Differential Eq Formulas + Derivations

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Separable Equations

Are in the form

$$N(y)y' = M(x).$$

Integrate both sides to get

$$\int N(y)\frac{dy}{dx}\,dx = \int M(x)\,dx.$$

Then $u = y(x) \implies du = y'(x)\,dx \implies$

$$\int N(u)\,du = \int M(x)\,dx.$$

Solve then back substitute.

Linear Equations

$$y' + P(x)y = f(x).$$

(can reduce to this by dividing by y' coefficient) Solutions:

$$y = \frac{\int e^{\int P(x)\,dx} f(x)\,dx + C}{e^{\int P(x)\,dx}}.$$

To get this get the integrating factor: $\mu = e^{\int P(x)\,dx}$. Note that $\mu' = P(x)\mu$. Multiplying the equation by μ yields

$$\mu y' + \mu P(x)y = \mu f(x).$$

Then reversing product rule on the left yields

$$\frac{d}{dx}(\mu y) = \mu f(x).$$

Integrating and dividing yields

$$y = \frac{\int \mu f(x)\,dx + C}{\mu},$$

ans substituting the expression for μ gives the desired result.

Exact Equations

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

If $M_y = N_x$, then M and N are the partial derivatives of a stream function. Let $\Psi(x,y)$ be a function such that $\Psi_x = M$ and $\Psi_y = N$. We can write the equation as

$$M(x,y) + N(x,y)\cdot y' = 0.$$

By chain rule, this is the same as

$$\frac{d}{dx}(\Psi(x,y)) = 0.$$

Thus,

$$\Psi(x,y) = C.$$

For some equations, M and N might not be partials of a stream function, but we can multiply by an integrating factor to make them partials of a stream function. There are two cases: when μ is only in x and when μ is only in y.

Suppose μ is only in y. Multiply by μ. Then we have

$$\frac{\partial}{\partial y}(\mu M) = \mu' M + \mu M_y = \mu N_x = \frac{\partial}{\partial x}(\mu N).$$

This is a linear differential equation in μ. Rewriting gives

$$\mu' = \left(\frac{N_x - M_y}{M}\right)\mu.$$

Using separation of variables yields

$$\mu = e^{\int \frac{N_x - M_y}{M}\,dy}.$$

If μ is only in y, then it's an integrating factor that can be used. In a similar spirit,

$$\mu = e^{\int \frac{M_y - N_x}{N}\, dx}$$

is an integrating factor if it's only in x.

Bernoulli Equations

$$y' + P(x)y = f(x)y^n.$$

Divide through by y^n to get

$$\frac{y'}{y^n} + P(x)\frac{1}{y^{n-1}} = f(x).$$

Let $u = \frac{1}{y^{n-1}}$. Then

$$u' = (1-n)\frac{1}{y^n} \cdot y'.$$

Substituting yields

$$\frac{1}{1-n} \cdot u' + P(x)u = f(x) \implies u' + (1-n)P(x)u = (1-n)f(x).$$

This is now a linear equation in u. Solve for u, then go back to y, done.

Homogenous Equations

Can be written as

$$y' = f\left(\frac{y}{x}\right) \text{ or } y' = f\left(\frac{x}{y}\right).$$

Will only show solution for first, second is identical. Let $u = \frac{y}{x}$. Then $y' = u + xu'$. Substituting yields

$$u + x\frac{du}{dx} = f(u).$$

Rewriting and using separation yields

$$\int \frac{1}{f(u) - u}\, du = \ln(x) + C.$$

Solve for u and go back to y, done.

Euler's Method

Let $y' = f(x, y)$. Given a point (x_1, y_1) that's on the curve of a solution, and a step h, an approximation for the point on the curve h after x_1 is

$$y_2 = y_1 + hf(x_1, y_n).$$

In general, so obtain an approximation of y_n given (x_{n-1}, y_{n-1}), use

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}).$$

Improved method:

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, y'_{n+1})),$$

where

$$y'_{n+1} = y_n + hf(x_n, y_n).$$

Homogenous Linear Equations with Constant Coefficients

Of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = 0.$$

Solutions come in the form e^{rt}, where is a root of the characteristic polynomial of the differential equation. If $r = \alpha \pm \beta i$ is complex, then the solution is

$$e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$

If r is real with multiplicity m, then the solutions are

$$c_0 e^{rx} + c_1 x e^{rx} + \cdots + c_{m-1} x^{m-1} e^{rx}.$$

If $r = \alpha \pm \beta i$ is complex with multiplicity m, then the solutions are

$$\begin{aligned} &e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) + \\ &xe^{\alpha x}(c_3 \cos(\beta x) + c_4 \sin(\beta x)) + \\ &\vdots \\ &+ x^{m-1} e^{\alpha x}(c_{2m-1} \cos(\beta x) + c_{2m} \sin(\beta x)). \end{aligned}$$

Over all solutions of the characteristic polynomial, add up all their corresponding differential equation solutions to get the final answer.

Nonhomogenous Linear Equations with Constant Coefficients

Of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = g(x).$$

Method of Undetermined Coefficients

First find the solution to the homogenous version of the equation, i.e.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = 0.$$

This is the complementary solution, y_c. Next, we need to find the particular solution y_p, which is based on $g(x)$. Basically guess the form of a solution with variable coefficients, plug into the equation and solve for the coefficients.

Examples:

- $g(x) = 4x + 5 \implies y_p = Ax + B$
- $g(x) = 7\cos(6x) \implies y_p = A\cos(6x) + B\sin(6x)$
- $g(x) = e^{-4x} \implies y_p = Ae^{-4x}$
- $g(x) = xe^{-5x} + 7e^{-5x} + 10x^2 e^{-5x} \implies y_p = (Ax^2 + Bx + C)e^{-5x}$

- $g(x) = x^2\cos(4x) + 20e^{7x}\cos(4x) \implies (Ax^2 + Bx + C)\cos(4x) + (Dx^2 + Ex + F)\sin(4x) + e^{7x}(G\cos(4x) + H\sin(4x))$

If a section of the particular solution is a section in the complementary solution, multiply the section in the particular solution by x. Keep doing until no longer part of the complementary solution.

Table for determining "trial" solutions of $ay''(x) + by'(x) + cy(x) = f(x)$

Case	$f(x)$ has a term that is a constant multiple of	trial solution $y_p(x)$ is a linear combination of terms of the form
I.	$H(x) = A_kx^k + \ldots + A_2x^2 + A_1x + A_0$, a polynomial of degree k	$x^m(A_kx^k + \ldots + A_2x^2 + A_1x + A_0)$, where m is the smallest integer such that none of the $(k+1)$ terms solves the corresponding homogeneous problem
II.	$e^{\alpha x}$	$Cx^me^{\alpha x}$, where m is the smallest integer such that $x^me^{\alpha x}$ does not solve the corresponding homogeneous problem
III.	$\cos(\beta x)$ or $\sin(\beta x)$	$x^m(A\cos(\beta x) + B\sin(\beta x))$, where m is the smallest integer such that neither of the two terms solves the corresponding homogeneous problem
IV.	$H(x)e^{\alpha x}$	$x^m(A_kx^k + \ldots + A_2x^2 + A_1x + A_0)e^{\alpha x}$, where m is the smallest integer such that none of the $(k+1)$ terms solves the corresponding homogeneous problem
V.	$H(x)\cos(\beta x)$ or $H(x)\sin(\beta x)$	$x^m(A_kx^k + \ldots + A_2x^2 + A_1x + A_0)\cos(\beta x) + x^m(Bx^k + \ldots + B_2x^2 + B_1x + B_0)\sin(\beta x)$, where m is the smallest integer such that none of those $2(k+1)$ terms solves the corresponding homogeneous problem
VI.	$e^{\alpha x}\cos(\beta x)$ or $e^{\alpha x}\sin(\beta x)$	$x^m(Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x))$, where m is the smallest integer such that neither of the two terms solves the corresponding homogeneous problem
VII.	$H(x)e^{\alpha x}\cos(\beta x)$ or $H(x)e^{\alpha x}\sin(\beta x)$	$x^m\left[(A_kx^k + \ldots + A_2x^2 + A_1x + A_0)e^{\alpha x}\cos(\beta x)\right] + x^m\left[(Bx^k + \ldots + B_2x^2 + B_1x + B_0)e^{\alpha x}\sin(\beta x)\right]$ where m is the smallest integer such that none of those $2(k+1)$ terms solves the corresponding homogeneous problem
	α may be any real number; β may be any real number; $H(x)$ must be a polynomial of degree k	

Variation of Parameters

$$y_c + y_p$$

where y_c is the solution to the complementary equation and

$$y_p = -y_1\int \frac{y_2g(t)}{y_1y_2' - y_1'y_2}\,dt + y_2\int \frac{y_1g(t)}{y_1y_2' - y_1'y_2}\,dt$$

where y_1 and y_2 are the fundamental solutions to the complementary equation. Note this works for any linear differential equation as long as we have the fundamental set of solutions (this version of variation of parameters applies to second order differential equations).

To derive this result, we look for u_1 and u_2 such that

$$y_p = u_1 y_1 + u_2 y_2$$

is a solution to the equation. Taking the derivative yields

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

Now we assume that $u_1 y_1' + u_2 y_2' = 0$ to make things easier. So now

$$y_p' = u_1 y_1' + u_2 y_2'.$$

Taking another derivative yields

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''.$$

Plugging these into the original differential equation $(p(t)y'' + q(t)y' + r(t)y = g(t))$ and rearranging yields

$$p\left(u'_1 y'_1 + u'_2 y'_2\right) + u_1\left(p y''_1 + q y'_1 + r y_1\right) + u_2\left(p y''_2 + q y'_2 + r y_2\right) = g.$$

Since y_1 and y_2 are fundamental solutions to the complementary equation, we have

$$u_1' y_1' + u_2' y_2' = \frac{g}{p}.$$

Assume $p = 1$ (this can be done by dividing out the leading coefficient in the original differential equation).

We now have the following system:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g. \end{aligned}$$

Solving for u_1' and u_2' yields

$$u_1' = -\frac{y_2 g}{y_1 y_2' - y_2 y_1'}$$

and

$$u_2' = \frac{y_1 g}{y_1 y_2' - y_2 y_1'}.$$

Note that $W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0$ since y_1 and y_2 are fundamental solutions. Thus, we have that

$$y_p = -y_1 \int \frac{y_2 g(t)}{y_1 y_2' - y_1' y_2}\, dt + y_2 \int \frac{y_1 g(t)}{y_1 y_2' - y_1' y_2}\, dt.$$

For nth order equations, do the same as above, just with more us and ys. We then obtain the following system of equations:

$$\begin{aligned}
{u'}_1 y_1 + {u'}_2 y_2 + \cdots + {u'}_n y_n &= 0 \\
{u'}_1 {y'}_1 + {u'}_2 {y'}_2 + \cdots + {u'}_n {y'}_n &= 0 \\
{u'}_1 {y''}_1 + {u'}_2 {y''}_2 + \cdots + {u'}_n {y''}_n &= 0 \\
&\vdots \\
{u'}_1 y_1^{(n-2)} + {u'}_2 y_2^{(n-2)} + \cdots + {u'}_n y_n^{(n-2)} &= 0 \\
{u'}_1 y_1^{(n-1)} + {u'}_2 y_2^{(n-1)} + \cdots + {u'}_n y_n^{(n-1)} &= g.
\end{aligned}$$

Using Cramer's Rule, we obtain that

$$u_i' = \frac{W_i}{W},$$

where W is the Wronskian of y_1, y_2, $\ldots$, y_3 and W_i is the determinant when you replace the ith column of the Wronskian with

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g \end{pmatrix}.$$

Then we have

$$y_p = \sum_{i=1}^{n} y_i \int \frac{W_i}{W}\, dt.$$

Cauchy-Euler Equations

$$x^2y'' + axy' + by = 0$$

Let a possible solution be $y = x^m$. Plugging this into the equation yields

$$x^2(m(m-1)x^{m-2}) + ax(mx^{m-1}) + bx^m = 0 \implies m(m-1) + am + b = 0 \implies$$

$$m^2 + (a-1)m + b = 0.$$

For two distinct roots we have

$$y = c_1x^{m_1} + c_2x^{m_2}.$$

For a repeated real root we have

$$y = c_1x^m \ln(x) + c_2x^m.$$

For complex roots $\alpha \pm \beta i$ we have

$$y = c_1x^\alpha \cos(\beta \ln(x)) + c_2x^\alpha \sin(\beta \ln(x)).$$