Linear Algebra Done Right Notes

Nikhil Reddy

16 February 2024

Contents

§1 Vector Spaces

Learn about the notion of a vector space.

§1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

$\mathbb{R}$ and $\mathbb{C}$ are defined as usual.
Properties of $\mathbb{C}$:

- $\mathbb{R}$ and $\mathbb{C}$ are abelian groups over addition and multiplication (commutative, associative, have identities, inverses, and are distributive)
- Actually, both are fields, so these are already implied

Example (Complex Commutativity)
$\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Solution. Spam commutativity for addition and multiplication. □

Definition (Elements in $\mathbb{F}^n$ and Operations)**.** $\mathbb{F}^n$ is all n-tuples

$$(x_1, x_2, \ldots, x_n)$$

with $x_i \in \mathbb{F}$. Addition is pointwise, i.e for $x, y \in \mathbb{F}^n$, we have that

$$x + y = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + x_2, \ldots, x_n + y_n).$$

This is commutative by $\mathbb{F}$ being a field. $0 \in \mathbb{F}^n$ is $0 = (0, 0, \ldots, 0)$ with n 0s. The additive inverse of x is $-x$, which is also taking the negative of every component. Scalar multiplication for $\lambda \in \mathbb{F}$ is

$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n).$$

Scalar multiplication is distributive over addition. Addition is distributive over scalar multiplication. Scalar multiplication is associative. Addition is associative. 1 is the identity for scalar multiplication. 0 is the identity for addition.

Exercises are spamming these definitions.

§1.2 Definition of Vector Space

We define addition and scalar multiplication in terms of the set V.

Definition (Addition, Scalar Multiplication)**.** Sentence is useless, meant to make bullets look nice

- An addition on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A scalar multiplication on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.

Definition. A vector space is a set V with addition and scalar multiplication with commutativity, associativity, additive identity, additive inverse, multiplicative identity, and distributive properties. Elements of a vector space are called vectors or points.

Example
$\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces, just verify the properties hold. $\mathbb{F}^\infty$ is also a vector space

Definition ($\mathbb{F}^S$). S is a set, then $\mathbb{F}^S$ is the set of functions from S to $\mathbb{F}$.

- For $f, g \in \mathbb{F}^S$, $f + g \in \mathbb{F}^S$ is the function defined by $f(x) + g(x)$ for all $x \in S$.
- For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, $\lambda f \in \mathbb{F}^S$ is defined as $\lambda f(x)$ for all $x \in$.

Example ($\mathbb{F}^S$ is vector space)
The 0 function $0(x) = 0$ for all $x \in \mathbb{F}$ is the additive identity. The additive inverse of $f(x)$ is $(-f)(x) = -f(x)$. All the other properties of a vector space hold by spamming axioms.

Example ($\mathbb{F}^n$ is a special case of above)
$S = \{1, 2, \ldots, n\}$, where each element maps to a element of $\mathbb{F}$.

Proposition (Unique Additive Identity)
A vector space has a unique additive identity.

Solution. 0, $0'$ are identities,

$$0' = 0' + 0 = 0 + 0' = 0.$$

$\square$

Proposition (Unique Additive Inverse)
Every element in a vector space has a unique additive inverse.

Solution. $v \in V$, with w, w' as inverses.

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

$\square$

Proposition
$0v = 0$ for every $v \in V$.

Solution. $0v = (0 + 0)v = 0v + 0v$. $\square$

> **Remark.** We have to use $0 = 0 + 0$ since we have to use distributive property to connect scalar multiplication and vector addition.

Proposition

$a0 = 0$ for scalar a.

Solution.

$$a0 = a(0 + 0) = a0 + a0.$$

□

Proposition

$(-1)v = v-$.

Solution.

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0.$$

Adding additive inverse of v to both sides gives (-1)v = -v, as desired. □

> **Remark.** A lot of the above stuff is axiomatic/definitional proofs, and more reminiscent of Tao's Real Analysis (should make notes for this too).

Problem (Exercise 1)**.** Prove that $-(-v) = -v$ for every $v \in V$.

Solution.

$$-(-v) + (-v) = (-1)(-v) + (-1)v = (-1)(-v + v) = (-1)0 = 0,$$

so $-(-v)$ is the additive inverse of $-v$, so $-(-v) = v$, as desired. □

Problem 1.1 (Exercise 3)**.** Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution. First existence. Adding $-v$ to both sides gives $3x = w - v$. Multiplying by $\frac{1}{3}$ on both sides gives $x = \frac{1}{3}(w - v)$.

Now uniqueness. Suppose y, y' both satisfy. Then $y = \frac{1}{3}(w - v) = y'$. □

Problem (Exervise 4)**.** The empty set is not a vector space. Why?

Solution. Doesn't satisfy additive identity. There are no elements, so there cannot exist an additive identity. □

Problem (Exercise 5)**.** Show that the additive inverse condition in the definition of a vector space can be replaced with

$$0v = 0 \text{ for all} v \in V.$$

Solution.

$$0v = (1 + (-1))v = 1v + (-1)v = v + (-1)v = 0,$$

so there exists a w such that $v + w = 0$, as desired. □

Problem. Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$?

Solution. No. We have that $(2-1)\infty = (1)\infty = \infty$ and $2\infty - 1\infty = \infty + (-\infty)0$, so it is not distributive, so it is not a vector space. □

§1.3 Subspaces

Definition (Subspace)**.** A subset U of V is called a subspace of V is U is also a vector space.

Example

$(x_1, x_2, 0)$ with $x_1, x_2 \in F$ is a subspace of $\mathbb{F}^3$.

Proposition (Conditions for Subspaces)

A $U \subset V$ is a subspace of V if and only if U satisfies the conditions below.

- Additive Identity: $0 \in U$
- Closed under addition
- Closer under scalar multiplication

Solution. If U is a subspace of V, then it satisfies the properties by definition.

First condition ensures additive identity. Second and third make sure addition and scalar work. Additive inverse holds by scalar multiplication by -1, and associativity and distributivity hold because that hold on V. □

Example (Subspaces)

We use subspace conditions to show all below as subspaces.

- If $b \in \mathbb{F}$ then
$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$
is a subspace if and only if and only if $b = 0$.
  1. When $b = 0$, we can easily verify all the subspace conditions hold.
  2. If we have a subspace, then $0 \in U$, so $0 = x_3 = 5(0) + b$ means b must be 0.
- The set of continuous real-valued functions on the interval $[0,1]$ is a subspace of $\mathbb{R}^{[0,1]}$.
  1. $f(x) = 0$ is the additive identity for $\mathbb{R}^{[0,1]}$, and it is continuous on $[0,1]$, so it is in U.
  2. Adding two functions in U gives a function in U. Same for scalar multiplication.
- The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
  1. $f(x) = 0$, $f \in U$.
  2. The sum of two differentiable functions is a differentiable function. Also closed under scalar multiplication.
- The set of differentiable functions real-valued functions f on the interval $(0,3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.
  1. If $b = 0$, conditions hold trivially.
  2. If U is a subspace, then $b = (f+g)'(2) = f'(2) + g'(2) = 2b$, so $b = 0$.
- The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^\infty$.
  1. $0 = (0, 0, \dots) \in \mathbb{C}^\infty$.
  2. Adding two sequences with limit 0 gives a sequence with limit 0. Same for scalar multiplication.

Definition (Sum of Subsets)**.** Suppose $U_1, U_2, \dots U_m$ are subsets of V. The sum of $U_1, U_2, \dots U_m$ is the set of all possible sums of elements in the subsets. So
$$U_1 + \dots + U_m = \{u_1 + \dots u_m \mid u_i \in U_i\}.$$

Example

We have $U = \{(x, 0, 0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}$ and $W = \{(0, y, 0) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}$. Then,
$$U + W = \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}.$$

Example

U, W are subsets of $\mathbb{F}^4$, $U = (x, x, y, y)$, $W = (x, x, x, y)$. Then

$$U + W = (x, x, y, z)$$

since when adding two elements from U and W, the sum always has equals first and second components. The sum of the third components can be arbitrary. Same for the fourth.

Proposition (Minimality of Subspace Sums)

If $U_1, \ldots, U_m$ are subspaces of V, then $U' = U_1 + \cdots + U_m$ is the smallest subspace of V containing U_i.

Solution. Clearly U' is a subspace. All the U_i are contained in U'. Also, in any subspace with U_i, we must have U' by closed addition. Thus we have minimality. □

Definition (Direct Sum)**.** Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $\sum U_i$ is called a direct sum if each element of $\sum U_i$ can we written in only one way as a sum of $\sum u_i$, where $u_i \in U_i$. If $\sum U_i$ is a direct sum, then we denote it with $U_1 \oplus U_2 \oplus \cdots \oplus U_m$.

Example

$U = (x, y, 0)$, $W = (0, 0, z)$. Then $U \oplus W = \mathbb{F}^3$.

Example (Nonexample)

$U_1 = (x, y, 0)$, $U_2 = (0, 0, z)$, $U_3 = (0, y, y)$. We have that $\mathbb{F}^3 = \sum U_i$ since we can write every vector in $\mathbb{F}^3$ as the sum of three vectors from each of the subsets. But $(0, 0, 0)$ can we written in two different ways, so it's not a direct sum.

Proposition (Direct Sum Condition)

U_i are subspaces of V. Then $W = \sum U_i$ is a direct sum if and only if the only way to write 0 is by taking 0 in each subspace.

Solution. If W is a direct sum, then by definition the only way to write 0 is by taking 0 from each U_i (0 is in each of these by subspace condition). Now suppose the only way to write 0 is to take $u_i = 0$. Now consider $v \in W$. Suppose there are two ways to write it,

$$v = \sum u_i = \sum v_i.$$

Subtracting gives $0 = \sum u_i - v_i$. We know that $u_i - v_i \in U_i$ because it is a subspace, and we also know that only way to write 0 is having all components equal 0. Thus $u_i = v_i$, so there is only one way to write each vector as a sum, as desired. □

Proposition (Direct Sum of Two Subspaces)
U, W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Solution. If $U + W$ is a direct sum, then for $v \in U \cap W$, we have that $0 = v + (-v)$, with $v \in U$ and $-1 \in W$. By unique representation, $v = 0$, so $U \cap W = \{0\}$. If $U \cap W = \{0\}$, then for $u \in U$ and $w \in W$ we have $0 = u + w$. We need to show $u = w = 0$. The equation implies $u = -w \in W$, so $u \in U \cap W$, which means $u = 0$, as desired. □

Problem (Exercise 1)**.** For each of the following subsets of $\mathbb{F}^3$, determine where it is a subspace of $\mathbb{F}^3$:

(a) $\{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) : x_1 x_2 x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) : x_1 = 5x_3\}$

Solution. Spam subspace conditions.

(a) Yes, it is closed under addition and scalar multiplication and has 0.

(b) No, does not have 0.

(c) Not closed under addition.

(d) Yes.

□

Problem (Exercise 5)**.** Is $\mathbb{R}^2$ a subspace of $\mathbb{C}^2$.

Solution. No, it is not closed under scalar multiplication. □

Problem (Exercise 7)**.** $U = \{(x, y) : x, y \in \mathbb{Z}\}$.

Problem (Exercise 8)**.** Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and scalar multiplication but is not a subspace.

Solution. $U = \{(x, y) : xy = 0\}$. □

Problem (Exercise 9)**.** Is the set of periodic functions $f : \mathbb{R} \to \mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$?

Solution. No. Consider $\sin(x)$ and $\sin(\pi x)$. When added, they do not form a periodic function, so the set is not closed under addition. □

Problem (Exercise 10)**.** If U_1 and U_2 are subspaces of V, then $U_1 \cap U_2$ is a subspace of V.

Solution. Let $W = U_1 \cap U_2$. 0 is in U_1 and U_2, so $0 \in W$. Consider a vector $w \in W$. Since $w \in U_1$, $\lambda w \in U_1$. Similarly for U_2. Thus W is closed under scalar multiplication. A similar argument can be used for addition, so W is a subspace of V. □

Problem (Exercise 12)**.** Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contains the other two.

Solution. Let U_1, U_2, be subspaces of V. If $U_1 \subset U_2$, then $U_1 \cup U_2 = U_2$, which is s subspace. Now suppose $U_1 \cup U_2 = W$ is a subspace. Suppose for the sake of contradiction $U_1 \nsubseteq U_2$. Pick $x \in U_1$, $x \notin U_2$ and $y \notin U_1$, $y \in U_2$. We know that $x + y = w \in W$ must be in U_1 or U_2. Suppose it's in U_1. Then, $y = w - x = w + (-1)x \in U_1$, a contradiction. The same applies to U_2. Thus, $U_1 \subseteq U_2$. □

Problem (Exercise 16)**.** Is the operation of addition on subspaces of V commutative?

Solution. Yes. Suppose we have U, W as subsapces of V. Then pick $u \in U$ and $w \in W$. We know that $u + w \in U + W$, but $u + w = w + u \in W + U$, so the operation is commutative. □

Problem (Exercise 17)**.** Is the operation of addition on subspaces of V associative?

Solution. Yes. Suppose we have $x \in X$, $y \in Y$, and $z \in Z$. Then, $(x+y)+z \in (X+Y)+Z$ and $x + (y + z) \in X + (Y + Z)$, so it is associative. □

Problem (Exercise 18)**.** Does subspace addition have an identity? Which subspaces have additive inverses?

Solution. Yes, $\{0\}$. Suppose you have a subspace $U \neq \{0\}$. For $U + W = \{0\}$, then $u + w = 0$ and this only works for all elements when $u = w = 0$. Thus, only $\{0\}$ has an additive inverse. □

Problem (Exercise 19)**.** Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution. $U_1 = \{0\}$, $U_2 = W$, W is any subspace of V that's not $\{0\}$. □

Problem (Exercise 20)**.** Suppose $U = (x, x, y, y)$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Solution. The subspace $W = (a, 0, 0, b)$ works. Note that we can write any element in $\mathbb{F}^4$ as a sum of elements from U and V, so $U + V = \mathbb{F}^4$. Note also that the only way two vectors in W and U are equal is when $x = y = a = b = 0$, or in other words, $U \cap W = \{0\}$. Thus, $U \oplus W = \mathbb{F}^4$. □

Problem (Exercise 21)**.** Suppose $U = \{(x, y, x + y, x - y, 2x)\}$. Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

Solution. $W = \{(0, 0, a, b, c)\}$ works. Note that two vectors in U and W are only equal when $x = y = a = b = c = 0$, so $U \cap W = \{0\}$, so $U + W$ is a direct sum. Note also that we can change a, b, and c such that $x + y$, $x - y$, and $2x$ can be whatever they need to be when two vectors are added together. Thus, $\mathbb{F}^5 = U + W$, so $\mathbb{F}^5 = U \oplus W$. □

Problem (Exercise 22)**.** U is the same as the previous problem. Find $W_1, W_2, W_3 \neq \{0\}$ such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution. $W_1 = (0, 0, a, 0, 0)$, $W_2 = (0, 0, 0, b, 0)$, $W_3 = (0, 0, 0, 0, c)$. The only way to write 0 as the sum of a vector from each subspace is by taking 0 from each subspace. Note also that we can write every vector in $\mathbb{F}^5$ as a sum of vectors from each subspace using the same reasoning as in exercise 21. Thus, $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$. □

Problem (Exercise 23)**.** Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W,$$

then $U_1 = U_2$.

Solution. $W = (x, x)$, $U_1 = (a, 0)$, $U_2 = (0, b)$. ☐

Problem (Exercise 24)**.** U_o is the set of real-valued odd functions. U_e is defined similarly. Show that $\mathbb{R}^{\mathbb{R}} = U_o \oplus U_e$.

Solution. Note that both U_o and U_e are subspaces, and that $U_o \cap U_e = \{0\}$. Note that for any $f \in \mathbb{R}^{\mathbb{R}}$, we can write an even function $e(x) = \frac{f(x)+f(-x)}{2}$ and an odd function $o(x) = \frac{f(x)-f(-x)}{2}$, so $U_o + U_e = \mathbb{R}^{\mathbb{R}}$. But we have $U_o \cap U_e = \{0\}$, so $U_o + U_e$ is a direct sum, so we are done. ☐

§2 Finite Dimensional Vector Spaces

§2.1 Span and Linear Independence

§2.1.1 Linear Combinations and Span

Definition (Linear Combination)**.** A linear combination of a list $v_1, v_2, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m$$

for $a_i \in \mathbb{F}$.

Definition (Span)**.** The set of all linear combinations of a list of vectors $v_1, v_2 \ldots, v_m$ is V is called the span of $v_1, \ldots, v_m$, denoted as $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

$$\operatorname{span}(v_1, \ldots v_m) = \{a_1v_1 + \cdots + a_mv_m : a_1, \ldots, a_m \in \mathbb{F}\}.$$

The span of the empty list $()$ is defined to be $\{0\}$. If $\operatorname{span}(v_1, \ldots, v_m)$ equals V, we say that $v_1, \ldots, v_m$ spans V.

Example

Examples of span:

- $(17, -4, 2) \in \operatorname{span}((2, 1, -3), (1, -2, 4))$.
- $(17, -4, 5) \in \operatorname{span}((2, 1, -3), (1, -2, 4))$.

Remark. I'm lazy so $\operatorname{span}(v_i)$ refers to $\operatorname{span}(v_1, \ldots, v_n)$, where context determines $v_1, \ldots, v_n$

Proposition (Span Size)

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Solution. $v_1, \ldots, v_m \in V$. $\operatorname{span}(v_i)$ is a subspace since 0 is in it and it's closed under addition and scalar multiplication (since it contains all linear combinations).

Any v_k is in $\operatorname{span}(v_i)$ ($a_k = 1$, everything else is 0), so $\operatorname{span}(v_i)$ contains each v_k. Since subspaces are closed under addition and scalar multiplication, any subspace with each v_k contains $\operatorname{span}(v_i)$, so we have our desired conclusion. □

Example

$$(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$$

spans $\mathbb{F}^n$ where there are n vectors in the list. Showing this is trivial.

Definition (Finite-Dimensional Vector Space)**.** A vector space is called finite-dimensional if some list of vectors in it spans the space.

Remark. Every list is finite by definition.

Definition (Polynomial)**.** A function $p : \mathbb{F} \to \mathbb{F}$ is called a polynomial with coefficients in $\mathbb{F}$ if there exist $a_0, \ldots, a_m$

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$. $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$. $\mathcal{P}_m(\mathbb{F})$ is the set of all polynomials with degree at most m ($p = 0$ has degree $-\infty$).

Example (Finite Dimensional Vector Space)
$\mathcal{P}_m(\mathbb{F})$ is a finite-demensional vector space, since $\mathcal{P}_m(\mathbb{F}) = \operatorname{span}(1, z, \ldots, z^m)$.

Definition (Infinite-Dimensional Vector Space)**.** A vector space is called infinite-dimensional if it is not finite-dimensional.

Example ($\mathcal{P}(\mathbb{F})$)
Let m be the highest degree of some llist of $\mathcal{P}(\mathbb{F})$. Every polynomial in the span of the list has degree at most m, so you can't create z^{m+1}, so no list spans $\mathcal{P}(\mathbb{F})$, so it is infinite dimensional.

§2.1.2 Linear Independence

Definition (Linearly Independent)**.** A list $v_1, \ldots, v_m$ of vectors in V is called linearly independent if the only solution to $\sum a_i v_i = 0$ is the trivial solution $a_i = 0$ for all i. The empty list $()$ is declared to be linearly independent.

A vector in the spanning list of a linearly independent list only has one representation (if it has two, subtract them, all coefficients must be 0, implies initial coefficients are the same).

Example (Linearly Independent Lists)
The only solution to all of these is the trivial solution.

(a) A list of one vector $v \in V$ is linearly dependent if and only if $v \neq 0$.

(b) A list of two vectors in V is linearly independent if and only if neither vector is a scalar multiple of the other.

(c) $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ is linearly independent in $\mathbb{F}^4$.

(d) The list $1, \ldots, z^m$ is linearly independent in $\mathcal{P}(\mathbb{F})$ for each nonnengative integer m.

Remark. It's easy to see if you remove a vector from a linearly independent list, it's still linearly independent. Suppose it wasn't. Then there exist $a_1, \ldots, a_k$ that are not all 0. But then you can just have $a_{k+1} = 0$ and have the original list be linearly dependent, a contradiction.

Definition (Linearly Dependent)**.** A list of vectors V is called linearly dependent if it is not linearly independent.

Example (Linearly Dependent Lists)

All of these have nontrivial solutions.

(a) $(2,3,1)$, $(1,-1,2)$, $(7,3,8)$ is linearly independent in $\mathbb{F}^3$.

(b) The list $(2,3,1)$, $(1,-1,2)$, $(7,3,c)$ is linearly dependent in $\mathbb{F}^3$ if and only if $c = 8$.

(c) If a vector in a list can be written as a linear combination of other vectors in the list, then the list is linearly dependent.

(d) Every list of vectors in V containing 0 is linearly dependent.

Lemma (Linear Dependence Lemma)

Suppose $v_1, \ldots, v_m$ is a linearly dependent in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold:

(a) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$.

(b) if the jth term is removed from $v_1, \ldots, v_m$, the span of the remaning list equals $\operatorname{span}(v_1, \ldots, v_m)$.

Solution. We have a nontrivial solution to $\sum a_i v_i = 0$. Let j be the largest index such that $a_j \neq 0$. Then

$$v_j = \sum -\frac{a_1}{a_j} v_i. \tag{1}$$

To prove (b), suppose $u \in \operatorname{span}(v_i)$. Then there exists c_i such that $\sum c_i v_i = m$. Replacing v_j with our expression in (1) shows u is in the span of the same list without v_j. □

Lemma

In a finite-dimensional vector sapce, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Solution. List A $u_1, \ldots, u_m$ is linearly independent in V. List B $w_1, \ldots, w_n$ spans V. We do the following procedure:

Step 1 Adjoining any vector from V to B gives a linearly dependent list, so by the linear dependence lemma, we can remove one of the vectors. If we have the list $u_1, w_1, \ldots w_n$, we can remove one of the w's.

Step j The B from step $j-1$ spans V. We do what we did in step 1. We know that we have to remove a w since $u_1, \ldots, u_j$ is linearly independent.

After step m, all the u's have been added. Thus, there are at least as many w's as u's.

□

Example

The list $(1,2,3)$, $(4,5,8)$, $(9,6,7)$, $(-3,2,8)$ is not linearly dependent in $\mathbb{R}^3$ since the list $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ spans $\mathbb{R}^3$, and linearly independent list $\leq$ spanning list.

Lemma

Every subspace of a finite-dimensioanl vector space is finite-dimensional.

Solution. We can keep adding vectors to our list of vectors in U such that v is not in the span of the previous list. This process will terminate, since this linearly independent list can't be longer than the spanning list of V. □

Problem (Exercise 1)**.** Suppose v_1, v_2, v_3, v_4 spans V. Prove that

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution. For any $u \in V$, we have that

$$u = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

We can rewrite this as

$$u = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4,$$

so these vectors do indeed span V.

□

Problem (Exercise 5)**.** Changing the field of the vector space scalars.

(a) Show that if we think of $\mathbb{C}$ as a real vector space, then $(1 + i, 1 - i)$ is linearly independent.

(b) Show that if we think of $\mathbb{C}$ as a complex vector space, then $(1 + i, 1 - i)$ is linearly dependent.

Solution. Easy.

(a) $1 + i$ isn't a scalar multiplt of $1 - i$ over the reals, so the list is linearly independent.

(b) We have $i(1 - i) + 1(1 + i) = 0$.

□

Problem (Exercise 6)**.** Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent in V.

Solution. Assume for the sake of contradiction it isn't. Then

$$0 = a_1(v_1-v_2)+a_2(v_2-v_3)+a_3(v_3-v_4)+a_4v_4 \implies a_1v_1 = (-a_1+a_2)v_2+(-a_2+a_3)v_3+(-a_3+a_4)v_4.$$

Since v_1, v_2, v_3, v_4 is linearly independent, we must have that $a_i = 0$, which contradicts the list being linearly dependent. Thus, the list is linearly independent. □

Problem (Exercise 7)**.** Prove or give a counterexample: If v_i is linearly independent in V, then

$$5v_1 - 4v_2, v_2, \dots, v_m$$

is linearly independent.

Solution. Suppose for the sake of contradiction the list isn't linearly independent. Then,

$$a_1(5v_1 - 4v_2) + a_2v_2 + \cdots + a_mv_m = 0 \implies 5a_1v_1 + (-4a_1 + a_2)v_2 + \cdots + a_mv_m = 0,$$

which implies that $a_i = 0$, a contradiction. □

Problem (Exercsie 8)**.** Prove or give a counterexample: If v_i is linearly independent in V and $0 \neq \lambda \in \mathbb{F}$ then λv_i is linearly independent.

Solution. Assume for the sake of contradiction the list isn't linearly independent. Then,

$$a_1\lambda v_1 + \cdots + a_m\lambda v_m = 0 \implies a_1v_1 + \cdots + a_mv_m = 0,$$

which implies $a_i = 0$, a contradiction. □

Problem (Exercise 9)**.** Prove or give a counterexample: If v_i and w_i are lineary independent lists in V, then so it $v_i + w_i$.

Solution. This is false. Pick a linearly independent list v_i and let $w_i = -v_i$. Then $v_i + w_i = 0$, so the list is linearly dependent. □

Problem (Exercise 10)**.** Suppose $v_1, \dots, v_m$ is a linearly dependent list in V and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \dots, v_m)$.

Solution. Since the second list is linearly dependent, we have that $\sum a_i(v_i + w) = 0$ for some a_i not all equal to 0. We can write this as $\sum a_iv_i + w\sum a_i = 0$. Note if $\sum a_i = 0$, then we would have $a_i = 0$, a contradiction, thus $\sum a_i \neq 0$. Then we can write w as

$$w = \frac{\sum -a_iv_i}{\sum a_i}.$$

□

Problem (Exercise 11)**.** Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \dots, v_m, w$ is linearly independent if and only iff $w \notin \operatorname{span}(v_1, \dots, v_m)$.

Solution. We prove the if direction first. Suppose for the sake of contradiction that if $w \notin \operatorname{span}(v_1, \dots, v_m)$ then $v_1, \dots, v_m, w$ is linearly dependent. Then we have

$$\sum a_iv_i + aw = 0 \implies w = \sum -\frac{a_i}{a}v_i,$$

a contradiction.

Suppose now for the sake of contradiction that if $v_1, \dots, v_m, w$ is linearly independent, then $w \in \operatorname{span}(v_1, \dots, v_m)$. We can write w as $\sum b_iv_i$, and so we can obtain a nontrivial solution to $\sum a_iv_i + aw$, namely $a_i = -b_i$, $a = 1$. □

Problem (Exercise 12 and 13)**.** Explain why no list of six polynomials is linearly independent in $\mathcal{P}_4(\mathbb{F})$ and no list of four polynomials spans $\mathcal{P}_4(\mathbb{F})$,

Solution. $\operatorname{span}(1, z, \ldots, z^4)$ spans the vector space and has 5 elements, thus a linearly dependent list must have at most 5 elements and a spanning list must have at least 5 elements. □

Problem (Exercise 14)**.** Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for any positive integer m.

Solution. If such a list exists, then at each m, v_{m+1} is not in the span by exercise 11. Now suppose V is infinite dimensional. Pick $v_1 \neq 0$. This clearly is linearly independent and can't span V since V has no finite spanning list. Now suppose we have k elements in the list. To find v_{k+1}, just pick an element not in $\operatorname{span}(v_1, \ldots, v_k)$. We know this will be linearly independent by exercise 11. □

Problem (Exercise 15)**.** Prove $\mathbb{F}^\infty$ is infinite-dimensional.

Solution. Our list of vectors is as follows:

$$(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, \ldots), \ldots$$

□

Problem (Exercise 17)**.** Suppose $p_0, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, \ldots, p_m$ is not linearly independent.

Solution. We can write each polynomial as $p_i = (z - 2)q_i$. Each q_i has degree at most $m - 1$. Since $\mathcal{P}_{m-1}(\mathbb{F})$ has a spanning list of length m, and our list has length $m + 1$, the q_i are not linearly independent, and so the original list isn't at well. □

§2.2 Bases

Definition (Basis)**.** A basis of V is a list of vectors that is linearly independent and spans V.

Example (Bases)

Examples of bases of vectors spaces.

(a) The list $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is the basis of $\mathbb{F}^n$, called the standard basis.

(b) $(1, 2), (3, 5)$ is a basis of $\mathbb{F}^2$.

(c) $(1, 1, 0), (0, 0, 1)$ is a basis of $\{(x, x, y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$.

(d) $(1, -1, 0), (1, 0, -1)$ is a basis of $\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$.

(e) $1, z, \ldots, z^m$ is the basis of $\mathcal{P}_M(\mathbb{F})$.

Proposition (Basis criteria)

A list $v_1, \dots, v_m$ of vectors in V is a basis of V if and only if every $v \in V$ can be uniquely written in the form $v = \sum a_i v_i$.

Solution. We can write every vector in such a way since the list spans V, and since it's linearly independent, each representation in the span of the list is unique. □

Proposition (Spanning list contains a basis)

Every spanning list in a vector space can be reduced to a basis of the vector space.

Solution. Start with a spanning list B equal to the list $v_1, \dots, v_m$.

Step 1 If $v_1 = 0$, delete it from B. Otherwise leave B unchanged.

Step j If v_j is in $\operatorname{span}(v_1, \dots, v_{j-1})$, delete v_j. Otherwise leave B unchanged.

The final list still spans V since each vector we removed was already in the span of the previous vectors. Note also that since no vectors in a list left are representable by other vectors in the list, we have a linearly independent list, and thus a basis. □

Proposition (Corollary)

Every finite-dimensional vector space has a basis.

Solution. By definition a finite-dimensional vector space has a spanning list. The previous result tells us it can be reduced to a basis. □

Proposition (Linearly independent list extends to a basis)

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Solution. $v_1, \dots, v_m$ is linearly independent in V and $w_1, \dots, w_n$ is a basis. Then we can reduce the list $v_1, \dots, v_m, w_1, \dots, w_n$ to a basis using the procedure above. None of the v_i get deleted since they're linearly independent. □

Proposition

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Solution. U is finite-dimensional, so we have a basis $u_1, \dots, u_m$. We can extend this basis to a basis of V with the list $v_1, \dots, v_m, w_1, \dots, w_n$. Let $W = \operatorname{span}(w_1, \dots, w_n)$. To show the direct sum, we need to show that $V = U + W$ and $U \cap W = \{0\}$.

Pick $v \in V$. We can write it as

$$v = \underbrace{a_1 v_1 + \cdots a_m u_m}_{u} + \underbrace{b_1 w_1 + \cdots b_n w_n}_{w},$$

which means $v = u + w$, where $u \in U$ and $w \in W$. Thus, $V = U + W$.
Now pick $v \in U \cap W$. We have

$$\sum a_i u_i = \sum b_i w_i \implies a_1 u_1 + \cdots a_m u_m - b_1 w_1 - \cdots - b_n w_n = 0.$$

Because this is a linear combination of the basis of V and the basis is linearly independent, $a_i, b_i = 0$, so $U \cap W = \{0\}$. Thus, we have the direct sum we wanted. □

Problem (Exercise 1)**.** Find all vector spaces with exactly one basis.

Solution. $\{0\}$ is the only one. □

Problem (Exercise 3)**.** Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

(a) Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Solution. Lemma spam.

(a) $(3, 1, 0, 0, 0)$, $(0, 0, 1, 7, 0)$, $(0, 0, 0, 0, 1)$. It's easy to see this spans U and is linearly independent, so it's a basis of U.

(b) We use the result of extending any linearly independent list to a basis. Add the standard basis for $\mathbb{R}^5$ to the list and then turn the spanning list into a linearly independent list. We end with

$$(3, 1, 0, 0, 0),\ (0, 0, 1, 7, 0),\ (0, 0, 0, 0, 1),\ (1, 0, 0, 0, 0),\ (0, 0, 1, 0, 0).$$

(c) By the subspace direct sum proposition, we know that

$$W = \operatorname{span}((1, 0, 0, 0, 0),\ (0, 0, 1, 0, 0))$$

. In other words,

$$W = \{(x, 0, y, 0, 0) \in \mathbb{R}^5 : x, y \in \mathbb{R}\}.$$

□

Problem (Exercise 3)**.** Let U be the subspace of $\mathbb{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

(a) Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbb{C}^5$.

(c) Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

Solution. Lemma spam.

(a) $(1,6,0,0,0)$, $(0,0,3,0,-1)$, $(0,0,0,3,-2)$. It's easy to see this spans U and is linearly independent, so it's a basis of U.

(b) We use the result of extending any linearly independent list to a basis. Add the standard basis for $\mathbb{C}^5$ to the list and then turn the spanning list into a linearly independent list. We end with

$$(1,6,0,0,0),\ (0,0,3,0,-1),\ (0,0,0,3,-2)\,(1,0,0,0,0),\ (0,0,1,0,0).$$

(c) By the subspace direct sum proposition, we know that

$$W = \operatorname{span}((1,0,0,0,0),\ (0,0,1,0,0))$$

. In other words,

$$W = \{(x,0,y,0,0) \in \mathbb{C}^5 : x,y \in \mathbb{C}\}.$$

□

Problem (Exercise 5)**.** Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials in the basis have degree 2.

Solution. $1, z, z^3, z^3 + z^2 + z + 1$. This is clearly linearly independent, and it's easy to show this spans $\mathcal{P}_3(\mathbb{F})$. □

Problem (Exercise 6)**.** Prove that is $v_1, \ldots, v_4$ is a basis of V, then so is $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$.

Solution. It's easy to see the list is linearly independent (see exercise 6 in 2.A). Suppose $v\ inV$. Then

$$v = \sum a_i v_i = a_1(v_1 + v_2) + (a_2 - a_1)(v_2 + v_3) + (a_3 - a_2 + a_1)(v_3 + v_4) + (a_4 - a_3 + a_2 - a_1)v_4.$$

□

Problem (Exercise 7)**.** Prove or give a counterexample: If $v_1, \ldots, v_4$ is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3, v_4 \notin U$, then v_1, v_2 is a basis of U.

Solution. $(1,1,0,0,)$, $(0,1,1,0)$, $(0,0,1,1)$, $(0,0,0,1)$ is basis of $\mathbb{R}^4$. The first two vectors belong to $(x,y,z,0)$ while the last two don't. Note the first two vectors don't span the subspace. □

Problem (Exercise 8)**.** Suppose U and W are subspaces of V such that $V = U \oplus W$. $u_1, \ldots, u_m$ and $w_1, \ldots, w_n$ as bases for U and W respectively. Prove that combination of those lists is a basis of V.

Solution. Note that for any $v \in V$, there's only one way to pick $u \in U$ and $W \in W$ such that $v = u + w$, so the representation of u using the combined list of vectors is unique. Thus, the list is a basis. □

§2.3 Dimension

Proposition

Any two bases of a finite-dimensional vector space have the same length.

Solution. Suppose V is finite-dimensional. Let B_1 and B_2 be two bases of V. Then B_1 is linearly independent in V and B_2 spans V, so the length of B_1 is at most the length of B_2. Interchanging roles we get the same inequality with the sign flipped, so the lengths are equal. □

Definition (Dimension)**.** The dimension of a finite-dimensional vector space is the length of any basis of the vector space, denoted by $\dim V$.

Example

$\dim \mathbb{F}^n = n$ because the standard basis has length n. $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$ because the standard basis consists of each degree up till m.

Proposition

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Solution. The basis of U is linearly independent in V, and the basis of V spans V. Thus, $\dim U \leq \dim V$. □

> **Remark.** Actually, if $\dim U = \dim V$, then $U = V$. We can show this by considering a length n basis in U. Pick a $v \in V$. It's either in U or not. If it isn't then the list appended with v is linearly independent, which is impossible, since $\dim V = n$. Surprisingly the book didn't mention this (at least not in this chapter) and made deciphering the solution for the polynomial example very confusing **grumble grumble**.

Proposition

V is finite-dimensional. Every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Solution. Suppose $\dim V = n$. We can extend the linearly independent list to a basis of V. However, the list has n elements, so the extension is trivial. Thus, the list is a basis. □

Example

Show that 1, $(x-5)^2, (x-5)^3$ is a basis of the subspace U of $\mathcal{P}_3(\mathbb{R})$ defined by

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0\}.$$

Clearly these polynomials are in U. It's also easy to see they are linearly independent. Thus, $\dim U \geq 3$. Since $\dim \mathcal{P}_3(\mathbb{R}) = 4$ and $U \neq \mathcal{P}_3(\mathbb{R})$, $\dim U = 3$. Thus, the list is a basis.

Proposition

V is finite-dimensional. Then every spanning list of vectors is V with length $\dim V$ is a basis of V.

Solution. Suppose $\dim V = n$ and $v_1, \ldots, v_n$ spans V. We can reduce the list to a basis. The reduction is trivial, and so the list is a basis. □

Proposition (Dimension of a sum)

If U_1 and U_2 are subspaces of finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Solution. Let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$, then the list is linearly independent in U_1 and U_2. Extending the basis in both yields $u_1, \ldots, u_m, v_1, \ldots, v_j$ and $u_1, \ldots, u_m, w_1, \ldots, w_k$, so $\dim U_1 = m + j$ and $\dim U_2 = m + k$. Now we just need to show that

$$u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$$

is a basis of $U_1 + U_2$. Clearly the span contains both subspaces, so we only need to show it's linearly independent.

Suppose

$$\sum a_i u_i + \sum b_i v_i + \sum c_i w_i = 0.$$

We can rewrite as

$$\sum c_i w_i = -\sum a_i u_i - \sum b_i v_i,$$

so $\sum c_i w_i \in U_1$. Since $w_i \in U_2$, $\sum c_i w_i \in U_1 \cap U_2$. Thus we can write the

$$\sum c_i w_i = \sum d_i u_i.$$

Since $w_1, \ldots, w_k, u_1, \ldots, u_m$ is linearly independent in U_2, $c_i, d_i = 0$. Thus,

$$\sum a_i u_i + \sum b_i v_i = 0.$$

Since $u_1, \ldots, u_m, v_1, \ldots, v_j$ is linearly independent in U_1, $a_i, b_i = 0$. Thus, we have our desired result.

□

Problem (Exercise 1)**.** Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove $U = V$.

Solution. See remark above. □

Problem (Exercise 2)**.** Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, all lines through the origin, and $\mathbb{R}^2$.

Solution. Clearly they're all subspaces. The only subspace with dimension 0 is $\{0\}$. For dimension 1, there's a basis vector u. All linear combinations of u create a line in the plane, and since $\{0\}$ is one of them, it is a line through the origin. Finally, a subspace with dimension 2 must be equal to $\mathbb{R}^2$. □

Problem (Exercise 3)**.** Show that the subspaces if $\mathbb{R}^3$ are precisely $\{0\}$, all lines in R^3 through the origin, all planes in $\mathbb{R}^3$ through the origin, and $\mathbb{R}^3$.

Solution. Clearly they're all subspaces. Again the only subspace with dimension 0 and $\{0\}$ and the only subspace with dimension 3 is $\mathbb{R}^3$. For dimension 1, pick nonzero vector in $\mathbb{R}^3$ to be our basis. By the same process as the last question, we conclude it's a line containing the origin. Now for dimension 2, pick two nonzero vectors that are not scalar multiples of each other, which implies they are a basis. Let the vectors in component form be (x_1, y_1, z_1) and (x_2, y_2, z_2). Any linear combination of these two amounts to $(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$. This is just the vector representation of a plane, and so we get that a plane is the only two dimensional subspace. □

Problem (Exercise 4)**.** Let $V = \mathcal{P}_4(\mathbb{F})$.

$$U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(6) = 0\}.$$

(a) Find a basis of U.

(b) Extend the basis in part (a) to a basis of V.

(c) Find a subspace W of V such that $V = U \oplus W$.

Solution. We love polynomials.

(a) Consider the list $z - 6, (z-6)^2, (z-6)^3, (z-6)^4$. It's easy to see they're linearly independent. Since the list has length 4, $\dim U \geq 4$. Since $\dim V = 5$ and $U \neq V$, $\dim U = 4$. Thus, the list is a basis of U.

(b) Extending with the standard basis, we obtain the following basis for V:

$$z - 6,\ (z-6)^2,\ (z-6)^3,\ (z-6)^4,\ 1.$$

(c) We know the basis of W is 1, so $W = \operatorname{span}(1) = c$ for some $c \in \mathbb{F}$, which means

$$W = \{c \in \mathcal{P}_4(\mathbb{F}) : c \in \mathbb{F}\}.$$

□

Problem (Exercise 5)**.** Let $V = \mathcal{P}_4(\mathbb{F})$.

$$U = \{p \in \mathcal{P}_4(\mathbb{F}) : p''(6) = 0\}.$$

(a) Find a basis of U.

(b) Extend the basis in part (a) to a basis of V.

(c) Find a subspace W of V such that $V = U \oplus W$.

Solution. We love polynomials.

(a) Consider the list $(z-6)^4, (z-6)^3, z-6, 1$. It's easy to see they're linearly independent. Note also that taking the second derivative of any linear combination leaves every term with $z-6$, so it is indeed in the subspace. Since the list has length 4, $\dim U \geq 4$. Since $\dim V = 5$ and $U \neq V$, $\dim U = 4$. Thus, the list is a basis of U.

(b) Extending with the standard basis, we obtain the following basis for V:

$$(z-6)^4,\ (z-6)^3,\ z-6,\ 1,\ z^2.$$

(c) We know the basis of W is z, so $W = \operatorname{span}(z) = cz$ for some $c \in \mathbb{F}$, which means

$$W = \{cz^2 \in \mathcal{P}_4(\mathbb{F}) : c \in \mathbb{F}\}.$$

□

Problem (Exercise 6)**.** Let $V = \mathcal{P}_4(\mathbb{F})$.

$$U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(2) = p(5)\}.$$

(a) Find a basis of U.

(b) Extend the basis in part (a) to a basis of V.

(c) Find a subspace W of V such that $V = U \oplus W$.

Solution. We love polynomials.

(a) Consider the list $1, (z-2)(z-5), (z-2)^2(z-5), (z-2)^2(z-5)^2$. Clearly they're linearly independent, and note they are closed under addition. Thus, $\dim U \geq 4$, and since $U \neq V$, $\dim U = 4$. Thus our list is a basis.

(b) Extending with the standard basis, we obtain the following basis for V:

$$1,\ (z-2)(z-5),\ (z-2)^2(z-5),\ (z-2)^2(z-5)^2,\ z.$$

(c) We know the basis of W is z, so $W = \operatorname{span}(z) = cz$ for some $c \in \mathbb{F}$, which means

$$W = \{cz \in \mathcal{P}_4(\mathbb{F}) : c \in \mathbb{F}\}.$$

□

Problem (Exercise 7)**.** Let $V = \mathcal{P}_4(\mathbb{F})$.

$$U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(2) = p(5) = p(6)\}.$$

(a) Find a basis of U.

(b) Extend the basis in part (a) to a basis of V.

(c) Find a subspace W of V such that $V = U \oplus W$.

Solution. We love polynomials.

(a) Consider the list $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$. Clearly the vectors are linearly independent. Moreover, for any $p \in U$, we can subtract off the value of $f(2)$, call it r, and then we can factor the polynomial. After adding back r, we get

$$(z-2)(z-5)(z-6)(z-k)+r = 1(z(z-2)(z-5)(z-6))-k((z-2)(z-5)(z-6))+f(2)(1),$$

so the list is a basis.

(b) Extending with the standard basis, we obtain the following basis for V:

$$1,\ (z-2)(z-5)(z-6),\ z(z-2)(z-5)(z-6),\ z, z^2.$$

(c) We know the basis of W is z, z^2, so $W = \operatorname{span}(z, z^2) = az^2 + bz$ for some $a, b \in \mathbb{F}$, which means

$$W = \{az^2 + bz \in \mathcal{P}_4(\mathbb{F}) : a,\ b \in \mathbb{F}\}.$$

□

Problem (Exercise 8)**.** Let $V = \mathcal{P}_4(\mathbb{R})$.

$$U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0\}.$$

(a) Find a basis of U.

(b) Extend the basis in part (a) to a basis of V.

(c) Find a subspace W of V such that $V = U \oplus W$.

Solution. We love polynomials.

(a) Let $p = ax^4 + bx^3 + cx^2 + dx + e$. Integrating, we see that the only condition on the coefficients is $3a + 5c + 15e = 0$. Now consider the following list of polynomials:

$$5x^4 - 1,\ x^3,\ 3x^2 - 1,\ x.$$

Clearly it's linearly independent, and each polynomial is in U. We can also see that any linear combination of the polynomials satisfies the condition. Thus, $\dim U \geq 4$, and since $U \neq V$, $\dim U = 4$, so the list is a basis.

(b) Extending with the standard basis, we obtain the following basis for V:

$$5x^4 - 1,\ x^3,\ 3x^2 - 1,\ x,\ 1.$$

(c) We know the basis of W is 1, so $W = \operatorname{span}(1) = c$ for some $c \in \mathbb{R}$, which means

$$W = \{c \in \mathcal{P}_4(\mathbb{R}) : c \in \mathbb{R}\}.$$

□

Problem (Exercise 9)**.** Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$, Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Solution. Note that we only need to show that the spans has a list of at least $m-1$ vectors that are linearly independent. Consider $v_2 - v_1, v_3 - v_2, \dots, v_m - v_{m-1}$. Note that $(v_i + w) - (v_{i-1} - w)$, so each is in the span. It's also easy to see that the list is linearly independent. Thus, $\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. ☐

Problem (Exercise 10)**.** Suppose $p_1, \dots, p_m \in \mathcal{P}(\mathbb{F})$ such that $\deg p_i = i$. Prove that $p_1, \dots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Solution. Note that the list is linearly independent, and since its length is the same as the dimension of the vector space, it is indeed a basis. ☐

Problem (Exercise 11)**.** Suppose U and W are subspaces $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Solution. By our dimension sum formula, we have that

$$8 = \dim \mathbb{R}^8 = \dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 8 - \dim(U \cap W).$$

Thus $\dim(U \cap W) = 0$, which implies that $U \cap W = \{0\}$, which implies the direct sum by direct sum conditions. ☐

Problem (Exercise 12)**.** Suppose U and W are both 5-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

Solution. By the dimension sum formula, we have that

$$\dim(U + W) = 10 - \dim(U \cap W).$$

If $\dim(U \cap W) = 0$, then the dimension of the sum of the two subspaces would be 10. However, that's impossible, since $\mathbb{R}^9$ has dimension 9. Thus $\dim(U \cap W) > 0$, so $U \cap W \neq \{0\}$. ☐

Problem (Exercise 13)**.** Suppose U and W are both 4-dimensional subspaces in $\mathbb{C}^6$. Prove that there exist two vector in $U \cap W$ such that neither is a scalar multiple of the other.

Solution. The condition is the same as saying the intersection has at least dimension two, which ensures that there exists two linearly independent vectors, which means they are not scalar multiples. Using our dimension sum formula, we have that

$$\dim(U + W) = 8 - \dim(U \cap W).$$

By the same reasoning as in the last problem, we need $\dim(U \cap W) \geq 2$, which is what we wanted.

☐

Problem (Exercise 14)**.** Suppose $U_1, \dots, U_m$ are finite-dimensional subspaces of V. Prove that $U_1, \dots, U_m$ is finite-dimensional and

$$\dim(U_1, \dots, U_m) \leq \dim U_1 + \dots + \dim U_m.$$

Solution. Note that $\dim U_1 + \dim U_2 \geq \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) = \dim(U_1 + U_2)$. Repeating this for each U_i yields our desired conclusion. Since the right hand side it finite, the left hand side is also finite, so we have $U_1, \dots, U_m$ is finite-dimensional. ☐

Problem (Exercise 15)**.** Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \dots, U_n$ such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

Solution. Let $v_1, \ldots, v_n$ be a basis for V and let $U_i = \operatorname{span}(v_i)$. Then, each vector in V can be uniquely represented as as linear combination of vectors from each U_i. Thus, we have the desired direct sum. □

Problem (Exercise 16)**.** Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

Solution. Let $\mathcal{U}_i$ be the basis of U_i. Since the subspace sum is a direct sum, each vector in the sum can be represented uniquely. Thus, we can pick a vector from each U_i such that their sum equals the vector in the subspace sum, which implies $\mathcal{P}_1, \ldots, \mathcal{P}_m$ is a basis for the direct sum. Since the cardinality of $\mathcal{P}_1, \ldots, \mathcal{P}_m = \dim U_1 + \cdots + \dim U_m$, we have out desired inequality, from which we immediately deduce the direct sum is finite-dimensional, since the right is finite. □

Problem (Exercise 17)**.** Prove or find a counter example:

$$\begin{aligned}\dim(U_1 + U_2 + U_3) = {} & \dim U_1 + \dim U_2 + \dim U_3 \\ & - \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) - \dim(U_3 \cap U_1) \\ & + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Solution. Use the subspaces $(x, 0)$, $(0, y)$, (z, z). □

§3 Linear Maps

This is a long chapter. Buckle up boys and girls.

§3.1 The Vector Space of Linear Maps

§3.1.1 Definition and Examples of Linear Maps

Definition (Linear Map)**.** A linear map from V to W is a function $T : V \to W$ with the following properties:

additivity $T(u + v) = Tu + Tv$ for all $u, v \in V$,

homogeneity $T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$.
The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.

Example (Zero)
$0 \in \mathcal{L}(V, W)$ is defined by

$$0v = 0.$$

0 on the left is the function itself, while the right is $0 \in W$.

Example (Identity)
The identity map, denoted I is the function on some vector space that takes each element to itself. Specifically, $I \in \mathcal{L}(V, V)$ is defined by

$$Iv = v.$$

Example (Differentiation)
Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ by

$$Dp = p'.$$

Example (Integration)
Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ by

$$Tp = \int_0^1 p(x)dx.$$

Example (Multiplication by x^2)
Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R})))$ by

$$T(p(x)) = x^2 p(x).$$

Example (Backward Shift)
Define $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty)$ by

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Example (From $\mathbb{R}^3$ to $\mathbb{R}^2$)
Define $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z).$$

Example (From $\mathbb{F}^n$ to $\mathbb{F}^m$)
Let $A_{j,k} \in F$ for $j \in [1, m]$ and $k \in [1, n]$, and define $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ by

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n).$$

In fact, every linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$ has this form.

Proposition
Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map such that

$$Tv_j = w_j.$$

Solution. First existence. Define $T : V \to W$ by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n.$$

It's easy to see it's a function from V to W since the first list is a basis of V. To get $Tv_j = w_j$, take $c_j = 1$ and everything else 0. Linear map conditions follow easily from addition and multiplication.

To prove uniqueness, suppose that $T \in \mathcal{L}(V, W)$ and $Tv_j = w_j$. By homogeneity, $T(c_jv_j) = c_jw_j$. Additivity implies that

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n.$$

Thus T us uniquely determined on span$(vlistv1n)$ by the equation above. Because the list is a basis, T is uniquely determined in V. □

§3.1.2 Algebraic Operations on $\mathcal{L}(V, W)$

Definition. Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. The sum $S + T$ and the product λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \text{ and } (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$. It's easy to see these are linear maps.

Proposition

With the operations above, $\mathcal{L}(V, W)$ is a vector space.

Solution. Commutativity, associativity, and distributivity are free. Additive identity is 0 map, multiplicative identity is 1, additive is inverse is the map defined as $(-1)T$. $\square$

Definition (Product of Linear Maps)**.** If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the product $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$ (i.e. function composition $S \circ T$). It's easy to verify this is a linear map.

Proposition

Bunch of axiomatic stuff that's easy to verify.

associativity

$$(T_1 T_2)T_3 = T_1(T_2 T_3).$$

identity

$$TI = IT = T.$$

I in the first part is identity on V, second part is identity on W.

distributive properties

$$(S_1 + S_2)T = S_1 T + S_2 T \text{ and } S(T_1 + T_2) = ST_1 + ST_2,$$

whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

Example (linear map multiplication is not commutative)

Suppose we have the differentiation map D from the set of all polynomials to itself, and let T be multiplication by x^2. Then

$$((TD)p)(x) = x^2 p'(x) \text{ and } ((DT)p)(x) = x^2 p'(x) + 2xp(x).$$

Proposition

Suppose T is a linear map from V to W. Then $T(0) = 0$.

Solution. We have

$$T(0) = T(0 + 0) = T(0) + T(0).$$

Adding the inverse of $T(0)$ to both sides gives us the desired equality. $\square$

Problem (Exercise 1)**.** Suppose $a, b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = T(2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

Solution. Only if is easy to show. Suppose T is linear. Then we have

$$T(\lambda v) = (2\lambda x - 4\lambda y + 3\lambda z + b, 6\lambda x + c\lambda^3 xyz) = (2\lambda x - 4\lambda y + 3\lambda z + \lambda b, 6\lambda x + c\lambda xyz) = \lambda T(v)$$

for all $\lambda \in \mathbb{R}$, which implies $b = c = 0$. □

Problem (Exercise 2)**.** Suppose $b, c \in \mathbb{R}$. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$T(p) = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)\, dx + c\sin(p(0))\right).$$

Show that T is linear if and only if $b = c = 0$.

Solution. Only if is easy to show. Suppose T is linear. Then we need

$$\begin{aligned} T(\lambda p) &= \left(3\lambda p(4) + 5\lambda p'(6) + b\lambda^2 p(1)p(2), \int_{-1}^{2} x^3 \lambda p(x)\, dx + c\sin(\lambda p(0))\right) \\ &= \left(3\lambda p(4) + 5\lambda p'(6) + b\lambda p(1)p(2), \lambda \int_{-1}^{2} x^3 p(x)\, dx + c\lambda \sin(p(0))\right) \\ &= \lambda T(p) \end{aligned}$$

for all $\lambda \in \mathbb{R}$, which implies $b = c = 0$.

□

Problem (Exercise 3)**.** Suppose $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbb{F}$ for $j \in [1, m]$ and $k \in [1, n]$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n).$$

for every $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Solution. Since a linear map is defined uniquely on its basis, we only need to find the scalars for the basis. We use the standard basis for $\mathbb{F}^n$. Let v_i equal the basis vector with 1 in the ith position. Suppose it is mapped to $w_i = (w_{1,i}, w_{2,i}, \ldots, w_{m,i})$. We let $A_{j,k} = w_{j,k}$. Indeed, we can see with these As that the $T(v_i) = w_i$. Thus we are done. □

Problem (Exercise 4)**.** Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Solution. Suppose $0 = \sum a_i v_i$. We have that

$$0 = T\left(\sum a_i v_i\right) = \sum a_i T(v_i).$$

This implies $a_i = 0$, which is what we wanted. □

Problem (Exercise 7)**.** Prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution. Let u be the basis of V, and let $\lambda u = Tu$. Since every vector in V can be written as $v = au$, we have $Tv = T(au) = a(Tu) = a\lambda u = \lambda v$. □

Problem (Exercise 10)**.** Suppose U us a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$. Define $T : V \to W$ by

$$T(v) = \begin{cases} S(v) & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \in U. \end{cases}$$

Prove that T is not a linear map on V.

Solution. Pick $v \in V$, $v \notin U$ and $u \in U$ such that $S(v) \neq 0$. Note that $u + v$ is not in U. Then, we have

$$0 = T(u + v) \neq T(u) + T(v) = S(v).$$

□

Problem (Exercise 11)**.** Show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $T(u) = S(u)$ for all $u \in U$.

Solution. Extend the basis of U, $u_1, \dots, u_m$, to a basis of V, $u_1, \dots, u_m, v_1, \dots, v_n$. Define $T(u_i) = S(u_i)$ and set the v_i to any vectors in W. Note that any $u \in U$ can be represented uniquely as a linear combination of the basis vectors. From this we obtain

$$T(u) = T\left(\sum a_i u_i\right) = \sum T(a_i u_i) = \sum a_i T(u_i) = \sum a_i S(u_i) = \sum S(a_i u_i) = S\left(\sum a_i u_i\right) = S(u),$$

so T is our desired map.

□

Problem (Exercise 12)**.** Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Solution. Let the basis of V be $v_1, \dots, v_m$. Let $x_1, \dots, x_m$ and $y_1, \dots, y_m$ be lists in W such that $T(v_i) = x_i$ and $S(v_i) = y_i$ for $T, S \in \mathcal{L}(V, W)$. Then we have $(T + S)(v_i) = x_i + y_i$. Note that no matter what linear combination of T and S is taken, the right side will never span all possible basis mappings from V to W, since the right side doesn't span W. This argument can be repeated for any finite number of maps in $\mathcal{L}(V, W)$, so the space is indeed infinite-dimensional. □

Problem (Exercise 13)**.** Suppose $v_1, \dots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \dots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $T(v_k) = w_k$.

Solution. Without loss of generality, since the vectors in V are linearly independent, let a_1 be part of a nontrivial solution such that a_1 is nonzero. Now pick a nonzero vector in W. Let $T(v_1) = w$ and $T(v_i) = 0$ for the rest of the vectors. Suppose such a map does exist. Then we have

$$0 = T(0) = T\left(\sum a_i v_i\right) = \sum a_i T(v_i) = a_1 w,$$

which is a contradiction. □

Problem (Exercise 14)**.** Suppose V is finite-dimensional with $\dim V \geq 2$. Prove that there exist $S, T \in \mathcal{L}(V, V)$ such that $ST \neq TS$.

Solution. Let $v_1, \ldots, v_n$ be the basis of V. Then define T as

$$T(v_1) = v_2$$

while every other vector in the basis maps to itself and define S as

$$S(v_2) = v_1$$

while every other vector in the basis maps to itself. Then we have

$$S(T(v_1)) = S(v_2) = v_1 \neq v_2 = T(v_1) = T(S(v_1)).$$

□

§3.2 Null Spaces and Ranges

§3.2.1 Null Spaces and Injectivity

Definition (Null Space)**.** For $T \in \mathcal{L}(V, W)$, the null space of T, denoted null T, is the subset of V consisting of those vectors that T maps to 0:

$$\text{null } T = \{v \in V : T(v) = 0\}.$$

Example (Null Spaces)

Pretty simple so far.

- If $T : V \to W$ is the zero map, then null $T = V$.
- $\varphi \in \mathcal{L}(\mathbb{C}^3, \mathbb{F})$ is defined by $\varphi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$. Then null $\varphi = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$. A basis of null φ is $(-2, 1, 0)$, $(-3, 0, 1)$.
- Let D be the differential map on real polynomials. The null space consists of functions who's derivative is 0, namely constants.
- Let T be the multiplication by x^2 on real polynomials. The null space consists of only the 0 polynomial.
- Null space of backward shift in $\mathbb{F}^\infty$ is $(a, 0, 0, \ldots)$.

Remark. Null space is also referred to as kernel.

Proposition (Null space is a subspace)

Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

Solution. We know that $T(0) = 0$ since T is a linear map. Suppose we have $u, v \in$ null T. Then

$$T(u + v) = T(u) + T(v) = 0$$

and

$$T(\lambda v) = \lambda T(v) = \lambda 0 = 0.$$

□

Definition (Injective)**.** A function $T : V \to W$ is injective if $T(u) = T(v)$ implies $u = v$.

Proposition (Injectivity condition)
The linear map $T : V \to W$ is injective if and only if null $T = \{0\}$.

Solution. Suppose T is injective, and suppose $v \in \text{null } T$. Then

$$T(v) = 0 = T(0),$$

which implies $v = 0$. Now suppose null $T = \{0\}$. Suppose $u, v \in V$ such that $T(u) = T(v)$. Then

$$0 = T(u) - T(v) = T(u - v),$$

which implies $u = v$. □

§3.2.2 Range and Surjectivity

Definition (Range)**.** For a function $T : V \to W$, the range of T is the subset of W consisting of those vectors that are of the form $T(v)$ for some $v \in V$:

$$\text{range } T = \{T(v) : v \in V\}.$$

Example (Range)
Pretty simple.

- If T is the zero map, then range $T = \{0\}$.
- $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ is defined by $T(x, y) = (2x, 5y, x + y)$. Then range $T = \{(2x, 5y, x + y) : x, y \in R\}$. A basis of range T is $(2, 0, 1)$, $(0, 5, 1)$.
- Let D be the differential map on real polynomials. Since every polynomial has an antiderivative, range $D = \mathcal{P}(\mathbb{R})$.

Proposition (Range is a subspace)
If $T \in \mathcal{L}(V, W)$, then the range T is a subspace of W.

Solution. We have $0 \in \text{range } T$. If $w_1, w_2 \in \text{range } T$, then there are $v_1, v_2 \in V$ such that $T(v_i) = w_i$. Thus

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2,$$

so range T is closed under addition. We can show similarly that it's closed under scalar multiplication. □

Definition (Surjective)**.** A function $T : V \to W$ is called surjective is its range equals W.

Example (Non-example)

The differential map $D : \mathcal{P}_5(\mathbb{R}) \to \mathcal{P}_5(\mathbb{R})$ is not surjective, since x^5 doesn't have a 5th degree or lower antiderivative. However, change the second vector space to $\mathcal{P}_4(\mathbb{R})$ and the map is then surjective.

§3.2.3 Fundamental Theorem of Linear Maps

Theorem (Fundamental Theorem of Linear Maps)

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Solution. Let $u_1, \ldots, u_m$ be a basis of null T. Then we can extend to a basis of V:

$$u_1, \ldots, u_m, v_1, \ldots, v_n.$$

Now we just need to show that a basis of range T is $Tv_1, \ldots, Tv_n$. Let $v \in V$. We can write

$$v = \sum^m a_i u_i + \sum^n b_i v_i.$$

Apply T to both sides to obtain

$$T(v) = \sum^n b_i T(v_i).$$

The u_i disappear since they're in the null space. This equation implies that $Tv_1, \ldots, Tv_n$ spans range T. Now we show it's linearly independent. Suppose

$$\sum^n c_i T(v_i) 0.$$

Then

$$T\left(\sum^n c_i v_i\right) = 0,$$

so the inside is in null T, which means

$$\sum^n c_i v_i = \sum^m d_i u_i.$$

Rearranging this yields the basis of V representing 0, which implies that $c_i = d_i = 0$, which is what we needed. Thus,

$$m + n = \dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = m + n.$$

□

Proposition

Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

Solution. Let $T \in \mathcal{L}(V,W)$. Then

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W \\ &> 0, \end{aligned}$$

so $\operatorname{null} T \neq \{0\}$, which implies it's not injective.

□

Proposition

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

Solution. Let $T \in \mathcal{L}(V,W)$. Then

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &\leq \dim V \\ &< \dim W, \end{aligned}$$

so T is not surjective.

□

Using the previous results, we now explore the solutions to systems of linear equations, homogenous and inhomogenous. First homogenous.

Fix positive integers m, n and let $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Consider the homogenous system of linear equations

$$\begin{gathered} \sum_{k=1}^{n} A_{1,k} x_k = 0 \\ \vdots \\ \sum_{k=1}^{n} A_{m,k} x_k = 0. \end{gathered}$$

Obviously the trivial solution works, so let's see if there are any others.

Define $T : \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k \right).$$

This is the same as asking when $T(v) = 0$ for $v \in \mathbb{F}^n$. Thus we want to know if $\operatorname{null} T$ is strictly bigger than $\{0\}$, which means T is not injective. Thus, we can conclude the following:

Proposition

A homogenous system of linear equations with more variables then equations has nonzero solutions.

Solution. The condition is the same as saying $n > m$. We know from an earlier result that if $\dim V > \dim W$ for a map $T : V \to W$, then T is not injective. Thus we have our desired conclusion. $\square$

Now inhomogenous. Again fix m, n and define the A's as before. For $c_1, \ldots, c_m \in \mathbb{F}$, consider the system of linear equations

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{m,k} x_k = c_m.$$

We want to know of there is some choice of c_i such that there are no solutions. Define $T : \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k \right).$$

The equation $T(v) = (c_1, \ldots, c_m)$ is the same as the system of equations for some $v \in \mathbb{F}^n$. Thus we want to know of range $T \neq \mathbb{F}^m$, which would mean T is not surjective. From here, we can easily conclude:

Proposition

An inhomogenous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Solution. The condition is the same as saying $n < m$. We know from an earlier results that of $\dim V < \dim W$ for a map $T : V \to W$, then T is not surjective. Thus we have our desired conclusion. $\square$

Problem (Exercise 1)**.** Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Solution. Consider the following map $T : \mathbb{R}^5 \to \mathbb{R}^2$ which is obviously linear:

$$T(a, b, c, d, e) = \{(a + b + c, d + e) \in \mathbb{R}^2\}.$$

Clearly the first vector space has dimension 5, and the second vector space has dimension 2. It's also to show that this definition for T has a range of $\mathbb{R}^2$, which means $\dim \operatorname{range} T = 2$, which implies that $\dim T = 5 - 2 = 3$. $\square$

Problem (Exercise 2)**.** Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\operatorname{range} S \subset \operatorname{null} T.$$

Prove that $(ST)^2 = 0$.

Solution. Note that $T(S(T(v))) = 0$ for any vector V, so we have $(ST)^2(v) = S(0) = 0$. $\square$

Problem (Exercise 3)**.** Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V?

(b) What property of T corresponds to $v_1, \ldots, v_m$ being linearly independent?

Solution. Condition spam.

(a) Since the v_i span V, the right side equals any vector in V, which means T is surjective.

(b) The only way the right side is 0 is if the $z_i = 0$, which implies that only $0 \in \mathbb{F}^m$ makes the output $0 \in V$. This implies that null $T = \{0\}$, which implies T is injective.

□

Problem (Exercise 4)**.** Show that

$$\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2\}$$

is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

Solution. Define $T : \mathbb{R}^5 \to \mathbb{R}^4$ as

$$T(a, b, c, d, e) = (a, b, 0, 0)$$

and $S : \mathbb{R}^5 \to \mathbb{R}^4$ as

$$T(a, b, c, d, e) = (0, 0, c, d).$$

Clearly both of these are linear maps. Note that the dimension of the range of both is 2, so the dimension of the null space is 3. Now note that

$$(T + S)(a, b, c, d, e) = (a, b, c, d).$$

The dimension of the range is 4, which means the dimension of the null space is 1, which means the set is not closed under addition, so it's not a subspace. □

Problem (Exercise 5)**.** Give an example of linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

Solution. Define $T : \mathbb{R}^4 \to \mathbb{R}^4$ as

$$T(w, x, y, z) = (y, z, 0, 0).$$

Note that this is a linear map. Note that

$$\operatorname{range} T = \{(y, z, 0, 0) \in \mathbb{R}^4\}.$$

This same subspace is also the null of T, so we have our desired map. □

Problem (Exercise 6)**.** Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

Solution. Suppose it did. Then $5 = \dim \mathbb{R}^5 = \dim \operatorname{null} T + \dim \operatorname{range} T = 2 \dim \operatorname{range} T$, which is impossible since the left is odd and the right is even. □

Problem (Exercise 7)**.** Suppose V and W are finite-dimensional with $2 \le \dim V \le \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution. Let $v_1, \ldots, v_m$ be a basis of V and $w_1, \ldots, w_n$ be a basis of W. Let the map T be defined as $T(v_i) = w_i$ on odd i and 0 otherwise and the map S be defined as $S(v_i) = w_i$ on even i and 0 otherwise. Clearly these maps are linear. Clearly $T + S$ is injective, so we just need to show T and S are not injective. However, this is clear from the definition of the maps. □

Problem (Exercise 8)**.** Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution. Define T and S to take $T(v_i) = 0$ for $i \in [n+1, m]$. Then, T takes odd i to w_i and the rest to 0, and S takes even i to w_i and the rest to 0. Clearly their sum is surjective. However, each only attains about half of a basis of W, so not all the vectors in W can be achieved. □

Problem (Exercise 9)**.** Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_m$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_m$ is linear independent in W.

Solution. We have

$$\sum a_i T(v_i) = 0 \implies T\left(\sum a_i v_i\right) = 0.$$

Since T is injective, null $T = \{0\}$, which implies that above $\sum a_i v_i = 0$, and since the v_i are linearly independent, this implies $a_i = 0$, as desired. □

Problem (Exercise 10)**.** Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $Tv_1, \ldots, Tv_n$ spans range T.

Solution. Pick $w \in \text{range } T$. We need

$$w = \sum a_i T(v_i) = T\left(\sum a_i v_i\right).$$

Let the vector $v \in V$ be the vector such that $T(v) = w$. Thus we have $v = \sum a_i v_i$, which has solutions since the list spans V. □

Problem (Exercise 11)**.** Suppose $S_1, \ldots s_n$ are injective linear maps such that $S_1 S_2 \cdots S_n$ makes sense. Prove that $S_1 S_2 \cdots S_n$ is injective.

Solution. Note that since S_i is injective, null $S_i = \{0\}$. This means that any nonzero vector will output a nonzero vector. Thus null $S_1 S_2 \cdots S_n = \{0\}$. □

Problem (Exercise 12)**.** Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \text{null } T = \{0\}$ and range $T = \{T(u) : u \in U\}$.

Solution. Let U be the subspace such that $V = \text{null } T \oplus U$. We know that since they form a direct sum, their intersection only contains 0. Note that for any $v \in V$, we can split it into a vector $u \in U$ and $n \in \text{null } T$ since they are a direct sum, which becomes $T(u + n) = T(u) + T(n) = T(u)$. So range T is the desired range. □

Problem (Exercise 13)**.** Suppose T is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\text{null } T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Solution. Note that $\dim \text{null } T = 2$, so $\dim \text{range } T = 2$. Since the output vector space has dimension 2, this implies that range $T = \mathbb{F}^2$, which implies T is surjective. □

Problem (Exercise 14)**.** Suppose U is a 3-dimensional subspace of $\mathbb{R}^8$ and that T is a linear map from $\mathbb{R}^8$ to $\mathbb{R}^5$ such that null $T = U$. Prove that T is surjective.

Solution. We have $\dim \mathbb{R}^8 = \dim \operatorname{null} T + \dim \operatorname{range} T \implies 8 - 3 = \dim \operatorname{range} T$. Since range T is a subspace of $\mathbb{R}^5$ and the range has the same dimension as $\mathbb{R}^5$, then we have range $T = \mathbb{R}^5$, which implies T is surjective. □

Problem (Exercise 15)**.** Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Solution. It's easy to see the subspace has dimension 2. Thus we have $\dim \mathbb{F}^5 = \dim \operatorname{null} T + \dim \operatorname{range} T \implies 5 - 2 = \dim \operatorname{range} T$. However, since range T is a subspace of $\mathbb{F}^2$, it has dimension at most 2, which is a contradiction. □

Problem (Exercise 16)**.** Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Solution. Let $v_1, \ldots, v_m$ be a basis of null T and $w_1, \ldots, w_n$ be a basis of range T. Note that there exist $u_i \in V$ such that $T(u_i) = w_i$. We now prove $v_1, \ldots, v_m, u_1, \ldots, u_n$ spans V. Note that for any $v \in V$, we have that for some $w \in \operatorname{range} T$

$$T(v) = w = \sum^n a_i w_i = \sum^n a_i T(u_i) = T\left(\sum^n a_i u_i\right).$$

Then we can move everything to one side to obtain

$$T\left(v - \sum^n a_i u_i\right) = 0,$$

which implies the argument is in null T, so

$$v = \sum^n a_i u_i + \sum^m b_i v_i,$$

as desired. In fact, from here we can prove the above list is a basis of V. Since V is finite dimensional, and since the length of our spanning list equals $\dim \operatorname{null} T + \dim \operatorname{range} T$, the list is the length of the dimension of V, so the list is a basis of V. □

Problem (Exercise 17)**.** Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Solution. We already have the if part as one of our results. Now suppose $\dim V \leq \dim W$. Let $w_1, \ldots, w_n$ be a basis in W and let $v_1, \ldots, v_m$ be a basis in V. We have $m \leq n$. Let T be the map such that $T(v_i) = w_i$ for $i \in [1, m]$. Note that $w_1, \ldots, w_m$ is linearly independent, so

$$\sum^m a_i w_i = 0$$

only when $a_i = 0$ This implies

$$\sum^m a_i T(v_i) = T\left(\sum^m a_i v_i\right) = 0,$$

which implies null $T = \{0\}$, which implies T is injective. □

Problem (Exercise 18)**.** Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim V \geq \dim W$.

Solution. We already have the if part as one of out results. Now suppose $\dim V \geq \dim W$. Let $w_1, \ldots, w_n$ be a basis in W and let $v_1, \ldots, v_m$ be a basis in $m \geq n$. Let $T(v_i) = w_i$ for $i \in [1, m]$ and map the rest of the vs to w_1. Note that we can represent any vector in W using the basis, so

$$\sum^n a_i w_i = w.$$

We can write this as

$$\sum^n a_i T(v_i) = T\left(\sum^n a_i v_i\right) = w,$$

so every $w \in W$ is in range T, so T is surjective.

□

Problem (Exercise 19)**.** Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution. Suppose such a T exists. Then we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \implies \dim V - \dim W \leq \dim V - \dim \operatorname{range} T = \dim U.$$

Now suppose $\dim U \geq \dim V - \dim W$. Rewriting, we have $\dim U + \dim W \geq \dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. Now pick any map such that for a basis of U, all the basis vectors go to 0. Then extend this basis to a basis of V and map the rest of the vectors to any nonzero vector in W. Then we have $\dim U = \dim \operatorname{null} T$, so the inequality becomes $\dim W \geq \dim \operatorname{range} T$, which is true. □

Problem (Exercise 20)**.** W is finite-dimensional and T is a linear map from V to W. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Solution. Suppose T is injective. Define $S' : \operatorname{range} T \to V$ such that $S'(T(v)) = v$. By injectivity, there's only one way to obtain $T(v)$ with a $v \in V$, so the map is well defined. It's easy to see the map is linear. Since range T is a subspace of W, we can extend S' to $S : W \to V$, and so we have our desired map.

Now suppose such a map S exists. Then $T(v) = T(u) \implies ST(v) = ST(u) \implies v = u$, so T is injective. □

Problem (Exercise 21)**.** V is finite-dimensional and T is a linear map from V to W. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

Solution. Note that W is finite-dimensional. Let w_i be a basis for W. Since T is surjective, $T(v_i) = w_i$ for some $v_i \in v$. Define $S(w_i) = v_i$. Then we have

$$T\left(S\left(\sum a_i w_i\right)\right) = T\left(\sum a_i v_i\right) = \sum a_i w_i,$$

so we have our desired map.

Now suppose such a map S exists. Then $T(S(w)) = w$, which implies range $T = W$. □

Problem (Exercise 22)**.** Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V,W)$ and $T \in \mathcal{L}(U,V)$. Prove that

$$\dim \operatorname{null} ST \le \dim \operatorname{null} S + \dim \operatorname{null} T.$$

Solution. Suppose the basis of null ST is $u_1, \dots, u_m$. Either $T(u_i) = 0$ or $S(T(u_i)) = 0$, where in the second option the inside is a nonzero vector in V. Let the first option correspond to list A and the second option correspond to list B. Extend A to A', a basis of U. Then, $|A| \le |A'| = \dim \operatorname{null} T$. Using the same idea, we can extend the image of B on T to a basis of V. Adding the inequalities gives the desired result. ☐

Problem (Exercise 23)**.** Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V,W)$ and $T \in \mathcal{L}(U,V)$. Prove that

$$\dim \operatorname{range} ST \le \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

Solution. It's clear that range ST is a subspace of range S, so $\dim \operatorname{range} ST \le \dim \operatorname{range} S$. Now suppose for the sake of contradiction that $\dim \operatorname{range} T < \dim \operatorname{range} ST$. We can rewrite ST as $S : \operatorname{range} T \to W$. Let range S be a subspace of W. Then $S : \operatorname{range} T \to \operatorname{range} S$ is surjective, which implies range $T >=$ range $S \ge$ range ST, a contradiction. Thus, we have our desired result. ☐

Problem (Exercise 24)**.** Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V,W)$. Prove that $T_1 \subset \operatorname{null} T_2$ if and only if there exists $S \in \mathcal{L}(V,V)$ such that $T_1 = T_2 S$.

Solution. Suppose there exists such an S, for any $v \in \operatorname{null} T_1$, we have

$$0 = \operatorname{null} ST_1 = \operatorname{null} T_2 = 0,$$

which implies null $T_1 \subset$ null T_2. Now suppose null $T_i \subset$ null T_2. Note that range T_1 is finite-dimensional. Let its basis be $T_1 v_1, \dots, T_m v_m$. This implies that $v_1, \dots, v_m$ are linearly independent. Let $K = \operatorname{span}(v_1, \dots, v_m)$. Thus, $V = K \oplus \operatorname{null} T_1$. Now we can write

$$v = \sum a_i T v_i + u$$

for $u \in \operatorname{null} T_1$. Now define S as $S(T_1(v_i)) = T_2(v_i)$. Since null $T_1 \subset$ null T_2, we have

$$T_2(v) = \sum a_i T_2(v_i) + T_2(u) = \sum a_i T_2(v_i).$$

We also have

$$S(T_1(v)) = \sum a_i S(T_1(v_i)) + S(T_1(u)) = \sum a_i T_2(v_i),$$

so we are done. ☐

§3.3 Matrices

Fun fun.

§3.3.1 Representing a Linear Map by a Matrix

Definition (Matrix)**.** An m-by$n$ matrix is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

$A_{j,k}$ denotes the entry in row j, column k of A.

Definition (Matrix of a linear map)**.** Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The matrix of T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries

$$T(v_k) = A_{1,k} w_1 + \cdots + A_{m,k} w_m$$

.

Example

Suppose $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ is defined by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y).$$

Then (using standard basis), we have $T(1,0) = (1,2,7)$ and $T(0,1) = (3,5,9)$, so

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}.$$

§3.3.2 Addition and Scalar Multiplication of Matrices

Definition (Matrix addition)**.** Just add component wise.

Proposition

S and T are linear maps from V to W. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$ (the bases are the same).

Solution. For basis vectors in V we have

$$T(v_i) = A_{i,1} w_1 + \cdots + A_{i,m} w_m$$

and

$$S(v_i) = B_{i,1} w_1 + \cdots + B_{i,m} w_m.$$

Then

$$(S+T)(v_i) = (A+B)_{i,1}w_1 + \cdots + (A+B)_{i,m}w_m.$$

This matches with the definition of matrix addition, so we're done.

□

Definition (Matrix scalar multiplication)**.** Scale entries by λ.

Proposition
$\mathcal{M}(\lambda T) = \lambda\mathcal{M}(T)$.

Solution. We have

$$\lambda T(v_i) = \lambda(A_{i,1}w_1 + \cdots + A_{i,m}w_m) = (\lambda A_{i,1})w_i.$$

This matches our definition of scalar multiplication, so we're done. □

Definition ($\mathbb{F}^{m,n}$)**.** Denotes all m by n matrices with entries in $\mathbb{F}$.

Proposition
$F^{m,n}$ is a vector space with dimension mn.

Solution. Clearly vector space, spanning list is clearly each matrix with a 1 entry and everything else 0. Also linearly independent since the only way to get 0 in one entry is to multiply corresponding matrix by 0. Number of such matrices is mn, done. □

§3.3.3 Matrix Multiplication

Basically forcing definition of matrix multiplication to have $\mathcal{M}(ST) = \mathcal{M}(S(T)) = \mathcal{M}(S)\mathcal{M}(T)$. We have $T : U \to V$, $S : V \to W$, $\dim V = n$, $\dim W = m$, $\dim U = p$.

Suppose $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = C$. For $1 \le k \le p$, we have

$$\begin{aligned}
(ST)u_k &= S\left(\sum_{r=1}^{n} C_{r,k}v_r\right) \\
&= \sum_{r=1}^{n} C_{r,k}S(v_r) \\
&= \sum_{r=1}^{n} C_{r,k}\sum_{j=1}^{m} A_{j,r}w_j \\
&= \sum_{j=1}^{m}\left(\left(\sum_{r=1}^{n} A_{j,r}C_{r,k}\right)w_j\right).
\end{aligned}$$

Now we can define it.

Definition (Matrix multiplication)**.** Suppose A is an m by n matrix and C is an n by p matrix. Then AC is defined to be the m by p matrix whose entry in row j, column k, is given by the following equation:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k}.$$

In other words, the entry in row j , column k, of AC is computed by taking row j of A and column k of C, multiplying together corresponding entries, and then summing.

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}.$$

Proposition

If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Solution. Our definition above forces this to work. □

Definition $(A_{j,\cdot}, A_{\cdot,k})$**.** Suppose A is an m by n matrix.

- If $1 \le j \le m$, then $A_{j,\cdot}$ denotes the 1 by n matrix consisting of row j of A.
- If $1 \le k \le n$, then $A_{\cdot,k}$ denotes the m by 1 matrix consisting of column k of A.

Example

If $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$, then $A_{2,\cdot}$ is row 2 of A and $A_{\cdot,2}$ is column 2 of A. In other words,

$$A_{2,\cdot} = \begin{pmatrix} 1 & 9 & 7 \end{pmatrix} \text{ and } A_{\cdot,2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.$$

Example

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = 2 \cdot 1 + 1 \cdot 2 = \begin{pmatrix} 4 \end{pmatrix} = 4.$$

Proposition (Way to think about matrix multiplication)

Row j times column k equals entry j,k.

A times column k equals column k in th product matrix.

If C is an n by 1 matrix and A is m by n, then the product is the sum of the scalar products of A with the entries of C.

Solution. Follows form definitions. □

Problem (Exercise 1)**.** V, W, linear map $T : V \to W$. Show that a matrix of T with respect to choices of bases has at least $\dim \operatorname{range} T$ nonzero entries.

Solution. Suppose $m = \dim \operatorname{range} T$. m vectors in the basis of W form range T. Since each w_i can be created within range T, at least one basis v_j in V must contain $w_i \in \operatorname{range} T$ in $T(v_j)$. ☐

§4 Polynomials

§5 Eigenvalues, Eigenvectors, and Invariant Subspaces

§6 Inner Product Spaces

§7 Operations on Inner Product Spaces

§8 Operators on Complex Vector Spaces

§9 Operators on Real Vector Spaces

§10 Trace and Determinant