Linear Algebra Notes

Contents

1. Vector Spaces

1.1. $\mathbb{R}^n$ and $\mathbb{C}^n$

$\mathbb{R}$ and $\mathbb{C}$ are defined as usual.

Example (Complex Commutativity): $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Definition 1.1.1 (Elements in $\mathbb{F}^n$ and Operations): $\mathbb{F}^n$ is all n-tuples

$$(x_1, x_2, ..., x_n)$$

with $x_i \in \mathbb{F}$.

Addition is pointwise. Scalar multiplication by λ multiplies each element by λ. If context is clear, $0 \in \mathbb{F}^n$ denotes $(0, 0, ..., 0)$, where there are n 0s.

1.2. Definition of Vector Space

Definition 1.2.1 (Vector Space): A vector space V is a set V with addition and scalar multiplication with commutativity, associativity, additive identity, additive inverse, multiplicative identity, and distributive properties. Elements of a vector space are called vectors or points.

Example: $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces, just verify the properties hold. $\mathbb{F}^\infty$ is also a vector space.

Definition 1.2.2 ($\mathbb{F}^S$): If S is a set, then $\mathbb{F}^S$ is the set of functions from S to $\mathbb{F}$.

Example ($\mathbb{F}^S$ is a vector space): The 0 function $0(x) = 0$ for all $x \in \mathbb{F}$ is the additive identity. The additive inverse is $(-f)(x) = -f(x)$. All other properties of vector spaces hold by spamming axioms. $\mathbb{F}^n$ is a special case of this, where $S = \{1, 2, ..., n\}$.

Proposition 1.2.1 (Unique additive identity): A vector space has a unique additive identity.

Proof: $0, 0'$ are identities.

$$0' = 0' + 0 = 0 + 0' = 0.$$

■

Proposition 1.2.2 (Unique additive inverse): Every element in a vector space has a unique additive inverse.

Proof: $v \in V$, with w, w' as inverses.

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

■

Proposition 1.2.3: $0v = 0$ for every $v \in V$.

Proof: $0v = (0 + 0)v = 0v + 0v$. ■

Remark: We have to use $0 = 0 + 0$ since we have to use the distributive property to connect scalar multiplication and vector addition.

Proposition 1.2.4: $a0 = 0$ for scalar a.

Proof: $a0 = a(0 + 0) = a0 + a0$. ■

Proposition 1.2.5: $(-1)v = -v$.

Proof:

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0.$$

■

Problems

Problem (Exercise 1): Prove that $-(-v) = v$ for every $v \in V$.

Solution:

$$-(-v) + (-v) = (-1)(-v) + (-1)v = (-1)(-v + v) = (-1)(0) = 0,$$

so $-(-v)$ is the additive inverse of $-v$, so $-(-v) = v$, as desired.

Problem (Exercise 3): Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution: First existence. Adding $-v$ to both sides gives $3x = w - v$. Multiplying by $\frac{1}{3}$ on both sides gives $x = \frac{1}{3}(w - v)$.

Now uniquness. Suppose y, y' both satisfy. Then $y = \frac{1}{3}(w - v) = y'$.

Problem (Exercise 4): The empty set is a not a vector space. Why?

Solution: Doesn't satsify additive identity. There are no elements, so there cannot exist and additive identity.

Problem (Exercise 5): Show that the additive inverse condition in the definition of a vector space can be replaced with

$$0v = 0 \text{ for all } v \in V.$$

Solution:

$$0v = (1 + (-1))v = 1v + (-1)v = v + (-1)v = 0,$$

so there exists a w such that $v + w = 0$, as desired.

Problem (Exercise 6): Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$?

Solution: No. We have that $(2 - 1)\infty = (1)\infty = \infty$ and $2\infty - 1\infty = \infty + (-\infty)0$, so it is not distributive, so it in not a vector space.

1.3. Subspaces

Definition 1.3.1 (Subspace): A subset U of V is called a subspace of V if U is also a vector space.

Example: $(x_1, x_2, 0)$ with $x_1, x_2 \in \mathbb{F}$ is a subspace of $\mathbb{F}^3$.

Proposition 1.3.1 (Conditions for subspaces): A $U \subset V$ is a subspace of V if and only if U satisfies the conditions below.

- Additive identity: $0 \in U$
- Closed under addition
- Closer under scalar multiplication

Solution: If U is a subspace of V, then it satisfies the properties by definition.

First condition ensures additive identity. Second and third make sure addition and scalar work. Additive inverse holds by scalar multiplication by -1, and associativity and distributivity hold because that holds on V.

Example:

- If $b \in \mathbb{F}$ then
$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$
is a subspace if and only if $b = 0$. When $b = 0$, we can easily verify all the subspace conditions hold. If we have a subspace, then $0 \in U$, so $0 = x_3 = 5(0) + b$ means b must be 0.
- The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$. $f(x) = 0$ is the additive identity for $\mathbb{R}^{[0,1]}$, and clearly addition and scalar multiplication are closed.
- The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
- The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.
- The set of all sequences of comples numbers with limit 0 is a subspace of $\mathbb{C}^\infty$.

Definition 1.3.2: Suppose $U_1, U_2, ..., U_m$ are subsets of V. The sum of $U_1, U_2, ..., U_m$ is the set of all possible sums of elements in the subsets. So
$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_i \in U_i\}.$$

Example: We have $U = \{(x, 0, 0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}$ and $W = \{(0, y, 0) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}$. Then,
$$U + W = \{(x, y, 0) \in \mathbb{F}^3 \mid x, y, \in \mathbb{F}\}.$$

Example: U, W are subsets in $\mathbb{F}^4$, $U = (x, x, y, y)$, $W = (x, x, x, y)$. Then
$$U + W = (x, x, y, z)$$
since when adding two elements from U and W, the sum always has equal first and second components. The sum of the third components can be arbitrary, and same for the fourth.

Proposition 1.3.2 (Minimality of subspace sums): If $U_1, ..., U_m$ are subspaces of V, then $U' = \sum U_i$ is the smallest subspace containing U_i.

Solution: Cleary U' is a subspace. All the U_i are contained in U'. Also, in any subspace with U_i, we must have U' by closed addition. Thus, we have minimmality.

Definition 1.3.3 (Direct Sum): Suppose $U_1, ..., U_m$ are subspaces of V. Tehn $\sum U_i$ is called a direct sum if each element of $\sum U_i$ can be written in only one way as a sum of $\sum u_i$, where $u_i \in U_i$. If $\sum U_i$ is a direct sum, then we denote is with $U_1 \oplus U_2 \oplus \cdots \oplus U_m$.

Example: $U = (x, y, 0), W = (0, 0, z)$. Then $U \oplus W = \mathbb{F}^3$.

Example (Nonexample): $U_1 = (x, y, 0), U_2 = (0, 0, z), U_3 = (0, y, y)$. We have that $\mathbb{F}^3 = \sum U_i$ since we can write every vector in $\mathbb{F}^3$ as the sum of three vectors from each of the subsets. But $(0, 0, 0)$ can be written in two different ways, so it's not a direct sum.

Proposition 1.3.3 (Direct Sum Condition): U_i are subspaces of V. Then $W = \sum U_i$ is a direct sum if and only if the only way to write 0 is by writing 0 in each subspace.

Solution: If W is a direct sum, then by definition the only way to write 0 is by taking 0 from each U_i (0 is in each of these by subspace condition). Now suppose the only way to write 0 is to take $u_i = 0$. Now consider $v \in W$. Suppose there are two ways to write it,

$$v = \sum u_i = \sum v_i.$$

Subtracting gives $0 = \sum u_i - v_i$. We know that $u_i - v_i \in U_i$ because it is a subapce, and we also know that the only way to write 0 is having all components equal 0. Thus, $u_i = v_i$, so there is only one way to write each vector as a sum, as desired.

Proposition 1.3.4 (Direct sum of two subspaces): U, W are subspaces of V. Then $U + W$ is a direct sum of and only if $U \cap W = \{0\}$.

Solution: If $U + W$ is a direct sum, then for $v \in U \cap W$, we have that $0 = v + (-v)$, with $v \in U$ and $-v \in W$. By unique representation, $v = 0$, so $U \cap W = \{0\}$. If $U \cap W = 0$, then for $u \in U$ and $w \in W$ we have $0 = u + w$. We need to show $u = w = 0$. The equation implies $u = -w \in W$, so $u \in U \cap W$, which means $u = 0$ as desired.

Problems

Problem (Exercise 1): For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subapce of $\mathbb{F}^3$:
- $\{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 0\}$
- $\{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 4\}$
- $\{(x_1, x_2, x_3) : x_1 x_2 x_3 = 0\}$
- $\{(x_1, x_2, x_3) : x_1 = 5x_3\}$

Solution:
- Yes, it is closed under addition and scalar multiplication and has 0.
- No, does not have 0.
- Not closed under addition.
- Yes.

Problem (Exercise 3): Show that the set of differentiable real valued functions on $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{-4,4}$.

Solution: $f \equiv 0$ clearly satisfies the additive identity. It's also easy to see it's closed under addition and scaling.

Problem (Exercise 4): Show that the set of continuous real valued functions on the interval $[0, 1]$ such that $\int_0^1 = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Solution: If $b = 0$, then the conclusion follows easily from verifying subspace conditions. If the set is a subspace, then for f we need $\int_0^1 2f = 2b = b$, which implies $b = 0$.

Problem (Exercise 5): Is $\mathbb{R}^2$ a subspace of $\mathbb{C}^2$.

Solution: No, it's not closed under scalar multiplication.

Problem (Exercise 6):
- Is $\{(a, b, c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?
- Is $\{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Solution:
- Yes, since over $\mathbb{R}$ we have $a^3 = b^3 \Rightarrow a = b$.
- No, since $(\omega, 1, 0) + (1, 1, 0) = (\omega + 1, 2, 0)$ is not in the set.

Problem (Exercise 7): Prove or give a counterexample: if U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses, then U is a subspace of $\mathbb{R}^2$.

Solution: $U = \{(x, y) : x, y, \in \mathbb{Z}\}$.

Problem (Exercise 8): Given an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and scalar multiplication but is not a subspace.

Solution: $U = \{(x, y) : xy = 0\}$

Problem (Exercise 9): Is the set of periodic functions $f : \mathbb{R} \to \mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$?

Solution: No. Consider $\sin(x)$ and $\sin(\pi x)$. When added, they do not form a periodic function, so the set is not closed under addition.

Problem (Exercise 10): If U_1 and U_2 are subspaces of V, then $U_1 \cap U_2$ is a subspace of V.

Solution: Let $W = U_1 \cap U_2$. 0 is in U_1 and U_2, so $0 \in W$. Consider a vector $w \in W$. Since $w \in U_1, \lambda w \in U_1$. Similarly for U_2. Thus W is closed under scalar multiplication. A similar argument can be used for addition, so W is a subspace of V.

Problem (Exercise 12): Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution: Let U_1, U_2, be subspaces of V. If $U_1 \subset U_2$, then $U_1 \cup U_2 = U_2$, which is a subspace. Now suppose $U_1 \cup U_2 = W$ is a subspace. Suppose for the sake of contradiction $U_1 \not\subset U_2$. Pick $x \in U_1$, $x \notin U_2$ and $y \notin U_1, y \in U_2$. We know that $x + y = w \in W$ must be in U_1 or U_2. Suppose its in U_1. Then, $y = w - x = w + (-1)x \in U_1$, a contradiction. The same applies to U_2. Thus, $U_1 \subseteq U_2$.

Problem (Exercise 19): Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution: $U_1 = \{0\}, U_2 = W, W$ is any subspace of V that's not $\{0\}$.

Problem (Exercise 20): Suppose $U = (x, x, y, y)$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Solution: The subspace $W = (a, 0, 0, b)$. Note that we can write any element in $\mathbb{F}^4$ as a sum of elements from U and V, so $U + v = \mathbb{F}^4$. Note also that the only way two vectors in W and U are equal is when $x = y = a = b = 0$, or in other words, $U \cap W = \{0\}$. Then, $U \oplus W = \mathbb{F}^4$.

Problem (Exercise 21): Suppose $U = \{(x, y, x + y, x - y, 2x)\}$. Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

Solution: $W = \{(0, 0, a, b, c)\}$

Problem (Exercise 22): U is the same as the previous problem. Find $W_1, W_2, W_3 \neq \{0\}$ such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution: $W_1 = (0, 0, a, 0, 0), W_2 = (0, 0, 0, b, 0), W_3 = (0, 0, 0, 0, c)$

Problem (Exercise 23): Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W,$$

then $U_1 = U_2$.

Solution: $W = (x, x), U_1 = (a, 0), U_2 = (0, b)$

Problem (Exercise 24): U_o is the set of real-valued functions. U_e is defined similarly. Show that $\mathbb{R}^{\mathbb{R}} = U_o \oplus U_e$.

Solution: Note that both U_o and U_e are subspaces, and that $U_o \cap U_e = \{0\}$. Note that for any $f \in \mathbb{R}^{\mathbb{R}}$, we can write an even function $e(x) = \frac{f(x)+f(-x)}{2}$ and an odd function $o(x) = \frac{f(x)-f(-x)}{2}$, so $U_o + U_e = \mathbb{R}^{\mathbb{R}}$. But we have $U_o \cap U_e = \{0\}$, so $U_o + U_e$ is a direct sum, so we're done.

2. Finite Dimensionl Vector Spaces

2.1. Linear Combinations and Span

Definition 2.1.1 (Linear Combination): A linear combination of a list $v_1, ..., v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_i \in \mathbb{F}$.

Example: $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$ while $(17, -4, 5)$ is not.

Definition 2.1.2 (Span): The set of all linear combinations of a list of vectors v_i in V is called the span of v_i, denoted by $\operatorname{span}(v_1, ..., v_m)$. In other words,

$$\operatorname{span}(v_1, ..., v_m) = \left\{ \sum a_i v_i : a_i \in \mathbb{F} \right\}.$$

The span of the empty list $()$ is defined to be $\{0\}$.

Proposition 2.1.1 (Span is the smallest containing subspace): The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Proof: Suppose v_i is a list of vectors in V. Note that the span is indeed a subspace since 0 is in the span and the same is closed under addition and scalar multiplication. Note that each v_k is also in the span.

Because subspaces are closed under scalar multiplication and addition, every subspace of V that contains each v_k contains $\operatorname{span}(v_1, ..., v_m)$. Thus $\operatorname{span}(v_1, ..., v_m)$ is the smallest subspace of V containing all vectors v_i. ■

Definition 2.1.3: If $\operatorname{span}(v_1, ..., v_m)$ equals V, we say that the list $v_1, ..., v_m$ *spans* V.

Example:

$$(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1)$$

spans $\mathbb{F}^n$, where there are n vectors in the list.

Definition 2.1.4 (Finite-dimensional vector space): A vector space is called finite-dimensional if some list of vectors in it spans the space.

Definition 2.1.5: $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$. $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m.

Definition 2.1.6 (Infinite dimensional vector space): A vector space is called infinite-dimensional if it is not finite-dimensional.

2.2. Linear Independence

Definition 2.2.1 (Linearly independent): A list $v_1, \ldots, v_m$ of vectors in V is called linearly independent if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that makes

$$\sum a_i v_i = 0$$

is $a_i = 0$ for all i. The empty list is also declared to be linearly independent.

Proposition 2.2.1: If some vectors are removed from a linearly independent list, the remaining list is also linearly independent.

Proof: If a list is linearly independent, then the only a_is that work are all 0. Suppose we remove some of the vectors. If the new list wasn't linearly dependent, then we could just drop 0s in front of the vectors we got rid of and have a non linearly independent list of vectors, which is a contradiction. ■

Lemma 2.2.1 (Linear dependence lemma): Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, \ldots, m\}$ such that

$$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}).$$

Furthermore, if k satsfies the condition above and the kth term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

Proof: Because the list $v_1, \ldots, v_m$ is linearly dependent, there exist numbers $a_1, \ldots, a_m \in \mathbb{F}$, not all 0, such that

$$\sum a_i v_i = 0.$$

Let k be the largest element of $\{1, ..., m\}$ such that $a_k \neq 0$. Then

$$v_k = -\frac{a_1}{a_k} v_1 - \cdots - \frac{a_{k-1}}{a_k} v_{k-1},$$

which proves that $v_k \in \operatorname{span}(v_1, ..., v_{k-1})$, as desired.

Now suppose k is any element of $\{1, ..., m\}$ such that $v_k \in \operatorname{span}(v_1, ..., v_{k-1})$. Let $b_1, ..., b_{k-1} \in \mathbb{F}$ such that

$$v_k = b_1 v_1 + \cdots + b_{k-1} v_{k-1}.$$

Suppose $u \in \operatorname{span}(v_1, ..., v_m)$. Then there exist $c_1, ..., c_m \in \mathbb{F}$ such that

$$u = c_1 v_1 + \cdots + c_m v_m.$$

In the equation above, we can replace v_k with the right side of the equation two above, which shows that u is in the span of the list obtained by removing the kth term from $v_1, ..., v_m$. Thus removing the kth term of the list $v_1, ..., v_m$ does not change the span of the list. ■

3. Linear Maps

4. Polynomials