M513 Equilibrium Statistical Mechanics and Kinetic Theory Ninnat Dangniam

Homework Assignment 1

DUE: Thursday 1 Dec 25 points

1. Monty Hall (10 points).

On the television show *Let's Make a Deal*, the host, Monty Hall, would show a contestant three doors. Behind one door was the grand prize, and behind the other two was nothing. The contestant chose one of the doors, say number 1, after which Monty would not open the chosen door yet. Instead, he opened one of the remaining doors, say number 3, revealing that there is nothing behind the door. The contestant was then offered the opportunity to switch to door number 2. Should he switch?

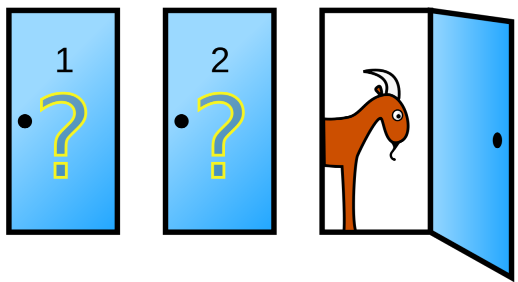

Figure 1: A popular rendition of the Monty Hall problem, with a car behind the correct door and goats behind the remaining doors. (Source: Wikipedia.)

Consider a generalization of the problem. There are N doors and only one door leads to the prize. The contestent guesses the door, Monty then opens n of the other doors, showing that none of them hides the prize, and the contestant is given the chance to switch to any of the $N - n - 1$ remaining doors.

(a) What is the probability $p(\checkmark)$ that the contestant's initial guess is right, and the probability $p(\text{✗})$ that the initial guess is wrong?

After Monty's revelation, Let G denotes the door of the contestant's initial **guess**, and let R denotes any one of the **remaining** doors.

(b) Given that the initial guess is right, what is the conditional probability $p(R|\checkmark)$ that the prize lies behind door R and the conditional probability $p(G|\checkmark)$, that the prize lies behind door G? What are the corresponding conditional probabilities, $p(R|\text{✗})$ and $p(G|\text{✗})$, given the that the initial guess is wrong?

(c) Since the contestant doesn't know whether his initial guess is right or wrong, what is relevant to his decision are the unconditioned probabilities $p(R)$ and $p(G)$. What are $p(R)$ and $p(G)$? For what values of n should the contestant change his initial guess to one of the remaining doors?

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2. Fourier transform (5 points).

Compute the characteristic function

$$\widetilde{\varphi}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \tag{1}$$

of a Gaussian distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right). \tag{2}$$

You may use the Gaussian integral $\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\pi/a}$. (Do you remember how to derive such an identity?)

Consider the Fourier reciprocity property; the broader the function $f(x)$ is in x, the narrower the Fourier transform $\widetilde{\varphi}(k)$ is in k. Is this true for this case of a Gaussian?

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3. Stirling's approximation (10 points).

(a) Show using integration by parts that

$$\int_{0}^{\infty} dx\, e^{-x} x^N \tag{3}$$

is an integral representation of the factorial $N!$.

(b) When N is large, the random variable $Y = (X - N)/\sqrt{N}$ looks like a Gaussian distribution with mean 0 and a unit variance. This motivates us to make the substitution $y = (x - N)/\sqrt{N}$. Show that the above expression can be rewritten as

$$N! = \left(\frac{N}{e}\right)^N \sqrt{N} \int_{-\sqrt{N}}^{\infty} dy\, e^{-y\sqrt{N}} \left(1 + \frac{y}{\sqrt{N}}\right)^N. \tag{4}$$

(c) Check that the integrand peaks at $y = 0$. Perform a saddle point approximation, that is, take the natural log of the integrand and expand the log around $y = 0$. By keeping only up to the quadratic term in the expansion, at the end you should get that $N!$ is proportional to the integral

$$\int_{-\sqrt{N}}^{\infty} dy\, e^{-y^2/2}. \tag{5}$$

Since the integrand far from $y = 0$ is negligibly small, we can extend the lower limit to $-\infty$ and perform the integration. Show that this gives us

$$N! \sim \sqrt{2\pi N} \left(\frac{N}{e}\right)^N. \tag{6}$$

(d) Check the accuracy of this asymptotic expression for $N = 5, 10, 20, 50, 100$. Also compare $\ln N!$ with $\ln[\sqrt{2\pi N}(N/e)^N]$ and convince yourself that the factor $\sqrt{2\pi N}$ can be neglected for large N, leaving $\ln N! \sim N \ln N - N$.
