Linear Algebra Notes

Contents

1. Vector Spaces

1.1. $\mathbb{R}^n$ and $\mathbb{C}^n$

$\mathbb{R}$ and $\mathbb{C}$ are defined as usual.

Example *(Complex Commutativity)*: $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Definition (Elements in $\mathbb{F}^n$ and Operations): $\mathbb{F}^n$ is all n-tuples

$$(x_1, x_2, ..., x_n)$$

with $x_i \in \mathbb{F}$.

Addition is pointwise. Scalar multiplication by λ multiplies each element by λ. If context is clear, $0 \in \mathbb{F}^n$ denotes $(0, 0, ..., 0)$, where there are n 0s.

1.2. Definition of Vector Space

Definition (Vector Space): A vector space V is a set V with addition and scalar multiplication with commutativity, associativity, additive identity, additive inverse, multiplicative identity, and distributive properties. Elements of a vector space are called vectors or points.

Example: $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces, just verify the properties hold. $\mathbb{F}^\infty$ is also a vector space.

Definition ($\mathbb{F}^S$): If S is a set, then $\mathbb{F}^S$ is the set of functions from S to $\mathbb{F}$.

Example *($\mathbb{F}^S$ is a vector space)*: The 0 function $0(x) = 0$ for all $x \in \mathbb{F}$ is the additive identity. The additive inverse is $(-f)(x) = -f(x)$. All other properties of vector spaces hold by spamming axioms. $\mathbb{F}^n$ is a special case of this, where $S = \{1, 2, ..., n\}$.

Proposition (Unique additive identity): A vector space has a unique additive identity.

Proof: $0, 0'$ are identities.

$$0' = 0' + 0 = 0 + 0' = 0.$$

■

Proposition (Unique additive inverse): Every element in a vector space has a unique additive inverse.

Proof: $v \in V$, with w, w' as inverses.

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

■

Proposition: $0v = 0$ for every $v \in V$.

Proof: $0v = (0 + 0)v = 0v + 0v$. ■

Remark: We have to use $0 = 0 + 0$ since we have to use the distributive property to connect scalar multiplication and vector addition.

Proposition: $a0 = 0$ for scalar a.

Proof: $a0 = a(0 + 0) = a0 + a0$. ■

Proposition: $(-1)v = -v$.

Proof:

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0.$$

■

1.2.1. Problems

Problem (Exercise 1): Prove that $-(-v) = v$ for every $v \in V$.

Solution:

$$-(-v) + (-v) = (-1)(-v) + (-1)v = (-1)(-v + v) = (-1)(0) = 0,$$

so $-(-v)$ is the additive inverse of $-v$, so $-(-v) = v$, as desired.

Problem (Exercise 3): Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution: First existence. Adding $-v$ to both sides gives $3x = w - v$. Multiplying by $\frac{1}{3}$ on both sides gives $x = \frac{1}{3}(w - v)$.

Now uniquness. Suppose y, y' both satisfy. Then $y = \frac{1}{3}(w - v) = y'$.

Problem (Exercise 4): The empty set is a not a vector space. Why?

Solution: Doesn't satsify additive identity. There are no elements, so there cannot exist and additive identity.

Problem (Exercise 5): Show that the additive inverse condition in the definition of a vector space can be replaced with

$$0v = 0 \text{ for all } v \in V.$$

Solution:

$$0v = (1 + (-1))v = 1v + (-1)v = v + (-1)v = 0,$$

so there exists a w such that $v + w = 0$, as desired.

Problem (Exercise 6): Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$?

Solution: No. We have that $(2 - 1)\infty = (1)\infty = \infty$ and $2\infty - 1\infty = \infty + (-\infty)0$, so it is not distributive, so it in not a vector space.

1.3. Subspaces

Definition (Subspace): A subset U of V is called a subspace of V if U is also a vector space.

Example: $(x_1, x_2, 0)$ with $x_1, x_2 \in \mathbb{F}$ is a subspace of $\mathbb{F}^3$.

Proposition (Conditions for subspaces): A $U \subset V$ is a subspace of V if and only if U satisfies the conditions below.
- Additive identity: $0 \in U$
- Closed under addition
- Closer under scalar multiplication

Solution: If U is a subspace of V, then it satisfies the properties by definition.

First condition ensures additive identity. Second and third make sure addition and scalar work. Additive inverse holds by scalar multiplication by -1, and associativity and distributivity hold because that holds on V.

Example:

- If $b \in \mathbb{F}$ then
$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$
is a subspace if and only if $b = 0$. When $b = 0$, we can easily verify all the subspace conditions hold. If we have a subspace, then $0 \in U$, so $0 = x_3 = 5(0) + b$ means b must be 0.
- The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$. $f(x) = 0$ is the additive identity for $\mathbb{R}^{[0,1]}$, and clearly addition and scalar multiplication are closed.
- The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
- The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.
- The set of all sequences of comples numbers with limit 0 is a subspace of $\mathbb{C}^{\infty}$.

Definition: Suppose $U_1, U_2, ..., U_m$ are subsets of V. The sum of $U_1, U_2, ..., U_m$ is the set of all possible sums of elements in the subsets. So
$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_i \in U_i\}.$$

Example: We have $U = \{(x, 0, 0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}$ and $W = \{(0, y, 0) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}$. Then,
$$U + W = \{(x, y, 0) \in \mathbb{F}^3 \mid x, y, \in \mathbb{F}\}.$$

Example: U, W are subsets in $\mathbb{F}^4$, $U = (x, x, y, y)$, $W = (x, x, x, y)$. Then
$$U + W = (x, x, y, z)$$
since when adding two elements from U and W, the sum always has equal first and second components. The sum of the third components can be arbitrary, and same for the fourth.

Proposition (Minimality of subspace sums): If $U_1, ..., U_m$ are subspaces of V, then $U' = \sum U_i$ is the smallest subspace containing U_i.

Solution: Cleary U' is a subspace. All the U_i are contained in U'. Also, in any subspace with U_i, we must have U' by closed addition. Thus, we have minimmality.

Definition (Direct Sum): Suppose $U_1, ..., U_m$ are subspaces of V. Tehn $\sum U_i$ is called a direct sum if each element of $\sum U_i$ can be written in only one way as a sum of $\sum u_i$, where $u_i \in U_i$. If $\sum U_i$ is a direct sum, then we denote is with $U_1 \oplus U_2 \oplus \cdots \oplus U_m$.

Example: $U = (x, y, 0), W = (0, 0, z)$. Then $U \oplus W = \mathbb{F}^3$.

Example *(Nonexample)*: $U_1 = (x, y, 0), U_2 = (0, 0, z), U_3 = (0, y, y)$. We have that $\mathbb{F}^3 = \sum U_i$ since we can write every vector in $\mathbb{F}^3$ as the sum of three vectors from each of the subsets. But $(0, 0, 0)$ can be written in two different ways, so it's not a direct sum.

Proposition (Direct Sum Condition): U_i are subspaces of V. Then $W = \sum U_i$ is a direct sum if and only if the only way to write 0 is by writing 0 in each subspace.

Solution: If W is a direct sum, then by definition the only way to write 0 is by taking 0 from each U_i (0 is in each of these by subspace condition). Now suppose the only way to write 0 is to take $u_i = 0$. Now consider $v \in W$. Suppose there are two ways to write it,

$$v = \sum u_i = \sum v_i.$$

Subtracting gives $0 = \sum u_i - v_i$. We know that $u_i - v_i \in U_i$ because it is a subapce, and we also know that the only way to write 0 is having all components equal 0. Thus, $u_i = v_i$, so there is only one way to write each vector as a sum, as desired.

Proposition (Direct sum of two subspaces): U, W are subspaces of V. Then $U + W$ is a direct sum of and only if $U \cap W = \{0\}$.

Solution: If $U + W$ is a direct sum, then for $v \in U \cap W$, we have that $0 = v + (-v)$, with $v \in U$ and $-v \in W$. By unique representation, $v = 0$, so $U \cap W = \{0\}$. If $U \cap W = 0$, then for $u \in U$ and $w \in W$ we have $0 = u + w$. We need to show $u = w = 0$. The equation implies $u = -w \in W$, so $u \in U \cap W$, which means $u = 0$ as desired.

1.3.1. Problems

Problem (Exercise 1): For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subapce of $\mathbb{F}^3$:
- $\{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 0\}$
- $\{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 4\}$
- $\{(x_1, x_2, x_3) : x_1 x_2 x_3 = 0\}$
- $\{(x_1, x_2, x_3) : x_1 = 5x_3\}$

Solution:
- Yes, it is closed under addition and scalar multiplication and has 0.
- No, does not have 0.
- Not closed under addition.
- Yes.

Problem (Exercise 3): Show that the set of differentiable real valued functions on $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{-4,4}$.

Solution: $f \equiv 0$ clearly satisfies the additive identity. It's also easy to see it's closed under addition and scaling.

Problem (Exercise 4): Show that the set of continuous real valued functions on the interval $[0, 1]$ such that $\int_0^1 = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Solution: If $b = 0$, then the conclusion follows easily from verifying subspace conditions. If the set is a subspace, then for f we need $\int_0^1 2f = 2b = b$, which implies $b = 0$.

Problem (Exercise 5): Is $\mathbb{R}^2$ a subspace of $\mathbb{C}^2$.

Solution: No, it's not closed under scalar multiplication.

Problem (Exercise 6):
- Is $\{(a, b, c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?
- Is $\{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Solution:
- Yes, since over $\mathbb{R}$ we have $a^3 = b^3 \Rightarrow a = b$.
- No, since $(\omega, 1, 0) + (1, 1, 0) = (\omega + 1, 2, 0)$ is not in the set.

Problem (Exercise 7): Prove or give a counterexample: if U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses, then U is a subspace of $\mathbb{R}^2$.

Solution: $U = \{(x, y) : x, y, \in \mathbb{Z}\}$.

Problem (Exercise 8): Given an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and scalar multiplication but is not a subspace.

Solution: $U = \{(x, y) : xy = 0\}$

Problem (Exercise 9): Is the set of periodic functions $f : \mathbb{R} \to \mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$?

Solution: No. Consider $\sin(x)$ and $\sin(\pi x)$. When added, they do not form a periodic function, so the set is not closed under addition.

Problem (Exercise 10): If U_1 and U_2 are subspaces of V, then $U_1 \cap U_2$ is a subspace of V.

Solution: Let $W = U_1 \cap U_2$. 0 is in U_1 and U_2, so $0 \in W$. Consider a vector $w \in W$. Since $w \in U_1$, $\lambda w \in U_1$. Similarly for U_2. Thus W is closed under scalar multiplication. A similar argument can be used for addition, so W is a subspace of V.

Problem (Exercise 12): Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution: Let U_1, U_2, be subspaces of V. If $U_1 \subset U_2$, then $U_1 \cup U_2 = U_2$, which is a subspace. Now suppose $U_1 \cup U_2 = W$ is a subspace. Suppose for the sake of contradiction $U_1 \not\subset U_2$. Pick $x \in U_1$, $x \notin U_2$ and $y \notin U_1, y \in U_2$. We know that $x + y = w \in W$ must be in U_1 or U_2. Suppose its in U_1. Then, $y = w - x = w + (-1)x \in U_1$, a contradiction. The same applies to U_2. Thus, $U_1 \subseteq U_2$.

Problem (Exercise 19): Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution: $U_1 = \{0\}, U_2 = W$, W is any subspace of V that's not $\{0\}$.

Problem (Exercise 20): Suppose $U = (x, x, y, y)$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Solution: The subspace $W = (a, 0, 0, b)$. Note that we can write any element in $\mathbb{F}^4$ as a sum of elements from U and V, so $U + v = \mathbb{F}^4$. Note also that the only way two vectors in W and U are equal is when $x = y = a = b = 0$, or in other words, $U \cap W = \{0\}$. Then, $U \oplus W = \mathbb{F}^4$.

Problem (Exercise 21): Suppose $U = \{(x, y, x + y, x - y, 2x)\}$. Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

Solution: $W = \{(0, 0, a, b, c)\}$

Problem (Exercise 22): U is the same as the previous problem. Find $W_1, W_2, W_3 \neq \{0\}$ such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution: $W_1 = (0, 0, a, 0, 0), W_2 = (0, 0, 0, b, 0), W_3 = (0, 0, 0, 0, c)$

Problem (Exercise 23): Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W,$$

then $U_1 = U_2$.

Solution: $W = (x, x), U_1 = (a, 0), U_2 = (0, b)$

Problem (Exercise 24): U_o is the set of real-valued functions. U_e is defined similarly. Show that $\mathbb{R}^{\mathbb{R}} = U_o \oplus U_e$.

Solution: Note that both U_o and U_e are subspaces, and that $U_o \cap U_e = \{0\}$. Note that for any $f \in \mathbb{R}^{\mathbb{R}}$, we can write an even function $e(x) = \frac{f(x)+f(-x)}{2}$ and an odd function $o(x) = \frac{f(x)-f(-x)}{2}$, so $U_o + U_e = \mathbb{R}^{\mathbb{R}}$. But we have $U_o \cap U_e = \{0\}$, so $U_o + U_e$ is a direct sum, so we're done.

2. Finite Dimensionl Vector Spaces

2.1. Span and Linear Independence

2.1.1. Linear Combinations and Span

Definition (Linear Combination): A linear combination of a list $v_1, ..., v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_i \in \mathbb{F}$.

Example: $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$ while $(17, -4, 5)$ is not.

Definition (Span): The set of all linear combinations of a list of vectors v_i in V is called the span of v_i, denoted by $\operatorname{span}(v_1, ..., v_m)$. In other words,

$$\operatorname{span}(v_1, ..., v_m) = \left\{\sum a_i v_i : a_i \in \mathbb{F}\right\}.$$

The span of the empty list $()$ is defined to be $\{0\}$.

Proposition (Span is the smallest containing subspace): The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Proof: Suppose v_i is a list of vectors in V. Note that the span is indeed a subspace since 0 is in the span and the same is closed under addition and scalar multiplication. Note that each v_k is also in the span.

Because subspaces are closed under scalar multiplication and addition, every subspace of V that contains each v_k contains $\operatorname{span}(v_1, ..., v_m)$. Thus $\operatorname{span}(v_1, ..., v_m)$ is the smallest subspace of V containing all vectors v_i. ■

Definition: If $\operatorname{span}(v_1, ..., v_m)$ equals V, we say that the list $v_1, ..., v_m$ *spans* V.

Example:

$$(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1)$$

spans $\mathbb{F}^n$, where there are n vectors in the list.

Definition (Finite-dimensional vector space): A vector space is called finite-dimensional if some list of vectors in it spans the space.

Definition: $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$. $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m.

Definition (Infinite dimensional vector space): A vector space is called infinite-dimensional if it is not finite-dimensional.

2.1.2. Linear Independence

Definition (Linearly independent): A list $v_1, ..., v_m$ of vectors in V is called linearly independent if the only choice of $a_1, ..., a_m \in \mathbb{F}$ that makes

$$\sum a_i v_i = 0$$

is $a_i = 0$ for all i. The empty list is also declared to be linearly independent.

Proposition: If some vectors are removed from a linearly independent list, the remaining list is also linearly independent.

Proof: If a list is linearly independent, then the only a_is that work are all 0. Suppose we remove some of the vectors. If the new list wasn't linearly dependent, then we could just drop 0s in front of the vectors we got rid of and have a non linearly independent list of vectors, which is a contradiction. ■

Lemma (Linear dependence lemma): Suppose $v_1, ..., v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, ..., m\}$ such that

$$v_k \in \operatorname{span}(v_1, ..., v_{k-1}).$$

Furthermore, if k satsfies the condition above and the kth term is removed from $v_1, ..., v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, ..., v_m)$.

Proof: Because the list $v_1, ..., v_m$ is linearly dependent, there exist numbers $a_1, ..., a_m \in \mathbb{F}$, not all 0, such that

$$\sum a_i v_i = 0.$$

Let k be the largest element of $\{1, ..., m\}$ such that $a_k \neq 0$. Then

$$v_k = -\frac{a_1}{a_k} v_1 - \cdots - \frac{a_{k-1}}{a_k} v_{k-1},$$

which proves that $v_k \in \operatorname{span}(v_1, ..., v_{k-1})$, as desired.

Now suppose k is any element of $\{1, ..., m\}$ such that $v_k \in \operatorname{span}(v_1, ..., v_{k-1})$. Let $b_1, ..., b_{k-1} \in \mathbb{F}$ such that

$$v_k = b_1 v_1 + \cdots + b_{k-1} v_{k-1}.$$

Suppose $u \in \operatorname{span}(v_1, ..., v_m)$. Then there exist $c_1, ..., c_m \in \mathbb{F}$ such that

$$u = c_1 v_1 + \cdots + c_m v_m.$$

In the equation above, we can replace v_k with the right side of the equation two above, which shows that u is in the span of the list obtained by removing the kth term from $v_1, ..., v_m$. Thus removing the kth term of the list $v_1, ..., v_m$ does not change the span of the list. ■

Lemma (length of linearly independent list $\leq$ length of spanning list): In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof: Suppose $u_1, ..., u_m$ is linearly dependent in V and $w_1, ..., w_n$ spans V.

First we append u_1 to the list of w_is. Since u_1 can be written as a linear combination of the w_i s, the new list is linearly dependent. By the linear dependence lemma, we can now take out one of the w_is (we can't take out u_1 since it's not 0).

We can continue this idea, appending u_k right after a_{k-1} and right before the w_is. Since the first k vectors are linearly independent, by the linear dependence lemma, we can remove one of the w_is and still have the same span. After adding all m vectors, the process stops. At each step as we add a u_i, the linear dependence lema implies that there is some w to remove. Thus there are at least as many w's as u's. ■

Proposition (Finite-dimensional subspaces): Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof: Suppose V is finite-dimensional and U is a subspace of V.

- Step 1: If $U = \{0\}$, then U is finite dimmensional and we are done. If $U \neq \{0\}$, then choose a nonzero vector $u_i \in U$.
- Step k: If $U = \operatorname{span}(u_1, ..., u_{k-1})$, then we are done. If $U \neq \operatorname{span}(u_1, ..., u_{k-1})$, then a choose a vector $u_k \in U$ such that $u_k \notin \operatorname{span}\ (u_1, ..., u_k)$.

After each step, as long as the process continues, we have constructed a list of vectors such that no vector in this list is in the span of the previous vectors. Thus after each step we have constructed a linearly independent list. This linearly independent list cannot be longer than any spanning list of V. Thus the process eventually terminates, which means U is finite-dimensional. ■

2.1.3. Problems

Problem (Exercise 1): Find a list of four distinct vectors in $\mathbb{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}.$$

Solution: $(1, -1, 0), (-1, 1, 0), (0, -1, 1), (0, 1, -1)$

Problem (Exercise 2): Prove or give a counterexample: if v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution: Note that any linear combination of the four vectors given is equal to a linear combination of the four original vectors, so the new list spans V.

Problem (Exercise 3): Suppose $v_1, ..., v_m$ is a list of vectors in V. For $k \in \{1, ..., m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $\operatorname{span}(v_1, ..., v_m) = \operatorname{span}(w_1, ..., w_m)$.

Solution:

$$\sum_{i=1}^{m} a_i w_i = \sum_{i=1}^{m} a_i \left(\sum_{j=1}^{i} v_j \right) = \sum_{k=1}^{m} \left(\sum_{i=1}^{m+1-k} a_{m+1-i} \right) v_k = \sum_{k=1}^{m} b_k v_k,$$

so any linear combination of w_is is a linear combination of v_is.

Problem (Exercise 11): Prove or give a counterexample: if $v_1, ..., v_m$ and $w_1, ..., w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, ..., v_m + w_m$ is linearly independent.

Solution: Take $w_i = -v_i$.

Problem (Exercise 17): Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, ...$ of vectors in V such that $v_1, ..., v_m$ is linearly independent for every positive integer m.

Solution: First suppose such a sequence exists. Then we know that $v_m \notin \operatorname{span}(v_1, ..., v_{m-1})$, since otherwise the list that included v_m would not be linearly independent. Thus, no list could span the entire space, which implies it is infinte-dimensional.

Now suppose the space is infinite-dimensional. This means no list spans the entire space. Thus, if we have a linearly independent list of size $m - 1$, there is some vector in V that's not in the span of the list, and appending that creates a new linearly independent list, which can keep going.

2.2. Bases

Definition (Basis): A basis of V is a list of vectors in V that is linearly independent and spans V.

Proposition (Criterion for basis): A list $v_1, ..., v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, ..., a_n \in \mathbb{F}$.

Proof: Basically uses ideas that led to linear independence. ■

Proposition: Every spanning list in a vector space can be reduced to a basis of the vector space.

Proof: If a vector v_k is in the span of the vectors $v_1, ..., v_{k-1}$, discard it. Keep doing this until you reach the end. The new list clearly must still span V, and by the linear dependence lemma, the new list is linearly independent, which means it's a basis. ■

This immediately implies every finite-dimensional vector space has a basis.

Proposition: Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Proof: Adjoin a spanning list to the vectors, then use proposition 2.2.2 to reduce to a basis (none of the original vectors get deleted since they were linearly independent). ■

Proposition: Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof: Because V is finite-dimensional, so is U. Thus there is a basis $u_1, ..., u_m$ of U. We can extend this list to a basis of V: $u_1, ..., u_m, w_1, ..., w_n$. Let $W = \operatorname{span}(w_1, ..., w_n)$.

To show $V = U + W$, note that for $v \in V$ we have

$$v = \sum a_i u_i + \sum b_i w_i.$$

The first sum is a vector in U, and the second is a vector in V, so we have $v \in U + W$, which means $V = U + W$.

Now suppose $v \in U \cap W$. Then we have

$$v = \sum a_i u_i = \sum b_i w_i \Rightarrow \sum a_i u_i - \sum b_i w_i = 0.$$

Since the list $u_1, ..., u_m, w_1, ..., w_n$ is linearly independent, this implies $a_i, b_i = 0$ for all i, so $U \cap W = \{0\}$. ∎

A simple consequence of this is that the extension of the list defines W needed to get $U \oplus W = V$.

2.2.1. Problems

Problem (Exercise 3): Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U. Then extend the basis to a basis in $\mathbb{R}^5$, then find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Solution: A basis is $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$. This is clearly linearly independent and clearly spans the subspace. We can extend the basis to

$$(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), (1, 0, 0, 0, 0), (0, 0, 1, 0, 0)$$

as a basis of $\mathbb{R}^5$. $W = \{(x, 0, 0, y, 0)\}$ is a W that works.

Problem (Exercise 4): Let U be the subspace of $\mathbb{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Repeat the procedure in the last problem.

Solution: A basis is $(1, 6, 0, 0, 0), \left(0, 0, 0, 1, -\frac{2}{3}\right), \left(0, 0, 1, 0, -\frac{1}{3}\right)$. We can extend this to

$$(1, 6, 0, 0, 0), \left(0, 0, 0, 1, -\frac{2}{3}\right), \left(0, 0, 1, 0, -\frac{1}{3}\right), (1, 0, 0, 0, 0), (0, 0, 0, 0, 1).$$

A W that work is $W = \{(x, 0, 0, 0, y)\}$.

Problem (Exercise 5): Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

Solution: Append a basis of W to a basis of V. Since $V = U + W$, any vector in V can be represented as a linear combination of vectors from this list, so the list spans V. Then just reduce the list down to a basis.

Problem (Exercise 7): Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Solution: Clearly the list spans V, since any linear combination of these vectors is just a linear combination of v_is. It also must be linearly independent since the v_is are linearly independent.

2.3. Dimension

Proposition (basis length does not depend on basis): Any two bases of a finite-dimensional vector space have the same length.

Proof: Follows from len(linearly independent) $\leq$ len(spanning). ■

Definition (dimension): The dimension of a finite-dimensional vector space, denoted as $\dim V$, is the length of any basis of the vector space.

Proposition (dimension of subspace): If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Proof: Pick a basis of U. This basis must also be linearly independent in V, so it can be extended to a basis of V, giving the desired inequality. ■

Proposition: Suppose V is finite-dimensional. Then evert linearly independent list of vectors in V of length $\dim V$ is a basis of V.

Proof: Suppose $\dim V = n$ and $v_1, \ldots, v_n$ is a linearly independent in V. Then it can be extended to a basis, but since a basis must have length n, no elements need to be added. Thus, the list is already a basis. ■

Corollary: Suppose that V is a finite-dimensional subspace and U is a subspace of V such that $\dim U = \dim V$. Then $U = V$.

Proof: Let $u_1, \ldots, u_n$ be a basis of U. Thus $n = \dim U = \dim V$. Thus $u_1, \ldots, u_n$ is a linearly independent list of vectors in V of length $\dim V$. Thus the list is a basis of V, which means every vector of V is a linear combination of vectors in the list, which means $U = V$. ■

Proposition: Suppose V is finite-dimensional. Then every spanning list of vectors in V of length $\dim V$ is a basis of V.

Proof: Suppose $\dim V = n$ and $v_1, \ldots, v_n$ spans V. The list $v_1, \ldots, v_n$ can be reduced to a basis of V. However, every basis of V has length n, so the reduction is trivial, thus the list is a basis of V. ■

Lemma (dimension of a sum): If V_1 and V_2 are subspaces of a finite-dimensional vector space, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

Proof: Let $v_1, ..., v_n$ be a basis of $V_1 \cap V_2$, so the $\dim(V_1 \cap V_2) = m$. Because v_is is a basis of $V_1 \cap V_2$, it is linearly independent in V_1, so we can extend it to a basis of V_1: $v_1, ..., v_n, u_1, u_j$. Similarly, extend it to a basis of V_2: $v_1, ..., v_n, w_1, ..., w_k$. Thus, $\dim V_1 = m + j$ and $\dim V_2 = m + k$. Showing that $v_i \cup u_i \cup w_i$ is basis of $V_1 + V_2$ will complete the proof.

Note that each vector in the list will be contained in $V_1 + V_2$, and that since part of the list is a basis of V_1 and the other part is a basis of V_2, every vector in $V_1 + V_2$ can be represented by the list, which means that list spans $V_1 + V_2$. All that remains to be shown is that the list is linearly independent.

Suppose

$$\sum a_i v_i + \sum b_i u_i + \sum c_i w_i = 0.$$

We can rewrite this as

$$\sum c_i w_i = -\sum a_i v_i - \sum b_i u_i.$$

The right side is a vector in V_1, and the left side is a linear combination of vectors in V_2, so both sides are in their intersection. Thus we can write

$$\sum c_i w_i = \sum d_i v_i.$$

However, v_i, w_i is the basis of V_2, so it's linearly independent, implying $c_i = d_i = 0$. Thus we have

$$\sum a_i v_i + \sum b_i u_i = 0.$$

Since these vectors are linearly independent in V_1, $a_i = b_i = 0$. ∎

2.3.1. Problems

Problem (Exercise 9): Suppose m is a positive integer and $p_0, ..., p_m \in \mathcal{P}(\mathbb{F})$ are such that p_k has degree k. Prove that $p_0, ..., p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Solution: Note that $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$. Thus we just need to show that the list is linearly independent. This is easy, since the only way they can sum to 0 is if p_m's coefficient is 0 (to get rid of x^m) and so on.

Problem (Exercise 11): Suppose $\dim U = \dim W = 4$ are subspaces of $\mathbb{C}^6$. Prove that there are two vectors in $U \cap W$ such that they are linearly independent.

Solution: Note that

$$6 \geq \dim(U + W) = 8 - \dim(U \cap W),$$

so $U \cap W$ must have dimension at least 2, implying the result.

Problem (Exercise 14): Suppose $\dim V = 10$ and $\dim V_1 = \dim V_2 = \dim V_3 = 7$ are subspaces of V. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Solution: First we have

$$\dim(V_1 \cap V_2 + V_3) = \dim(V_1 \cap V_2) + \dim V_3 - \dim(V_1 \cap V_2 \cap V_3).$$

Note that $\dim(V_1 + V_2) = 14 - \dim(V_1 \cap V_2)$. Plugging this in, we have

$$\dim(V_1 \cap V_2 + V_3) = 21 - \dim(V_1 + V_2) - \dim(V_1 \cap V_2 \cap V_3).$$

Note that the left side is at most 10, so we have

$$\dim(V_1 + V_2) + \dim(V_1 \cap V_2 \cap V_3) \geq 11.$$

If the second term on the left was zero, then $\dim(V_1 + V_2)$ would need to be at least 11, which is impossible since they are both subspaces of a 10 dimensional vector space. Thus $\dim(V_1 \cap V_2 \cap V_3) \geq 1$, which implies the intersection is not equal to $\{0\}$.

Problem (Exercise 17): Suppose that $V_1, \ldots, V_m$ are finite-dimensional subspaces of V. Prove that

$$\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

Solution: Easy by induction.

Problem (Exercise 20): Prove that if V_1, V_2, and V_3 are finite-dimensional subspaces of a vector space, then

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3$$

$$- \frac{\dim(V_1 \cap V_2) + \dim(V_2 \cap V_3) + \dim(V_3 \cap V_1)}{3}$$

$$- \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_2 + V_3) \cap V_1) + \dim((V_3 + V_1) \cap V_2)}{3}.$$

Solution: Note that

$$\dim(V_1 + V_2 + V_3) = \dim(V_1 + V_2) + \dim V_3 - \dim((V_1 + V_2) \cap V_3) =$$

$$\dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) + \dim V_3 - \dim((V_1 + V_2) \cap V_3).$$

Sum cyclically and divide by 3 to get the desired result.

3. Linear Maps

General maps T are assumed to be in $\mathcal{L}(V, W)$.

Basis of V and W are $v_1, \dots, v_n$ and $w_1, \dots, w_m$ unless stated otherwise.

3.1. Vector Space of Linear Maps

3.1.1. Definition of Linear Maps

Definition (Linear map): A linear map from V to W is a function $T : V \to W$ such that $T(u + v) = Tu + Tv$ and $T(\lambda v) = \lambda(Tv)$. The set of linear maps from V to W is denoted by $\mathcal{L}(V, W)$, and if $W = V$, then it is denoted as $\mathcal{L}(V)$.

Lemma (Linear map lemma): Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_k = w_k$$

for each k.

Proof: First existence. Define $T : V \to W$ by

$$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n,$$

where c_i are arbitrary scalars. Since the vs form a basis, this is indeed a function from V to W. Setting $c_k = 1$ and everything else to 0 for each k shows $Tv_k = w_k$. Showing this is a linear map is very easy.

Now uniquness. Suppose $T \in \mathcal{L}(V, W)$ and that $Tv_k = w_k$. Then we have $T(c_k v_k) = c_k w_k$. Additivity implies

$$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n.$$

Thus T is uniquely determined on $\operatorname{span}(v_1, \dots, v_n)$ by the equation above. Since the vs are a basis, T is uniquely determined on V. ∎

3.1.2. Algebraic Operations on $\mathcal{L}(V, W)$

Definition: Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. Then $S + T$ is defined as

$$(S + T)(v) = Sv + Tv$$

and λT is defined as

$$(\lambda T)(v) = \lambda(Tv).$$

This definition of addition and scalar multiplication makes $\mathcal{L}(V, W)$ a vector space, which is easy to verify.

Definition (product of linear maps): If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the product $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

Associativity, identity, and distributivity all apply to the products of linear maps.

Proposition: Suppose T is a linear maps. Then $T(0) = 0$.

Solution:

$$T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0.$$

3.1.3. Problems

Problem (Exercise 4): Suppose $T \in \mathcal{L}(V, W)$ and $v_1, ..., v_m$ is a list of vectors in V such that $Tv_1, ..., Tv_m$ is a linearly independent list in W. Prove that $v_1, ..., v_m$ is linearly independent.

Solution:

$$\sum c_i T v_i = 0 \Rightarrow T\left(\sum c_i v_i\right) = 0.$$

Problem (Exercise 7): Prove that if $\dim V = 1$ and $T \in \mathcal{L}(v)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution: Let w be the basis of V. Then every vector in V can be written as cw for some $c \in \mathbb{F}$. Suppose $Tw = c_0 w$. Then multiplying by any scalar yields $Tv = c_0 v$ where $v \in V$, so we're done.

Problem (Exercise 11): Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the idenityt if and only if $ST = TS$ for all $S \in \mathcal{L}(V)$.

Solution: Suppose T is a scalar multiple of the identity. Then the equation easily follows. Now suppose T is not a scalar multiple of the identity. There exists v such that $Tv = u \neq \lambda v$ for any $\lambda \in \mathbb{F}$. Thus, u and v are linearly independent. The result holds trivially for $\dim V = 1$, so assume $\dim V \geq 2$. Then u and v are part of a basis of V. Let $Sv = v$ and $Su = 0$. Then

$$S(Tv) = Su = 0 \neq u = Tv = T(Sv).$$

Problem (Exercise 13): Show that a linear map can be extended from a subspace of V to V.

Solution: Let U be the subspace. Let $Tu_i = w_i$ be the outputs of the basis vectors of U. Extend the basis of U to V. Let the new vectors in the basis map to any vector in the output vector space, done.

3.2. Null Spaces and Ranges

3.2.1. Null Space and Injectivity

Definition (null space): For $T \in \mathcal{L}(V, W)$, the null space if T, denoted null T, is

$$\text{null } T = \{v \in V : Tv = 0\}.$$

Proposition: null T is a subspace of V.

Proof: Since $T(0) = 0$, $0 \in \text{null } T$. Note that if $u, v \in \text{null } T$ then $T(u + v) = Tu + Tv = 0$, and similarly with scalars, so it is indeed a subspace. ■

Definition (injective): Standard definition for injective functions.

Proposition: T is injective if and only if null $T = \{0\}$.

Proof: First suppose T is injective. We already know that $0 \in \text{null } T$. Now suppose $v \in \text{null } T$. Then $Tv = 0 = T(0)$, which implies that $v = 0$.

Now suppose null $T = 0$. Pick u, v such that $Tu = Tv$. Then $T(u - v) = 0$, which implies $u - v \in \text{null } T$, which implies $u = v$. ■

3.2.2. Range and Surjectivity

Definition (range): The range of T is the following subset of W:

$$\text{range } T = \{Tv : v \in V\}.$$

Proposition: range T is a subspace of W.

Proof: Cleary $0 \in \text{range } T$. Note that for $w_1, w_2 \in W$ we have

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$$

for some v_1, v_2, so $w_1 + w_2$ is in range T. Similarly for scalars. ■

Definition (surjective): $T \in \mathcal{L}(V, W)$ is surjective if range $T = W$.

3.2.3. Fundamental Theorem of Linear Maps

Theorem (fundamental theorem of linear maps): Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Proof: Let $u_1, ..., u_m$ be a basis of null T, extend it to a basis of V:

$$u_1, ..., u_m, v_1, ..., v_n.$$

Thus we need to show $\dim \operatorname{range} T = n$. We show that Tv_i form a basis of range T.

Pick $v \in V$. We can write it as

$$v = \sum c_i u_i + \sum d_i v_i.$$

Applying T to both sides yields

$$Tv = \sum c_i T v_i,$$

where all the u's disappear because they are in null T. This equation implies Tv_i spans range T, which also implies range T is finite dimensional.

To show the list is linearly independent, we have

$$\sum c_i T v_i = 0 \Rightarrow \sum c_i v_i \in \operatorname{null} T.$$

Thus we have

$$\sum c_i v_i = \sum d_i u_i,$$

and since $u_1, ..., u_m, v_1, ..., v_m$ is linearly independent, $c_i = d_i = 0$, done. ■

Proposition: Suppose V and W are finite-dimensional vector spaces.
- If $\dim V > \dim W$, then there is no injective linear map from V to W.
- If $\dim V < \dim W$, then there is no surjective linear map from V to W.

Proof:
- $\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T \geq \dim V - \dim W > 0$.
- $\dim \operatorname{range} T = \dim V - \dim \operatorname{null} T \leq \dim V < \dim W$.

■

3.2.4. Problems

Problem (Exercise 11): Suppose V is finite dimensional. Prove that there exists a subspace U of V such that

$$U \cap \operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{Tu : u \in U\}.$$

Solution: Take U such that $V = U \oplus \operatorname{null} T$, which exists and satisfies the first condition. Let the basis of null T be represented by N, and extend it to a basis of V with the list M. Note that $\operatorname{span}(M) = U$. Thus, any vector in V will have the vectors in N that constitute it become 0, while every vector in M will map to Tu, implying range T is equal to the range of T on U.

Problem (Exercise 12): Suppose T is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Solution: Note that $\dim \operatorname{null} T = 2$. Since $\dim \mathbb{F}^4 = 4$, we have $\dim \operatorname{range} T = 2$. Since range $T = 2$, $\dim \mathbb{F}^2 = 2$, and since range T is a subspace of $\mathbb{F}^2$, we must have range $T = \mathbb{F}^2$, so T is surjective.

Problem (Exercise 15): Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Solution: Let T be the map. Let $v_1, ..., v_m$ be the basis of null T, and let $w_1, ..., w_n$ be the basis of range T. Since w_i's are linearly independent, we have that $u_1, ..., u_n$ are linearly independent, where $Tu_i = w_i$. Now pick some vector $v \in V$ not in null T. We have $Tv \in \operatorname{range} T$, so $Tv = \sum a_i w_i$ for some choice of a's. Replacing the w's with Tu's yields $Tv = T(\sum a_i u_i)$. Since $v \notin$ null T, we've essentially changed T to T', where null $T' = \{0\}$, so we have $v = \sum a_i v_i$, which implies there exists a spanning list for V, which means it's finite-dimensional.

Problem (Exercise 21): Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W. Prove that $\{v \in V : Tv \in U\}$ is a subspace of V and

$$\dim\{v \in V : Tv \in U\} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$

Solution: Denote the set as U'. Checking it's a subspace is easy. Note that null $T \subseteq U'$, since $0 \in U$. Note also that $U \cap \operatorname{range} T$ is the subspace give by just applying T to U'. Now define $T' \in \mathcal{L}(U', W)$ such that $T'v = Tv$ for $v \in U'$ (essentially restricting T to U'). We have null $T =$ null T' and range $T' = U \cap \operatorname{range} T$, so applying the fundamental theorem of linear maps to T' yields

$$\dim U' = \dim \text{null } T + \dim(U \cap \text{range } T).$$

3.3. Matrices

3.3.1. Representing a Linear Map by a Matrix

Definition (matrix): An m-by-n matrix is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

$A_{j,k}$ denote the entry in row j, column k of A.

Definition (matrix of a linear map): Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The matrix of T with respect to these bases is the m by n matrix $\mathcal{M}(T)$ whose entries are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases are note clear from context, then the notation $\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$ is used.

3.3.2. Addition and Scalar Multiplication of Matrices

Definition (matrix addition and scalar multiplication): Matrices of the same dimensions are added pointwise, and a matrix being multiplied by a scalar has the scalar distributed to all entries.

Proposition: $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$ and $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

Proof: For the the first one, note that on a basis vector v_k, $(S + T)v_k$ will be equal to the sum of the representations of Tv_k and Sv_k by a basis of W. Thus their coefficients add up, which corresponds to the adding of elements in the matrices pointwise. Similarly for scalar mulitplication. ■

Proposition: Suppose m and n are positibe integers. Then $\mathbb{F}^{m,n}$, the set of all m by n matrices with entries in $\mathbb{F}$, is a vector space of dimension mn.

Proof: It's clear that $\mathbb{F}^{m,n}$ is a vector space. For the dimension, note that the list of the matrices that have 1 in one slot and 0 in the rest for all slots spans the space and is linearly independent. ■

3.3.3. Matrix Multiplication

Definition (matrix multiplication): Suppose A is an m by n matrix and B is an n by p matrix. Then AB is defined to be the m by p matrix whose entry in row j, column k, is given by

$$(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r} B_{r,k}.$$

Thus the entry in row j, column k, of AB is computed by taking row j of A and column k of B, multiplying together corresponding entries, and then summing.

Proposition: If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Proof: This follows because matrix multiplication was defined for this to be true. ■

Definition ($A_{j,\cdot}, A_{\cdot,k}$): Suppose A is an m by n matrix. $A_{j,\cdot}$ represents the 1 by n matrix consisting of row j of A. $A_{\cdot,k}$ represents the 1 by k matrix consisting of column k of A.

Proposition: Suppose A is an m by n matrix and B is an n by p matrix. Then

$$(AB)_{j,k} = A_{j,\cdot} \cdot B_{\cdot,k}$$

Proof: By definition, we have

$$(AB)_{j,k} = A_{j,1} B_{1,k} + \cdots + A_{j,n} B_{n,k}.$$

By definition this is equal to $A_{j,\cdot} \cdot B_{\cdot,k}$. ■

Proposition: Suppose A is an m by n matrix and B is an n by p matrix. Then

$$(AB)_{\cdot,k} = AB_{\cdot,k}$$

Proof: The entry in row j of $(AB)_{\cdot,k}$ is the left side of the previous result and the entry in row j is the right side if the previous result. ■

Proposition (linear combination of columns): Suppose A is an m by n matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. Then

$$Ab = b_1 A_{\cdot,1} + \dots + b_n A_{\cdot,n}$$

Proof: The entry in row k of the m by 1 product is by definition

$$A_{k,1} b_1 + \dots + A_{k,n} b_n.$$

This holds for all k, so we can replace $A_{k,i}$ with $A_{\cdot,i}$. ■

Analogous results of the last two propositions hold for the rows of matrices.

Proposition: Suppose C is an m by c matrix and R is a c by n matrix.

- If $k \in \{1, ..., n\}$, then column k of CR is a linear combination of the columns of C, with the coefficients of this linear combination coming from column k of R.
- If $k \in \{1, ..., m\}$, then row j of CR is a linear combination of the rows of R, with the coefficients of this linear combination coming from row j of C.

Proof: Column k of CR equals $CR_{\cdot,k}$ by 3.3.3.3, which equals the linear combination of the columns of C with coefficients coming from $R_{\cdot,k}$ by 3.3.3.4. Proving the second bullet uses the analogous row results. ■

3.3.4. Column-Row Factorization and Rank of a Matrix

Definition (column rank, row rank): Suppose A is an m by n matrix with entries in $\mathbb{F}$.

- The column rank of A is the dimension of the span of the columns of A in $\mathbb{F}^{m,1}$.
- The row rank of A is the dimension of the span of the rows of A in $\mathbb{F}^{1,n}$.

Example: Suppose

$$A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix}.$$

The column rank of A is the dimension of

$$\operatorname{span}\left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix}\right)$$

in $\mathbb{F}^{2,1}$. The first two vectors are linearly independent, so the span has dimension at least two. Since $\dim \mathbb{F}^{2,1} = 1$, the span of the list must be 2, so the column rank of A is two.

The row rank of A is the dimension if

$$\operatorname{span}((4\ \ 7\ \ 1\ \ 8), (3\ \ 5\ \ 2\ \ 9))$$

in $\mathbb{F}^{1,4}$. Both vectors are linearly independent, so the dimension of the span is 2.

Definition (transpose): The transpose of a matrix A, denoted by A^t, is the matrix obtained from A by interchanging rows and columns. Specifically, if A is an m by n matrix, then A^t is the n by m matrix whose entries are given by the equation

$$(A^t)_{k,j} = A_{j,k}.$$

Example: If $A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix}$, then $A^t = \begin{pmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{pmatrix}$.

Proposition (column-row factorization): Suppose A is an m by n matrix with entries in $\mathbb{F}$ and column rank $c \geq 1$. Then there exist an m by c matrix and a c by n matrix R, both with entries in $\mathbb{F}$, such that $A = CR$.

Proof: Each column of A is an m by 1 matrix. The list $A_{\cdot,1}, ..., A_{\cdot,n}$ of columns of A can be reduced to a basis of the span of the columns of A. This basis has length c by the definition of column rank. The c columns in this basis can be put together to form an m by c matrix C.

If $k \in \{1, ..., n\}$, then column k of A is a linear combination of the columns of C. Make the coefficients of this linear combination into column k of a c by n matrix that we call R. Then $A = CR$. ■

Proposition (column rank equals row rank): Suppose $A \in \mathbb{F}^{m,n}$. Then the column rank of A equals the row rank of A.

Proof: Let c denote the column rank of A. Let $A = CR$ be the column-row factorization of A, where C is an m by c matrix and R is a c by n matrix. We know every row of A is a linear combination of the rows of R. Because R has c rows, this implies that the row rank of A is less than or equal to the column rank c of A. To prove the inequality in the other direction, apply the result in the previous paragraph to A^t, getting

$$\begin{aligned} \text{column rank of } A &= \text{row rank of } A^t \\ &\leq \text{column rank of } A^t \\ &= \text{row rank of } A. \end{aligned}$$

Thus the column rank equals the row rank. ■

3.3.5. Problems

Problem (Exercise 15): Prove that if A is an m by n matrix and C is an n by p matrix, then

$$(AC)^t = C^t A^t.$$

Solution: Note that

$$(AC)^t_{j,k} = (AC)_{k,j} = \sum_{r=1}^{n} A_{k,r}C_{r,j} = \sum_{r=1}^{n} C^t_{j,r}A^t_{r,k} = \left(C^tA^t\right)_{j,k}.$$

Problem (Exercise 17): Suppose $T \in \mathcal{L}(V)$, and $u_1, ..., u_n$ and $v_1, ..., v_n$ are bases of V. Prove that the following are equivalent.

1. T is injective
2. The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$.
3. The columns of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$.
4. The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{1,n}$.
5. The rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, ..., u_n), (v_1, ..., v_n))$.

Solution: 2 and 3 are equivalent since a spanning/linearly independent list implies the list is a basis in a vector space where the list equals the dimension. Similarly for 4 and 5. Note also that the last 4 are equivalent since column rank = row rank. Now we just need to show 1 ⇔ 2. Denote $\mathcal{M}(T) = A$.

First suppose the columns are linearly independent. Then the only linear combination of columns that equals the 0 matrix is when all the coefficients are 0. This means that $T(u)$ is only 0 for an input vector u when the linear combination of the basis vectors that make it up have coefficients all 0. This means that $\dim \operatorname{null} T = 0$, which means T is injective. Now suppose the columns were not linearly independent. Then some linear combination of some of the columns equals another column. WLOG let these columns be $A_{\cdot,1}, ..., A_{\cdot,c}$. Then

$$\sum_{i=1}^{c} a_i A_{\cdot,i} = A_{\cdot,k}$$

for some $n \geq k \geq c$. This is equivalent to saying that

$$T\left(\sum_{i=1}^{c} a_i v_i - v_k\right) = 0.$$

Since the v's are a basis, the vector on the left is nonzero, so $\dim \operatorname{null} T > 0$, so T is not injective.

3.4. Invertibility and Isomorphisms

3.4.1. Invertible Linear Maps

Definition (invertible): A linear map $T \in \mathcal{L}(V, W)$ is invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the indentity operator on V and TS equals the identity operator on W.

Proposition (inverse is unique): An invertible linear map has a unique inverse, denoted by T^{-1}.

Proof: If S_1, S_2 are inverses, then

$$S_1 = S_1 I = S_1(TS_2) = (S_1T)S_2 = IS_2 = S_2.$$

■

Proposition: A linear map is invertible if and only if it is injective and surjective.

Proof: Suppose $T \in \mathcal{L}(V, W)$. First suppose T is invertible. Suppose $Tu = Tv$. Then we have

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v,$$

which implies injectivity. Now let $w \in W$. We have $w = T(T^{-1}w)$, so range $T = W$, implying surjectivity.

Now suppose T is injective and surjective. For each $w \in W$, define $S(w)$ to be the unique element of V such that $T(S(w)) = w$ (exists by injective + surjective). This definition implies that $T \circ S$ equals the indentity operator on W. Also note that for $v \in V$ we have

$$T((S \circ T)v) = (T \circ S)(Tv) = I(Tv) = Tv,$$

so $(S \circ T)v = v$ by injectivity. Thus $S \circ T$ is the indentity operator on V.

To prove that S is a linear map, just throw $S(w)$ inside T and use linearity and scalar multiplication. ■

Proposition: Suppose that V and W are finite-dimensional vector spaces, $\dim V = \dim W$, and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible} \Leftrightarrow T \text{ is injective} \Leftrightarrow T \text{ is surjective}.$$

Proof: If T is injective, by the fundamental theorem of linear maps we have

$$\dim \operatorname{range} T = \dim V - \dim \operatorname{null} T = \dim V = \dim W,$$

which means T is surjective. If T is surjective, then we can similarly show that $\dim \operatorname{null} T = 0$, so T is injective. Thus if we have injectivity or surjectivity, then both hold, which implies T is invertible. ■

Proposition: Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(W, V)$. Then $ST = I$ if and only if $TS = I$.

Proof: First suppose $ST = I$. If $v \in V$ and $Tv = 0$, then

$$v = Iv = (ST)v = S(Tv) = S(0) = 0.$$

Thus T is injective, and since $\dim V = \dim W$, T is invertible. Multiplying $ST = I$ by T^{-1} on the right yields $S = T^{-1}$, which implies $TS = TT^{-1} = I$, as desired. The other direction is proved basically identically. ■

3.4.2. Isomoprhic Vector Spaces

Definition (isomorphic): An isomorphism is an invertible linear map. Two vector spaces are called isomorphic if there is an isomorphism from one vector space onto the other one.

Proposition: Two finite-dimensional vector spaces over $\mathbb{F}$ are isomoprhic if and only if they have the same dimension.

Proof: First suppose V and W are isomoprhic. Then there exists an isomorphism T from V to W. Since T is invertible, we have $\operatorname{null} T = \{0\}$ and $\operatorname{range} T = 0$. Thus we have $\dim V = \dim W$ by the fundamental theorem of linear maps.

Now suppose V and W have the same dimension. Let $v_1, ..., v_n$ be a basis of V and $w_1, ..., w_n$ be a basis of W. Let

$$T\left(\sum c_i v_i\right) = \sum c_i w_i.$$

Note that T is surjective since the right side spans V, and $\operatorname{null} T = 0$ since the right side is linearly independent. Thus T is an isomorphism, meaning V and W are isomorphic. ■

Proposition ($\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$ are isomoprhic): Suppose $v_1, ..., v_n$ and $w_1, ..., w_n$ are a basis of V and W respectively. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$, where $\mathcal{M}$ is the linear map that gives the matrix for T with respect to the bases.

Proof: First we show $\mathcal{M}$ is injective. If $T \in \mathcal{L}(V, W)$ and $\mathcal{M}(T) = 0$, then $Tv_i = 0$ for all i. Since v_i's are a basis of V, this implies $T = 0$, so $\operatorname{null} M = \{0\}$.

Now surjectivity. Pick $A \in \mathbb{F}^{m,n}$. By the linear map lemma, there exists $T \in \mathcal{L}(V, W)$ such that

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j$$

for all $k = 1, ..., n$. Because $\mathcal{M}(T)$ equals A, the range of $\mathcal{M}$ equals $\mathbb{F}^{m,n}$ as desired. ■

Corollary: Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.

Proof: Follows since $\dim \mathbb{F}^{m,n} = mn$ and $\mathbb{F}^{m,n}$ is isomorphic to $\mathcal{L}(V, W)$. ■

3.4.3. Linear Maps Thought of as Matrix Multiplication

Definition (matrix of vector): Suppose $v \in V$ and $v_1, ..., v_n$ is a basis of V. The matrix of v with respect to the basis is in n by 1 matrix

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

where $b_1, ..., b_n$ are the scalars such that

$$v = b_1 v_1 + \cdots + b_n v_n.$$

This matrix is denoted $\mathcal{M}(v)$.

Proposition ($\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$): Suppose $T \in \mathcal{L}(V, W)$ and $v_1, ..., v_n$ is basis of V and $w_1, ..., w_m$ is basis of W. Let $1 \leq k \leq n$. Then k^{th} column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$ equals $\mathcal{M}(Tv_k)$.

Proof: Note that the k^{th} column of $\mathcal{M}(T)$ is defined at the scalars that make up Tv_k in W, so this follows by definition. ■

Proposition: Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Then we have

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v).$$

Proof: Suppose $v = \sum b_i v_i$. Thus

$$Tv = \sum b_i T v_i.$$

Then we have

$$\begin{aligned} \mathcal{M}(Tv) &= b_1\mathcal{M}(Tv_1) + \cdots + b_n\mathcal{M}(Tv_n) \\ &= b_1\mathcal{M}(T)_{\cdot,1} + \cdots + b_n\mathcal{M}(T)_{\cdot,n} \\ &= \mathcal{M}(T)\mathcal{M}(v). \end{aligned}$$

The last equality comes from the linear combination of columns proposition earlier. ■

Proposition (dim range T = column rank of $\mathcal{M}(T)$): Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then dim range T equals the column rank of $\mathcal{M}(T)$.

Proof: The linear map that takes $w \in W$ to $\mathcal{M}(w)$ is an isomorphism from W onto $\mathbb{F}^{m,1}$. The restriction of this isomorphism to range T (which equals $\operatorname{span}(Tv_1, \ldots, Tv_n)$) is an isomorphism from range T onto $\operatorname{span}(\mathcal{M}(Tv_1), \ldots, \mathcal{M}(Tv_n))$. For each $k \in \{1, \ldots, n\}$, the m by 1 matrix $\mathcal{M}(Tv_k)$ equals column k of $\mathcal{M}(T)$. Thus

$$\dim \operatorname{range} T = \text{column rank of } \mathcal{M}(T),$$

as desired. ■

3.4.4. Change of Basis

If $T \in \mathcal{L}(V)$, and we are using the same basis for input and output, then we generally have $\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(T, (v_1, \ldots, v_n), (v_1, \ldots, v_n))$.

Definition (identity matrix): Suppose n is a positive integer. The n by n matrix

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

with 1's on the diagonal and 0's everywhere else is called the identity matrix and denoted as I.

Definition (invertible): A square matrix A is called invertible if there is a square matrix B of the same size such that $AB = BA = I$. B is called the inverse of A and denoted by A^{-1}.

Corollary:

$$(AC)^{-1} = C^{-1}A^{-1}$$

and

$$(A^{-1})^{-1} = A.$$

Proof: Easy to verify. ■

Proposition: Suppose $T \in \mathcal{L}(U, W)$ and $S \in \mathcal{L}(V, W)$. If $u_1, \ldots, u_m$ is a basis of U, $v_1, \ldots, v_n$ is a basis of V, and $w_1, \ldots, w_p$ is a basis of W, then

$$\begin{aligned} &\mathcal{M}\big(ST, (u_1, \ldots, u_m), (w_1, \ldots, w_p)\big) \\ &= \mathcal{M}\big(S, (v_1, \ldots, v_n), (w_1, \ldots, w_p)\big)\mathcal{M}(T, (u_1, \ldots, u_m), (v_1, \ldots, v_n)). \end{aligned}$$

Proof: Follows from definition of matrix multiplication (this proposition was noted noted earlier, there the bases of the matrices are just explicitly written out). ■

Proposition (matrix of identity operator with respect to two bases): Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then the matrices

$$\mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)) \text{ and } \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))$$

are invertible, and each is the inverse of each other.

Proof: In the matrix multiplication proposition above, replace S and T with the identity map I, getting

$$I = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))\mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n)),$$

where the I on the left is the identity matrix. Swapping the bases yields

$$I = \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))\mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)).$$

By the definition of invertibility, we are done. ■

Example: Consifer the bases $(4, 2), (5, 3)$ and $(1, 0), (0, 1)$ of $\mathbb{F}^2$. Because $I(4, 2) = 4(1, 0) + 2(0, 1)$ and $I(5, 3) = 5(1, 0) + 3(0, 1)$, we have

$$\mathcal{M}(I, ((4, 2), (5, 3)), ((1, 0), (0, 1))) = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}.$$

The inverse of the matrix is

$$\begin{pmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{pmatrix}.$$

Thus by the previous result we have

$$\mathcal{M}(I, ((1, 0), (0, 1)), ((4, 2), (5, 3))) = \begin{pmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{pmatrix}.$$

Theorem (change of basis formula): Suppose $T \in \mathcal{L}(V)$. Suppose $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V. Let

$$A = \mathcal{M}(T, (u_1, \dots, u_n)) \text{ and } B = \mathcal{M}(T, (v_1, \dots, v_n))$$

and $C = \mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$. Then

$$A = C^{-1} BC.$$

Proof: In the matrix multiplication proposition, replace w_k with u_k are replace S with I to obtain

$$A = C^{-1} \mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n)),$$

where C^{-1} was obtained using the proposition above.

Use the multiplication proposition again, this time replacing w_k with v_k. Also replace T with I and S with T to obtain

$$\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n)) = BC.$$

Subbing this into the first equation we got yields the desired result. ■

Proposition (matrix of inverse equals inverse of matrix): Suppose $v_1, \dots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$ is invertible. Then $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$, where both matrices are with respect to the basis $v_1, \dots, v_n$.

Proof: By the matrix multiplication proposition, we have

$$\mathcal{M}(T^{-1}T) = \mathcal{M}(T^{-1})\mathcal{M}(T) \Rightarrow \mathcal{M}(T)^{-1} = \mathcal{M}(T^{-1}).$$

■

3.4.5. Problems

Problem (Exercise 3): Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

1. T is invertible.
2. $Tv_1, \dots, Tv_n$ is a basis of V for every basis $v_1, \dots, v_n$ of V.
3. $Tv_1, \dots, Tv_n$ is a basis of V for some basis $v_1, \dots, v_n$ of V.

Solution: 2 obviously implies 3. To show 3 implies 2, note that any other basis of V can be written in terms of $v_1, \dots, v_n$, and thus the outputs on those new basis vectors must be a basis, otherwise we would get a contradiction about $v_1, \dots, v_n$ being basis vectors.

Note that 1 implies 2 since that means T is injective and surjective, so range $T = V$, so the basis vector outputs will span V. 2 implies 1 since it implies null $T = \{0\}$, so T in injective, which means it's invertible.

Problem (Exercise 11): Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \Leftrightarrow S \text{ and } T \text{ are invertible.}$$

Solution: Suppose ST is invertible. Then null $ST = \{0\}$ and range $ST = V$. Note that for nonzero v, we must have $Tv \neq 0$, so T is invertible. Since T is invertible, T is also surjective, so S must also be injective in order for range $ST = V$, giving the desired result. If both S and T are invertible, then the null space of both is 0, so range $ST = V$.

3.5. Products and Quotients of Vector Spaces

3.5.1. Products of Vector Spaces

Definition (product of vector spaces): Suppose $V_1, ..., V_m$ are vector spaces over $\mathbb{F}$. The product $V_1 \times \cdots \times V_m$ is defined by

$$V_1 \times \cdots \times V_m = \{(v_1, ..., v_m) : v_i \in V_i\}.$$

Addition and scalar multiplication are defined pointwise.

Proposition: Product of vector spaces over $\mathbb{F}$ is a vector space over $\mathbb{F}$.

Proof: Easy to verify. ■

Proposition: Suppose $V_1, ..., V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \cdots \times V_m) = \sum \dim V_i.$$

Proof: Choose a basis for each V_i. Consider the list of vectors in the product space that only consists of a single vector over all vectors from each basis. This is a basis of the product space, and has length $\sum \dim V_i$. ■

Proposition: Suppose $V_1, ..., V_m$ are subspaces of V. Define a linear map $\Gamma : V_1 \times \cdots \times V_m \to V_1 + \cdots + V_m$ by

$$\Gamma(v_1, ..., v_m) = v_1 + \cdots + v_m.$$

Then $V_1 + \cdots + V_m$ is a direct sum if and only if Γ is injective.

Solution: Γ is injective if and only if null $\Gamma = \{0\}$. Thus the only way to get 0 is if we take each $v_i = 0$, which means $V_1 + \cdots + V_m$ is a direct sum.

Proposition: Suppose V is finite-dimensional and $V_1, ..., V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if

$$\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m.$$

Solution: Γ in the previous result is surjective. Thus Γ is injective if and only if

$$\dim(V_1 + \cdots + V_m) = \dim(V_1 \times \cdots \times V_m) = \sum \dim V_i.$$

3.5.2. Quotient Spaces

Definition ($v + U$): Suppose $v \in V$ and $U \subseteq V$. Then $v + U$ is the subset of V defined by

$$v + U = \{v + u : u \in U\}.$$

Definition (translate): For $v \in V$ and U a subset of V, the set $v + U$ is said to be a translate of U.

Definition (quotient space): Suppose U is subspace of V. Then the quotient space V/U is the set of all translates of U:

$$V/U = \{v + U : v \in V\}.$$

Example *(quotient spaces)*:
- If $U = \{(x, 2x) \in \mathbb{R}^2\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ that have slope 2.
- If U is a line in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all lines in $\mathbb{R}^3$ parallel to U.
- If U is a plane in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all planes in $\mathbb{R}^3$ parallel to U.

Proposition (two translates of a subspace are equal or disjoint): Suppose U is a subspace of V and $v, w \in V$. Then

$$v - w \in U \Leftrightarrow v + U = w + U \Leftrightarrow (v + U) \cap (w + U) \neq \emptyset.$$

Proof: First suppose $v - w \in U$. For $u \in U$ we have

$$v + u = w + ((v - w) + u) \in w + U.$$

Thus $v + U \subseteq w + U$. Similarly we have $w + U \subseteq v + U$. Thus $v + U = w + U$.

If the two sets are equal then they're obviously nondisjoint.

Now suppose the last statement is true. Thus there exist $u_1, u_2 \in U$ such that

$$v + u_1 = w + u_2.$$

Thus $v - w = u_2 - u_1 \in U$. ■

Definition (addition and scalar multiplication on V/U): Suppose U is a subspace of V. Then addition and scalar multiplication are defined on V/U by

$$(v + U) + (w + U) = (v + w) + U$$
$$\lambda(v + U) = (\lambda v) + U$$

for all $v, w \in V$ and all $\lambda \in \mathbb{F}$.

Proposition (quotient space is a vector space): Suppose U is a subspace of V. Then v/U, with the operations of addition and scalar multiplication defined above, is a vector space.

Proof: The potential problem with the definition above is that the translate representation is not unique. Suppose $v_1, v_2, w_1, w_2 \in V$ are such that

$$v_1 + U = v_2 + U \text{ and } w_1 + U = w_2 + U.$$

To show that the definition of addition on V/U above makes sense, we must show that $(v_1 + w_1) + U = (v_2 + w_2) + U$. We have

$$v_1 - v_2, w_1 - w_2 \in U.$$

Thus we have $(v_1 - v_2) + (w_1 - w_2) = (v_1 + w_1) - (v_2 + w_2) \in U$. This we have

$$(v_1 + w_1) + U = (v_2 + w_2) + U,$$

as desired. We show the same thing for scalar multiplication.

Note that identity of V/U is $0 + U = U$, and the inverse of $v + U$ is $(-v) + U$. ■

Definition (quotient map): Suppose U is a subspace of V. The quotient map $\pi : V \to V/U$ is the linear map defined by

$$\pi(v) = v + U$$

for each $v \in V$.

Proposition (dimension of quotient space): Suppose V is finite-dimensional and U is a subspace of V. Then $\dim V/U = \dim V - \dim U$.

Proof: Let π denote the quotient map. If $v \in V$, then $v + U = 0 + U$ if and only if $v \in U$, which means null $\pi = U$. The definition of π implies range $\pi = V/U$ (V/U is the space of all translates of U, and the right side of the quotient map can take on all $v \in V$). By the fundamental theorem of linear maps, we have $\dim V = \dim U + \dim V/U$, which yields the desired result.

■

Definition: Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T} : V/(\text{null } T) \to W$ by

$$\tilde{T}(v + \text{null } T) = Tv.$$

Proposition (null space and range of $\tilde{T}$): suppose $T \in \mathcal{L}(V, W)$. Then

1. $\tilde{T} \circ \pi = T$, where π is the quotient map of V into $V/(\text{null } T)$;
2. $\tilde{T}$ is injective;
3. range $\tilde{T} =$ range T;
4. $V/(\text{null } T)$ and range T are isomoprhic vector spaces.

Proof:

1. For $v \in V$, we have $\tilde{T}(\pi(v))\pi = \tilde{T}(v + \text{null } T) = Tv$, as desired.
2. Suppose $v \in V$ and $\tilde{T}(v + \text{null } T) = 0$. Then $Tv = 0$. Thus $v \in \text{null } T$. This means $v + \text{null } T = 0 + \text{null } T$. This implies that null $\tilde{T} = \{0 + \text{null } T\}$. Thus $\tilde{T}$ is injective.
3. By definition range $\tilde{T} =$ range T.
4. 2 and 3 imply that if we think of $\tilde{T}$ as mapping into range T, then $\tilde{T}$ is an isomorphism from $V/(\text{null } T)$ onto range T.

■

3.5.3. Problems

Problem (Exercise 2): Suppose that $V_1, \dots, V_m$ are vector spaces such that $V_1 \times \dots \times V_m$ is finite-dimensional. Prove that V_k is finite dimensional for each k.

Solution: WLOG we show V_1 is finite-dimensional, as the rest follow similarly. Note that $V_1 \times \{0\} \times \dots \times \{0\}$ is a subspace of $V_1 \times \dots \times V_m$, so it is finite dimensional. Consider

$$T : V_1 \times \{0\} \times \dots \times \{0\} \to V_1$$

such that

$$T((v_1, 0, \dots, 0)) = v_1,$$

where $v_1 \in V_1$. Note that the right side spans V_1, so range $T = V_1$, and also note that for the right side to be 0, $v_1 = 0$, so null $T = \{0\}$, meaning T is an isomorphism. Thus, since $V_1 \times \{0\} \times \dots \times \{0\}$ is finite-dimensional, V_1 must also be finite-dimensional.

Problem (Exercise 6): Suppose $v, x \in V$ and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Solution: Note that $0 \in U$, so we have $v = x + w \to v - x = w \in W$. This implies $v + W = x + W$. Similarly we have $v + U = x + U$. Both of these imply that $v + W = v + U$. Since both subspaces are translated by the same vector, the initial two subspaces must be equal.

Problem (Exercise 9): Prove that a nonempty subset A of V is a translate of some subspace V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.

Solution: Suppose $\lambda v + (1 - \lambda)w \in A$ for all v, w, λ. Fix $x \in A$ and consider $(-x) + A$. Pick $u \in (-x) + A$ and note that

$$\lambda u = \lambda v - \lambda x = -x + \lambda v + (1 - \lambda)x \in (-x) + A,$$

so $(-x) + A$ is closed under scalar mulitplication.

Now consider $a, b \in (-x) + A$. We have $\frac{a+b}{2} = -x + \frac{v+w}{2} \in (-x) + A$ by taking $\lambda = \frac{1}{2}$. Note that $0 \in (-x) + A$ since $x \in A$. Thus $(-x) + A$ is a subspace of V, so A is a translate of a subspace.

For the other direction, let $A = x + U$ for some subspace U of V. Note that

$$\lambda(x + v) + (1 - \lambda)(x + w) = x + (\lambda v + \lambda w + w) \in A$$

for $v, w \in U$, as desired.

Problem (Exercise 14): Suppose U and W are subspaces of V and $V = U \oplus W$. Suppose $w_1, ..., w_m$ is a basis of W. Prove that $w_1 + U, ..., w_m + U$ is a basis of V / U.

Solution: Suppose we have $w + U \in V / U$. Note that $U \cap W = \{0\}$. Thus, if $w \notin U$, then $w \in W$. If $w \in U$, then take $c_i = 0$ for each basis vector. Otherwise, $w = \sum c_i w_i$, so take those c_i for each basis vector.

Problem (Exercise 16): Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$, and $\varphi \neq 0$. Prove that $\dim V / (\text{null } \varphi) = 1$.

Solution: Note that $V / (\text{null } \varphi)$ and range φ are isomorphic. Since $\varphi \neq 0$, range $\varphi = \mathbb{F}$, so $\dim V / (\text{null } \varphi) = \dim \text{range } \varphi = 1$.

3.6. Duality

3.6.1. Dual space and Dual Map

Definition (linear functional): A *linear functional* on V is an element of $\mathcal{L}(V, \mathbb{F})$.

Definition (dual space): The *dual space* of V, denote by V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbb{F})$.

Proposition ($\dim V' = \dim V$): Suppose V is finite-dimensional. Then V' is also finite dimensional and

$$\dim V' = \dim V.$$

Proof: We have

$$\dim V' = \dim \mathcal{L}(V, \mathbb{F}) = (\dim V)(\dim \mathbb{F}) = \dim V.$$

■

Definition (dual basis): If $v_1, ..., v_n$ is a basis of V, then the *dual basis* of $v_1, ..., v_n$ is the list $\varphi_1, ..., \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j. \end{cases}$$

Proposition: Suppose $v_1, ..., v_n$ is a basis of V and $\varphi_1, ..., \varphi_n$ is the dual basis. Then

$$v = \varphi_1(v)v_1 + \cdots + \varphi_{n(v)}v_n$$

for each $v \in V$.

Proof: We can write $v = \sum c_i v_i$. Applying φ_j to each side yields

$$\varphi_j(v) = c_j.$$

Thus we can replace c_j with $\varphi_j(v)$, giving us the desired result. ■

Proposition: Suppose V is finite-dimensional. Then the dual basis of V is a basis of V'.

Proof: Suppose $v_1, ..., v_n$ is a basis of V. Let $\varphi_1, ..., \varphi_n$ be the dual basis. Suppose

$$\sum a_i \varphi_i = 0.$$

We have

$$\left(\sum a_i \varphi_i\right)(v_k) = a_k$$

for each k. The first equation implies $a_k = 0$ for all k, so the φ's are linearly independent. Since $\dim V' = n$, $\varphi_1, ..., \varphi_n$ must be a basis. ■

Definition (dual map): Suppose $T \in \mathcal{L}(V, W)$. The *dual map* of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by

$$T'(\varphi) = \varphi \circ T.$$

Proposition (algebraic properties of dual maps): Suppose $T \in \mathcal{L}(V, W)$. Then
- $(S + T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$,
- $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$,
- $(ST)' = T'S'$ for all $S \in \mathcal{L}(W, U)$.

Proof:
- $(S + T)'(\varphi) = \varphi \circ (S + T) = \varphi \circ S + \varphi \circ T = S'(\varphi) + T'(\varphi)$.
- $(\lambda T)'(\varphi) = \varphi \circ (\lambda T) = \lambda(\varphi \circ T) = \lambda T'(\varphi)$.
- Suppose $\varphi \in U'$. Then

$$(ST)'(\varphi) = \varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi)) = (T'S')(\varphi).$$

■

3.6.2. Null Space and Range of Dual Linear map

Definition (annihilator): For $U \subseteq V$, the annihilator of U, denoted by U^0, is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

Proposition: Suppose $U \subseteq V$. Then U^0 is a subspace of V'.

Proof: Note that $0 \in U^0$.

Suppose $\varphi, \psi \in U^0$. For $u \in U$ we have

$$(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0 + 0 = 0,$$

so U^0 is closed under addition. We can similarly show that U^0 is closed under scalar multiplication. ■

Proposition (dimension of the annihlator): Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^0 = \dim V - \dim U.$$

Below are two proofs, with the first being slicker.

Proof: Let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ for each $u \in U$. The fundamental theorem of linear maps on i' yields

$$\dim V' = \dim \operatorname{null} i' + \dim \operatorname{range} i'.$$

Suppose $i'(\varphi) = 0$. We have $i'(\varphi) = \varphi \circ i = 0$. Note that since range $i = U$, φ must be 0 on all of U, implying $\varphi \in U^0$. We also have $\dim V' = \dim V$, so we can write

$$\dim V = \dim U^0 + \dim \operatorname{range} i'.$$

If $\varphi \in U'$, then φ can be extended to a linear functional ψ on V. The definition of i' shows that $i'(\psi) = \varphi$. Thus $\varphi \in \operatorname{range} i'$, which implies that range $i' = U'$. Thus $\dim \operatorname{range} i' = \dim U' = \dim U$, and putting everthing together we have

$$\dim V = \dim U^0 + \dim U.$$

■

Proof: Let $u_1, \ldots, u_m$ be a basis of U and extend it to $u_1, \ldots, u_m, \ldots, u_n$ a basis of V. let $\varphi_1, \ldots, \varphi_n$ be the dual basis. We will show that $\varphi_{m+1}, \ldots, \varphi_n$ is a basis of U^0. Note that these maps are linear independent since they are part of the dual basis, so we just need to show that they span U^0.

Suppose $\varphi \in \operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n)$. We can write $\varphi = \sum_{i=m+1}^{n} c_i \varphi_i$. For any $\sum_{i=1}^{m} a_i u_i = u \in U$, we have

$$\left(\sum_{i=m+1}^{n} c_i \varphi_i\right)\left(\sum_{i=1}^{m} a_i u_i\right) = 0,$$

so $\operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n) \subseteq U^0$.

Now suppose $\varphi \in U^0$. We can write it as $\varphi = \sum_{i=1}^{n} c_i \varphi_i$. Note that since $u_i \in U$ for $i = 1, \ldots, m$ and since $\varphi \in U^0$, we have

$$0 = \varphi(u_i) = c_i.$$

Thus $\varphi = \sum_{i=m+1}^{n} c_i \varphi_i$, so $\varphi \in \operatorname{span}\left(\varphi_{m_1}, \ldots, \varphi_n\right)$. Thus $U^0 \subseteq \operatorname{span}(\varphi_{m+1}, \ldots \varphi(n))$, implying the desired result. ■

Proposition: Suppose V is finite-dimensional and U is a subspace of V. Then
- $U^0 = \{0\} \iff U = V$;
- $U^0 = V' \iff U = \{0\}$.

Proof: For the first bullet point, we have

$$\begin{aligned} U^0 = \{0\} &\iff \dim U^0 = 0 \\ &\iff \dim U = \dim V \\ &\iff U = V. \end{aligned}$$

Similarly, we have

$$\begin{aligned} U^0 = V' &\iff \dim U^0 = \dim V' \\ &\iff \dim U^0 = \dim V \\ &\iff \dim U = 0 \\ &\iff U = \{0\}. \end{aligned}$$

■

Proposition (null space of T'): Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then
- $\text{null } T' = (\text{range } T)^0$;
- $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$.

Proof: First suppose $\varphi \in \text{null } T'$. Thus $0 = T'(\varphi) = \varphi \circ T$. Hence

$$0 = (\varphi \circ T)(v) = \varphi(Tv) \text{ for every } v \in V.$$

Thus $\varphi \in (\text{range } T)^0$, which implies $\text{null } T' \subseteq (\text{range } T)^0$. Now suppose $\varphi \in (\text{range } T)^0$. Thus $\varphi(Tv) = 0$ for every vector $v \in V$. Hence $0 = \varphi \circ T = T'(\varphi)$, whcih implies $\varphi \in \text{null } T'$, which again implies $(\text{range } T)^0 \subseteq \text{null } T'$, so we're done.

The second bullet follows easily:

$$\begin{aligned} \dim \text{null } T' &= \dim (\text{range } T)^0 \\ &= \dim W - \dim \text{range } T \\ &= \dim W - (\dim V - \dim \text{null } T) \\ &= \dim \text{null } T + \dim W - \dim V. \end{aligned}$$

■

Proposition: Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is surjective} \iff T' \text{ is injective}.$$

Proof:

$$\begin{aligned}
T \in \mathcal{L}(V, W) \text{ is surjective} &\Longleftrightarrow \text{range } T = W \\
&\Longleftrightarrow (\text{range } T)^0 = \{0\} \\
&\Longleftrightarrow \text{null } T' = \{0\} \\
&\Longleftrightarrow T' \text{ is injective.}
\end{aligned}$$

■

Proposition (range of T'): Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then
- $\dim \text{range } T' = \dim \text{range } T$;
- $\text{range } T' = (\text{null } T)^0$.

Proof: We have

$$\begin{aligned}
\dim \text{range } T' &= \dim W' - \dim \text{null } T' \\
&= \dim W - \dim (\text{range } T)^0 \\
&= \dim \text{range } T.
\end{aligned}$$

For the second bullet, first suppose $\varphi \in \text{range } T'$. Thus there exists $\psi \in W'$ such that $\varphi = T'(\psi)$. If $v \in \text{null } T$, then

$$\varphi(v) = (T'(\psi))(v) = (\varphi \circ T)(v) = \psi(Tv) = \psi(0) = 0.$$

Thuse $\varphi \in (\text{null } T)^0$. This implies that $\text{range } T' \subseteq (\text{null } T)^0$.

To complete the proof, we show that $\text{range } T'$ and $(\text{null } T)^0$ have the same dimension. Note that

$$\begin{aligned}
\dim \text{range } T' &= \dim \text{range } T \\
&= \dim V - \dim \text{null } T \\
&= \dim (\text{null } T)^0,
\end{aligned}$$

where the first equality comes from the first bullet. ■

Proposition: Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is injective} \Longleftrightarrow T' \text{ is surjective.}$$

Proof:

$$\begin{aligned}
T \text{ is injective} &\Longleftrightarrow \text{null } T = \{0\} \\
&\Longleftrightarrow (\text{null } T)^0 = V' \\
&\Longleftrightarrow \text{range } T' = V'.
\end{aligned}$$

■

3.6.3. Matrix of Dual of Linear map

Proposition: Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$\mathcal{M}(T') = (\mathcal{M}(T))^t.$$

Proof: Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$. Suppose $1 \leq j \leq m$ and $1 \leq k \leq n$. From the definition of $\mathcal{M}(T')$ we have

$$T'(\psi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r.$$

The left side is $\psi_j \circ T$. Thus applying both sides of the equation to v_k (a basis vector) yields

$$\begin{aligned}(\psi_j \circ T)(v_k) &= \sum_{r=1}^{n} C_{r,j} \varphi_{r(v_k)} \\ &= C_{k,j}.\end{aligned}$$

We also have

$$\begin{aligned}(\psi_j \circ T)(v_k) &= \psi_{j(Tv_k)} \\ &= \psi_j \left(\sum_{r=1}^{m} A_{r,k} w_r \right) \\ &= \sum_{r=1}^{m} A_{r,k} \psi_j(w_r) = A_{j,k}.\end{aligned}$$

Thus we have $C_{k,j} = A_{j,k}$, which means $C = A^t$, giving us the desired result. ■

Heres another proof that column rank equals row rank.

Proposition: Suppose $A \in \mathbb{F}^{m,n}$. Then the column rank of A equals the row rank of A.

Proof: Define $T : \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$. Thus $\mathcal{M}(T) = A$, where $\mathcal{M}(T)$ is with respect to the standard bases of $\mathbb{F}^{n,1}$ and $\mathbb{F}^{m,1}$. Now

$$\begin{aligned}\text{column rank of } A &= \text{column rank of } \mathcal{M}(T) \\ &= \dim \text{range } T \\ &= \dim \text{range } T' \\ &= \text{column ranke of } \mathcal{M}(T') \\ &= \text{column rank of } A^t \\ &= \text{row rank of } A.\end{aligned}$$

■

3.6.4. Problems

Problem (Exercise 1): Explain why each linear functional is surjective or is the zero map.

Solution: If a linear functional is not the 0 map, then $\dim \operatorname{range} \varphi \geq 1$, and since $\dim \mathbb{F} = 1$, $\operatorname{range} \varphi = \mathbb{F}$.

Problem (Exercise 3): Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Solution: Pick a surjective map φ (must exist by exercise 1). Suppose $\varphi(v) = a$. Then the map $\frac{1}{a}\varphi$ works.

Problem (Exercise 4): Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

Solution: Note that since $U \neq V$, $\dim U < \dim V$. Thus, $\dim U^0 = \dim V - \dim U > 0$, so there exists nonzero maps in the annihilator of U.

Problem (Exercise 5): Suppose $T \in \mathcal{L}(V, W)$ and $w_1, \ldots, w_m$ is a basis of range T. Hence for each $v \in V$, there exist unique numbers $\varphi_1(v), \ldots, \varphi_{m(v)}$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,$$

thus defining functions $\varphi_1, \ldots, \varphi_m$ from V to $\mathbb{F}$. Show that each of the functions $\varphi_1, \ldots, \varphi_m$ is a linear functional on V.

Solution: Clearly $\varphi_{i(0)} = 0$, since $T(0) = 0 = \sum \varphi_i(0)w_i$.

We have

$$T(v + w) = \sum \varphi_i(v + w)w_i = \sum(\varphi_i(v) + \varphi_i(w))w_i = Tv + Tw,$$

so φ_i is additive. Similarly, its easy to show that φ_i is homogenous, so they are indeed linear functionals.

Problem (Exercise 10): Suppose m is a positive integer.
- Show that $1, x - 5, \ldots, (x - 5)^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$.
- What is the dual basis in the first bullet?

Solution: Clearly the list is a basis since the degree of every polynomial is different. We have

$$\varphi_k(p) = \frac{p^k(5)}{k!}.$$

Problem (Exercise 11): Suppose $v_1, ..., v_n$ is basis of V and $\varphi_1, ..., \varphi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

Solution: Since linear maps are entirely determined on the basis, we just need to show this equation holds for each basis vector. Plug in v_i to both sides of the equation to obtain

$$\psi(v_i) = \psi(v_1)\varphi_1(v_i) + \cdots + \psi(v_n)\varphi_n(v_i) = \psi(v_i)\varphi_i(v_i) = \psi_{v_i}.$$

Problem (Exercise 13): Show that the dual map of the identity operator on V is the identity operator on V'.

Solution: We have

$$I'(\varphi) = \varphi \circ I = \varphi.$$

Problem (Exercise 14): Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z).$$

Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbb{R}^2$ and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of $\mathbb{R}^3$.

- Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.
- Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.

Solution: Note that $T'(\varphi_1)(x, y, z) = 4x + 5y + 6z$. Similarly, $T'(\varphi_2)(x, y, z) = 7x + 8y + 9z$. Thus, we can write

$$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3 \text{ and } T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3.$$

Problem (Exercise 15): Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by

$$Tp = x^2 p(x) + p''(x).$$

- Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbb{R})$.
- Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = \int_0^1 p$. Evaluate $(T'(\varphi))(x^3)$.

Solution: For the first bullet, we have

$$T'(\varphi)(p) = (\varphi \circ T)(p) = \varphi(x^2 p(x) + p''(x)) = 8p(4) + 16p'(4) + p'''(4).$$

For the second bullet, we have

$$T'(\varphi)(x^3) = (\varphi \circ T)(x^3) = \varphi(x^5 + 6x) = \int_0^1 x^5 + 6x \,\mathrm{d}x = \frac{19}{6}.$$

Problem (Exercise 16): Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$T' = 0 \Longleftrightarrow T = 0.$$

Solution: Suppose $T' = 0$. Thus we have

$$\varphi \circ T = 0$$

for any $\varphi \in W'$. If $(\text{range } T)^0 = W'$, then range $T = \{0\}$, which implies $T = 0$, so now we can assume $(\text{range } T)^0 \neq W'$. Pick $\varphi \notin (\text{range } T)^0$. Thus if T is nonzero, there exists some v such that $\varphi(Tv) \neq 0$, but this is impossible. Thus, $T = 0$.

Now suppose $T = 0$. Then we have

$$T'(\varphi) = \varphi \circ T = \varphi(0) = 0$$

for all $\varphi \in W'$, so $T' = 0$.

Problem (Exercise 33): Suppose U is a subspace of V. Let $\pi : V \to V/U$ be the quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

- Show that π' is injective.
- Show theat range $\pi' = U^0$.
- Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

Solution: π' is injective if and only if π is surjective, and π is surjective by definition.

Suppose $\varphi \in (V/U)'$. For any $u \in U$, we have

$$\pi'(\varphi)(u) = (\varphi \circ \pi)(u) = \varphi(u + U) = \varphi(0 + U) = 0.$$

Thus for any $\varphi \in (V/U)$, $\pi'(\varphi) \in U^0$, so range $\pi' \subseteq U^0$.

Now suppose $\psi \in U^0$. Thus, $\psi(u) = 0$ for any $u \in U$. Define $\varphi \in (V/U)'$ as $\varphi(v + U) = \psi(v)$. It's easy to verify that φ is a linear map. For any $v \in V$, we have

$$\pi'(\varphi)(v) = (\varphi \circ \pi)(v) = \varphi(v + U) = \psi(v).$$

Thus there exists $\varphi \in (V/U')$ such that $\pi'(\varphi) = \psi$. This implies $U^0 \subseteq \text{range } \pi'$, giving us range $\pi' = U^0$.

If we consider π' as a map from $(V/U)'$ to U^0, then π' is both injective and surjective. Thus π' is an isomorphism from $(V/U)'$ onto U^0.

4. Polynomials

Pretty easy stuff.

Below is a linear algebra proof of the division "algorithm" for polynomials.

Proposition: Suppose that $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that

$$p = sq + r$$

and $\deg r < \deg s$.

Proof: Let $n = \deg p$ and let $m = \deg s$. If $n < m$, then take $q = 0$ and $r = p$, which satisfty all the necessary conditions. Now we can assume $n \geq m$.

The list

$$1, z, ..., z^{m-1}, s, zs, ..., z^{n-m}s$$

is linearly independent in $\mathcal{P}_n(\mathbb{F})$ because each polynomial has a different degree. Since $\dim \mathcal{P}_n(\mathbb{F}) = n + 1$ and the list has length $n + 1$, the list is a basis of $\mathcal{P}_n(\mathbb{F})$.

Because $p \in \mathcal{P}_n(\mathbb{F})$ and the list is a basis, there exist unique constans $a_0, a_1, ..., a_{m-1}, b_0, b_1, ..., b_{n-m}$ such that

$$\begin{aligned} p &= a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + b_0 s + b_1 zs + \cdots + b_{n-m} z^{n-m} s \\ &= \underbrace{a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}}_{r} + s\underbrace{(b_1 + b_1 z + \cdots + b_{n-m} z^{n-m})}_{q}. \end{aligned}$$

These choices of r and q satsify all the conditions. ■

4.1. Problems

Problem (Exercise 7): Suppose that m is a nonnegative integer, $z_1, ..., z_{m+1}$ are distinct elements of $\mathbb{F}$, and $w_1, ..., w_{m+1} \in \mathbb{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_{m(\mathbb{F})}$ such that

$$p(z_k) = w_k$$

for each $k = 1, ..., m + 1$.

Solution: Define the map $T : \mathcal{P}_{m(\mathbb{F})} \to \mathbb{F}^{m+1}$ by

$$T(p) = (p(z_1), ..., p(z_{m+1})).$$

Note that $T(0) = 0$ and that T is both additive and homogenous. Next we show T is injective. Suppose $T(p) = (0, ..., 0)$ and p is nonzero. Thus implies that p has $m + 1$ distinct roots. However, this is impossible, since p has degree at most m. Thus $p = 0$, meaning T is injective. Since the input and output space have the same dimension, this implies that T is invertible. Thus, given a unique output $(w_1, ..., w_{m+1})$, there exists a unique polynomial p such that $T(p) = (w_1, ..., w_{m+1})$.

5. Eigenvalues and Eigenvectors

5.1. Invariant Subspaces

5.1.1. Eigenvalues

Definition (operator): A linear map from a vector space to itself is called an *operator*.

Definition (invariant subspace): Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant* under T if $Tu \in U$ for every $u \in U$.

Definition (eigenvalue): Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an *eigenvalue* of T is there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Proposition (equivalent conditions to be an eigenvalue): Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent.
1. λ is an eigenvalue of T.
2. $T - \lambda I$ is not injective.
3. $T - \lambda I$ is not surjective.
4. $T - \lambda I$ is not invertible.

Proof: 1 and 2. are equivalent since $Tv = \lambda v \Rightarrow Tv - \lambda Iv = (T - \lambda I)(v) = 0$. The last three are all equivalent by previous results. ■

Definition (eigenvector): Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvector of T. A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Proposition (linearly independent eigenvectors): Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

Proof: Suppose this is false. Then there exists a smallest positive integer m such that there exists a linearly dependent list $v_1, \ldots, v_m$ of eigenvectors of T corresponding to distinct eigenvalues

$\lambda_1, \ldots, \lambda_m$ of T. Thus there exist $a_1, \ldots, a_m \in \mathbb{F}$, none of which are 0 (because of the minimmality of m) such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Apply $T - \lambda_m I$ to both sides to obtain

$$a_1(\lambda_1 - \lambda_m)v_1 + \cdots + a_{m-1}(\lambda_{m-1} - \lambda_m)v_{m-1} = 0.$$

Because the eigenvalues are distinct, none of the coefficients of the vectors are 0, so this contradicts the minimmality of m. ■

Proposition: Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Proof: Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $T \in \mathcal{L}(V)$. Then the corresponding eigenvectors $v_1, \ldots, v_m$ are linearly independent, which implies $m \leq \dim V$. ■

5.1.2. Polynomials Applied to Operators

T^m denotes $\underbrace{T \cdots T}_{m \text{ times}}$, with $T^0 = I$ and $T^{-m} = (T^{-1})^m$ if T is invertible.

For a $p \in \mathcal{P}(\mathbb{F})$, $p(T)$ denotes

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

We have $(pq)(T) = p(T)q(T) = q(T)p(T)$ (formal proof omitted because obvious).

Proposition: Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then null $p(T)$ and range $p(T)$ are invariant under T.

Proof: Suppose $u \in \text{null } p(T)$. Then $p(T)u = 0$. Thus

$$(p(T))(Tu) = T(p(T)u) = T(0) = 0.$$

Thus $Tu \in \text{null } p(T)$, so it is invariant under T.

Suppose $u \in \text{range } p(T)$. Then there exists $v \in V$ such that $u = p(T)v$. Thus

$$Tu = T(p(T)v) = p(T)(Tv),$$

so it is invariant under T. ■

5.1.3. Problems

5.2. The Minimal polynomial

Theorem (existence of eigenvalues): Evert operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

Proof: Let $\dim V = n$ and $T \in \mathcal{L}(V)$. Choose $v \in V$ with $v \neq 0$. Then

$$v, Tv, T^2 v, ..., T^n v$$

is not linearly independent since it has length $n + 1$. Thus there exists a nonconstant polynomial p of smallest degree such that

$$p(T)v = 0.$$

By the funamental theorem of algebra, we can write

$$p(z) = (z - \lambda)q(z).$$

This implies

$$0 = p(T)v = (T - \lambda I)(q(T)v).$$

Because q has smaller degree than p, $q(T)v \neq 0$. Thus the equation above implies that λ is an eigenvalue with eigenvector $q(T)v$. ■