Real Analysis Solutions

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Chapter 3: Sequences

3.1

For $\varepsilon = .0005$, there is some N such that for all $n > N$, $|a_n - .001| < .0005$, which implies that a_n is positive for $n > N$. Thus, finitely many terms are negative.

3.2

(a) Even terms are 0, odd terms are 1

(b) No such sequence exists. Suppose otherwise, and assume it converges to a. Pick $0 < \varepsilon < |a|$. Then, there exists N such that for all $n > N$, $|a_n - a| < \varepsilon$. However, since there are infinitely many 0s, there's some $k > N$ such that $a_k = 0$, in which case we get $|0 - a| < \varepsilon$, a contradiction.

(c) No such sequence exists. Suppose otherwise, and assume it converges to $a < 0$. Pick $0 < \varepsilon < |a|$. Then, there exists N such that for all $n > N$, $|a_n - a| \varepsilon$. Unwinding, this yields

$$a - \varepsilon < a_n < a + \varepsilon < 0,$$

a contradiction.

(d) $(a_n) = \frac{\sqrt{2}}{2^n}$

3.4

(a) Fix any $\varepsilon > 0$. Set $N = \frac{1}{\varepsilon^2}$. Then for any $n > N$

$$\left|\left(7 - \frac{1}{\sqrt{n}}\right) - 7\right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{(1/\varepsilon^2)}} = \varepsilon.$$

(b) Fix any $\varepsilon > 0$. Set $N = \frac{12}{25\varepsilon} - \frac{1}{5}$. Then for any $n > N$

$$\left|\left(\frac{2n-2}{5n+1}\right) - \frac{2}{5}\right| = \frac{12}{25n+5} < \frac{12}{25N+5} = \frac{12}{25\left(\frac{12}{25\varepsilon} - \frac{1}{5}\right) + 5} = \varepsilon.$$

(c) Fix any $\varepsilon > 0$. Set $N = \left(\sqrt{\frac{1}{\varepsilon^2} - \frac{51}{4}} - \frac{1}{2}\right)^2$. Then for any $n > N$

$$\left|\left(7 - \frac{1}{\sqrt{n + \sqrt{n} + 13}}\right) - 7\right| = \frac{1}{\sqrt{n + \sqrt{n} + 13}} < \frac{1}{\sqrt{N + \sqrt{N} + 13}} = \varepsilon.$$

3.5

$(a_n) = -\frac{1}{n}$

3.7

(a) No, $\varepsilon = 0.5$ has no working N.

(b) Will never come within .5 of a limit.

(c) Eventually constant.

3.8

(a) Fix $\varepsilon/2 > 0$. There exists some N_1 such that for all $n > N_1$, $|a_n - a| < \varepsilon/2$. Similarly, there exists some N_2 such that for all $n > N_2$, $|b_n - b| < \varepsilon/2$. Pick $N = \max\{N_1, N_2\}$. Then, for any $n > N$,

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(b) If $c = 0$, then it holds true. For nonzero c, fix $\varepsilon/|c| > 0$. There exists from N such that for all $n > N$, $|a_n - a| < \varepsilon/|c|$. Then, for any $n > N$,

$$|c \cdot a_n - c \cdot a| = |c| \cdot |a_n - a| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

3.9

Fix $M > 0$. Then there exists N_1 and N_2 such that for all $n > N_1$, $a_n > \frac{M}{2}$, and for all $n > N_2$, $b_n > \frac{M}{2}$. Let $N = \max\{N_1, N_2\}$. Then, for all $n > N$,

$$a_n + b_n > \frac{M}{2} + \frac{M}{2} = M.$$

3.10

Fix $\varepsilon > 0$. Then there exists N_1 such that $|a_{2n} - L| < \epsilon$ for all $n > N_1$ and there exists N_2 such that $|a_{2n-1} - L| < \epsilon$ for all $n > N_2$. Now let $N = \max\{N_1, N_2\}$. Then, for any $k > N$, a_k will be within ε of L, since it will satisfy the other two inequalities, regardless of the parity of k.

3.11

False. Take $(a_n) - \frac{1}{n}$.

3.13

(a) Basically the same as 3.8 (a)

(b) We show that $\left(\frac{1}{b_n}\right) \to \frac{1}{b}$, and then the general rule follows from multiplication rule. Since b_n, converges, b_n is bounded. Let C be its bound. Then, there's some N such that for all $n > N$, $|b_n - b| < C\,|b|\,\varepsilon$, where $\varepsilon > 0$ is fixed. Then, for any $n > N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| < \frac{1}{C\,|b|} \cdot C\,|b|\,\varepsilon = \varepsilon.$$

3.15

$(a_n) = (-1)^n$, $(b_n) = (-1)^{n+1}$

3.16

$(a_n) = (-1)^n$, $(b_n) = (-1)^{n+1}$

3.17

It's not possible. For sufficiently large n, b_n will be come much larger than a_n minus any limiting value, eclipsing any ε.

3.18

(a) $|a_n| \le C$. Fix $\varepsilon > 0$. Then there exists N such that for any $n > N$, $|b_n - 0| < \varepsilon / C$. Then, for any $n > N$,

$$|a_n b_n - 0| = |a_n| \cdot |b_n - 0| < C \cdot \frac{\varepsilon}{C} = \varepsilon.$$

(b) $(a_n) = n^2$, $(b_n) = \frac{1}{n}$

3.21

Yes. Use Bolzano-Weierstrass on the subsequence.

3.22

(a) Suppose $L > M$. Then pick $0 < \varepsilon < L - M$. Then, $|a_n - L| \ge L - M > \varepsilon$, so the initial hypothesis can't be true.

(b) Suppose $L > M$. Pick $0 < \varepsilon < (L - M)/2$. Then its easy to show that $a_n > b_n$, a contradiction.

3.23

(a) Reverse triangle inequality.

(b) $(a_n) = (-1)^n$

3.25

Taking the negative of the sequence. This makes it monotone increasing and bounded above. Note that the infinum becomes the supremum, so the monotone convergence theorem for increasing sequences finishes it off.

3.26

Do the same as above.

3.27

Every even term is 1, and every odd term is n.

3.29

Note that for some K we have $|a_i - a| < \frac{\varepsilon}{2}$ for all $i > K$. Let $M = \max |a_i - a|$. Then for some N we have $\frac{KM}{n} < \frac{\varepsilon}{2}$ for all $n > N$. Now we have

$$\begin{aligned}
|b_n - a| &= \sum_{i=1}^{n} \frac{|a_i - a|}{n} \\
&= \sum_{i=1}^{K} \frac{|a_i - a|}{n} + \sum_{i=K+1}^{n} \frac{|a_i - a|}{n} \\
&< \frac{KM}{n} + \frac{(n-K)\varepsilon}{2n} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

3.30

Note that b_n is monotone decreasing, since the supremum of the sequence can never go up. Since (a_n) is bounded, b_n must also be bounded, therefore b_n converges by monotone convergence.

3.32

(a_n) is a subsequence of (a_n).

3.33

Let $M > 0$. Then there exists N such that for all $n > N$, $a_n > M$. Now suppose (a_{n_k}) is a subsequence of (a_n). Note that for any k, $n_k > k$, so clearly $a_{n_k} > M$ holds for all $k > N$.

3.34

$(a_n) = (-1)^n n$, $(b_n) = (-1)^n n$

3.35

$(a_n) = (-1)^n n$

3.37

(a) Induction on n

(b) Show increasing with induction on n, monotone boundedness shows it converges.

(c) We just need so show that $\sup(\{a_n : n \in \mathbb{N}\}) = 2$, after which the monotone convergence theorem will give us the desired result. We already showed 2 is an upper bound. Pick some $\varepsilon > 0$. We need to show there exists some N such that for all $n > N$, $a_n > 2 - \varepsilon$. Sub in the expression for a_n, square, subtract two, there the right side will be less than $2 - \varepsilon$. Rewrite the right as $2 - \varepsilon_1$, and repeat. Eventually it will be less than 0, and the number of iterations gives N.

3.38

$(a_n) = n$

3.39

Since a_n and b_n are Cauchy, we have

$$|a_m - a_n|, |b_m - b_n| < \frac{\varepsilon}{2}$$

for all $m, n > K$ for some K. Then we have

$$\begin{aligned} |c_m - c_n| &= ||a_m - b_m| - |b_n - a_n|| \\ &\le |a_m - b_m + b_n - a_n| \\ &= |(a_m - a_n) - (b_m - b_n)| \\ &\le |a_m - a_n| + |b_m - b_n| \le \varepsilon. \end{aligned}$$

3.40

(a) None exist, since a Cauchy sequence is bounded.

(b) $(a_n) = n^{(-1)^n}$

3.43

Note that since a_n converges, it must be bounded, which implies $\sqrt{a_n}$ is bounded, say by C. For every $n > N$ for some N we have

$$|a_n - L| < \varepsilon(C + L) \implies \left|\sqrt{a_n} - \sqrt{L}\right|(C + L) < \varepsilon(C + L) \implies \left|\sqrt{a_n} - \sqrt{L}\right| < \varepsilon.$$

3.44

This is just the contrapositive of every subsequence converging to the same limit implies the sequence converges to that limit.

3.45

Note that $0 \le b_n \le a_n$ and that (b_n) is increasing. Thus it converges.

Chapter 4: Series

4.1

(a) Converges conditionally

(b) Diverges

(c) Converges

(d) Converges

(e) Diverges

(f) Diverges

(g) Converges conditionally

(h) Diverges

(i) Diverges

4.2

s_n alternates between 0 and 1, so the series diverges.

4.6

We have $s_n = a_1 - a_{n+1}$, and since a_n approaches infinity, the partial sums approach a_1.

4.7

$(a_n) = (-1)^n \frac{1}{\sqrt{n}}$

4.17

Let $b_n = \left|\frac{a_{k+1}}{a_k}\right|$ for convenience. Pick a such that $r < q < 1$. Since $b_n \to r$, we have

$$|b_n - r| < q - r$$

for $n \geq N$ for some N. Rewriting this yields

$$2r - q < b_n < q.$$

Expanding the right side of the inequality gives

$$|a_{n+1}| \leq |a_n| \, q.$$

Now we can show the sum converges absolutely. Rewrite as

$$\sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_k| \, .$$

Note that $|a_k| < |a_N| \, q^{k-N}$ for $k \geq N$. Thus we have the sum is less than

$$\sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_N| \, q^{k-N}.$$

The second sum is a convergent geometric series, and the first sum is finite, and since the original sum is less than this one, it must converge.

4.18

First we prove the series converges when $\rho < 1$. Note for some N we have

$$\left|a_n^{1/n} - \rho\right| < \frac{1-\rho}{2}$$

for all $n \geq N$. This is the same as

$$\frac{3\rho - 1}{2} < a_n^{1/n} < \frac{\rho + 1}{2},$$

which implies $a_n < \left(\frac{\rho+1}{2}\right)^n$ for $n \geq N$. Thus we have

$$\sum_{k=1}^{\infty} a_k < \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{\infty} \left(\frac{\rho+1}{2}\right)^k.$$

By geometric series the right side is finite, so the left side must also be finite.

Now suppose $\rho > 1$. Note for some N we have

$$\left|a_n^{1/n} - \rho\right| < \frac{\rho-1}{2}$$

for all $n \geq N$. This is the same as

$$\frac{p\rho+1}{2} < a_n^{1/n} < \frac{3\rho-1}{2},$$

which implies $a_n > \left(\frac{\rho+1}{2}\right)^n$ for $n \geq N$. Thus we have

$$\sum_{k=1}^{\infty} a_k > \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{\infty} \left(\frac{\rho+1}{2}\right)^k.$$

The right side diverges by geometric series, so the left side must diverge as well.

4.19

$a_n = \sqrt{n}$.