Topology Notes

Contents

1. Topological Spaces and Continuity

Definition (topology): A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X that satsify the following properties:

a) $\emptyset$ and X are in $\mathcal{T}$.
b) The union of an arbitrary collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.
c) The union of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

Then the pair $(X, \mathcal{T})$ is a *topological space*, sometimes just denoted X if $\mathcal{T}$ is clear from context. The sets in $\mathcal{T}$ are called the *open sets* of X.

***Example** (standard topology)*: The standard topology $\mathcal{T}$ on $\mathbb{R}^n$ is given by

$$\{U \subseteq \mathbb{R}^n : \text{for all } x \in U, \text{there exists } \varepsilon > 0 \text{ such that } B_\varepsilon(x) \subseteq U\}.$$

***Example** ((in)discrete topology)*: Two obvious topologies for any set X are given by the discrete topology $\mathcal{P}(X)$ (every subset of X) and the indiscrete topology $\{\emptyset, X\}$.

***Example** (particular point topology)*: Fix $p \in X$. Then

$$\{U \subseteq X : p \in U\}$$

is a topology on X.

***Example** (co-finite topology)*: Let X be any nonempty set. Then the co-finite topology is given by

$$\mathcal{T} = \{U \subseteq X : X \backslash U \text{ is fininte}\} \cup \{\emptyset\}.$$

Clearly a) is satisfied, since $|X \backslash X| = 0$. Suppose $\{V_\alpha\}_{\alpha \in J}$ is a collection of sets in the $\mathcal{T}$, indexed by some jet J. Then

$$X \backslash \bigcup_{\alpha \in J} V_\alpha \subseteq X \backslash V_{\alpha'},$$

where $\alpha' \in J$ is arbitrary. Since the right side is finite, so is the left, so $\mathcal{T}$ is closed under arbitrary unions, and so b) is satisfied. Now suppose $V_1, \ldots, V_n$ are a collection of sets in $\mathcal{T}$. Then

$$X \backslash \bigcap_{i=1}^{n} V_i = \bigcup_{i=1}^{n} X \backslash V_i.$$

The right side if the union of finite sides, so it's finite, and thus the left is as well. Thus $\mathcal{T}$ is closed under finite intersection, so c) is satisfied. Thus $\mathcal{T}$ is indeed a topology.

***Example** (co-countable topology)*: Let X be a nonemtpy set. Then the co-countable topology is given by

$$\mathcal{T} = \{U \subseteq X : X \backslash U \text{ is countable}\} \cup \{\emptyset\}.$$

Remark: To show a topology is closed under finite intersection, we only have to show that it's closed under pairwise intersection, after which finite intersection follows by induction.

Definition (finer/coarser): Let X be a set, and suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X. Then $\mathcal{T}_1$ is *refined* by $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$. We also say that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$, or that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$.

Example: On $\mathbb{R}^n$, the discrete topology refines the usual topology, which refines the indiscrete topology.

Example: Topologies are not necessarily comparable, for example the topology that contains all subsets that contained a fixed point x is neither of a subset of nor contains the usual topology on $\mathbb{R}^n$.

1.1. Topological Bases

Given a collection of sets $\mathcal{A}$, we use the shorthand $\bigcup \mathcal{A}$ to denote $\bigcup_{U \in \mathcal{A}} U$.

Definition (basis): If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X that satisfies the following properties:

a) For each $x \in X$, at least one basis element contains B contains x (this is equivalent to $\bigcup \mathcal{B} = X$).
b) If $x \in B_1 \cap B_2$ of two basis elements B_1, B_2, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Basis elements are called *basis open sets.*

Proposition (topology generated by basis): Suppose $\mathcal{B}$ is a basis of X. Let

$$\mathcal{T} := \{U \subseteq X : \text{for all } x \in U, \text{there exists } B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}.$$

Then $\mathcal{T}$ is a topology, called the *topology generated by* $\mathcal{B}$, and is denoted $\mathcal{T}_{\mathcal{B}}$.

Proof: Clearly $\emptyset, X \in \mathcal{T}$, since the condition on the collection holds vacuously for $\emptyset$, and the first condition of a basis implies that X also satisfies the condition on the collection.

Now suppose $\{V_\alpha\} \subseteq \mathcal{T}$. Let $x \in \bigcup_\alpha V_\alpha$. Then $x \in V_{\alpha'}$ for some α. Since $V_{\alpha'} \in \mathcal{T}$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq V_{\alpha'} \subseteq \bigcup_\alpha V_\alpha$. This holds for all x in the union, so $\bigcup_\alpha V_\alpha \in \mathcal{T}$, so $\mathcal{T}$ is closed under arbitrary union.

Now suppose $V_1, V_2 \in \mathcal{T}$, and let $x \in V_1 \cap V_2$. Then $x_1 \in V_1, V_2$, and so by the collection condition, there exists $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq V_1, x \in B_2 \subseteq V_2$. This implies $x \in B_1 \cap B_2 \subseteq V_1 \cap V_2$. Then by the second condition for a basis, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap$

$B_2 \subseteq V_1 \cap V_2$. Since this holds for all x in the intersection, $V_1 \cap V_2 \in \mathcal{T}$, and so $\mathcal{T}$ is closed under pairwise intersection, and thus finite intersection. Thus $\mathcal{T}$ is a topology, as desired. ■

Proposition (equivalent way to generate topology from basis): Suppose $\mathcal{B}$ is a basis of X. Then $\mathcal{T}_{\mathcal{B}}$ is given by

$$\mathcal{T}' = \left\{\bigcup \mathcal{C} : \mathcal{C} \subseteq \mathcal{B}\right\}.$$

Proof: Suppose $V \in \mathcal{T}'$. Then $V = \bigcup_\alpha B_\alpha$ for $\{B_\alpha\} \subseteq \mathcal{B}$. Let $x \in V$. Then $x \in B_{\alpha'}$, for some α', and thus $x \in B_{\alpha'} \subseteq V$. This holds for all $x \in V$, so $V \in \mathcal{T}_{\mathcal{B}}$. Thus $\mathcal{T}' \subseteq \mathcal{T}_{\mathcal{B}}$. Now suppose $V \in \mathcal{T}_{\mathcal{B}}$. By definition, for each $x \in V$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq V$. Thus $\bigcup_{x \in V} B_x = V$, so $V \in \mathcal{T}'$, and thus $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}'$. Since we have both inclusions, the collections are equal, as desired. ■

Example *(discrete topology basis)*: For X, the basis $\mathcal{B} = \{\{x\} : x \in X\}$ generates the discrete topology. This is because the union of any collection of elements in $\mathcal{B}$ gives a subset of X, which is in the discrete topology, and any subset can be written as the union of its individual elements.

Example *(standard topology basis)*: The set of open intervals in $\mathbb{R}$ forms a basis for the standard topology.

Example: $\mathcal{T}_{\mathcal{T}} = \mathcal{T}$.

Example *(lower limit topology)*: Let $\mathcal{B} = \{[a, b) \subseteq \mathbb{R} : a < b\}$. Then $\mathcal{B}$ is a basis of $\mathbb{R}$, and generates a topology $\mathcal{T}$ known as the lower limit topology. The topological space $(\mathbb{R}, \mathcal{T})$ is often called the *Sorgenfrey line*.

Proposition (basis from topology): Suppose X is a topological space, and let $\mathcal{C}$ be a collection of open sets of X such that for each open $U \subseteq X$ and for each $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof: First we show that $\mathcal{C}$ is a basis for X. Clearly $\mathcal{C}$ covers X, since for $x \in X \subseteq X$, the condition implies there exists $C \in \mathcal{C}$ that contains x. Now suppose $C_1, C_2 \in \mathcal{C}$, and let $x \in C_1 \cap C_2$. Since these two are open, there intersection is open, so by the condition, there exists an open $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Thus the second condition for a basis is satisfied.

Now we show that this basis actually generates the topology on X. Suppose V is open in X. Then for each $x \in V$, there exists $C_x \in \mathcal{C}$ that contains x, and thus we can write $V = \bigcup_{x \in V} C_x$, so $V \in \mathcal{T}_{\mathcal{C}}$. Now suppose V is the union of a subcollection of $\mathcal{C}$. Since they're all open sets, their union is open in X, and so the open sets generated by $\mathcal{T}_{\mathcal{C}}$ are open in X. Thus the topologies are the same. ■

Proposition (comparing bases): Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the following are equivalent:

a) $\mathcal{T}'$ is finer than $\mathcal{T}$.
b) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there exists $B' \in \mathcal{T}'$ such that $x \in B' \subseteq B$.

Proof: Suppose $\mathcal{T}' \supseteq \mathcal{T}$, $x \in X$, and $B \in \mathcal{B}$ contains x. Then B will be in $\mathcal{T}$, since it's a basis element for it, and thus $B \in \mathcal{T}'$. Thus we can take $B' = B$. Since this holds for arbitrary x and B, a) implies b).

Now suppose b) is true, and let $V \in \mathcal{T}$. Then we can write $V = \bigcup_\alpha B_\alpha$ for $\{B_\alpha\} \subseteq \mathcal{T}$. Note that we can write $B_\alpha = \bigcup_{\gamma_\alpha} B'_{\gamma_\alpha}$ for $\left\{B'_{\gamma_\alpha}\right\} \subseteq \mathcal{B}'$ by condition b), so we have $V = \bigcup_\alpha \left(\bigcup_{\gamma_\alpha} B'_{\gamma_\alpha}\right)$. Since this is the union of a collection of basis elements in $\mathcal{B}'$, V is an open set in $\mathcal{T}'$. Thus $\mathcal{T} \subseteq \mathcal{T}'$, as desired. ■

Definition (subbasis): A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

Proposition: Let $\mathcal{B}$ be the collection of all finite intersections of elements in a subbasis $\mathcal{S}$ for a topology on X, or more explicity

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S} \right\}.$$

Then $\mathcal{B}$ is a basis for a topology on X.

Proof: Since $\mathcal{S}$ covers X, clearly so does $\mathcal{B}$. Now suppose $B_1 = S_1 \cap \dots \cap S_m$ and $B_2 = S'_1 \cap \dots \cap S'_n$ are elements in $\mathcal{B}$, and suppose $x \in B_1 \cap B_2$. Then $x \in S_i, S'_j$ for some $i \leq m, j \leq n$. In particular, $x \in S_i \cap S'_j \subseteq B_1 \cap B_2$, and so $\mathcal{B}$ satisfies the second basis condition. ■

Example: Subasses are basically a way to express bases. For example, the subbasis $\{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$ generates the basis of open intervals in $\mathbb{R}$, which is a basis for the standard topology on $\mathbb{R}$.

1.2. Product Topology on $X \times Y$

Definition (product topology): Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V : U \text{ and } V \text{ are open in } X \text{ and } Y \text{ respectively}\}.$$

It's easy to check that $\mathcal{B}$ is indeed a basis.

Example: Suppose $\mathbb{R}$ is equipped with the standard topology. Then the $\mathbb{R} \times \mathbb{R}$ product topology is generated by a basis of what are essentially unions of rectangles in the plane.

Proposition (product of bases): If $\mathcal{B}_1$ is a basis on X and $\mathcal{B}_2$ is a basis on Y, then

$$\mathcal{B}' = \{B_1 \times B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$$

is a basis for the product topology of $X \times Y$.

Proof: Clearly $\mathcal{B}'$ covers $X \times Y$, since $\mathcal{B}_1$ covers X and $\mathcal{B}_2$ covers Y, so we can take the product of all the covering sets. Similarly the second condition for a basis follows by applying it to each individual basis. We note this basis generates the same topology as the product topology since we can first form any open $U \in X$ by forming it using sets in $\mathcal{B}_1$ times some set $B_2 \subseteq V \subseteq Y$, then do the same thing for $\mathcal{B}_2$. This implies that $\mathcal{B} \subseteq \mathcal{T}_{\mathcal{B}'}$, and so will generate the product topology. ∎

Definition (projections): Let $\pi_1 : X \times Y \to X$ be defined by

$$\pi_1(x, y) := x$$

and $\pi_2 : X \times Y \to Y$ be defined by

$$\pi_2(x, y) := y.$$

Proposition: The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof: Note that $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$. Suppose $A \in X$ is open and $B \in Y$ is open. Then $\pi_1^{-1}(A) \cap \pi_2^{-1}(B) = A \times B$, so the set of finite intersections of elements in $\mathcal{S}$ contain the basis for the product topology on $X \times Y$. Clearly any set of finite intersections of sets from $\mathcal{S}$ will also product $U \times V$ for open U and V in X and Y respectively, so $\mathcal{S}$ does generate the basis for the product topology, as desired. ∎

1.3. Subspace Topology

Definition (subspace topology): Let $(X, \mathcal{T})$ be a topological space. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is the *subspace topology* on Y.

Again it's easy to check that this is a toplogy.

Example: Let $Y = (0, 1) \cup [2, 3]$ and suppose $\mathbb{R}$ is equipped with the standard topology. Then the subspace topology induced on Y is just the open sets of $\mathbb{R}$ intersected with Y. Thus $Y \cap (0, 1) = (0, 1)$ is open in Y, and $Y \cap (1, 4) = [2, 3]$ is also open in Y.

Proposition (basis of subspace topology): If $\mathcal{B}$ is a basis for the topology of X, then

$$\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$$

is a basis for the subspace topology on a subset $Y \subseteq X$

Proof: It's easy to see that this is a basis. Then note that since the union of a collection of sets in $\mathcal{B}$ can make any open $U \in X$, the union of those same sets intersected with Y can make $Y \cap U$. Thus this basis generates the subspace topology on Y, as desired. ■

Proposition (open sets relative to space and subspace): Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof: Since U is open in Y, it can be written as $V \cap Y$, where V is open in X. Since Y is open in X, the intersection of these two sets is open in X, and thus U is open in X. ■

Proposition (subspace of product): If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof: The set $U \times V$ is a basis element in $X \times Y$, for open $U \in X, V \in Y$. Thus $(A \times B) \cap (U \times V)$ is a basis element of the subspace topology on $A \times B$. Then we have $(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$, and since $A \cap U, B \cap V$ are open sets for the subspace topologies on A and B, their product is a basis element for the product topology on $A \times B$. Thus the two have the same bases, and thus generate the same topology on $A \times B$. ■

1.4. Closed Sets and Limit Points

1.4.1. Closed Sets

Definition (closed): A subset A of a topological space X is said to be *closed* if $X \backslash A$ is open.

Proposition: In a topological space X,
a) $\emptyset$ and X are closed,
b) Arbitrary intersections of closed sets are closed,
c) Finite unions of closed sets are closed.

Proof: Follows from open set rules and DeMorgan's laws. ∎

Proposition: Let Y be subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof: Suppose A is closed in Y. Then $Y \backslash A$ is open in Y. Thus $Y \backslash A = Y \cap V$, where $V \subseteq X$ is open. Then $X \backslash V$ is closed in X, and $A = Y \cap (X \backslash V)$, as desired.

Now suppose $A = Y \cap C$, where C is closed in X. Then $X \backslash C$ is open in X, so $Y \cap X \backslash C = Y \backslash A$ is open in Y by definition of the subspace toplogy. Thus A is closed in Y. ∎

Proposition (closed sets relative to space and subspace): Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Proof: We have that $Y \backslash A$ is open in Y, and thus can be written as $Y \cap V$ for some open $V \subseteq X$. Since $X \backslash Y$ is open in X, the set $X \backslash Y \cup V = Y \backslash A$ is open in X. Thus A is closed in Y, as desired. ∎

1.4.2. Closure and Interior of a Set

Definition (interior): Given a subset A of a topological space X, the *interior* of A is defined as the union of all open sets contained in A, and is denote $\mathrm{int}(A)$.

Definition (closure): Given a subset A of a topological space X, the *closure* of A is defined as the intersection of all closed sets containing A, and is denoted $\overline{A}$.

Proposition: Let Y be a subspace of X, let A be a subset of Y, let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.

Proof: By definition, we have

$$\overline{A} = \bigcap \{C \subseteq X : C \text{ is closed in } X\}.$$

Letting A' denote the closure of A in Y, we again have by definition

$$A' = \bigcap \{C' \subseteq Y : C' \text{ is closed in } Y\}.$$

Since closed sets in Y can be written as the intersection of Y with a closed set in X, we can write the above as

$$A' = \bigcap \{C \cap Y : C \text{ is closed in } X\} = Y \cap \left(\bigcap \{C \subseteq X : C \text{ is closed in } X\}\right) = Y \cap \overline{A},$$

where the first equality holds since closed sets in X give rise to closed sets in Y by intersecting them with Y. ∎

Proposition (closure from open sets and basis): Let A be a subset of the topological space X.

a) Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.
b) If $\mathcal{B}$ is a basis for the topology on X, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Proof:

a) Suppose $x \in \overline{A}$. If $x \in A$, then any open set containing x trivially intersects A at x, so suppose $x \notin A$. If U is an open set containing x that doesn't intersect A, then $X \backslash U$ is a closed set not containing x that has A as a subset. But then by the definition of closure, that would imply that $x \notin \overline{A}$, contradiction. Thus every open set containing x intersects A.

   Now suppose every open set contaning x intersects A, and suppose $x \notin \overline{A}$. Thus there exists some closed set C not containing x that contains A. Then $X \backslash C$ is an open set containing x that doesn't intersect A, contradiction.

b) This follows from part a). Namely, the if direction follows since basis elements are open, and the only if direction follows since the basis elements can be unioned to make any open set.

∎

Definition (topological neighborhood): If U is an open set of a topological space X containing x, then U is a *neighborhood* of x.

Corollary (restatement of previous proposition): $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

Example: Conside the subspace $Y = (0, 1]$ of $\mathbb{R}$, and the subset $\left(0, \frac{1}{2}\right)$. Then its closure in $\mathbb{R}$ is $\left[0, \frac{1}{2}\right]$, and its closure in Y is $\left(0, \frac{1}{2}\right]$.

Example: Consider $A = (1, 2)$ in the lower limit topology. Note that the open set $[2, 3)$ contains 2 but doesn't intersect $(1, 2)$, so $2 \notin \overline{A}$. However, every open set in the basis of the topology (the interval $[a, b)$) that contains 1 clearly intersects A, so $1 \in \overline{A}$. Thus $\overline{A} = [1, 2)$.

Definition (dense): A subset D of a topological space X is *dense* in X if $\overline{D} = X$.

Proposition: If $(X, \mathcal{T})$ is a topological space and $D \subseteq X$, show that D is dense in X if and only if every nonemtpy open set of X intersects D.

Proof: Suppose D is dense, and let $x \in X$ be arbitrary. Since $x \in \overline{D} = X$, every neighborhood of x intersects D. Since every nonempty open set is a neighborhood of some point x, every nonempty open set intersects D, as desired.

Now suppose every nonempty open set intersects D, and fix $x \in X$. Then every neighborhood of x intersects D, and so $x \in \overline{D}$. This holds for all x, so $\overline{D} = X$. ■

Definition (boundary): Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then the *boundary* of A, denote ∂A, is the set

$$\partial A = \{x \in X : \text{every neighborhood of } x \text{ intersects } A \text{ and } X \setminus A\}.$$

Proposition: Let A be a subset of a topological space X. Then

$$\partial A = \overline{A} \cap \overline{X \setminus A} = \overline{A} \setminus \operatorname{int}(A).$$

Proof: Suppose $x \in \partial(A)$. Then every neighborhood of x intersects A, so by the previous corollary, $x \in \overline{A}$. Similar logic yields $x \in \overline{X \setminus A}$, and so $\partial A \subseteq \overline{A} \cap \overline{X \setminus A}$. If x is in the intersection of the closure and the complements closure, then again by the corollary every open neighborhood must intersect A and $X \setminus A$, so by definition $x \in \partial A$. Thus $\partial A \supseteq \overline{A} \cap \overline{X \setminus A}$. Thus the sets are equal.

Suppose $x \in \partial A$. Then it every neighborhood intersects A, and so $x \in \overline{A}$. Note also that $x \notin \operatorname{int}(A)$, since otherwise there would exist a neighborhood of x contained in A, and so wouldn't intersect $X \setminus A$, thus not being in the boundary. Thus $\partial A \subseteq \overline{A} \setminus \operatorname{int}(A)$. Reversing the reasoning implies the reverse inclusion, so the sets are equal. ■

Proposition: Let A be a subset of a topological space X. Then $\overline{A} = A \cup \partial A$ and $\text{int}(A) = A \backslash \partial A$.

Proof: Suppose $x \in \overline{A}$. If it's not in A, then every neighborhood must intersect both A (as otherwise it wouldn't be in the closure) and $X \backslash A$ (namely at x), and thus $x \in \partial A$, implying $\overline{A} \subseteq A \cup \partial A$. Now suppose $x \in A \cup \partial A$. If $x \in A$, then $x \in \overline{A}$. If $x \in \partial A$, then by definition every neighborhood of x intersects A, so $x \in \overline{A}$. This $A \cup \partial A \subseteq \overline{A}$.

Suppose $x \in \text{int}(A)$. Then there exists a neighborhood of x contained in A. Since this neighborhood doesn't intersect $X \backslash A$, $x \notin \partial A$. Thus $x \in A \backslash \partial A$, so $\text{int}(A) \subseteq A \backslash \partial A$. Now suppose $x \in A$ by $x \notin \partial A$. Then some neighborhood of x does not intersect $X \backslash A$, and thus will be contained in A, implying $x \in \text{int}(A)$. Thus $\text{int}(A) \supseteq A \backslash \partial A$. ■

Proposition: Let A be a subset of a topological space X. Then $X = \text{int}(A) \sqcup \partial A \sqcup \text{int}(X \backslash A)$, where $\sqcup$ means a disjoint union.

Proof: Clearly subsets of A and $X \backslash A$ will be disjoint, so the interiors of both are disjoint. Any $x \in \partial A$ will also be disjoint from both by definition. If $x \in X$, then it has to be in one of these, as either some neighborhood lies in A, some neighborhood lies in $X \backslash A$, or all neighborhoods intersect both. ■

1.4.3. Limit Points

Definition (limit point): If A is a subset of a topological space X, and if $x \in X$, then x is a *limit point* of A if every neighborhood of x intersects A at a point other than x.

Remark: This is equivalent to saying that $x \in \overline{A \backslash \{x\}}$.

Example: Consider $A = (0, 1] \cup \{2\}$. Then $[0, 1]$ is the set of limit points.

Proposition: Let A be a subset of a topological space X, and let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.

Proof: Suppose $x \in \overline{A}$. If $x \in A$, then clearly $x \in A \cup A'$. If $x \notin A$, then every neighborhood of x must intersect A at a point other than x, since otherwise x wouldn't be in the closure. Thus $x \in A'$, and so $\overline{A} \subseteq A \cup A'$.

Now suppose $x \in A \cup A'$. Clearly $x \in A \Rightarrow x \in \overline{A}$, so suppose $x \in A'$. Then every neighborhood of of x intersects A by definition, so $x \in \overline{A}$, meaning $A \cup A' \subseteq \overline{A}$. We have the inclusion in both directions, so the sets are equal. ■

Corollary: A subset of a topological space is closed if and only if it contains all its limit points.

Proof: If A is closed, then $\overline{A} = A$, since it's the smallest closed set that contains A, and so must contain all its limit points. If A contains all its limit points, then by the previous proposition, $\overline{A} = A \cup A' = A$, and thus is closed. ■

1.4.4. Hausdorff Spaces

Hausdorff spaces are nice in the sense that in these spaces, points are topologically distinguishable because open sets can "separate" them. In particular, sequence convergence in Hausdorff spaces makes much more sense, as they can only converge to at most one point.

Definition (Hausdorff space): A topological space X is called a *Hausdorff space* (or said to be *Hausdorff*) if for each pair of distinct $x_1, x_2 \in X$, there exist neighborhoods of x_1 and x_2 that are disjoint.

Proposition: Every finite set in a Hausdorff space X is closed.

Proof: We show that $\{x\}$ is closed. The general claim follows by showing it for singletons and then taking finite unions. Suppose $y \neq x$. Then since X is Hausdorff, there exists a neighborhood of x that is disjoint from a neighborhood of y. Since the neighborhood of y clearly doesn't intersect $\{x\}$, y is not a limit point of $\{x\}$, and thus won't be contained in $\overline{\{x\}}$. This holds for all $y \neq x$, so $\overline{\{x\}} = \{x\}$. ■

Definition (topological convergence): Suppose $(x_n) \in X$ is a sequence in a topological space. Then x_n converges to $x \in X$ if for every neighborhood U of x, there exists a positive integer N such that $x_n \in U$ for all $n \geq N$.

Proposition: If X is Hausdorff, then a sequence converges to at most one point.

Proof: Suppose (x_n) converges to distinct points $a, b \in X$. Since X is Hausdorff, there exists a neighborhood A of a and B of b such that A and B are disjoint. By the definition of convergence, there exists N_a such that $x_n \in A$ for all $n \geq N_a$ and N_b such that $x_n \in B$ for all $n \geq N_b$. Letting $N = \max\{N_a, N_b\}$, we have that $x_n \in A, B$ for all $n \geq N$, but this is impossible since A and B are disjoint. ■

Proposition: The product of two Hausdorff spaces is Hausdorff, and the subspace of a Hausdorff space is Hausdorff.

Proof: Follows easily from definitions. ■

1.5. Continuous Functions

Definition (continuous): Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *continuous* of for each open subset V of Y, the set $f^{-1}(V)$ is open in X.

Definition (continuous at a point): Let X and Y be topological spaces, and let $x \in X$. If for each neighborhood V of $f(x)$ in Y there is a neighborhood U of x such that $f(U) \subseteq V$, then f is *continuous at* x.

It's clear that if the inverse image of basis elements is open, then f is also continuous, since open sets are the union of basis elements, and the inverse image distributes over unions.

Example: Let $\mathbb{R}_\ell$ denote $\mathbb{R}$ with the lower limit topology. Then the identity function $\mathrm{id} : \mathbb{R} \to \mathbb{R}_\ell$ is not continuous, since the inverse image of $[0, 1)$, which is open in $\mathbb{R}_\ell$, is not open in $\mathbb{R}$. However, $\mathrm{id} : \mathbb{R}_\ell \to \mathbb{R}$ is continuous.

Proposition: Let X and Y be topological spaces, and let $f : X \to Y$. Then the following are equivalent:
a) f is continuous.
b) For every subset A of X, we have $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.
c) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.
d) f is continuous at x for all $x \in X$.

Proof: First we show that a) $\Longleftrightarrow$ d). Suppose a) is true, and fix $x \in X$. Let V be any neighborhood of $f(x)$. Then by continuity, $f^{-1}(V)$ is open in X. Since clearly it will contian x, it's an open neighborhood of x, and by definition, $f(f^{-1}(V)) \subseteq V$. Thus f is continuous at x. Now suppose f is continuous at x for all $x \in X$, and let V be any open set in Y. If $f^{-1}(V) = \emptyset$, then clearly the pre-image is open in X. Otherwise, there exists $x \in X$ such that $f(x) \in V$. Then by continuity at a point, there exists a neighborhood U_x of x in X such that $f(U_x) \subseteq V$. Thus $U_x \subseteq f^{-1}(V)$. Then for all $x \in f^{-1}(V)$, we can union their neighborhoods together, and obtain an open set $U = f^{-1}(V)$. This holds for all open V in Y, so f is continuous.

First suppose a) is true, and suppose $y \in f\left(\overline{A}\right)$. Thus $f(x) = y$ for some $x \in \overline{A}$. Let V be a neighborhood of y. Then $f^{-1}(V)$ is a neighborhood of x. Thus it will intersect A (since x is in the closure of A), which implies that V intersects $f(A)$. Since this holds for any neighborhood of y, we have $y \in \overline{f(A)}$, and thus $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.

Now suppose b) is true. Suppose B is closed in Y, and let $A = f^{-1}(B)$. Then $f(A) = f(f^{-1}(B)) \subseteq B$. Thus, for $x \in \overline{A}$, we have

$$f(x) \in f\left(\overline{A}\right) \subseteq \overline{f(A)} \subseteq \overline{B} = B.$$

Thus $x \in f^{-1}(B)$, so $\overline{A} \subseteq A \Rightarrow \overline{A} = A$. Thus A is closed in X.

Now suppose c) is true. If V is open in Y, then $Y \backslash V$ is closed in Y. Then $f^{-1}(Y \backslash V) = f^{-1}(Y) \backslash f^{-1}(V) = X \backslash f^{-1}(V)$ is closed in X, which implies that $f^{-1}(V)$ is open. Thus f is continuous.■

Proposition (continuity preserves convergence): Let $f : X \to Y$ be continuous and suppose $(x_n) \in X$ converges to X. Then $(f(x_n)) \in Y$ converges to $f(x)$.

Proof: Let V be an open set in Y containing $f(x)$. Then the inverse image is an open set containing x, and thus by convergence, contains $(x_n)_{n \geq N}$ for some N. Thus V contains $(f(x_n))_{n \geq N}$. Since V is arbitrary, we do indeed have that $f(x_n) \to f(x)$. ■

Proposition (continuous functions): Let X, Y, Z be topological spaces.

a) If $f : X \to Y$ is constant, then f is continuous.
b) If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.
c) If $f : X \to Y$ and $g : Y \to Z$ are continuouos, then $g \circ f : X \to Z$ is continuous.
d) If $f : X \to Y$ is continuous, and if A is a subspace of X, then $f|_A : A \to Y$ is continuous.
e) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range is continuous. If Z is a space have Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.
f) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.

Proof:
a) If an open $V \subseteq Y$ contains the constant, then $f^{-1}(V) = X$ is open, otherwise $f^{-1}(V) = \emptyset$ is open, so f is continuous.
b) If $V \subseteq X$ is open, then $f^{-1}(V) = A \cap V$, which is open in A by definition.
c) If $V \subseteq Z$ is open, then $g^{-1}(V)$ is open in Y, so $f^{-1}(g^{-1})(V) = (g \circ f)^{-1}(V)$ is open in X.
d) Follows from using b) and c).
e) Let $V \subseteq Z$ be open. Then it can be written as $U \cap Z$ for open U in Y, so $f^{-1}(U) \cap f^{-1}(Z) = f^{-1}(U) \cap X = f^{-1}(U)$ is open in X. Then since $f^{-1}(U) = g^{-1}(V)$ (which follows easily by checking inclusions), we see that h is continuous. To show that after expanding the range a map is still continuous, apply b) and c) using the inclusion map as the outer function.
f) Let V be open in Y. Then $f^{-1}(V) \cap U_\alpha = \left(f|_{U_\alpha}\right)^{-1}(V)$, which is open in U_α by continuity. Since U_α is open, the set is open in X as well. Then since $f^{-1}(V) = \bigcup_\alpha (f^{-1}(V) \cup U_\alpha)$, $f^{-1}(V)$ is open as well.

■

Lemma (pasting lemma): Let $X = A \cup B$, where A, B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.

Proof: Suppose C is closed in Y. Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. By the equivalent formulations of continuity, $f^{-1}(C)$ is closed in A, and thus closed in X. Similarly, $g^{-1}(C)$ is closed in X, so $h^{-1}(C)$ is closed in X. Thus h is continuous. $\blacksquare$

Remark: The lemma also works if A and B are open.

Proposition (maps into products): Let $f : A \to X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous if and only if $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.

Proof: Suppose f is continuous, and let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps. Note that $\pi_1^{-1}(U) = U \times Y$ is open if U is open, and similarly for π_2. Thus they're both continuous, so the compositions $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$ are also continuous.

Now suppose each component function is continuous, and suppose W is open in $X \times Y$. Then we can write it as $\bigcup_\alpha U_\alpha \times V_\alpha$ with U_α, V_α open in X, Y. Then note that $f^{-1}(W) = \bigcup_\alpha f^{-1}(U_\alpha \times V_\alpha)$. If we can show that each of these is open in A, we're done.

By continuity, $f_1^{-1}(U_\alpha) \cup f_2^{-1}(V_\alpha)$ is open. Note that if $x \in f^{-1}(U_\alpha \times V_\alpha)$, then $f_1(x) \in U_\alpha, f_2(x) \in V_\alpha$, so x is also in the union. If f is in the union, then by the reverse logic $x \in f^{-1}(U_\alpha \times V_\alpha)$. Thus $f_1^{-1}(U_\alpha) \cup f_2^{-1}(V_\alpha) = f^{-1}(U_\alpha \times V_\alpha)$, so we're done. $\blacksquare$

Definition (homemorphism): Let X and Y be topological spaces, and suppose $f : X \to Y$ is a bijection. If both f and the inverse f^{-1} of f are continuous, then f is called a *homeomorphism*, and the spaces X and Y are said to be *homeomorphic*.

Homeomorphisms essentially give a bijection between the open sets of topological spaces, so if something is true for open sets in one space, then it will be true for the open sets in the other space. Such properties are called *topological properties*, as they're preserved under homeomorphisms.

1.6. More Important Topologies

Here we discuss a few more important topologies.

1.6.1. Product Topology

Definition (J-tuple): Let J be an index set. Given a set X, we define a J-tuple of elements of X to be a function $\boldsymbol{x} : J \to X$. If α is an element of J, we often denote $\boldsymbol{x}(\alpha)$ by x_α (i.e. the coordinate of $\boldsymbol{x}$). We denote the the entire with

$$(x_\alpha)_{\alpha \in J},$$

or just (x_α) is J is understood.

This is just a generalization of coordinate notation where the index set can be uncountable.

Definition (projection mapping): Define the function $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ by

$$\pi_\beta\left((x_\alpha)_{\alpha \in J}\right) = x_\beta.$$

This is the *projection mapping* associated with index β.

This is also just a generalization of the regular projection maps.

Definition (box topology): Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. For the cartesian product $\prod_{\alpha \in J} X_\alpha$, let

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha : U_\alpha \text{ is open in } X_\alpha \right\}$$

be a basis. Then the topology generated by this basis is the *box topology*.

Definition (product topology): Let β denote the collection

$$\mathcal{S}_\beta = \{\pi^{-1}(U_\beta) : U_\beta \text{ is open in } X_\beta\},$$

and let

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta.$$

The topology generated by the subbasis $\mathcal{S}$ is called the *product topology*.

Note that a basis element generated by the subbasis above is given by

$$B = \bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}\left(U_{\beta_i}\right),$$

where U_{β_i} are open in X_{β_i}. If the product of the sets is a finite product, then the two bases are equal, so the box topology is equal to the product topology. However, if the product is infinite, then the basis elements of the product topology are a subset of the basis elements of the box topology. This is because a basis element in the product topology looks like a product of finitely many open sets, while the rest of the coordinates are the full spaces X_α. Basis elements in the box topology can have arbitrarily many coordinates be proper subsets of the spaces.

Proposition: Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in J} B_\alpha,$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, is a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many α and $B_\alpha = X_\alpha$ for all remaining indices, is a basis for the product topology.

Proof: The first part follows since we can make an open set $U_\alpha \subseteq X_\alpha$ using $\mathcal{B}_\alpha$, take the product over all such open sets and get the basis for the box topology. The second one is just the basis generated by the subbasis in the definition of the product topology. ■

Proposition: Let A_α be a subspace of X_α for all $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given either the box topology or the product topology.

Proof: This follows easily, since every open set can be written as $A_\alpha \cap U_\alpha$, where U_α is open in X_α, and then taking the product yields that every every open set in $\prod A_\alpha$ can be written as $\prod A_\alpha \cap \prod U_\alpha$. ■

Proposition: If each space X_α is Hausdorff, then $\prod X_\alpha$ is Hausdorff in both the box and product topologies.

Proof: First we look at the box topology. Suppose $(x_\alpha), (y_\alpha) \in \prod X_\alpha$ are distinct. Since each X_α is Haussdorff, there are open sets $U_\alpha, V_\alpha \subseteq X_\alpha$ such that $x_\alpha \in U_\alpha, y_\alpha \in V_\alpha$, and U_α and V_α are disjoint for all α. Then $(x_\alpha) \in \prod U_\alpha, (y_\alpha) \in \prod V_\alpha$, and the two products are disjoint, so the product space is Hausdorff.

For the product topology, we only need to pick disjoint open sets for some coordinate, and we can take X_α for the rest. The product of the sets will still be disjoint, and indeed the product will be open, since only one coordinate in each is not the entire space. ■

Proposition: Let $\{X_\alpha\}$ be an indexed fmaily of spaces, and let $A_\alpha \subseteq X_\alpha$ for each α. If $\prod X_\alpha$ has either the product or the box topology, then

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}.$$

Proof: Suppose (x_α) is a point in $\prod \overline{A_\alpha}$, and let $U = \prod U_\alpha$ be a basis element in either topology that contains (x_α). Since $x_\alpha \in \overline{A_\alpha}$, the set $U_\alpha \cap A_\alpha$ is nonemtpy, so some point y_α lies in it. Then $(y_\alpha) \in U \cap \prod A_\alpha$. Since U is arbitrary, we have that $(x_\alpha) \in \overline{\prod A_\alpha}$.

Now suppose $(x_\alpha) \in \overline{\prod A_\alpha}$. Pick some index β, and let V_β be an arbitrary open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, and since it contains (x_α), the intersection of it and $\prod A_a$ will contain some point (y_α). Then $y_\beta \in V_\beta$, so $V_\beta \cap \prod A_\beta \neq \emptyset$, so $x_\beta \in \overline{A_\beta}$. ■

So far, the topologies have acted the same (and as we already noted, are the same for finite products), but the next result shows that we often prefer the product topology, as it does not hold for the box topology.

Proposition: Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof: Note that the projection map from the product to X_β is continuous. Then if f is continuous, $\pi_\beta \circ f = f_\beta$ is continuous.

Now suppose each coordinate function is continuous. We show that each subbasis element has an open inverse image, from which the result follows by intersecting and union to get the full topology. A subbasis element is $\pi^{-1}(V_\beta)$, where V_β is open in X_β. Then $f^{-1}(\pi^{-1}(V_\beta)) = f_\beta^{-1}(V_\beta)$, which is open by the continuity of the coordinate functions, so we're done. ■

The reason this doesn't work in the box topology is that open sets where infinitely many coordinates are not the full space can't be made via finite intersections.

Example *(failure of the box topology)*: Consider the function $f : \mathbb{R} \to \mathbb{R}^\omega$ given by $f(t) = (t, t, t, ...)$. Then each coordinate function $f_n(t) = t$ is continuous. However, consider in $\mathbb{R}^\omega$ the open set $(-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots$. The inverse image of this set is $\{0\}$, which is not open in $\mathbb{R}$.

From here on out, when we use product spaces, they will be given the product topology unless otherwise stated.

1.6.2. Metric Topology (NOT COMPLETE)

Definition (metric): A *metric* on a set X is a function

$$d : X \times X \to \mathbb{R}$$

having the following properties:
a) $d(x, y) \geq 0$ for all $x, y \in X$, and equality holds if and only if $x = y$.
b) $d(x, y) = d(y, x)$ for all $x, y \in X$.
c) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Definition (metric topology): If d is a metric on the set X, then the collection of all ε-balls $B_d(x, \varepsilon)$ (or denoted $B_\varepsilon(x)$ when the metric is understood) for all $x \in X$ and $\varepsilon > 0$, is a basis for the *metric topology* on X induced by d.

The first basis condition is obvious. For the second, let $B_1 = B_{\varepsilon_1}(x_1)$, $B_2 = B_{\varepsilon_2}(x_2)$, and suppose $x \in B_1 \cap B_2$. Then $B_{\min(d(x,x_1),d(x,x_2))}(x)$ is contained in the intersection, which can be checked with the triangle inequality.

Proposition: A set U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_\delta(y, \delta) \subseteq U$.

Proof: One direction follows from taking union for each ball around each point in U. The other direction follows from the triangle inequality. ■

Definition (metrizable): If X is a topological space, X is said to be *metrizable* if there exists a metric d on the set X that induces the topology of X. A *metric space* is a metrizable space X together with a specific metric d that gives the topology of X.

Example: On a set X, define $d(x, y) = 1$ if $x \neq y$, and $d(x, y) = 0$ if $x = y$. It's easy to check this is a metric. Then this metric induces the discrete metric, as $B_{\frac{1}{2}}(x) = \{x\}$ for all $x \in X$, and then taking unions gives you every subset of X.

Example: Let $d(x, y) = |x - y|$ on $\mathbb{R}$. This is the standard metric, and induces the standard topology on $\mathbb{R}$.

Proposition: Let d and d' be two metric on the set X, and let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon).$$

Proof: Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Since $B_d(x, \varepsilon) \in \mathcal{T}$, it will also be in $\mathcal{T}'$. Then the previous proposition gives us the desired conclusion. Now suppose the condition holds. For a given basis element B of $\mathcal{T}$, we can find a ball $B_d(x, \varepsilon)$ within it. Then from the condition, there exists δ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$. Then the lemma about comparing bases applies. ■

1.6.3. Quotient Topology (DO WHEN LEARNING ALG TOP)

1.7. Interesting Problems

Problem: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be a set with the property that for every $x \in A$, there is an open set $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq A$. Show that A is open.

Solution: We claim that $\bigcup_{x \in A} U_x = A$. Clearly if y is in the left, then it's in the right, and if y is in the right, then it will be contained in U_y, and so will be in the left. Thus A is the union of open sets, and thus is also open.

Problem: Let $\{\mathcal{T}_\alpha : \alpha \in I\}$ be a collection of topologies on X. Prove that there is a unique finest topology that is refined by all the $\mathcal{T}_\alpha$'s.

Solution: Let $\mathcal{T} = \bigcap_{\alpha \in I} \mathcal{T}_\alpha$. If $\mathcal{T}$ is a topology, that it's clear that $\mathcal{T} \subseteq \mathcal{T}_\alpha$ for all $\alpha \in I$, and so is refined by all of those topologies. Further, if $\mathcal{T}'$ is refined by those same topologies, then $V \in \mathcal{T}' \Rightarrow V \in \mathcal{T}_\alpha \forall \alpha \in I \Rightarrow V \in \mathcal{T} \Rightarrow \mathcal{T}' \subseteq \mathcal{T}$, and so $\mathcal{T}$ is the finest possible topology.

Clearly $\emptyset, X \in \mathcal{T}$. Suppose $\{U_\beta\}_{\beta \in J}$ is a collection of sets in $\mathcal{T}$. Then by definition, $\{U_\beta\} \subseteq T_\alpha$ for all $\alpha \in I$, and thus $\bigcup_{\beta \in J} U_\beta \in \mathcal{T}_\alpha$ for all $\alpha \in I$. Thus $\bigcup_{\beta \in J} U_\beta \in \mathcal{T}$, so $\mathcal{T}$ is closed under arbitrary unions. We do the same thing to get that $\mathcal{T}$ is clsoed under finite intersections, so it is indeed a topology.

Problem: Let $X = [0,1]^{[0,1]}$ be the set of all functions $f : [0,1] \to [0,1]$. Given $A \subseteq [0,1]$, let

$$U_A := \{f \in X : f(x) = 0 \text{ for all } x \in A\}.$$

Show that $\mathcal{B} := \{U_A : A \subseteq [0,1]\}$ is a basis for a topology on X.

Solution: Clearly $\mathcal{B}$ covers X, since a function with zero set Z will be contained in U_Z. Now suppose $U_A, U_B \in \mathcal{B}$ and $f \in U_A \cap U_B$. Thus $f(x) = 0$ for all $x \in A \cap B$, which means $f \in U_{A \cap B} \subseteq U_A \cap U_B$, so $\mathcal{B}$ satisfies the second basis condition.

Problem: Let $\mathcal{B}$ be a basis on X, and let $\mathcal{T}_\mathcal{B}$ be the topology it generates. Show that $\mathcal{T}_\mathcal{B}$ is the intersection of all topologies that contain $\mathcal{B}$.

Solution: Let

$$\mathcal{T}' = \bigcap_{\substack{\mathcal{T} \text{ is a topology on } X \\ \mathcal{B} \subseteq \mathcal{T}}} \mathcal{T}.$$

Clearly $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}'$, since any topology that contains $\mathcal{B}$ must contain all possible unions of elements of $\mathcal{B}$, which generates $\mathcal{T}_{\mathcal{B}}$. Now suppose $V \in \mathcal{T}'$. Since clearly $\mathcal{T}_{\mathcal{B}}$ satisfies the conditions of the intersection, this implies that $V \in \mathcal{T}_{\mathcal{B}}$, and since we have the inclusion both ways, the topologies are equal, as desired.

Problem: Let $\{\mathcal{T}_\alpha : \alpha \in I\}$ be a collection of topologies on a set X. Prove that there is a unique coarsest topology that refines all the $\mathcal{T}_\alpha$'s.

Solution: We know we can create a basis for each topology, so let $\mathcal{B}_\alpha$ denote the basis that generates $\mathcal{T}_\alpha$. Let $\mathcal{B} = \bigcup_{\alpha \in I} \mathcal{B}_\alpha$. We claim that $\mathcal{T}_{\mathcal{B}}$ is the desired topology. Fix $\mathcal{T}_\alpha$. For any $x \in X$ and any $B_\alpha \in \mathcal{B}_\alpha$, we can pick $\mathcal{B}_\alpha \in \mathcal{B}$ and obtain $x \in B_\alpha \subseteq B_\alpha$. Then the proposition about comparing bases implies that $\mathcal{T}_{\mathcal{B}} \supseteq \mathcal{T}_\alpha$.

Now suppose $\mathcal{T}'$ also refines the collection of topologies. If $V \in \mathcal{T}_{\mathcal{B}}$, then it can be written as

$$\bigcup_{\alpha \in I} \left(\bigcup_{\beta \in J_\alpha} B_{\alpha,\beta} \right),$$

where the inner union is the union of a collection of sets from $\mathcal{B}_\alpha$. Since $\mathcal{T}'$ refines all the topologies, it will contain each of these basis elements, and thus their union, implying that $V \in \mathcal{T}'$. Thus $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}'$, implying it's the coarsest possible topology.

Problem: Fix an infinite subset A of $\mathbb{Z}$ such that $\mathbb{Z} \backslash A$ is also infinite. Construct a topology on $\mathbb{Z}$ in which:

a) A is open.
b) Singletons are never open.
c) For any pair of distinct integer m and n, there are disjoint open sets U and V such that $m \in U$ and $n \in V$.

Solution: We create a basis $\mathcal{B}$ as follows: First add A and $\mathbb{Z} \backslash A$ to the basis. Since A is countable, we can rewrite its elements as $\{a_1, a_2, \dots\}$. Now add the sets

$$A^r_{2^k} := \{a_n : n \equiv r \pmod{2^k}\}$$

to $\mathcal{B}$. Add similar sets $B^r_{2^k}$ using $\mathbb{Z} \backslash A$ instead. Note that this basis covers $\mathbb{Z}$, since any element in $\mathbb{Z}$ is either in A or $\mathbb{Z} \backslash A$. Then it will be one of a_n or b_m, at which point it's clear that $a_n \in A^n_{2^n}$ and $b_m \in B^m_{2^m}$. Also sets are either disjoint or subsets of each other (since we're creating open sets by partitioning existing open sets), so the second condition for a basis is satisfied. Then the topology generated by $\mathcal{B}$ satisfies the problem conditions, which is easy to check.

Remark: The idea for this construction came from the fact that any topology that satisfies the conditions cannot have finite open sets, since otherwise we could continue to intersect sets containing m and n with the disjoint U and V and obtain smaller open sets until we create a singleton. Thus every open set must be infinite. Then we can declare $\mathbb{Z} \setminus A$ to be open, and we now have basically have to find a topology on A and $\mathbb{Z} \setminus A$ that satisfies the condition, but with free reign to choose the initial infinite subset.

Problem: Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Solution: If we let C denote the limit points of $A \cup B$, then clearly the limit points of A and B are contained C. If $x \in C$, then every neighborhood of x intersects $A \cup B$ at a point other than x. Thus x must either be a limit point of A or B, since otherwise it wouldn't be a limit point of either, and thus would not be a limit point of $A \cup B$. Thus $\overline{A \cup B} = A \cup B \cup C = A \cup B \cup A_{\text{limit points}} \cup B_{\text{limit points}} = \overline{A} \cup \overline{B}$.

Problem: Show that the union of dense sets is dense.

Solution: Follows from $\overline{A \cup B} = \overline{A} \cup \overline{B}$ (we leave one set and union everything else and make it B).

Problem: Show that a subset A of a topological space X is open if and only if $\operatorname{int}(A) = A$.

Solution: Suppose A is open. Since by definition $\operatorname{int}(A)$ is the union of all open subsets of A, we must have $\operatorname{int}(A) = A$. Now suppose $\operatorname{int}(A) = A$. Thus for every $x \in A$, by definition there exists a neighborhood of x contained in A. Then by the first problem, A is open.

Problem: Let $(X, \mathcal{T})$ be a topological space, and let D_1, D_2 be dense open subsets of X. Prove that $D_1 \cap D_2$ is dense and open.

Solution: Let U be open in X. Then, since D_1 is dense and open, $U \cap D_1$ is nonempty and open. Doing the same with D_2 yields that $U \cap D_1 \cap D_2$ is nonempty, so $D_1 \cap D_2$ is dense. Since both are open, their intersection will also be open.

Problem: Suppose $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Solution: No. Take $f : \mathbb{R} \to \mathbb{R}$ with the usual topology defined by $f(x) = 0$, which is clearly continuous, since the inverse image is either $\emptyset$ or $\mathbb{R}$. Then for $A = \mathbb{R}$, $f(A) = \{0\}$. For any $x \in \mathbb{R}$, it's a limit point of $\mathbb{R}$, but $f(x) = 0$ is not a limit point of $f(A)$, since any open set containing 0 doesn't intersect $f(A)$ anywhere else.

Problem: Let X and X' denote a single set in the two topolgies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X' \to X$ be the identity function.
a) Show that i is continuous $\iff \mathcal{T}'$ is finer then $\mathcal{T}$.
b) Show that i is a homemorphism $\iff \mathcal{T}' = \mathcal{T}$.

Solution:

a) Suppose i is continuous. Since $f^{-1}(V) = V$, by continuity V is open in X' if its open in X. Thus every open set in X is open in X', so $\mathcal{T} \subseteq \mathcal{T}'$. The other follows by the reverse logic.
b) If i is a homeomorphism, then that implies V is open in X if and only if it's open in X'.

Problem: Let $x_1, x_2, \ldots$ be a sequence in $\prod X_\alpha$ with the product topology. Show that this sequence converges to x if and only if $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converges to $\pi_\alpha(x)$.

Solution: Since the projection map is continuous, we already know that the convergence of the product space sequence implies the convergence of the coordinate sequences.

Now suppose each coordinate sequence converges. Consider a subbasis element $\pi_\beta^{-1}(V_\beta)$, where V_β is open in X_β and contains $\pi_\beta(x)$. By convergence, V_β contains $\left(\pi_\beta(x_n)\right)_{n \geq N}$ for some N. Thus $\pi_\beta^{-1}(V_\beta)$ contains $(x_n)_{n \geq N}$. This holds for arbitrary V_β and any index β. Since this holds for all the subbasis element, it will hold for any neighborhood of x, so we're done.

Problem: Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequence that are eventually zero. Find the closure of $\mathbb{R}^\infty$ in both the product and box topology.

Solution: First we show that $\mathbb{R}^\infty$ is dense in the product topology. The basic open sets are

$$U_1 \times \cdots \times U_{N-1} \times \mathbb{R} \times \cdots,$$

where each U_i is open in $\mathbb{R}$. For the first $N - 1$ sets, we can pick an element from there, and then for all the $\mathbb{R}$ components, we can choose 0. This creates an element of $\mathbb{R}^\infty$. Since this works for arbitrary basis open sets, the intersection of any open set and $\mathbb{R}^\infty$ is nonempty, and thus $\mathbb{R}^\infty$ is dense in $\mathbb{R}^\omega$, implying $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

Now we do the box topology. Consider some sequence $(x_1, x_2, \ldots)$ that's nonzero for infinitely many terms. Then for those nonzero terms, we can enclose them in an open interval that doesn't contain 0. This gives a basic open set in the box topology that doesn't intersect $\mathbb{R}^\infty$. Thus $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$.

2. Connectedness and Compactness

2.1. Connected Spaces

Definition (separation/connected): Let X be a topological space. A *separation* of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be *connected* if there does not exist a separation of X.

Note that the if X is disconnected, the sets U and V are clopen, since $X \setminus U = V$ is closed and vice versa. Conversely, if X has a non trivial clopen subset, then that set and its complement form a separation.