenous Equations

Can be written as

$$y' = f\left(\frac{y}{x}\right) \text{ or } y' = f\left(\frac{x}{y}\right).$$

Will only show solution for first, second is identical. Let $u = \frac{y}{x}$. Then $y' = u + xu'$. Substituting yields

$$u + x\frac{du}{dx} = f(u).$$

Rewriting and using separation yields

$$\int \frac{1}{f(u) - u}\, du = \ln(x) + C.$$

Solve for u and go back to y, done.

Euler's Method

Let $y' = f(x, y)$. Given a point (x_1, y_1) that's on the curve of a solution, and a step h, an approximation for the point on the curve h after x_1 is

$$y_2 = y_1 + hf(x_1, y_n).$$

In general, so obtain an approximation of y_n given (x_{n-1}, y_{n-1}), use

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}).$$

Improved method:

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, y'_{n+1})),$$

where

$$y'_{n+1} = y_n + hf(x_n, y_n).$$

Homogenous Linear Equations with Constant Coefficients

Of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = 0.$$

Solutions come in the form e^{rt}, where is a root of the characteristic polynomial of the differential equation. If $r = \alpha \pm \beta i$ is complex, then the solution is

$$e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$

If r is real with multiplicity m, then the solutions are

$$c_0 e^{rx} + c_1 x e^{rx} + \cdots + c_{m-1} x^{m-1} e^{rx}.$$

If $r = \alpha \pm \beta i$ is complex with multiplicity m, then the solutions are

$$\begin{aligned}
&e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) + \\
&xe^{\alpha x}(c_3 \cos(\beta x) + c_4 \sin(\beta x)) + \\
&\vdots \\
&+ x^{m-1} e^{\alpha x}(c_{2m-1} \cos(\beta x) + c_{2m} \sin(\beta x)).
\end{aligned}$$

Over all solutions of the characteristic polynomial, add up all their corresponding differential equation solutions to get the final answer.

Nonhomogenous Linear Equations with Constant Coefficients

Of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = g(x).$$

Method of Undetermined Coefficients

First find the solution to the homogenous version of the equation, i.e.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = 0.$$

This is the complementary solution, y_c. Next, we need to find the particular solution y_p, which is based on $g(x)$. Basically guess the form of a solution with variable coefficients, plug into the equation and solve for the coefficients.

Examples:

- $g(x) = 4x + 5 \implies y_p = Ax + B$
- $g(x) = 7\cos(6x) \implies y_p = A\cos(6x) + B\sin(6x)$
- $g(x) = e^{-4x} \implies y_p = Ae^{-4x}$
- $g(x) = xe^{-5x} + 7e^{-5x} + 10x^2 e^{-5x} \implies y_p = (Ax^2 + Bx + C)e^{-5x}$

- $g(x) = x^2 \cos(4x) + 20e^{7x} \cos(4x) \implies (Ax^2 + Bx + C)\cos(4x) + (Dx^2 + Ex + F)\sin(4x) + e^{7x}(G\cos(4x) + H\sin(4x))$

If a section of the particular solution is a section in the complementary solution, multiply the section in the particular solution by x. Keep doing until no longer part of the complementary solution.